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Research article

Blow up behavior of minimizers for a fractional p-Laplace problem with external potentials and mass critical nonlinearity

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Abstract: In this paper, we studied the fractional p-Laplace problem with a general potential and a mass-critical nonlinearity. By using constrained variational methods, we established the existence and nonexistence of minimizers for this problem. Moreover, we investigated the blow-up behaviour of non-negative minimizers to the above equation. Finally, under a polynomial-type potential, we obtained the optimal rate of blow-up through some subtle energy estimates.

Keywords: fractional p-Laplacian; mass critical; minimizers; blow up
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1. Introduction

In this paper, we consider the following equation involving the fractional p-Laplacian:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u + b|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

subject to the mass constraint

$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x = 1,$$

where $N > ps$, $s \in (0,1)$, $p > 1$, $q = p + \frac{sp^2}{N}$, $\lambda \in \mathbb{R} \setminus \{0\}$, $b > 0$ are parameters, and $V(x) \in C(\mathbb{R}^N)$. The operator $(-\Delta)_p^s$ is the fractional p-Laplacian, defined by

$$(-\Delta)_p^s u(x) = C_{N,s,p} P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \mathrm{d}y,$$

if u is smooth enough, where $C_{N,s,p}$ is a normalization constant, $P.V.$ denotes the Cauchy principle value see [3].

If $p = N = 2$, and $s = 1$, (1.1) reduces to the Gross–Pitaevskii (GP) equation, which is independent of time. Gross [10] and Pitaevskii [23] independently introduced this equation during the study of Bose-Einstein condensates. There are many studies on Eq (1.1) with various potential $V(x)$, for example, [11, 22, 27]. Guo and Seiringer [11] proved that if $V(x)$ satisfies the following condition:

$$(V_1') \;\; 0 \leqslant V(x) \in L^{\infty}_{loc}(\mathbb{R}^N), \;\; \lim_{|x|\to\infty} V(x) = \infty \;\text{ and }\; \inf_{x\in\mathbb{R}^N} V(x) = 0,$$

there is a normalized solution of (1.1) for all $b < b^*$, where $b^* > 0$ is a critical value. Furthermore, they showed that if

$$V(x) = k(x) \prod_{i=1}^{m} |x - x_i|^{r_i}, \tag{1.2}$$

$x_i \neq x_j$ if $i \neq j, 0 < C \leqslant k(x) \leqslant \frac{1}{C}$ for all $x \in \mathbb{R}^N$, the solutions blow up at some point x_{i_1} with $q_{i_1} = \max\{q_1, \cdots, q_m\}$. Subsequently, Wang and Zhao [27] extended this result to periodic potentials. Regarding the related work on singular potential, please refer to [13, 22].

When $p = 2$, $(-\Delta)^s_p$ reduces to $(-\Delta)^s$, which is the linear fractional Laplace operator. We remark that in recent years, the research on nonlinear problems involving fractional and non-local operators has become extremely popular. The fractional Laplace operators play a fundamental role in describing various phenomena such as physics, biology, finance, phase transitions, game theory, image processing, Lévy processes, and optimization, see [14, 25, 29] for more backgrounds and related results. Du et al. [8] considered the existence, nonexistence, and mass concentration of L^2-normalized solutions for nonlinear fractional Schrödinger equation with trapping potential:

$$(-\Delta)^s u + V(x)u = \mu u + af(u), \quad x \in \mathbb{R}^N,$$

where $N \geqslant 2$, $0 < s < 1$. Moreover, under an appropriate condition of V, they conducted a detailed analysis of the blow-up behavior of the minimizers in the mass critical case. Liu et al. [15] showed the existence and concentration behavior of L^2 constrainted minimizers of the mass-critical fractional Schrödinger energy functional with a ring-shaped potential. A similar problem for the mass subcritical case has been studied in [17, 28].

Due to the nonlinearity of the operator and its non-local properties, the fractional p-Laplace problem has received more and more attention in recent years. Many authors have established several existence and regularity results. For example, Castro, Kuusi, and Palatucci [5] obtained the interior Hölder regularity results for fractional p-minimizers. By using barrier arguments, Iannizzotto, Mosconi, and Squassina [12] proved C^{α}-regularity up to the boundary for the weak solutions of a Dirichlet problem driven by the fractional p-Laplace operator. Pezzo and Quaas [6] showed the existence and nonexistence of an unbounded sequence of eigenvalues for the fractional p-Laplacian with weight in $\mathbb{R}^N$ and extended the decay result to the positive solutions of a Schrödinger type equation. Lou, Qin, and Liu [18] proved the existence and non-existence of the solutions with prescribed L^p-norm for a fractional p-Laplacian equation

$$(-\Delta)^s_p u - |u|^{r-2}u = \lambda |u|^{p-2}u, \quad x \in \mathbb{R}^N, \quad \lambda \in \mathbb{R},$$

where $N > ps$ with $s \in (0, 1)$, $1 < p < r < p^*_s := pN/(N - ps)$.

However, the results regarding the existence and blow-up behavior of normalized solutions to the problem (1.1), the literature seems quite incomplete. The purpose of this article is to consider the existence and blow-up behavior of non-negative solutions to the nonlinear Schrödinger equation involving fractional p-Laplacian and external potential. To be precise, we intend to extend the results obtained in [8, 16], in which authors respectively investigated a fractional Kirchhoff equation and a fractional Schrödinger equation. Due to the fact that the operator $(-\Delta)_p^s$ is not linear when $p \neq 2$, there will be more technical difficulties in studying our problem. In addition, some useful techniques developed for studying fractional Laplace problems are not suitable for solving problems such as (1.1). In fact, we cannot utilize the powerful framework provided by Caffarelli and Silvestre harmonic extensions [4], nor can we utilize various tools such as commutator estimation, strong barrier, and density estimation [19, 20].

The rest of this paper is organized as follows. Section 2 is dedicated to some preliminary notations and lemmas, and state the main results of this paper. In Section 3, we derive the existence and nonexistence of minimizers for problem (1.1) and give the proof of Theorems 2.5. Section 4 is devoted to proving the concentration results of problems (1.1), i.e., Theorems 2.6 and 2.7.

2. Preliminaries and main results

In this section, we recall some results on the fractional Sobolev spaces and provide some lemmas that will be frequently used in the rest of this article.

Define $D^{s,p}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with

$$\|u\|^p_{D^{s,p}(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(u)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.$$

The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows:

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(u)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y < \infty \right\},$$

endowed with the norm

$$\|u\|^p_{W^{s,p}(\mathbb{R}^N)} = \|u\|^p_{D^{s,p}(\mathbb{R}^N)} + \|u\|^p_{L^p(\mathbb{R}^N)}.$$

To accurately state our results, we first introduced the main assumptions about potential $V(x)$:
(V_1) $V(x) \in C(\mathbb{R}^N)$, $\lim\limits_{|x|\to\infty} V(x) = \infty$, and $\inf_{x\in\mathbb{R}^N} V(x) = 0$.

We consider the following constraint minimization problem:

$$m(b) := \inf_{u\in\mathcal{S}_1} J_b(u), \tag{2.1}$$

where

$$J_b(u) := \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x - \frac{b}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x,$$

and

$$\mathcal{S}_1 = \left\{ u \in X, \int_{\mathbb{R}^N} |u|^p dx = 1 \right\}$$

with

$$X=\left\{u\in W^{s,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x<+\infty\right\}$$

equipped with the norm

$$\|u\|_X:=\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}.$$

For any $u\in\mathcal{S}_1$ fixed, it is easy to see that $u_t(x)=t^{\frac{N}{p}}u(tx)\in\mathcal{S}_1$, but

$$\begin{aligned}J_b(u_t)=&\frac{t^{ps}}{p}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y+\frac{1}{p}\int_{\mathbb{R}^N}V\left(\frac{x}{t}\right)|u|^p\mathrm{d}x-\frac{bt^{\frac{Nq}{p}-N}}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x\\ &\to-\infty,\end{aligned}$$

as $t\to\infty$, if $q>p+\frac{p^2s}{N}$.

We begin to recall the following embeddings of the fractional Sobolev spaces.

Lemma 2.1. [7] *Let $N>ps$, $s\in(0,1)$, $p>1$. Then there exists a sharp constant $S_*>0$ such that for any $u\in D^{s,p}(\mathbb{R}^N)$*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)}\leqslant S_*^{-1}\|u\|_{D^{s,p}(\mathbb{R}^N)},$$

where $p_s^*=\frac{pN}{N-ps}$.

Lemma 2.2. [2] *Assume that $V\in L^\infty_{loc}(\mathbb{R}^N)$ with $\lim_{|x|\to\infty}V(x)=\infty$. Then for all $p\leqslant r<p_s^*$, the embedding $X\hookrightarrow L^r(\mathbb{R}^N)$ is compact.*

According to [9, 24], we can obtain the following vanishing lemma for fractional Sobolev space.

Lemma 2.3. [2] *Let $N>ps$. If $\{u_n\}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_R(z)}|u_n|^p\mathrm{d}x=0,$$

where $R>0$, then $u_n\to0$ in $L^r(\mathbb{R}^N)$, for all $r\in(p,p_s^*)$.

Next, let us introduce the following fractional Gagliardo–Nirenberg–Sobolev inequality in [21].

Lemma 2.4. [21] *For $u\in W^{s,p}(\mathbb{R}^N)$, $N>ps$, $s\in(0,1)$, $p>1$, $q=p+\frac{p^2s}{N}$ and b^* given in (2.11), there holds*

$$\int_{\mathbb{R}^N}|u|^q\mathrm{d}x\leqslant\frac{N+ps}{Nb^*}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\cdot\left(\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\right)^{\frac{ps}{N}}. \tag{2.2}$$

Moreover, the equality is attained at $u(x)=\tau_1Q(\tau_2x)$ with some $\tau_1,\tau_2\in\mathbb{R}\setminus\{0\}$ and $Q\in\mathcal{M}$.

Before presenting the main results of this article, we recall some results in [3, 18]. For $q=p+\frac{p^2s}{N}$, up to translations, the fractional p-Laplacian equation

$$(-\Delta)_p^su+\frac{ps}{N}|u|^{p-2}u=|u|^{q-2}u,\quad x\in\mathbb{R}^N, \tag{2.3}$$

where $N > ps$, $0 < s < 1$, has a unique ground state solution (see [18]). Motivated by [18], we consider the following energy functional:

$$E(u) = \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \frac{s}{N}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x - \frac{1}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x. \tag{2.4}$$

It is easy to conclude that u is a weak solution of (2.3) if and only if u is a critical point of E, that is,

$$\langle E'(u), \phi\rangle = 0, \quad \text{for all } \phi \in W^{s,p}(\mathbb{R}^N).$$

We denote by $\mathcal{N}$ the set of all nontrivial weak solutions to (2.3), that is,

$$\mathcal{N} := \left\{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} : \langle E'(u), \phi\rangle = 0, \quad \text{for all } \phi \in W^{s,p}(\mathbb{R}^N)\right\}.$$

For $u \in \mathcal{N}$, taking $\phi = u$, we derive that

$$\langle E'(u), u\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \frac{ps}{N}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x - \int_{\mathbb{R}^N}|u|^q\mathrm{d}x = 0. \tag{2.5}$$

From [1], we obtain the following Pohozaev identity:

$$\frac{N-ps}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + s\int_{\mathbb{R}^N}|u|^p\mathrm{d}x - \frac{N}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x = 0. \tag{2.6}$$

Combining (2.5) and (2.6), we deduce that

$$\int_{\mathbb{R}^N}|u|^q\mathrm{d}x = \left(1+\frac{ps}{N}\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = \left(1+\frac{ps}{N}\right)\int_{\mathbb{R}^N}|u|^p\mathrm{d}x. \tag{2.7}$$

Recall that $Q \in W^{s,p}(\mathbb{R}^N)$ is a ground state solution to (2.3) if it is the least energy solution among all nontrivial solutions to (2.3). Thereby, $u \in \mathcal{N}$; from (2.7), we derive that

$$\begin{aligned} Q \in \mathcal{M} :&= \left\{u \in \mathcal{N} : E(u) = \inf_{v\in\mathcal{N}} E(v)\right\} \\ &= \left\{u \in \mathcal{N} : E(u) = \inf_{v\in\mathcal{N}} \frac{s}{N}\int_{\mathbb{R}^N}|v|^p\mathrm{d}x\right\}. \end{aligned} \tag{2.8}$$

Moreover, similar to [3], it is easy to check that there exists $C > 0$ such that

$$|Q(x)| \leqslant \frac{C}{1+|x|^{N+ps}}, \quad \text{for } x \in \mathbb{R}^N, \tag{2.9}$$

and

$$|\partial_{x_j}Q(x)| \leqslant \frac{C}{1+|x|^{N+ps}}, \quad \text{for } x \in \mathbb{R}^N, j = 1, 2\cdots, N. \tag{2.10}$$

Let

$$b^* = \left(\int_{\mathbb{R}^N}|Q|^p\,\mathrm{d}x\right)^{\frac{ps}{N}}. \tag{2.11}$$

We obtain the following results.

Theorem 2.5. *Let $N > ps$, $s \in (0,1)$, $p > 1$, $q = p + \frac{sp^2}{N}$, and suppose that $V(x)$ satisfies the condition (V_1) and let $Q \in \mathcal{M}$. Then,*

(i) Problem (2.1) *admits at least one minimizer if $b \in [0, b^*)$;*

(ii) Problem (2.1) *admits no minimizer if $b \in [b^*, +\infty)$ and $m(b) = -\infty$ if $b \in (b^*, +\infty)$.*

Moreover, $m(b) > 0$ if $b \in (-\infty, b^*)$ and $\lim_{b \nearrow b^*} m(b) = m(b^*) = 0$.

Remark 2.1. *Theorem 2.5 gives a full classification on the existence and nonexistence of minimizers of problem* (2.1)*. We point out that the threshold b^* defined in* (2.11) *is unconcerned with the choice of $Q \in \mathcal{M}$. Indeed, let e_0 be the least energy of* (2.4)*, i.e., $E(Q) = e_0$, together with* (2.8)*, we derive that $\int_{\mathbb{R}^N} |Q|^p \mathrm{d}x = Ne_0$, which yields that b^* is independent of $Q \in \mathcal{M}$.*

The next result concerns the behavior of the minimizer u_b of (2.1) when b gets close to b^* from below.

Theorem 2.6. *Let $N > ps$, $s \in (0,1)$, $p > 1$, $q = p + \frac{sp^2}{N}$, and suppose that $V(x)$ satisfies the condition (V_1), and let $u_b \geqslant 0$ be a minimizer of problem* (2.1)*. Then, for any sequence of $\{b_n\}$ with $b_n \nearrow b^*$, as $n \to \infty$,*

(i)

$$\varepsilon_n := \left(\iint_{\mathbb{R}^{2N}} \frac{|u_{b_n}(x) - u_{b_n}(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right)^{-\frac{1}{ps}} \to 0; \tag{2.12}$$

(ii) Let $\tilde{y}_{b_n}$ be a global maximum point of $u_{b_n}(x)$; there holds

$$\lim_{b \nearrow b^*} \mathrm{dist}(\tilde{y}_{b_n}, \mathcal{V}) = 0,$$

where $\mathcal{V} := \left\{ x \in \mathbb{R}^N : V(x) = 0 \right\}$;

(iii) There exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$ such that

$$\lim_{n \to \infty} \varepsilon_n^{\frac{N}{p}} u_{b_n}(\varepsilon_n x + \tilde{y}_{b_n}) = \frac{Q(x)}{(b^*)^{N/(p^2 s)}}, \ \ \text{in } W^{s,p}(\mathbb{R}^N),$$

where $\tilde{y}_{b_n}$ is a global maximum point of u_{b_n} and $\lim_{n\to\infty} \tilde{y}_{b_n} = x_0 \in \mathcal{V}$.

In what follows, we shall suppose that the trapping potential V is a polynomial-type trapping potential satisfying (1.2). Let $Q \in \mathcal{M}$ and $z_0 \in \mathbb{R}^N$ be such that

$$\int_{\mathbb{R}^N} |x + z_0|^r |Q(x)|^p \mathrm{d}x = \inf_{z \in \mathbb{R}^N} \int_{\mathbb{R}^N} |x+z|^r |Q(x)|^p \mathrm{d}x, \ r = \max\{r_1, \cdots, r_m\}. \tag{2.13}$$

Define

$$\sigma_i = \int_{\mathbb{R}^N} |x + z_0|^r |Q(x)|^p \mathrm{d}x \cdot \lim_{x \to x_i} \frac{V(x)}{|x - x_i|^{r_i}} \in (0, \infty], \tag{2.14}$$

and

$$\sigma = \min\{\sigma_1, \cdots, \sigma_m\}, \ \ B := \{x_i : \sigma_i = \sigma\}. \tag{2.15}$$

Theorem 2.7. *Let $N > ps$, $s \in (0,1)$, $p > 1$, $q = p + \frac{sp^2}{N}$, and assume that $V(x)$ satisfies* (1.2)*,* (2.13)–(2.15)*, and let b_n be the convergent subsequence in Theorem 2.6. Then,* *(i) For $m(b_n)$ defined in* (2.1)*, it holds*

$$m(b_n) \approx \frac{(b^* - b_n)^{\frac{r}{ps+r}}}{p(b^*)^{\frac{N+r}{ps+r}}} \sigma^{\frac{ps}{ps+r}} \left(\left(\frac{r}{p}\right)^{\frac{ps}{ps+r}} + \left(\frac{p}{r}\right)^{\frac{r}{ps+r}} \right),$$

as $k \to \infty$, where $f(b_n) \approx g(b_n)$ indicates that $f/g \to 1$ as $n \to \infty$;

(ii) Let ε_n be given in (2.12)*; then*

$$\varepsilon_n \approx \delta_n := (b^*)^{\frac{N-ps}{ps(ps+r)}} (b^* - b_n)^{\frac{1}{ps+r}} \sigma^{-\frac{1}{ps+r}} \left(\frac{p}{r}\right)^{\frac{1}{ps+r}}, \tag{2.16}$$

and

$$\lim_{n\to\infty} \delta_n^{\frac{N}{p}} u_{b_n}(\delta_n x + \tilde{y}_{b_n}) = \frac{Q(x)}{(b^*)^{\frac{N}{sp^2}}} \text{ in } W^{s,p}(\mathbb{R}^N).$$

Moreover,

$$\lim_{n\to\infty} \tilde{y}_{b_n} = x_0 \in \mathcal{V},$$

where $\tilde{y}_{b_n}$ is a global maximum point of u_{b_n}.

Remark 2.2. *It is necessary to point out several difficulties encountered in Theorems 2.6 and 2.7:*

(i) In contrast to the case $p = 2$ in [8], the nonlinearity of the nonlocal operator renders some of the ideas presented in [8] inapplicable to our problem.

(ii) Our results hinge on precise energy estimations. Nevertheless, the fractional Laplace operator has distinct properties compared to the Laplace operator. As a consequence, the trial functions formulated in [25, 26, 29] are not suitable for our study. Therefore, we are compelled to reconstruct suitable trial functions to analyze the energy situation (refer to Lemmas 4.2 and 5.1).

Throughout the paper, we use the following notations:

- $L^q(\mathbb{R}^N)$ denotes the Lebesgue space with the norm

$$\|u\|_{L^q(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^q dx\right)^{1/q}.$$

- For any $x \in \mathbb{R}^N$ and $R > 0$, $B_R(x) := \{y \in \mathbb{R}^N : |y - x| < R\}$.
- C indicates positive numbers that may be different in different lines.

3. Existence and nonexistence of minimizers

In this section, we investigate the existence and non-existence of the minimizers of (2.1) and provide the proof for Theorems 2.5.

Lemma 3.1. *Let $0 \leqslant v_0 \in W^{s,p}(\mathbb{R}^N)$ satisfy the equation*

$$(-\Delta)_p^s v_0 + \frac{ps}{N} v_0^{p-1} = b^* v_0^{q-1}, \tag{3.1}$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} |v_0|^p \mathrm{d}x = 1. \tag{3.2}$$

Then, v_0 satisfies the equality of (2.2) *and $v_0 = (b^*)^{-\frac{N}{sp^2}} Q(x)$ for some $Q \in \mathcal{M}$.*

Proof. It follows from (3.1) and (3.2) that

$$\int_{\mathbb{R}^N} |v_0|^q \mathrm{d}x = \frac{N+ps}{Nb^*},$$

which implies that v_0 satisfies the equality of (2.2). By Lemma 2.4, there exist $\tau_1, \tau_2 > 0$ such that

$$v_0 = \tau_1 Q(\tau_2 x),$$

for some $Q \in \mathcal{M}$. According to (3.1), (3.2), (2.7), and (2.11), we deduce that $\tau_1 = (b^*)^{-\frac{N}{sp^2}}$ and $\tau_2 = 1$, which completes the proof. □

Lemma 3.2. *If u_b is a minimizer of problem* (2.1)*, then u_b is a ground state solution of* (1.1) *for some $\lambda = \lambda_b$.*

Proof. Let u_b be a minimizer of problem (2.1); by applying Lagrange's multipliers rule, there exists a Lagrange multiplier λ_b such that

$$(-\Delta)_p^s u_b + V(x)|u_b|^{p-2}u_b = \lambda_b |u_b|^{p-2}u_b + b|u_b|^{q-2}u_b. \tag{3.3}$$

Set

$$I_b(u) = \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \frac{1}{p}\int_{\mathbb{R}^N} [V(x)-\lambda_b]|u|^p \mathrm{d}x - \frac{b}{q}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x. \tag{3.4}$$

Now, we need to show that $I_b(u_b) \leqslant I_b(v)$ for all nontrivial weak solutions v of (3.3). From (3.3), we derive that

$$\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} [V(x)-\lambda_b]|v|^p \mathrm{d}x = b\int_{\mathbb{R}^N} |v|^q \mathrm{d}x. \tag{3.5}$$

Together with (3.4), we conclude that

$$I_b(v) = \left(\frac{1}{p} - \frac{1}{q}\right) b \int_{\mathbb{R}^N} |v|^q \mathrm{d}x = \frac{bs}{N+ps}\int_{\mathbb{R}^N} |v|^q \mathrm{d}x.$$

Let $a = \|v\|_{L^p(\mathbb{R}^N)}$ and $\tilde{v} = \frac{v(x)}{a}$, then $\|\tilde{v}\|_{L^p(\mathbb{R}^N)} = 1$. Since u_b is a minimizer of (2.1), we obtain that

$$J_b(\tilde{v}) \geqslant J_b(u_b),$$

which indicates that

$$I_b(\tilde{v}) = J_b(\tilde{v}) - \frac{\lambda_b}{p}\int_{\mathbb{R}^N} |\tilde{v}|^p \mathrm{d}x \geqslant J_b(u_b) - \frac{\lambda_b}{p}\int_{\mathbb{R}^N} |u_b|^p \mathrm{d}x = I_b(u_b). \tag{3.6}$$

Meanwhile, from (3.5), we deduce that

$$I_b(\tilde{v}) = \left(\frac{1}{pa^p} - \frac{1}{qa^q}\right) b \int_{\mathbb{R}^N} |v|^q \mathrm{d}x \leqslant \frac{bs}{N+ps}\int_{\mathbb{R}^N} |v|^q \mathrm{d}x = I_b(v).$$

This, together with (3.6), implies that $I_b(v) \geqslant I_b(u_b)$. □

The following results will be very important in this work. Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\phi = 1$ on $B_1(0)$, $\phi = 0$ on $\mathbb{R}^N \backslash B_2(0)$, $0 \le \phi \le 1$, and $|\nabla \phi| \le C$. Define

$$Q_t(x) = \phi(x/t)Q(x) \tag{3.7}$$

for any $t > 0$, where $Q(x)$ is a ground state solution of (2.3).

Lemma 3.3. *Let $N > ps$, $s \in (0,1)$, $p > 1$. Then*

$$\iint_{\mathbb{R}^{2N}} \frac{|Q_t(x) - Q_t(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \leqslant \iint_{\mathbb{R}^{2N}} \frac{|Q(x) - Q(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + O(t^\kappa),$$

where $\kappa = \min\{2ps, (p-2)N + sp^2\}$, as $t \to \infty$.

Proof. First, we claim that the following two cases hold true:
Case 1. For all $x \in \mathbb{R}^N$ and $y \in B_t^c(0)$, with $|x - y| \leqslant t/2$,

$$|Q_t(x) - Q_t(y)| \leqslant Ct^{-N-ps}|x-y|. \tag{3.8}$$

Case 2. For all $x, y \in B_t^c(0)$,

$$|Q_t(x) - Q_t(y)| \leqslant Ct^{-N-ps} \min\{1, |x-y|\}, \tag{3.9}$$

for all $t \geqslant 2$ and some $C > 0$.

Let us prove Case 1. For all $x \in \mathbb{R}^N$ and $y \in B_t^c(0)$, with $|x-y| \leqslant t/2$, let

$$\eta = \tau x + (1-\tau)y$$

for some $\tau \in [0,1]$. Then,

$$|\eta| = |y + \tau(x-y)| \geqslant |y| - \tau|x-y| \geqslant t - \tau\frac{t}{2} \geqslant \frac{t}{2},$$

which, together with (2.10), implies that $|\nabla Q(\eta)| \leqslant Ct^{-N-ps}$. Thus, one can see that

$$|Q_t(x) - Q_t(y)| \leqslant Ct^{-N-ps}|x-y|,$$

which yields (3.8). Next, we show Case 2. For all $x, y \in B_t^c(0)$, if $|x-y| \leqslant 1$, then Case 2 is derived from Case 1, since $t \geqslant 2$. So, we should assume that $|x - y| > 1$. From (2.9), we infer that

$$|Q_t(x) - Q_t(y)| \leqslant |Q(x)| + |Q(y)| \leqslant Ct^{-N-ps},$$

which yields (3.9).

Set

$$D_1 := \left\{(x,y) \in \mathbb{R}^{2N} : x \in B_t(0), y \in B_t^c(0), |x-y| > \frac{t}{2}\right\},$$

$$D_2 := \left\{(x,y) \in \mathbb{R}^{2N} : x \in B_t(0), y \in B_t^c(0), |x-y| \leqslant \frac{t}{2}\right\}.$$

According to (3.7), we derive that

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y &= \iint_{B_t(0)\times B_t(0)} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &+ 2\iint_{D_1} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + 2\iint_{D_2} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &+ \iint_{B_t^c(0)\times B_t^c(0)} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.10}$$

From (3.7) and (3.9), we deduce that

$$\begin{aligned}&\iint_{B_t^c(0)\times B_t^c(0)} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ \leqslant & Ct^{-pN-p^2s}\iint_{B_{2t}(0)\times\mathbb{R}^N} \frac{\min\{1,|x-y|^p\}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = O\left(t^{-(p-1)N-sp^2}\right).\end{aligned} \tag{3.11}$$

By (3.8), setting $\zeta = x-y$, we see that

$$\begin{aligned}&\iint_{D_2} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ \leqslant & Ct^{-pN-p^2s}\iint_{\{x\in B_t(0), y\in B_t^c(0), |x-y|\leqslant \frac{t}{2}\}} \frac{|x-y|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ \leqslant & Ct^{-pN-p^2s}\int_{\{|x|\leqslant t\}}\mathrm{d}x\int_{\{|\zeta|\leqslant t/2\}}\frac{1}{|\zeta|^{N+ps-p}}d\zeta = O\left(t^{-(p-1)N-p^2s-ps+p}\right).\end{aligned} \tag{3.12}$$

Recalling that $Q_t(x) = Q(x)$ for all $x\in B_t(0)$, we obtain that for all $(x,y)\in D_1$,

$$\begin{aligned}&|Q_t(x)-Q_t(y)|^p = |Q(x)-Q(y)+Q(y)-Q_t(y)|^p \\ \leqslant & |Q(x)-Q(y)|^p + |Q(y)-Q_t(y)|^p + C_p|Q(x)-Q(y)|^{p-1}|Q(y)-Q_t(y)| \\ &+ C_p|Q(x)-Q(y)||Q(y)-Q_t(y)|^{p-1},\end{aligned}$$

by using the inequality $(\alpha+\beta)^p \leqslant \alpha^p+\beta^p+C_p\alpha^{p-1}\beta+C_p\alpha\beta^{p-1}$, where $\alpha,\beta>0$, $p>1$ and $C_p = C(p) > 0$.

By a direct computation, it follows that

$$\begin{aligned}&\iint_{D_1} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ \leqslant & \iint_{D_1} \frac{|Q(x)-Q(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + \iint_{D_1} \frac{|Q(y)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &+ C_p\iint_{D_1} \frac{|Q(x)-Q(y)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \\ &+ C_p\iint_{D_1} \frac{|Q(x)-Q(y)|^{p-1}|Q(y)-Q_t(y)|}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.13}$$

•Estimate of $\iint_{D_1}\frac{|Q(y)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y$.

From (2.9), it follows that

$$\begin{aligned}&\iint_{D_1}\frac{|Q(y)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\leqslant 2^p\iint_{D_1}\frac{|Q(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ \leqslant&Ct^{-pN-sp^2}\iint_{\{x\in B_t(0),y\in B_t^c(0),|x-y|>\frac{t}{2}\}}\frac{1}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ \leqslant&Ct^{-pN-sp^2}\int_{\{|x|\leqslant t\}}\mathrm{d}x\int_{\{|\zeta|>t/2\}}\frac{1}{|\zeta|^{N+ps}}d\zeta\\ =&O(t^{-pN-sp^2-ps+N}),\end{aligned}\tag{3.14}$$

as $t\to\infty$.

•Estimate of $\iint_{D_1}\frac{|Q(x)-Q(y)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y$.

Note that, by (2.9), for all $(x,y)\in D_1$,

$$|Q(x)||Q(y)|^{p-1}\leqslant Ct^{-(p-1)N-p(p-1)s}.$$

Therefore,

$$\begin{aligned}&\iint_{D_1}\frac{|Q(x)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ \leqslant&2^{p-1}\iint_{D_1}\frac{|Q(x)||Q(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ \leqslant&Ct^{-(p-1)N-p(p-1)s}\iint_{D_1}\frac{1}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y=O(t^{-(p-2)N-sp^2}).\end{aligned}\tag{3.15}$$

Similarly,

$$\begin{aligned}\iint_{D_1}\frac{|Q(y)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\leqslant&2^{p-1}\iint_{D_1}\frac{|Q(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ =&O(t^{-pN-sp^2-ps+N}).\end{aligned}\tag{3.16}$$

Combining (3.15) and (3.16), we conclude that

$$\begin{aligned}&\iint_{D_1}\frac{|Q(x)-Q(y)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ \leqslant&\iint_{D_1}\frac{|Q(x)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y+\iint_{D_1}\frac{|Q(y)||Q(y)-Q_t(y)|^{p-1}}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ =&O(t^{-(p-2)N-sp^2}),\end{aligned}\tag{3.17}$$

as $t\to\infty$.

•Estimate of $\iint_{D_1}\frac{|Q(x)-Q(y)|^{p-1}|Q(y)-Q_t(y)|}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y$.

Similarly, by (2.9), for all $(x,y)\in D_1$,

$$\begin{aligned}|Q(x)-Q(y)|^{p-1}|Q(y)|\leqslant&C|Q(x)|^{p-1}|Q(y)|+C|Q(y)|^{p-1}|Q(y)|\\ \leqslant&C(t^{-N-ps}+t^{-Np-sp^2})\end{aligned}\tag{3.18}$$

Then, from (3.18), it follows that

$$\begin{aligned}&\iint_{D_1} \frac{|Q(x)-Q(y)|^{p-1}|Q(y)-Q_t(y)|}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\\ \leqslant &Ct^{-2ps}+Ct^{-pN-sp^2-ps+N}=O(t^{-2ps}),\end{aligned} \tag{3.19}$$

as $t \to \infty$.

Combining (3.13), (3.14), (3.17), and (3.19), we conclude that

$$\iint_{D_1} \frac{|Q_t(x)-Q_t(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \leqslant \iint_{D_1} \frac{|Q(x)-Q(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y + O(t^{-\kappa}).$$

Together with (3.10)–(3.12), we deduce the desired result.

□

Proof of Theorem 2.5. (i) For all $u \in X$ with $\|u\|_{L^p(\mathbb{R}^N)} = 1$, from (2.2) and (V_1), we conclude that, if $b \in [0, b^*)$,

$$\begin{aligned}J_b(u) \geqslant&\frac{1}{p}\left(1-\frac{b}{b^*}\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y+\frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\\ \geqslant&\frac{1}{p}\left(1-\frac{b}{b^*}\right)\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y \geqslant 0.\end{aligned} \tag{3.20}$$

Hence, $m(b)$ is well-defined. Let $\{u_n\} \subset X$ be a minimizing sequence satisfying

$$\lim_{n\to\infty} J_b(u_n) = m(b), \ \ \|u_n\|_{L^p(\mathbb{R}^N)} = 1.$$

According to (3.20), there exist $C_1, C_2 > 0$ such that

$$\left|\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right| \leqslant C_1, \ \ \left|\int_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x\right| \leqslant C_2$$

for all $n \in \mathbb{N}$. Thus, $\{u_n\}$ is bounded in X. From Lemma 2.2, there exists a subsequence, still denoted by $\{u_n\}$ and $u \in X$, such that $u_n \rightharpoonup u$, weakly in X, and $u_n \to u$, strongly in $L^r(\mathbb{R}^N)$, for $p \leqslant r < p_s^*$ as $n \to \infty$. Using the weak lower semi-continuity of J_b, we infer that $J_b(u) = m(b)$ and $\|u\|_{L^p(\mathbb{R}^N)} = 1$, which means u is a minimizer of $m(b)$.

(ii) Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\phi = 1$ on $B_1(0)$, $\phi = 0$ on $\mathbb{R}^N \backslash B_2(0)$, $0 \leq \phi \leq 1$, and $|\nabla\phi| \leq C$. Motivated by [28], for $x_0 \in \mathbb{R}^N$, we set a trial function as follows:

$$u_t := A_t \frac{t^{\frac{N}{p}}}{\|Q\|_{L^p(\mathbb{R}^N)}}\phi\Big(\frac{x-x_0}{t}\Big)Q(t(x-x_0)),$$

where Q is a ground state solution of (2.3) and $A_t > 0$ is chosen such that $\|u_t\|_{L^p(\mathbb{R}^N)} = 1$. We first show that $\lim_{t\to\infty} A_t = 1$. In fact, by (2.9), we have

$$\frac{1}{A_t^p} = \frac{\int_{\{|x|\leqslant t^2\}} Q^p(x)\mathrm{d}x + \int_{\{t^2<|x|\leqslant 2t^2\}} Q^p(x)\mathrm{d}x}{\|Q\|^p_{L^p(\mathbb{R}^N)}} = 1 + O(t^{-2(p-1)N-2ps}),$$

as $t \to \infty$. By a direct computation, we conclude that as $t \to \infty$,

$$\int_{\mathbb{R}^N} |u_t|^q \mathrm{d}x = \frac{A_t^q t^{ps}}{\|Q\|_{L^p(\mathbb{R}^N)}^q} \left[\int_{\mathbb{R}^N} |Q|^q \mathrm{d}x + O(t^{-2qN-2pqs+2N}) \right]. \tag{3.21}$$

From Lemma 3.3, (2.7) and (3.21), we obtain that

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u_t(x)-u_t(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y - \frac{1}{p}\frac{Nb}{N+ps} \int_{\mathbb{R}^N} |u_t|^q \mathrm{d}x \\ \leqslant &\frac{A_t^p t^{ps}}{p\|Q\|_{L^p(\mathbb{R}^N)}^p} \left[\iint_{\mathbb{R}^{2N}} \frac{|Q(x)-Q(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \right. \\ &\left. - \frac{Nb}{(N+ps)\|Q\|_{L^p(\mathbb{R}^N)}^{sp^2/N}} \int_{\mathbb{R}^N} |Q|^q \mathrm{d}x + O(t^{-\kappa}) \right] \\ = &\frac{t^{ps}}{p} \left[\left(1 - \frac{b}{b^*}\right) + O(t^{-\kappa}) \right], \end{aligned} \tag{3.22}$$

as $t \to \infty$. By a direct computation, we can deduce that

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} V(x)|u_t(x)|^p \mathrm{d}x = V(x_0) \text{ for a.e. } x \in \mathbb{R}^N. \tag{3.23}$$

For $b > b^*$, combining (3.22) and (3.23), we can conclude that

$$m(b) \leqslant \lim_{t\infty} J_b(u_t) = -\infty,$$

which yields the nonexistence of minimizers of (2.1).

For $b = b^*$, choosing $x_0 \in \mathbb{R}^N$ such that $V(x_0) = 0$, (3.22) and (3.23) lead to $m(b^*) \leqslant 0$. From (3.20), one can see that $m(b^*) \geqslant 0$, which implies $m(b^*) = 0$. Assume by contradiction that there exists a minimizer $\bar{u} \in X$ for $m(b^*) = 0$ with $\|\bar{u}\|_{L^p(\mathbb{R}^N)} = 1$. Then, we can obtain that

$$\int_{\mathbb{R}^N} V(x)|\bar{u}(x)|^p \mathrm{d}x = \inf_{x\in\mathbb{R}^N} V(x) = 0,$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{|\bar{u}(x)-\bar{u}(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = \frac{Nb}{N+ps} \int_{\mathbb{R}^N} |\bar{u}|^q \mathrm{d}x.$$

These yield a contradiction, because from the first equation, it can be inferred that $\bar{u}$ must have compact support, but the second equation means that it must be equal to the translation and scaling of Q.

Finally, we need to prove that $\lim_{b\nearrow b^*} m(b) = 0$. Indeed, taking $x_0 \in \mathbb{R}^N$ such that $V(x_0) = 0$, and letting $t = (b^* - b)^{-\frac{1}{ps+1}}$, we can easily obtain that $\limsup_{b\nearrow b^*} m(b) \leqslant 0$ from (3.22) and (3.23). □

4. Blow up behavior of the ground state

In this section, we focus on the blow-up behavior of the non-negative minimizers of (2.1) as $b \nearrow b^*$. For any given sequence b_n with $b_n \nearrow b^*$ as $n \to \infty$, for the sake of convenience, we denote that

$$u_n = u_{b_n},\ \ w_n = w_{b_n},\ \ \varepsilon_n = \varepsilon_{b_n},\ \ z_n = z_{b_n},\ \ y_n = y_{b_n},\ \ \bar{w}_n = \bar{w}_{b_n}.$$

Lemma 4.1. *Assume that $V(x)$ satisfies (V_1). Let u_n be a nonnegative minimizer for $m(b_n)$ with $b_n \nearrow b^*$. Then,*

$$\varepsilon_n := \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{-\frac{1}{ps}} \to 0.$$

Proof. According to $m(b^*) = 0$, we can deduce that

$$\int_{\mathbb{R}^N} V(x)|u_n|^p\mathrm{d}x \to 0,\ \ \text{as } b_n \nearrow b^*. \tag{4.1}$$

Next, we claim that

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y = +\infty. \tag{4.2}$$

If (4.2) does not hold true, together with (4.1), we conclude that there exists a subsequence $\{b_n\}$ with $b_n \nearrow b^*$ such that $\{u_n\}$ is bounded in X. Thanks to Lemma 2.2, there exists $u_0 \in X$ such that, up to a subsequence,

$$u_n \rightharpoonup u_0 \geq 0, \quad \text{in } W^{s,p}(\mathbb{R}^N), \quad u_n \to u_0 \ \text{ in } L^r(\mathbb{R}^N) \text{ for } r \in [p, p_s^*),\ \text{as}\ \ n \to \infty.$$

Using the same strategy as proof of Theorem 2.5-(i), we can obtain that u_0 is a minimizer of $m(b^*)$. This contradicts the fact $m(b^*)$ is not attained, which is proved in Theorem 2.5-(ii). □

Lemma 4.2. *Assume that $V(x)$ satisfies (V_1). Let u_n be a nonnegative minimizer for $m(b_n)$ with $b_n \nearrow b^*$. Then, there exists a sequence $\{z_n\}, R_0$, and $\delta > 0$ such that the function*

$$w_n(x) := \varepsilon_n^{\frac{N}{p}} u_n(\varepsilon_n x + \varepsilon_n z_n) \tag{4.3}$$

satisfies

$$\liminf_{b\nearrow b^*} \int_{B_{R_0}(0)} |w_n|^2\mathrm{d}x \geqslant \delta > 0, \tag{4.4}$$

where ε_n is given in (2.12). Moreover, for any sequence $b_n \nearrow b^*$, there exists a subsequence, still denoted by $\{b_n\}$, such that $y_n := \varepsilon_n z_n \to x_0$ and $x_0 \in \mathbb{R}^N$ is the global minimum point of $V(x)$, i.e., $V(x_0) = 0$. Particularly, for any $\varrho > 0$, it holds

$$u_n(x) = \frac{1}{\varepsilon_n^{\frac{N}{p}}} w_n\left(\frac{x-y_n}{\varepsilon_n}\right) \to 0,\ \ for\ any\ x \in B_\varrho^c(x_0),\ as\ n \to \infty. \tag{4.5}$$

Proof. The proof is done in six steps.
Step 1. We claim that (4.4) holds.

Define $\bar{w}_n(x) = \varepsilon_n^{\frac{N}{p}} u_n(\varepsilon_n x)$, where ε_n is given in (2.12). Then

$$\iint_{\mathbb{R}^{2N}} \frac{|\bar{w}_n(x) - \bar{w}_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} |\bar{w}_n|^p \mathrm{d}x = 1. \tag{4.6}$$

From (2.2), we see that

$$\begin{aligned} 0 \leqslant & \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y - \frac{Nb}{N+ps} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x \\ = & \varepsilon_n^{-ps} - \frac{Nb}{N+ps} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x \leqslant \frac{m(b)}{p}. \end{aligned}$$

According to Lemma 4.1 and Theorem 2.5-(ii), we know that

$$\varepsilon_n \to 0, \ \text{ and } \ m(b_n) \to 0, \ \text{ as } n \to \infty.$$

Thus,

$$\int_{\mathbb{R}^N} |\bar{w}_n|^q \mathrm{d}x = \varepsilon_n^{ps} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x \to \frac{N+ps}{Nb^*}, \quad \text{as } n \to \infty. \tag{4.7}$$

Next, we claim that there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ and $R_0, \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_{R_0}(z_n)} |\bar{w}_n|^p \mathrm{d}x \geqslant \delta > 0. \tag{4.8}$$

Assume by contradiction that for any $R > 0$, there exists a subsequence $\{\bar{w}_n\}$ such that

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_R(z)} |\bar{w}_n|^p \mathrm{d}x = 0.$$

From Lemma 2.3, we obtain that $\bar{w}_n \to 0$ in $L^r(\mathbb{R}^N)$ for $r \in (p, p_s^*)$. This contradicts (4.7). Hence, (4.4) follows directly from (4.8).

Step 2. We will show that $y_n := \varepsilon_n z_n \to x_0$, as $n \to +\infty$, and $V(x_0) = 0$.

By Lemma 4.1, we see that

$$\varepsilon_n \to 0, \quad as\ n \to +\infty. \tag{4.9}$$

Thanks to (4.3) and (4.9), we infer that

$$\int_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n z_n)|w_n|^p \mathrm{d}x \to 0, \ \text{ as } n \to +\infty,$$

which yields that

$$0 = \liminf_{n\to+\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n z_n)|w_n|^p \mathrm{d}x \geqslant \liminf_{n\to+\infty} \int_{B_R(0)} V(\varepsilon_n x + \varepsilon_n z_n)|w_n|^p \mathrm{d}x.$$

Due to the fact that $\lim_{|x|\to\infty} V(x) = \infty$ and (4.4), we derive that $\{y_n\}$ is bounded in $\mathbb{R}^N$. Then, up to a subsequence, there exists an $x_0 \in \mathbb{R}^N$ such that $y_n \to x_0$, as $n \to +\infty$. From (4.4) and Fatou's Lemma, one sees that

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n z_n)|w_n|^p \mathrm{d}x \geqslant V(x_0) \liminf_{n\to+\infty} \int_{B_{R_0}(0)} |w_n|^p \mathrm{d}x \geqslant V(x_0)\delta.$$

This shows $V(x_0) = 0$.

Step 3. We come to prove that there exists $w_0 \in X$ with $w_0 \geqslant 0$ and $w_0 \not\equiv 0$ such that

$$w_n \to w_0 \text{ in } X, \quad \text{as } n \to +\infty.$$

Since u_n is a non-negative minimizer for $m(b_n)$, we deduce that

$$(-\Delta)_p^s u_n + V(x) v_n^{p-1} = \lambda_n u_n^{p-1} + b_n u_n^{q-1}, \ \text{ in } \mathbb{R}^N, \tag{4.10}$$

where $\lambda_n := \lambda_{b_n} \in \mathbb{R}$ is a Lagrange multiplier. Moreover, we obtain

$$\lambda_n = \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u_n|^p \mathrm{d}x - b_n \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x. \tag{4.11}$$

From (2.12), (4.3), and (4.6), we conclude that,

$$\begin{cases} \iint_{\mathbb{R}^{2N}} \frac{|w_n(x)-w_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} |w_n|^p \mathrm{d}x = 1, \\ \int_{\mathbb{R}^N} |w_n|^q \mathrm{d}x = \varepsilon_n^{ps} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x \to \frac{N+ps}{Nb^*}, \quad \text{as } n \to \infty. \end{cases} \tag{4.12}$$

Therefore, by (4.11) and (4.12), we obtain that,

$$\lambda_n \varepsilon_n^{ps} \to -\frac{ps}{N}, \ \text{ as } n \to +\infty. \tag{4.13}$$

Using (4.3) and (4.10), we derive that

$$(-\Delta)_p^s w_n + \varepsilon_n^{ps} V(\varepsilon_n x + \varepsilon_n z_n) w_n^{p-1} = \varepsilon_n^{ps} \lambda_n w_n^{p-1} + b_n w_n^{q-1}, \ \text{ in } \mathbb{R}^N, \tag{4.14}$$

(4.12) implies that $\{w_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$. Therefore, there exists $w_0 \in W^{s,p}(\mathbb{R}^N)$ such that up to a subsequence,

$$w_n \rightharpoonup w_0 \geq 0, \quad \text{in } W^{s,p}(\mathbb{R}^N), \text{as } n \to \infty. \tag{4.15}$$

By (4.13)–(4.15), we infer that w_0 is the weak solution to

$$(-\Delta)_p^s w_0 + \frac{ps}{N} w_0^{p-1} = b^* w_0^{q-1} \quad \text{in } \mathbb{R}^N. \tag{4.16}$$

Together with Lemma 3.1, we obtain that

$$w_0 = \frac{Q(x)}{(b^*)^{\frac{N}{p^2 s}}}.$$

Obviously,

$$\iint_{\mathbb{R}^{2N}} \frac{|w_0(x) - w_0(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} |w_0|^p \mathrm{d}x = 1, \tag{4.17}$$

which yields

$$w_n \to w_0 \ \text{ in } W^{s,p}(\mathbb{R}^N),$$

recalling (4.12) and (4.15).

Step 4. Our aim is to show that $\|w_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ uniformly for $n \in \mathbb{N}$.

For $\tau \geq 1$ and $T > 0$, define

$$w_{T,n}(x) = \begin{cases} w_n(x), & w_n(x) \leq T, \\ T, & w_n(x) > T, \end{cases}$$

and $\eta(w_n) = w_n w_{T,n}^{p(\tau-1)}$. First, we show $\eta(w_n) \in W^{s,p}(\mathbb{R}^N)$. We define $B_1 := \{x \in \mathbb{R}^N : |w_n(x)| \leq T\}$ and $B_2 := \{x \in \mathbb{R}^N : |w_n(x)| > T\}$. It follows from the definition of $\eta(w_n)$ that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\eta(w_n)(x) - \eta(w_n)(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \\
=&\iint_{B_1 \times B_1} \frac{|\eta(w_n)(x) - \eta(w_n)(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + 2\iint_{B_1 \times B_2} \frac{|\eta(w_n)(x) - \eta(w_n)(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \\
&+ \iint_{B_2 \times B_2} \frac{|\eta(w_n)(x) - \eta(w_n)(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \\
\leq& CT^{p^2(\tau-1)} \iint_{\mathbb{R}^{2N}} \frac{|w_n(x) - w_n(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y,
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x)|\eta(w_n)(x)|^p \mathrm{d}x &= \int_{B_1} V(x)|\eta(w_n)(x)|^p \mathrm{d}x + \int_{B_2} V(x)|\eta(w_n)(x)|^p \mathrm{d}x \\
&\leqslant CT^{p^2(\tau-1)} \int_{\mathbb{R}^N} V(x)|w_n(x)|^p \mathrm{d}x,
\end{aligned}$$

which yields that $\eta(w_n) \in X$. Consider the function

$$\Lambda(t) := \frac{t^p}{p}, \ \ \Gamma(t) := \int_0^t [\eta'(\xi)]^{\frac{1}{p}} \mathrm{d}\xi.$$

We note that

$$\Gamma(w_n) \geqslant \frac{1}{\tau} w_n w_{T,n}^{\tau-1}.$$

Therefore, by Lemma 2.1 and the above inequality, we can deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|\Gamma(w_n)(x) - \Gamma(w_n)(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y \geqslant S_*^{-1} \left(\int_{\mathbb{R}^N} |\Gamma(w_n)|^{p_s^*} \mathrm{d}x \right)^{\frac{p}{p_s^*}} \\
\geqslant& \frac{S_*^{-1}}{\tau^p} \left(\int_{\mathbb{R}^N} |w_n w_{T,n}^{\tau-1}|^{p_s^*} \mathrm{d}x \right)^{\frac{p}{p_s^*}}.
\end{aligned} \tag{4.18}$$

For fixed $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha > \beta$, using the Jensen inequality, we obtain that

$$\begin{aligned}
\Lambda'(\alpha - \beta)(\eta(\alpha) - \eta(\beta)) =& (\alpha-\beta)^{p-1}(\eta(\alpha) - \eta(\beta)) = (\alpha-\beta)^{p-1} \int_\beta^\alpha \eta'(\xi) \mathrm{d}\xi \\
=& (\alpha-\beta)^{p-1} \int_\beta^\alpha (\Gamma'(\xi))^p \mathrm{d}\xi \geqslant \left(\int_\beta^\alpha (\Gamma'(\xi)) \mathrm{d}\xi \right)^p \\
=& (\Gamma(\alpha) - \Gamma(\beta))^p.
\end{aligned}$$

Similarly, we can prove that the above inequality holds when $\alpha \leqslant \beta$, which implies

$$|\Gamma(\alpha)-\Gamma(\beta)|^p \leqslant \Lambda'(\alpha-\beta)(\eta(\alpha)-\eta(\beta)).$$

Hence, we conclude that

$$\begin{aligned}&|\Gamma(w_n(x))-\Gamma(w_n(y))|^p\\ \leqslant&|w_n(x)-w_n(y)|^{p-2}(w_n(x)-w_n(y))(w_nw_{T,n}^{p(\tau-1)}(x)-w_nw_{T,n}^{p(\tau-1)}(y)).\end{aligned} \tag{4.19}$$

Taking $\eta(w_n)$ as a test function in (1.1), and using (4.19), we derive that

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|\Gamma(w_n)(x)-\Gamma(w_n)(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}V(x)|w_n|^pw_{T,n}^{p(\tau-1)}\mathrm{d}x\\ \leqslant&\lambda_n\int_{\mathbb{R}^N}|w_n|^pw_{T,n}^{p(\tau-1)}\mathrm{d}x+b_n\int_{\mathbb{R}^N}|w_n|^qw_{T,n}^{p(\tau-1)}\mathrm{d}x.\end{aligned}$$

Together with (4.18), we can infer that

$$\left(\int_{\mathbb{R}^N}|w_nw_{T,n}^{\tau-1}|^{p_s^*}\mathrm{d}x\right)^{\frac{p}{p_s^*}} \leqslant C\tau^p\left(\int_{\mathbb{R}^N}|w_n|^pw_{T,n}^{p(\tau-1)}\mathrm{d}x+\int_{\mathbb{R}^N}|w_n|^qw_{T,n}^{p(\tau-1)}\mathrm{d}x\right). \tag{4.20}$$

Let $\tau_1 = \frac{p_s^*}{p}$, and take $L>0$. Taking into account of $0 \leqslant w_{T,n}(x) \leqslant w_n$, and using the Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}|w_n|^qw_{T,n}^{p(\tau-1)}\mathrm{d}x &= \int_{\mathbb{R}^N}|w_n|^{q-p}(w_nw_{T,n}^{(p_s^*-p)/p})^p\mathrm{d}x\\ &= \int_{\{w_n>L\}}|w_n|^{q-p}(w_nw_{T,n}^{(p_s^*-p)/p})^p\mathrm{d}x+\int_{\{w_n\leqslant L\}}|w_n|^{q-p}(w_nw_{T,n}^{(p_s^*-p)/p})^p\mathrm{d}x\\ &\leqslant \left(\int_{\{w_n>L\}}|w_n|^p\mathrm{d}x\right)^{ps/N}\left(\int_{\{w_n>L\}}(w_nw_{T,n}^{(p_s^*-p)/p})^{p_s^*}\mathrm{d}x\right)^{p/p_s^*}\\ &\quad + L^{q-p}\int_{\{w_n\leqslant L\}}|w_n|^{p_s^*}\mathrm{d}x.\end{aligned}$$

Since $w_n \in L^p(\mathbb{R}^N)$, it is easy to see that for any $L>0$ large enough,

$$\left(\int_{\{w_n>L\}}|w_n|^p\mathrm{d}x\right)^{ps/N} \leqslant \frac{1}{2C\tau^p},$$

and then, we conclude that

$$\int_{\mathbb{R}^N}|w_n|^qw_{T,n}^{p(\tau_1-1)}\mathrm{d}x \leqslant \frac{1}{2C\tau_1^p}\left(\int_{\mathbb{R}^N}(w_nw_{T,n}^{(p_s^*-p)/p})^{p_s^*}\mathrm{d}x\right)^{p/p_s^*}+L^{q-p}\int_{\mathbb{R}^N}|w_n|^{p_s^*}\mathrm{d}x.$$

Together with (4.20), one sees that

$$\left(\int_{\mathbb{R}^N}|w_nw_{T,n}^{\tau_1-1}|^{p_s^*}\mathrm{d}x\right)^{\frac{p}{p_s^*}} \leqslant C\tau_1^pL^{q-p}\int_{\mathbb{R}^N}|w_n|^{p_s^*}\mathrm{d}x < \infty.$$

Letting $T \to \infty$, it follows that $w_n \in L^{(p_s^*)^2/p}(\mathbb{R}^N)$.

Assuming $\tau > \tau_1$ and taking the limit as $T \to \infty$ in (4.20), one gets

$$\left(\int_{\mathbb{R}^N} |w_n|^{\tau p_s^*} \mathrm{d}x\right)^{\frac{p}{p_s^*}} \leqslant C\tau^p \left(\int_{\mathbb{R}^N} |w_n|^{p\tau} \mathrm{d}x + \int_{\mathbb{R}^N} |w_n|^{q+p(\tau-1)} \mathrm{d}x\right),$$

which yields

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |w_n|^{\tau p_s^*} \mathrm{d}x\right)^{\frac{1}{p_s^*(\tau-1)}} \\ \leqslant&(C\tau^p)^{\frac{1}{\tau-1}} \left(\int_{\mathbb{R}^N} |w_n|^{p\tau} \mathrm{d}x + \int_{\mathbb{R}^N} |w_n|^{q+p(\tau-1)} \mathrm{d}x\right)^{\frac{1}{p(\tau-1)}}.\end{aligned} \tag{4.21}$$

Set $w_n^{p\tau} = w_n^l w_n^m$, $w_n^{q+p(\tau-1)} = w_n^{l_1} w_n^{m_1}$, where $l = \frac{(p_s^*-p)p_s^*}{p(\tau-1)}$, $m = p\tau - l$, $l_1 = \frac{(p_s^*-q)p_s^*}{p(\tau-1)}$, $m_1 = q + p(\tau-1) - l_1$. Then, $\tau > \tau_1$ implies that $0 < l, l_1 < p_s^*$. Applying Young's inequality, we can infer that

$$\begin{aligned}\int_{\mathbb{R}^N} |w_n|^{p\tau} \mathrm{d}x \leqslant& \frac{l}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{p_s^*} \mathrm{d}x + \frac{p_s^* - l}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{\frac{mp_s^*}{p_s^*-l}} \mathrm{d}x \\ =&\frac{l}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{p_s^*} \mathrm{d}x + \frac{p_s^* - l}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{p_s^*+p(\tau-1)} \mathrm{d}x \\ \leqslant& C\left(1 + \int_{\mathbb{R}^N} |w_n|^{p_s^*+p(\tau-1)} \mathrm{d}x\right),\end{aligned} \tag{4.22}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |w_n|^{q+p(\tau-1)} \mathrm{d}x \leqslant& \frac{l_1}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{p_s^*} \mathrm{d}x + \frac{p_s^* - l_1}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{\frac{m_1 p_s^*}{p_s^*-l_1}} \mathrm{d}x \\ =&\frac{l}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{p_s^*} \mathrm{d}x + \frac{p_s^* - l}{p_s^*} \int_{\mathbb{R}^N} |w_n|^{p_s^*+p(\tau-1)} \mathrm{d}x \\ \leqslant& C\left(1 + \int_{\mathbb{R}^N} |w_n|^{p_s^*+p(\tau-1)} \mathrm{d}x\right).\end{aligned} \tag{4.23}$$

It then follows from (4.21)–(4.23) that

$$\left(\int_{\mathbb{R}^N} |w_n|^{\tau p_s^*} \mathrm{d}x\right)^{\frac{1}{p_s^*(\tau-1)}} \leqslant (C\tau^p)^{\frac{1}{\tau-1}} \left(1 + \int_{\mathbb{R}^N} |w_n|^{p_s^*+p(\tau-1)} \mathrm{d}x\right)^{\frac{1}{p(\tau-1)}}.$$

Iterating this process, we infer that

$$\left(\int_{\mathbb{R}^N} |w_n|^{\tau_{i+1} p_s^*} \mathrm{d}x\right)^{\frac{1}{p_s^*(\tau_{i+1}-1)}} \leqslant (C\tau_{i+1}^p)^{\frac{1}{\tau_{i+1}-1}} \left(1 + \int_{\mathbb{R}^N} |w_n|^{p_s^*\tau_i} \mathrm{d}x\right)^{\frac{1}{p_s^*(\tau_i-1)}},$$

where $p_s^* + p(\tau_{i+1} - 1) = p_s^*\tau_i$ and $\tau_1 = p_s^*/p$. Then, $\tau_{i+1} = \left(\frac{p_s^*}{p}\right)(\tau_i - 1) + 1$. Let $C_{i+1} = C\tau_{i+1}^p$ and

$$K_i = \left(1 + \int_{\mathbb{R}^N} |w_n|^{p_s^*\tau_i} \mathrm{d}x\right)^{\frac{1}{p_s^*(\tau_i-1)}},$$

Therefore, there exists $C > 0$ independent of i such that

$$K_{i+1} \leqslant \prod_{j=2}^{i+1} C_j^{\frac{1}{\tau_{j-1}}} K_1 \leqslant CK_1,$$

which shows

$$\|w_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C.$$

Step 5. We will show $w_n(x) \to 0$ as $|x| \to +\infty$ uniformly for large n.
(4.14) can be rewritten as

$$(-\Delta)_p^s w_n = h_n(x) \quad \text{in } \mathbb{R}^N,$$

where $h_n(x) = -\varepsilon_n^{ps} V(\varepsilon_n x + \varepsilon_n z_n) w_n^{p-1} + \varepsilon_n^{ps} \lambda_n w_n^{p-1} + b_n w_n^{q-1}$. By Step 4, it is obvious that $h_n \in L^\infty(\mathbb{R}^N)$ and $\|h_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. Then, applying [12, corollary 5.5], we can obtain that $w_n \in C^{0,\alpha}(\mathbb{R}^N)$ for some $\alpha > 0$. Therefore, we obtain $w_n(x) \to 0$ as $|x| \to +\infty$ uniformly for large n.
Step 6. We will prove that (4.5) holds.

According to [6, Lemma 7.1], there exists a function γ such that

$$0 < \gamma \leq \frac{C}{1 + |x|^{N+ps}},$$

and

$$(-\Delta)_p^s \gamma + \frac{1}{2}\gamma^{p-1} = 0, \quad \text{in } \mathbb{R}^N \setminus B_{R_1}(0),$$

for some $R_1 > 0$. Furthermore, since Step 5 and (4.13), there exists $R_2 > 0$ sufficiently large such that for large n,

$$(-\Delta)_p^s w_n + \frac{1}{2} w_n^{p-1} \leqslant (-\Delta)_p^s w_n + \varepsilon_n^{ps} V(\varepsilon_n x + \varepsilon_n z_n) w_n^{p-1} - \varepsilon_n^{ps} \lambda_n w_n^{p-1} - b_n w_n^{q-1} = 0,$$

for $|x| \geq R_2$. Arguing as in the proof of [3, Theorem 1.1], we see that

$$w_n(x) \leqslant \frac{C}{1 + |x|^{N+ps}}, \quad \text{uniformly for large } n. \tag{4.24}$$

For any $x \in B_\varrho^c(x_0)$, we obtain

$$\frac{|x - y_n|}{\varepsilon_n} \geqslant \frac{|x - x_0|}{2\varepsilon_n} \geqslant \frac{\varrho}{2\varepsilon_n} \to +\infty, \quad \text{as } n \to +\infty. \tag{4.25}$$

Consequently, by (4.3), (4.24) and (4.25), we deduce that

$$\begin{aligned} u_n(x) =& \frac{1}{\varepsilon_n^{p/N}} w_n\left(\frac{x - y_n}{\varepsilon_n}\right) \leqslant \frac{1}{\varepsilon_n^{p/N}} \frac{C}{1 + \left|\frac{x - y_n}{\varepsilon_n}\right|^{N+ps}} \\ \leqslant& \frac{1}{\varepsilon_n} \frac{C}{1 + \left|\frac{\varrho}{2\varepsilon_n}\right|^{2+2s}} \to 0, \quad x \in B_\varrho^c(x_0). \end{aligned}$$

Thus, (4.5) holds true. □

Proof of Theorem 2.6. (i) Noting Lemma 4.1, (2.12) follows immediately.

(ii) Let $\tilde{y}_n$ be a global maximum point of u_n, then, by Lemma 4.2 Step 2, we derive that $\tilde{y}_n \to x_0$ as $n \to +\infty$ with $V(x_0) = 0$.

(iii) According to (4.10) and (4.13), one sees that

$$u_n(\tilde{y}_n) \geqslant \frac{C}{\varepsilon_n^{N/p}}.$$

Now, let us consider the function

$$\tilde{w}_n := \varepsilon_n^{N/p} u_n(\varepsilon_n x + \tilde{y}_n). \tag{4.26}$$

Using the same arguments in the proof of Step 3 in Lemma 4.2, we see that

$$(-\Delta)_p^s \tilde{w}_n + \varepsilon_n^{ps} V(\varepsilon_n x + \tilde{y}_n) \tilde{w}_n^{p-1} = \varepsilon_n^{ps} \lambda_n \tilde{w}_n^{p-1} + b_n \tilde{w}_n^{q-1}, \quad \text{in } \mathbb{R}^N,$$

and there exists $\tilde{w}_0 \in W^{s,p}(\mathbb{R}^N)$ such that, up to a subsequence,

$$\tilde{w}_n \rightharpoonup \tilde{w}_0 > 0,$$

in $W^{s,p}(\mathbb{R}^N)$, as $n \to +\infty$, where $\tilde{w}_0$ satisfies (4.16) and (4.17).

Taking account of Lemma 3.1, we conclude that

$$\tilde{w}_0 = \frac{Q(x)}{(b^*)^{\frac{N}{p^2 s}}}.$$

□

5. Blow up behavior for polynomial type potential

This section is dedicated to proving Theorem 2.7. In the following content, we always indicate b_n to be a convergent subsequence in Theorem 2.6. We first give an upper bound of $m(b)$.

Lemma 5.1. *Suppose $V(x)$ satisfies (V_2). Then for n large,*

$$0 \leqslant m(b_n) \leqslant \frac{(b^* - b_n)^{\frac{r}{ps+r}}}{p(b^*)^{\frac{N+r}{ps+r}}} \sigma^{\frac{ps}{ps+r}} \left(\left(\frac{r}{p}\right)^{\frac{ps}{ps+r}} + \left(\frac{p}{r}\right)^{\frac{ps}{ps+r}} + o_n(1) \right),$$

where σ is given by (2.15).

Proof. Let $Q \in \mathcal{M}$ and $z_0 \in \mathbb{R}^N$ be such that (2.13) holds. For $x_0 \in B$, take

$$u(x) := A_t \frac{t^{\frac{N}{p}}}{\|Q\|_{L^p(\mathbb{R}^N)}} \phi\Big(\frac{x - x_0}{t}\Big) Q(t(x - x_0) - z_0)$$

as a trial function. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x \\
=&A_t^p \int_{\mathbb{R}^N} V(x)\frac{t^N}{\|Q\|^p_{L^p(\mathbb{R}^N)}}\phi^p\Big(\frac{x-x_0}{t}\Big)Q^p(t(x-x_0)-z_0)\mathrm{d}x \\
=&\frac{A_t^p}{\|Q\|^p_{L^p(\mathbb{R}^N)}}\int_{\mathbb{R}^N} t^N|x-x_0|^r\phi^p\Big(\frac{x-x_0}{t}\Big)Q^p(t(x-x_0)-z_0)\frac{V(x)}{|x-x_0|^r}\mathrm{d}x \\
\leqslant&\frac{A_t^p t^{-r}}{\|Q\|^p_{L^p(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}|x|^r Q^p(x-z_0)\mathrm{d}x \lim_{x\to x_0}\frac{V(x)}{|x-x_0|^r}+o(1)\right) \\
\leqslant&A_t^p t^{-r}(b^*)^{-\frac{N}{ps}}(\sigma+o(1)),
\end{aligned} \tag{5.1}$$

where $o(1)\to 0$ as $t\to\infty$. Gathering (5.1), (3.22) and (3.23), we infer that for large t,

$$0\leqslant m(b_n)\leqslant J_{b_n}(u)\leqslant \frac{t^{ps}}{p}\frac{b^*-b_n}{b^*}+\frac{t^{-r}}{p}A_t^p(b^*)^{-\frac{N}{ps}}(\sigma+o(1))+O(t^{-\kappa}).$$

Taking $t=\left(\frac{(b^*)^{1-\frac{N}{ps}}\sigma r}{(b^*-b_n)p}\right)^{\frac{1}{ps+r}}$, then $t\to\infty$ as $b_n\nearrow b^*$. Hence,

$$0\leqslant m(b_n)\leqslant \frac{(b^*-b_n)^{\frac{r}{ps+r}}}{p(b^*)^{\frac{N+r}{ps+r}}}\sigma^{\frac{ps}{ps+r}}\left[\left(\frac{r}{p}\right)^{\frac{ps}{ps+r}}+\left(\frac{p}{r}\right)^{\frac{r}{ps+r}}+o_n(1)\right],$$

as $n\to\infty$. □

Proof of Theorem 2.7. (i) Consider the function

$$\hat{w}_n := \delta_n^{N/p}u_n(\delta_n x+\tilde{y}_n). \tag{5.2}$$

From (3.20) and (5.2), one gets that

$$\begin{aligned}
m(b_n)=&J_{b_n}(u_n) \\
\geqslant&\frac{1}{p}\left(1-\frac{b}{b^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u_n|^p\mathrm{d}x \\
=&\frac{1}{p}\left(1-\frac{b}{b^*}\right)\delta_n^{-ps}+\frac{1}{p}\int_{\mathbb{R}^N}V(\delta_n x+\tilde{y}_n)|\hat{w}_n|^p\mathrm{d}x.
\end{aligned} \tag{5.3}$$

By Theorem 2.6, we can assume that $\tilde{y}_n\to x_i$ with $x_i\in B$ as $n\to\infty$. Define

$$\tilde{V}_n(x):=\frac{V(\delta_n x+\tilde{y}_n)}{a_i|\delta_n x+\tilde{y}_n-x_i|^{r_i}},$$

where $a_i=\lim\limits_{x\to x_i}\frac{V(x)}{|x-x_i|^{r_i}}$. Then, $\tilde{V}_n(x)\to 1$ a.e. in $x\in\mathbb{R}^N$ as $n\to\infty$. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^N}V(\delta_n x+\tilde{y}_n)|\hat{w}_n|^p\mathrm{d}x=&\int_{\mathbb{R}^N}\tilde{V}_n a_i|\delta_n x+\tilde{y}_n-x_i|^{r_i}|\hat{w}_n|^p\mathrm{d}x \\
=&\delta_n^{r_i}\int_{\mathbb{R}^N}\tilde{V}_n a_i\left|x+\frac{\tilde{y}_n-x_i}{\delta_n}\right|^{r_i}|\hat{w}_n|^p\mathrm{d}x.
\end{aligned}$$

We will show that $\limsup_{n\to\infty} \frac{|\tilde{y}_n - x_i|}{\delta_n} < \infty$ and $r_i = r$. In fact, if not, set

$$\chi_n := \int_{\mathbb{R}^N} \tilde{V}_n a_i \left| x + \frac{\tilde{y}_n - x_i}{\delta_n} \right|^{r_i} |\hat{w}_n|^p \mathrm{d}x,$$

then, $\chi_n \to \infty$ if $\frac{|\tilde{y}_n - x_i|}{\delta_n} \to \infty$. Furthermore, using (5.3), we obtain that

$$\begin{aligned} m(b_n) \geqslant & \frac{1}{p}\left(1 - \frac{b}{b^*}\right)\delta_n^{-ps} + \frac{1}{p}\delta_n^{r_i}\chi_n \\ \geqslant & \frac{1}{p}\left(\frac{b^* - b_n}{b^*}\right)^{\frac{r_i}{ps+r_i}} \chi_n^{\frac{ps}{ps+r_i}} \left[\left(\frac{r}{p}\right)^{\frac{ps}{ps+r}} + \left(\frac{p}{r}\right)^{\frac{r}{ps+r}}\right], \end{aligned} \tag{5.4}$$

which contradicts Lemma 5.1. Therefore,

$$\limsup_{n\to\infty} \frac{|\tilde{y}_n - x_i|}{\delta_n} < \infty.$$

Together with Fatou's Lemma

$$\liminf_{n\to+\infty} \chi_n > 0.$$

Together with Fatou's lemma, Lemma 5.1, we observe that $r_i = r$ and

$$\begin{aligned} & \liminf_{n\to\infty} \int_{\mathbb{R}^N} \tilde{V}_n \left| x + \frac{\tilde{y}_n - x_i}{\delta_n} \right|^{r} |\hat{w}_n|^p \mathrm{d}x \lim_{x\to x_i} \frac{V(x)}{|x - x_i|^r} \\ \geqslant & \frac{1}{(b^*)^{\frac{N}{ps}}} \int_{\mathbb{R}^N} |x + z_0|^r |Q(x)|^p \mathrm{d}x \cdot \lim_{x\to x_i} \frac{V(x)}{|x - x_i|^{r_i}} \geqslant \frac{1}{(b^*)^{\frac{N}{ps}}}\sigma. \end{aligned} \tag{5.5}$$

Then, from (5.4) and (5.5), we obtain that

$$\begin{aligned} m(b_n) \geqslant & \frac{1}{p}\left(1 - \frac{b_n}{b^*}\right)\delta_n^{-ps} + \frac{1}{p}\delta_n^{r}\frac{1}{(b^*)^{\frac{N}{ps}}}\sigma \\ \geqslant & \frac{(b^* - b_n)^{\frac{r}{ps+r}}}{p(b^*)^{\frac{N+r}{ps+r}}}\sigma^{\frac{ps}{ps+r}}\left(\left(\frac{r}{p}\right)^{\frac{ps}{ps+r}} + \left(\frac{p}{r}\right)^{\frac{r}{ps+r}}\right), \end{aligned}$$

where the equality holds in the last inequality if and only if

$$\delta_n = (b^*)^{\frac{N-ps}{ps(ps+r)}}(b^* - b_n)^{\frac{1}{ps+r}}\sigma^{-\frac{1}{ps+r}}\left(\frac{p}{r}\right)^{\frac{1}{ps+r}}.$$

(ii) From Lemma 5.1, it follows that the inequality (5.5) is indeed an equality, which yields $x_0 \in B$. Next, we suffice to prove that

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = 1,$$

where ε_n is given by (2.16). If not, there exists a subsequence $\{\varepsilon_n\}$, such that

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n} = \theta \neq 1,$$

where $0 \leqslant \theta \leqslant \infty$. By (5.4), for n large, we observe that

$$\begin{aligned} m(b_n) \geqslant & \frac{1}{p}\left(1-\frac{b}{b^*}\right)\varepsilon_n^{-ps}+\frac{1}{p}\varepsilon_n^r\frac{1}{(b^*)^{\frac{N}{ps}}}\sigma \\ > & \frac{(b^*-b_n)^{\frac{r}{ps+r}}}{p(b^*)^{\frac{N+r}{ps+r}}}\sigma^{\frac{ps}{ps+r}}\left[\left(\frac{r}{p}\right)^{\frac{ps}{ps+r}}+\left(\frac{p}{r}\right)^{\frac{r}{ps+r}}\right], \end{aligned}$$

which contradicts Lemma 5.1.

□

6. Conclusions

In this paper, we investigated the existence, nonexistence, and blow-up behavior of minimizers for a fractional p-Laplacian problem with external potentials and mass-critical nonlinearity. By employing constrained variational methods and refined energy estimates, we established a critical threshold b^* that determines the existence of minimizers. Specifically, minimizers exist for $b < b^*$ and cease to exist for $b \geq b^*$. Furthermore, we analyzed the concentration phenomena of minimizers as b approaches b^*, demonstrating that they concentrate at points where the potential $V(x)$ attains its minimum.

Our work extends previous results for linear fractional operators to the nonlinear fractional p-Laplacian case, addressing significant technical challenges due to the operator's nonlinear and nonlocal nature. These findings contribute to the broader understanding of nonlocal partial differential equations and have potential applications in fields such as physics, optimization, and phase transitions. Future research could explore more general potentials, multi-peak solutions, and further applications in stochastic processes.

Author contributions

Conceptualization, H. B. Chen, J. Yang; writing—original draft preparation, X. Y. Zhang; writing— review and editing, J. Yang. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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