AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 3581–3596.
DOI:10.3934/math.2025165
Received: 26 November 2024
Revised: 13 February 2025
Accepted: 19 February 2025
Published: 24 February 2025

Research article

Approximate controllability of evolution hemivariational inequalities under nonlocal conditions

Yongwei Jia*

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: Jiayongwei9608@163.com.

Abstract: This article utilizes fixed-point theory and nonsmooth analysis to study the existence and approximate controllability of mild solutions for a class of nonlinear evolution hemivariational inequalities with nonlocal conditions in Hilbert spaces. In fact, our main results encompass a broader range of related issues. Finally, we use an example to illustrate that our results are valuable.

Keywords: approximate controllability; generalized gradient; nonlocal problem
Mathematics Subject Classification: Primary 35R70; Secondary 49J53

1. Introduction

In this paper, we study the approximate controllability of evolutionary hemivariational inequality

$$\begin{cases} \langle -u'(t) + \mathcal{N}(t)u(t) + B\mu(t), \rho\rangle_Y + J^0(t, u(t); \rho) \geq 0, \quad \forall\, t \in I,\ \forall\, \rho \in Y, \\ u(0) = \sum\limits_{k=1}^{m} d_k u(t_k), \end{cases} \tag{1.1}$$

with under nonlocal conditions in Y, where Y is a Hilbert space whose scalar product is denoted by $\langle\cdot,\cdot\rangle_Y$, $I = [0, b], b > 0$, and $\{\mathcal{N}(t) : t \in I\}$ is a family of linear operators on Y such that the domain of $N(t)$ does not depend on t. $B : U \to Y$ is a bounded linear operator, where U is a Hilbert space. The control function μ takes values in $L^2(I, U)$, $0 < t_1 < t_2 < t_3 < \cdots < t_m < b, m \in \mathbb{N}$, and d_k are real numbers, $d_k \neq 0, k = 1, 2, \ldots, m$. $J^0(t, \cdot, ; \cdot)$ represents the generalized directional derivative of a locally Lipschitz function $J(t, \cdot) : Y \longrightarrow \mathbb{R}$.

In recent years, hemivariational inequalities have been powerful tools for solving physics, engineering, and optimization problems, and they are also an important and interesting topic in the field of mathematics. Because of its important uniqueness, this field has received widespread attention. At the same time, this has also led many scholars to obtain solutions to hemivariational inequality problems under many conditions, see [1–3].

Control problems are widely used in the fields of mechanics, physics, and aerodynamics. Therefore, many research works focus on optimal control problems of hemivariational inequalities, and a series of progress has been made. For example, when Necas and others studied friction problems, Malla and Nassif used pseudo-variant results to overcome difficulties when studying crystal tube problems. Not only that, hemivariational inequalities are also a powerful tool for studying game theory and equilibrium theory in economics and transportation.

Problems with non-local conditions are driven by physical problems. The application of non-local initial conditions in physics is significantly better than $u(0) = u_0$ in terms of application range and application effect. For example, Deng [4] discussed phenomenon of a very small amount of gas in the tube under non-local conditions. In this scenario, we permit additional measurements at t_k that are more accurate than the measurement taken at $t = 0$. As a result, nonlocal conditions may be more appropriate than the conventional initial condition $u(0) = u_0$ for accurately characterizing specific physical phenomena. For insights on the significance of nonlocal conditions, refer to [5–7].

In 2000, Migorski and Ochal [8] used the variational method to discuss the control problem of parabolic hemivariational inequalities

$$\begin{cases} w'(t) + \mathcal{N}(t)w(t) + \lambda(t) = f(t) + B(t)\mu(t), \quad \forall\, t \in (0, T), \\ w(0) = w_0, \\ \lambda(y, t) \in \hat{\beta}(y, t, u(y, t), w(y, t)), \quad \forall\, (x, t) \in \Omega \times (0, T), \end{cases}$$

where y and μ denote the control function, and Ω is subset of $\mathbb{R}^3$. The lower-order term $\hat{\beta}$ is multivalued and discontinuous.

Nonlinear control theory is a branch of control theory that primarily focuses on the control of nonlinear systems. Nonlinear systems are widely prevalent in real-world applications, such as mechanical systems, biological systems, electrical systems, and others; see references [9–11]. Fixed point theorems are one of the powerful tools for solving control problems of nonlinear systems, and good results have been obtained (see, e.g., [12–14]). At present, there are few studies on the approximate controllability of hemivariational inequality control problems, and the earliest dates can be traced back to [15]. In 2015, Liu Zhenhai and Liu Xiuwen [15] presented the notion of mild solutions for hemivariational inequalities. By using the fixed point theorem of multivalued maps, the approximate controllability of the hemivariational inequality

$$\begin{cases} \langle -y'(t) + \mathcal{N}y(t) + Bu(t), \rho\rangle_Y + J^0(t, y(t); \rho) \geq 0, \quad \forall\, t \in I = [0, b],\ \forall\, \rho \in Y, \\ y(0) = y_0, \end{cases}$$

is obtained, where $\mathcal{N} : D(\mathcal{N}) \subseteq Y \to Y$ is the infinitesimal generator of a C_0-semigroup $T(t)$ on Y, and Ω is a subset of $\mathbb{R}^3$. In the same year, Liu Zhenhai, Liu Xiuwen, and Motreanu [16] used unstable point theory and non-smooth analysis to discuss the approximate controllability of the following variational inequality:

$$\begin{cases} \langle -y'(t) + \mathcal{N}(t)y(t) + Bu(t), \rho\rangle_Y + J^0(t, y(t); \rho) \geq 0, \quad \forall\, t \in I = [0, b],\ \forall\, \rho \in Y, \\ y(0) = y_0, \end{cases}$$

where $\mathcal{N}(t)$ is a linear operator on Y and is independent of time t.

Evolution inequalities with nonlocal conditions can better describe practical problems. Therefore, based on [16], this paper further discusses the problem (1.1).

In the second section, we give some necessary definitions and concepts. The first major result in Section 3 obtains the existence of mild solutions to inclusion problem (2.1), which is more general than inclusion problem (1.1). Then, we show that inclusion problem (2.1) is approximately controllable. Finally, we use an example to show that our main result is appropriate.

2. Preliminaries and basic definitions

Below we will introduce some symbols, definitions, and some basic concepts as necessary.

Let $\|\cdot\|_X$ denote the norm of the Banach space X, X^* represent the dual of X, and $\langle\cdot,\cdot\rangle$ represents the duality paring between X^* and X. By $C(I,X)$, denote the Banach space of continuous functions from I into X equipped with the norm $\|u\|_{C(I,X)} = \sup\limits_{t\in J}\|u(t)\|_X$. $\mathcal{L}(E,H)$ denotes bounded linear operators from E to H, where E and H are Banach spaces. Next, we give some definitions of multivalued maps. Readers can refer to [17] for more details. The set of all non-empty subsets of X is represented by $\mathcal{P}(X)$. We will use the notation

$$\mathcal{P}_{cp,c\varsigma}(X) = \{\Pi \in \mathcal{P}(X) : \Pi \text{ is compact and convex}\}.$$

If, for any $u \in X$, $Q(u)$ is closed (convex), then the multivalued map $Q : E \to \mathcal{P}(X)$ is closed (convex) valued. Q is u.s.c. (upper semicontinuous), $Q(u)$ is a nonempty, closed subset of X, and the set Λ containing $Q(u)$, there is an open neighborhood $\mathcal{M}$ of u such that $Q(\mathcal{M}) \subseteq \Lambda$, where Λ is open set of E. If $Q(\Lambda)$ is relatively compact for any bounded subset $\Lambda \subset X$, then Q is said to be completely continuous. For a given separable metric space (X,d) and measure space (Π,Σ), if for each closed set $V \subseteq X$, $Q^{-1}(V) = \{t \in \Pi : Q(t) \cap V \neq \emptyset\} \in \Sigma$, then the multivalued mapping $Q : \Pi \to \mathcal{P}(X)$ is called measurable.

The following theorem appearing in [18] will be used to prove the main result.

Theorem 2.1. *If Π is a subspace of Banach space X and Π is convex, closed, and nonempty, with $0 \in \Pi$, and $\Gamma : \Pi \to P_{cp,c\varsigma}(\Pi)$ is an upper semicontinuous multivalued map from a bounded set to a relatively compact set, then one of the two statements below is true:*

(i) The set $\Lambda = \{u \in \Pi : u \in \eta\Gamma(u), \eta \in (0,1)\}$ is unbounded.

(ii) There is a $u \in \Pi$ such that $u \in \Gamma(u)$, i.e., Γ has a fixed point.

Let us understand some basic concepts of non-smooth analysis (for more concepts, readers can refer to [19]). We suppose that the function $l : X \to \mathbb{R}$ is locally Lipschitz. Let

$$l^0(u;g) := \limsup_{\lambda\to 0^+,\xi\to u} \frac{l(\xi+\lambda g) - l(\xi)}{\lambda}$$

represent the definition of the generalized directional derivative of l in the direction g at $u \in X$.

The Clarke's subdifferential or generalized gradient of l at $u \in X$ is the subset of X^* given by

$$\partial l(u) := \{u^* \in X^* : l^0(u;g) \geq \langle u^*, g\rangle,\ \forall\, g \in X\}.$$

Lemma 2.1. *[19] If the function $l : X \to \mathbb{R}$ is locally Lipschitz, then the follwing conditions are satisfied:*

(i) for any $u, g \in X$, one has $l^0(u,g) = \max\{\langle u^*, g\rangle : x^* \in \partial l(u)\}$;

(ii) for all $u \in X, \partial l(u)$ is a convex, nonempty, and weak-compact subset of X^* and $\|u^*\|_{X^*} \leq \alpha$ for any $u^* \in \partial l(u)$, where $\alpha > 0$ is the Lipschitz constant of l near u.

Next, we consider the hemivariational inclusion problem under nonlocal conditions:

$$\begin{cases} u'(t) \in \mathcal{N}(t)u(t) + B\mu(t) + K(t, u(t)), \quad t \in I = [0, b], \\ u(0) = \sum_{k=1}^{m} d_k u(t_k), \end{cases} \tag{2.1}$$

where $K : I \times Y \to 2^Y$ is a multivalued map. 2^Y represents the power set of Y, which means that 2^Y is the set composed of all subsets of Y.

Below, we will suppose that the conditions from [20] hold:

(B1) the domain $D(\mathcal{N}(t))$ of $\mathcal{N}(t)(t \in I)$ is independent of time t and it is dense in Y;

(B2) for all $t \in I, \kappa \in \mathbb{C}$, the resolvent $R(\kappa, \mathcal{N}(t))$ of $\mathcal{N}(t)$ exists, with Re $\kappa \leq 0$, and there exists a positive number C_1 such that

$$\|R(\kappa, \mathcal{N}(t))\| \leq \frac{C_1}{|\kappa| + 1};$$

(B3) there exist $\delta \in (0, 1]$ and $C_2 > 0$ such that

$$\|\mathcal{N}(t) - \mathcal{N}(\gamma)\mathcal{N}^{-1}(\xi)\| \leq C_2 |t - \gamma|^{\delta}, \quad \forall\, t, \gamma, \xi \in I;$$

(B4) for all $t \in I$, there is a $\kappa \in \rho(\mathcal{N}(t))$ such that the resolvent $R(\kappa, \mathcal{N}(t))$ is a compact operator, where $\rho(\mathcal{N}(t))$ is the resolvent set of $\mathcal{N}(t)$.

Definition 2.1. *If the following conditions are satisfied:*

(i) $\mathcal{T}(s, s) = \mathcal{I}, \mathcal{T}(t, r)T(r, s) = \mathcal{T}(t, s)$ *for* $0 \leq s \leq r \leq t \leq b$,

(ii) $(t, s) \mapsto \mathcal{T}(t, s)$ *is strongly continuous for* $0 \leq s \leq t \leq b$,

then family of bounded linear operators $\mathcal{T}(t, s)$ *on Banach space X is called evolution system, where* $0 \leq s \leq t \leq b$.

Lemma 2.2. *[16] If conditions* $(B1)$–$(B3)$ *are satisfied, then there exists a unique evolution system* $\mathcal{T}(t, s)$ *that satisfies the following conditions:*

(i) $\|\mathcal{T}(t, s)\| \leq M$ *and M is a positive constant.*

(ii) $\mathcal{T}(t, s) : Y \to D(\mathcal{N}(t))$ *and* $t \to \mathcal{T}(t, s)$ *is strongly differentiable in Y,* $\frac{\partial}{\partial t}\mathcal{T}(t, s)$ *is strongly continuous, where* $0 \leq s \leq t \leq b$.

In addition,

$$\frac{\partial}{\partial t}\mathcal{T}(t, s) + \mathcal{N}(t)\mathcal{T}(t, s) = 0,$$

$$\|\frac{\partial}{\partial t}\mathcal{T}(t, s)\| = \|\mathcal{N}(t)\mathcal{T}(t, s)\| \leq \frac{M}{t - s},$$

$$\|\mathcal{N}(t)\mathcal{T}(t, s)\mathcal{N}^{-}(s)\| \leq M.$$

(iii) For any $g \in D(\mathcal{N}(t))$,

$$\frac{\partial}{\partial s}\mathcal{T}(t, s)g = \mathcal{T}(t, s)\mathcal{N}(s)g,$$

and $\mathcal{T}(t, s)g$ *is differentiable with respect to s.*

Lemma 2.3. *[21] If* $\mathcal{N}(t)$ *satisfies hypotheses* $(B1)$–$(B4)$ *and* $\mathcal{T}(t, s)$ *is the evolution system generated by* $\mathcal{N}(t)$*, then* $\mathcal{T}(t, s)$ *is a compact operator whenever* $t - s > 0$.

The following conditions will be used to prove the main results:

(C1) $\sum_{k=1}^{m} |d_k| < \frac{1}{M}$.

(C2) $\|B\| \leq M_1$, where $M_1 > 0$ is a constant.

From Lemma 2.2 and conditions (C1), we get that

$$\| \sum_{k=1}^{m} d_k \mathcal{T}(t_k, 0)\| \leq M \sum_{k=1}^{m} |d_k| < 1.$$

From the above formula and operator spectrum theorem, we get that

$$\mathcal{O} := (I - \sum_{k=1}^{m} d_k \mathcal{T}(t_k, 0))^{-1}$$

is bounded and $D(\mathcal{O}) = X$. Using Neumann expression,

$$\mathcal{O} = \sum_{n=0}^{\infty} (\sum_{k=1}^{m} d_k \mathcal{T}(t_k, 0))^n.$$

Hence,

$$\|\mathcal{O}\| \leq \sum_{n=0}^{\infty} \| \sum_{k=1}^{m} d_k \mathcal{T}(t_k, 0)\|^n = \frac{1}{1 - \| \sum_{k=1}^{m} d_k \mathcal{T}(t_k, 0)\|} \leq \frac{1}{1 - M \sum_{k=1}^{m} |d_k|}. \tag{2.2}$$

For every $\mathcal{H} \in L^2(I, Y)$, we discuss the following nonlocal problem for linear evolutionary equation:

$$\begin{cases} u'(t) = \mathcal{N}(t)u(t) + \mathcal{H}(t), \quad t \in I, \\ u(0) = \sum_{k=1}^{m} d_k u(t_k). \end{cases} \tag{2.3}$$

Lemma 2.4. *[20] If the conditions* $(B1)$–$(B3)$ *are satisfied, then system* (2.3) *has a unique mild solution* $u \in C(I, Y)$, *expressed by the following formula:*

$$u(t) = \int_0^b \mathcal{G}(t, s)\mathcal{H}(s)ds,$$

where

$$\mathcal{G}(t, s) = \sum_{k=1}^{m} \mathcal{Y}_{t_k}(s)\mathcal{T}(t, 0)\mathcal{O}\mathcal{T}(t_k, s) + \mathcal{Y}_t(s)\mathcal{T}(t, s), \quad t, s \in [0, b], \tag{2.4}$$

$$\mathcal{Y}_{t_k}(s) = \begin{cases} d_k, & s \in [0, t_k), \\ 0, & s \in [t_k, b], \end{cases} \quad \mathcal{Y}_t(s) = \begin{cases} 1, & s \in [0, t), \\ 0, & s \in [t, b]. \end{cases} \tag{2.5}$$

Definition 2.2. *Given* $\mu \in L^2(I, U)$. *If there exists* $f \in L^2(I, Y)$ *such that* $f(t) \in K(t, u(t))$ *for all* $t \in I$ *and*

$$u(t) = \int_0^b \mathcal{G}(t, s)(B\mu(s) + f(s))ds,$$

then $u \in C(I, Y)$ *is called the mild solution of problem* (2.1).

In fact, from the generalized gradient, we know that problem (1.1) is equivalent to the problem

$$\begin{cases} u'(t) \in \mathcal{N}(t)u(t) + B\mu(t) + \partial J(t, u(t)), \quad \forall\, t \in I, \\ u(0) = \sum\limits_{k=1}^{m} d_k u(t_k), \end{cases} \tag{2.6}$$

where ∂J denotes the generalized Clarke's subdifferential of $J(t, \cdot) : Y \to \mathbb{R}$.

Definition 2.3. *Given $\mu \in L^2(I, U)$, if there exists $f \in L^2(I, Y)$ such that $f(t) \in \partial J(t, u(t))$ for all $t \in I$ and*

$$u(t) = \int_0^b \mathcal{G}(t, s)(B\mu(s) + f(s))ds,$$

then $u \in C(I, Y)$ is called the mild solution of inequality (1.1).

Let $\mathcal{K}_b(K) = \{u(b) \in Y : u(\cdot)$ be a mild solution of system (2.1), control function $\mu \in L^2(I, U)$ with $u(0) \in Y\}$.

Definition 2.4. *If $\overline{\mathcal{K}_b(K)} = Y$, then problem* (2.1) *is called approximately controllable on I, where $\overline{\mathcal{K}_b(K)}$ is the closure of $\mathcal{K}_b(K)$.*

3. Existence of mild solutions

For convenience, make the following three assumptions for the multivalued map $K : I \times Y \to 2^Y$:

$H_1(K)$ K is jointly measurable on $I \times Y$ and has convex, nonempty, and weakly compact values.

$H_2(K)$ For all $t \in I$, if $u_n \to u$ in Y and $h_n \to h$ weakly in Y with $h_n \in K(t, u_n)$, i.e., $K(t, \cdot)$ has a strongly-weakly closed graph, then $h \in K(t, u)$.

$H_3(K)$ There is a positive constant c and a function $a \in L^2(I, \mathbb{R})$ such that

$$\|K(t, u)\|_Y := \sup\{\|\chi\|_Y : \chi \in K(t, u)\} \leq a(t) + c\|u\|_Y, \ \forall\, t \in I, u \in Y.$$

Now we define the multivalued map $\mathcal{M} : L^2(I, Y) \to 2^{L^2(I,Y)}$ as

$$\mathcal{M}(u) = \{f \in L^2(I, Y) : f(t) \in K(t, u(t)), \ \forall\, t \in I, u \in L^2(I, Y)\}.$$

Lemma 3.1. *[22] If the conditions $H_1(K)$–$H_3(K)$ are satisfied, then for any function $u \in L^2(I, Y)$, the set $\mathcal{M}(u)$ is convex, nonempty, and weakly compact.*

Lemma 3.2. *[23] If the conditions $H_1(K)$–$H_3(K)$ are satisfied, then operator $\mathcal{M}$ satisfies the following: if $u_n \to u$ in $L^2(I, Y)$, $f_n \to f$ weakly in $L^2(I, Y)$, and $f_n \in \mathcal{M}(u_n)$, then $f \in \mathcal{M}(u)$.*

Theorem 3.1. *Assume that conditions $(B1)$–$(B4)$, $(C1)$–$(C2)$ and $H_1(K)$–$H_3(K)$ are satisfied. Then, for every $\mu \in L^2(I, U)$, the system* (2.1) *has a mild solution on I.*

Proof. Based on Lemma 3.1, define a multivalued mapping $\Gamma : C(I, Y) \to 2^{C(I,Y)}$,

$$\Gamma(u) = \{h \in C(I, Y) : h(t) = \int_0^b \mathcal{G}(t, s)(B\mu(s) + f(s))ds, \ f \in \mathcal{M}(u)\}. \tag{3.1}$$

Step 1. Γ is bounded, i.e., it maps bounded sets into bounded sets in $C(I, Y)$.

For each $r \geq MN_2(MN_1+1)$, let $B_r := \{u \in C(I,Y) : \|u\|_{C(I,Y)} \leq r\}$. Set $\varphi \in \Gamma(u)$ with $u \in B_r$. Then, there exists $f \in \mathcal{M}(u)$ such that

$$\varphi(t) = \int_0^b \mathcal{G}(t,s)(B\mu(s) + f(s))ds, \quad \forall\, t \in I, \tag{3.2}$$

where

$$N_1 = \frac{\sum_{k=1}^{m} |d_k|}{1 - M\sum_{k=1}^{m} |d_k|}, \quad N_2 = \int_0^b (\|a\|_{L^2(I,\mathbb{R})} + cr + M_1 \cdot \|\mu\|_{L^2(I,U)})ds. \tag{3.3}$$

By Lemma 2.2 and $H_3(K)$, for any $t \in I$, we know that

$$\begin{aligned}
\|\varphi(t)\|_Y \leq& \int_0^b \|\mathcal{G}(t,s)\| \cdot \|B\mu(s) + f(s)\|_Y ds \\
=& \int_0^b \|\sum_{k=1}^{m} \mathcal{Y}_{t_k}(s)\mathcal{T}(t,0)\mathcal{O}\mathcal{T}(t_k,s) + \mathcal{Y}_t\mathcal{T}(t,s)\| \cdot \|B\mu(s) + f(s)\|_Y ds \\
\leq& M \cdot \|\mathcal{O}\| \cdot \int_0^b \sum_{k=1}^{m} |\mathcal{Y}_{t_k}(s)| \cdot \|\mathcal{T}(t_k,s)\| \cdot \|f(s) + B\mu(s)\|_Y ds \\
\leq& M^2 N_1 \int_0^{t_k} ((\|a\|_{L^2(I,\mathbb{R})} + c\|u(s))\|_Y + M_1 \cdot \|\mu\|_{L^2(I,U)})ds \\
&+ M \int_0^t ((\|a\|_{L^2(I,\mathbb{R})} + c\|u(s))\|_Y + M_1 \cdot \|\mu\|_{L^2(I,U)})ds \\
\leq& M^2 N_1 \int_0^{t_k} ((\|a\|_{L^2(I,\mathbb{R})} + cr + M_1 \cdot \|\mu\|_{L^2(I,U)})ds \\
&+ M \int_0^t ((\|a\|_{L^2(I,\mathbb{R})} + cr + M_1 \cdot \|\mu\|_{L^2(I,U)})ds \\
\leq& MN_2(MN_1 + 1) \\
\leq& r.
\end{aligned}$$

Step 2. For any $r > 0, \Gamma(B_r) := \cup\{\Gamma(u) : x \in B_r\}$ is equicontinuous. Using condition $H_3(K)$, for $0 < t_1 < t_2 \leq b$ and $\varepsilon > 0$,

$$\begin{aligned}
\|\varphi(t_2) - \varphi(t_1)\|_Y =& \|\int_0^b (\mathcal{G}(t_2,s) - \mathcal{G}(t_1,s)) \cdot (B\mu(s) + f(s))ds\|_Y \\
=& \|\int_0^b (\mathcal{T}(t_2,0) - \mathcal{T}(t_1,0)) \sum_{k=1}^{m} \mathcal{Y}_{t_k}(s)\mathcal{O}\mathcal{T}(t_k,s)(B\mu(s) + f(s))ds \\
&+ \int_0^b (\mathcal{Y}_{t_2}(s)\mathcal{T}(t_2,s) - \mathcal{Y}_{t_1}(s)\mathcal{T}(t_1,s))(B\mu(s) + f(s))ds \\
&+ \int_0^b \mathcal{Y}_{t_1}(s)\mathcal{T}(t_2,s)(B\mu(s) + f(s))ds - \int_0^b \mathcal{Y}_{t_2}(s)\mathcal{T}(t_2,s)(B\mu(s) + f(s))ds\|_Y \\
=& \|\int_0^b (\mathcal{T}(t_2,0) - \mathcal{T}(t_1,0)) \sum_{k=1}^{m} \mathcal{Y}_{t_k}(s)\mathcal{O}\mathcal{T}(t_k,s)(B\mu(s) + f(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^b \mathcal{Y}_{t_1}(s)(\mathcal{T}(t_2,s)-\mathcal{T}(t_1,s))(B\mu(s)+f(s))ds\\
&+\int_0^b (\mathcal{Y}_{t_2}(s)-\mathcal{Y}_{t_1}(s))\mathcal{T}(t_2,s)(B\mu(s)+f(s))ds\|_Y\\
\le&\|(\mathcal{T}(t_2,0)-\mathcal{T}(t_1,0))\int_0^b \sum_{k=1}^m \mathcal{Y}_{t_k}(s)\mathcal{O}\mathcal{T}(t,s)(B\mu(s)+f(s))ds\|_Y\\
&+\int_0^{t_1} \|\mathcal{T}(t_2,s)-\mathcal{T}(t_1,s)\|\cdot(\|a\|_{L^2(I,\mathbb{R})}+c\|u(s)\|_H+M_1\cdot\|\mu\|_{L^2(I,U)})ds\\
&+M\int_{t_1}^{t_2} (\|a\|_{L^2(I,\mathbb{R})}+c\|u(s)\|_Y M_1\cdot\|\mu\|_{L^2(I,U)})ds\\
=&I_1+I_2+I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \|(\mathcal{T}(t_2,0)-\mathcal{T}(t_1,0))\int_0^b \sum_{k=1}^m \mathcal{Y}_{t_k}(s)\mathcal{O}\mathcal{T}(t,s)(B\mu(s)+f(s))ds\|_Y,\\
I_2 &= \int_0^{t_1} \|(\mathcal{T}(t_2,s)-\mathcal{T}(t_1,s))\|\cdot(\|a\|_{L^2(I,\mathbb{R})}+c\|u(s)\|_Y+\|\mu\|_{L^2(I,U)})ds, \qquad (3.4)\\
I_3 &= M\int_{t_1}^{t_2} (\|a\|_{L^2(I,\mathbb{R})}+c\|u(s)\|_Y M_1\|\mu\|_{L^2(I,U)})ds.
\end{aligned}
$$

For I_1, applying Lemma 2.2, and Eqs (2.2), (2.4), (2.5), (3.3), and (3.4) along with hypothesis $H_3(K)$, we know that

$$
\begin{aligned}
\|\int_0^b \sum_{k=1}^m \mathcal{Y}_{t_k}(s)\mathcal{O}\mathcal{T}(t_k,s)(B\mu(s)+f(s))ds\|_Y &\le MN_1\int_0^{t_k} (\|a\|_{L^2(I,\mathbb{R})}+c\|u\|_{L^2(I,Y)}+M_1\|\mu\|_{L^2(I,U)})ds\\
&\le MN_1N_2. \qquad (3.5)
\end{aligned}
$$

Therefore, we get that $I_1 \to 0$ as $t_2 - t_1 \to 0$. According to the compactness of $\mathcal{T}(t,s)$, we can easily get that $I_2, I_3 \to 0$ as $t_2 \to t_1$.

Setp 3. Γ is completely continuous.

First, we will prove $\Gamma(B_r)$ is relatively compact in $C(I,Y)$. Let

$$\Phi(t) := \{\varphi(t) : \varphi \in \Gamma(B_r),\ \forall\, t \in I\}.$$

From (3.1), we can see that

$$
\begin{aligned}
(\Gamma\varphi)(0) &= \int_0^b \mathcal{G}(0,s)\cdot(B\mu(s)+f(s))ds\\
&= \int_0^b \sum_{k=1}^m \mathcal{Y}_{t_k}\mathcal{T}(0,0)\mathcal{O}\mathcal{T}(t_k,s)(B\mu(s)+f(s))ds\\
&= \sum_{k=1}^m d_k\mathcal{O}\int_0^{t_k} \mathcal{T}(t_k,s)(B\mu(s)+f(s))ds. \qquad (3.6)
\end{aligned}
$$

For every $0 < \varepsilon < t_k$ and $\varphi \in \Gamma(B_r)$, we can define the operator

$$(\Gamma^{\varepsilon}\varphi)(0) = \mathcal{T}(t_k, t_k - \frac{\varepsilon}{2})\mathcal{T}(t_k - \frac{\varepsilon}{2}, t_k - \varepsilon) \sum_{k=1}^{m} d_k O \int_0^{t_k-\varepsilon} \mathcal{T}(t_k - \varepsilon, s)(B\mu(s) + f(s))ds,$$

From the compactness of $\mathcal{T}(t_k - \frac{\varepsilon}{2}, t_k - \varepsilon)$, $\{(\Gamma^{\varepsilon}\varphi)(0)\}$ is relatively compact in Y. In addition,

$$\begin{aligned}
\|(\Gamma^{\varepsilon}\varphi)(0) - (\Gamma\varphi)(0)\|_H \leq & \|\mathcal{T}(t_k, t_k - \frac{\varepsilon}{2})\mathcal{T}(t_k - \frac{\varepsilon}{2}, t_k - \varepsilon) \sum_{k=1}^{m} d_k O \int_0^{t_k-\varepsilon} \mathcal{T}(t_k - \varepsilon, s)(B\mu(s) + f(s))ds \\
& - \sum_{k=1}^{m} d_k O \int_0^{t_k-\varepsilon} \mathcal{T}(t_k, s)(B\mu(s) + f(s))ds\|_Y \\
& + \|\sum_{k=1}^{m} d_k O \int_{t_k-\varepsilon}^{t_k} \mathcal{T}(t_k, \varepsilon)(B\mu(s) + f(s))ds\|_Y \\
\leq & MN_1 \int_{t_k-\varepsilon}^{t_k} \|B\mu(s) + f(s)\|_Y ds \\
\leq & MN_1 (\int_{t_k-\varepsilon}^{t_k} \|a\|_{L^2(I,\mathbb{R})} ds + cr + M_1 \|\mu\|_{L^2(I,U)} ds) \\
& \to 0 \quad \text{as} \quad (\varepsilon \to 0).
\end{aligned} \tag{3.7}$$

Therefore, we prove the existence of relatively compact sets $\{(\Gamma^{\varepsilon}\varphi)\}(0)$ that are arbitrarily close to $\{(\Gamma\varphi)(0) : \varphi \in \Gamma(B_r)\}$, and we can easily see that $(\Gamma\varphi)(0)$ is relatively compact in Y. We define the operator $\Gamma^{\varepsilon}\varphi$ as

$$(\Gamma^{\varepsilon}\varphi)(t) = \sum_{k=1}^{m} d_k \mathcal{T}(t,0) O \int_0^{t_k} \mathcal{T}(t_k, s)(B\mu(s) + f(s))ds + \mathcal{T}(t, t - \frac{\varepsilon}{2})\mathcal{T}(t - \frac{\varepsilon}{2}, t - \varepsilon) \int_0^{t-\varepsilon} \mathcal{T}(t - \varepsilon, s)(B\mu(s) + f(s))ds.$$

From the compactness $\mathcal{T}(t - \frac{\varepsilon}{2}, t - \varepsilon)$ and $\mathcal{T}(t, 0)$ in Y, we can see that $\{(\Gamma^{\varepsilon}\varphi)(t) : \varphi \in B_r\}$ is relatively compact in Y for any $\varepsilon \in (0, t)$. Using the same method as (3.7), we get that there are relatively compact sets $\{(\Gamma^{\varepsilon}\varphi)(t) : \varphi \in B_r\}$ arbitrarily close to $\{(\Gamma\varphi) : \varphi \in B_r\}$ that are also relatively compact in Y. Hence, $\{(\Gamma\varphi)(t) : \varphi \in B_r\}$ is also relatively compact in Y for $t \in I$. Therefore, the set $\{(\Gamma\varphi)(t) : \varphi \in B_r\}$ is relatively compact in Y for $t \in I$. Moreover, this means that $\Phi(t)$ is relatively compact in Y. The proof of this step can be completed by using the Arzela-Ascoli theorem combined with step 2.

Step 4. Γ has a closed graph.

We suppose that $u_n \to u$ in $C(I, Y)$ and $\varphi_n \to \varphi^*$ in $C(I, Y)$ with $\varphi_n \in \Gamma(u_n)$. If $\varphi_n \in \Gamma(u)$ and there exists $f_n \in \mathcal{M}(u_n)$ such that

$$\varphi_n(t) = \int_0^b \mathcal{G}(t, s)(B\mu(s) + f_n(s))ds, \tag{3.8}$$

then we can see from condition $H_3(K)$ that $\{f_n\}_{n\geq 1}$ is bounded in $L^2(I, Y)$. Therefore, we can assume that

$$f_n \to f^* \text{ weakly in } L^2(I, Y). \tag{3.9}$$

By the compactness of $\mathcal{T}(t,s)$, (3.8), and (3.9), we know that

$$\varphi_n(t) \to \int_0^b \mathcal{G}(t,s)(B\mu(s) + f^*(s))ds. \tag{3.10}$$

Because $\varphi_n \to \varphi^*$ in $C(I,Y)$ and $f_n \in \mathcal{M}(u_n)$, from Lemma 3.2 and (3.10), we know that $f^* \in \mathcal{M}(u^*)$. Therefore, we can prove that $\varphi^* \in \Gamma(u^*)$.

Step 5. A priori estimate.

It can be seen from [23, Proposition 3.1.2] that Γ is u.s.c. From the above four steps, it can be concluded that Γ satisfies all the conditions of Theorem 2.1.

We assume that

$$\Lambda := \{\mathcal{T} \in C(I,Y) : u \in \eta\Gamma(u),\ 0 < \eta < 1\}.$$

Let $u \in \Lambda, 0 < \eta < 1$. There is an $f \in \mathcal{M}(u)$ such that

$$u(t) = \eta \int_0^b \mathcal{G}(t,s)(B\mu(s) + f(s))ds.$$

By condition $H_3(K)$,

$$\|u(t)\|_Y \le \|\eta \int_0^b \mathcal{G}(t,s)(B\mu(s) + f(s))ds\|_Y \le \| \int_0^b \mathcal{G}(t,s)(B\mu(s) + f(s))ds\|_Y \le MN_2(MN_1 + 1),$$

for all $t \in J$.

From Theorem 2.1, Γ has a fixed point, which means that this fixed point is the mild solution of system (2.1). □

4. Approximate controllability results

For convenience, we first introduce the non-bounded linear operator

$$\Gamma_0^b = \int_0^b \mathcal{G}(b,s)BB^*\mathcal{G}^*(b,s)ds$$

on Y, where B^* is the adjoint operator of B,

$$\mathcal{G}^*(b,s) = \sum_{k=1}^{m} \mathcal{Y}_{t_k}(s)T^*(b,0)O^*\mathcal{T}^*(t_k,s) + \mathcal{Y}_b(s)\mathcal{T}^*(b,s), \quad s \in [0,b],$$

where O^* and $\mathcal{T}^*(t,s)$ denote the adjoint operators of O and $\mathcal{T}(t,s)$. Hence, the inverse of $\varepsilon\mathcal{I} + \Gamma_0^b$ exists, i.e., the resolvent

$$R(\varepsilon, -\Gamma_0^b) = (\varepsilon\mathcal{I} + \Gamma_0^b)^{-1},\ \forall\, \varepsilon > 0.$$

From Lemma 3.1, it easy to see that for each $u \in C(I,Y) \subset L^2(I,Y)$, $\mathcal{M}(u) \neq \emptyset$ is satisfied. Therefore, for any $\varepsilon > 0$, let $\Gamma_\varepsilon : C(I,Y) \to 2^{C(I,Y)}$ as follows:

$$\Gamma_\varepsilon(u) = \{h \in C(I,Y) : h(t) = \int_0^b \mathcal{G}(t,s)(B\mu_\varepsilon(s) + f(s))ds, f \in \mathcal{M}(u)\},$$

where the control function $\mu_\varepsilon(t) = B^*\mathcal{G}^*(b,s)R(\varepsilon,\Gamma_0^b)p(u_\varepsilon(\cdot))$ with

$$p(u_\varepsilon(\cdot)) = u_b - \int_0^b \mathcal{G}(b,s)f^\varepsilon(s)ds. \tag{4.1}$$

The following assumptions will be used in the subsequent proof process.

$H_4(K)$: There exists a function $a \in L^2(J,\mathbb{R})$ such that

$$\|K(t,u)\|_Y := \sup\{\|\chi\|_Y H : \chi \in K(t,u) \le a(t), \forall\, t \in I\}.$$

Theorem 4.1. *Assume that conditions* $(B1)$–$(B4)$, $(C1)$–$(C2)$, $H_1(K)$, $H_2(K)$, *and* $H_4(K)$ *are satisfied. Then, system* (2.1) *is approximately controllable on* I.

Proof. Since condition $H_4(K)$ implies condition $H_3(K)$, it follows from Theorem 3.1 that Γ_ε has a fixed point $u_\varepsilon \in C(I,Y)$. This means that $u_\varepsilon \in C(I,Y)$ is a mild solution of (2.1) for every $\varepsilon > 0$. Hence there exists $f^\varepsilon \in \mathcal{M}(u^\varepsilon)$ such that

$$u_\varepsilon(t) = \int_0^b \mathcal{G}(t,s)(f^\varepsilon(s) + BB^*\mathcal{G}^*(b,t)R(\varepsilon,-\Gamma_0^b p(u_\varepsilon(\cdot))))ds. \tag{4.2}$$

Hence,

$$\begin{aligned} u_\varepsilon(b) &= \int_0^b \mathcal{G}(a,s)(f^\varepsilon(s) + BB^*\mathcal{G}^*(b,t)R(\varepsilon,-\Gamma_0^b p(u_\varepsilon(\cdot))))ds \\ &= u_b - p(u_\varepsilon(\cdot)) + \int_0^b \mathcal{G}(b,s)BB^*\mathcal{G}^*(b,s)R(\varepsilon,\Gamma_0^b)p(u_\varepsilon(\cdot))ds \\ &= u_b - p(u_\varepsilon(\cdot)) + \int_0^b \mathcal{G}(b,s)BB^*\mathcal{G}^*(b,s)R(\varepsilon,\Gamma_0^b)p(u_\varepsilon(\cdot))ds \\ &= u_b - (\varepsilon\mathcal{I} + \Gamma_0^b)R(\varepsilon,\Gamma_0^b)p(u_\varepsilon(\cdot)) + \Gamma_0^b R(\varepsilon,\Gamma_0^b)p(u_\varepsilon(\cdot)) \\ &= u_b - \varepsilon R(\varepsilon,\Gamma_0^b)p(u_\varepsilon(\cdot)) \end{aligned} \tag{4.3}$$

with $p(u_\varepsilon(\cdot)) = u_b - \int_0^b \mathcal{G}(b,s)f^\varepsilon(s)ds$.

From condition H_4(K), we can see

$$\int_0^b \|f^\varepsilon(s)\|_Y ds \le \|a\|_{L^2(I,\mathbb{R})}\sqrt{b}.$$

Using (4.2) and Gronwall's inequality, it is easy to see that $\{f^\varepsilon\} \in L^2(I,Y)$ is bounded. Therefore, there exists a subsequence converging weakly to f in $L^2(I,Y)$.

Let

$$z := u_b - \int_0^b \mathcal{G}(b,s)f(s)ds. \tag{4.4}$$

Therefore, from (4.1) and (4.4), we get that

$$\|p(u_\varepsilon) - z\|)_Y \le \|\int_0^b \mathcal{G}(b,s)(f^\varepsilon(s) - f(s))ds\|_Y. \tag{4.5}$$

Because $\mathcal{T}(t,s)$ is compact, $\mathcal{G}(t,s)$ is also compact. This means that

$$g(t) \to \int_0^t \mathcal{G}(t,s)g(s)ds,\ \ \forall\, t \in I$$

is compact and

$$\int_0^b \mathcal{G}(t,s)(f^{\varepsilon}(s) - f(s))ds \to 0 \quad (as\ \varepsilon \to 0). \tag{4.6}$$

Hence, from (4.5) and (4.6), we have

$$\|p(u_{\varepsilon}) - z\|_Y \to 0,\ \text{as } \varepsilon \to 0. \tag{4.7}$$

Hence, from (4.3) and (4.7), we get that

$$\begin{aligned} \|u_{\varepsilon}(b) - u_b\|_Y = &\|\varepsilon R(\varepsilon, \Gamma_0^b)p(u_{\varepsilon})\|_Y. \\ \le&\|\varepsilon R(\varepsilon, \Gamma_0^b)z\|_Y + \|\varepsilon R(\varepsilon, \Gamma_0^b)\| \cdot \|p(u_{\varepsilon}) - z\|_Y \\ \le&\|\varepsilon R(\varepsilon, \Gamma_0^b)z\|_Y + \|p(u_{\varepsilon}) - z\|_Y \\ &\to 0, \quad as \quad \varepsilon \to 0. \end{aligned} \tag{4.8}$$

Consider the arbitrariness of $u_b \in Y$, such that problem (2.1) is approximately controllable on I. □

Now we consider the controlling inequality (1.1). Assume that the function $J : I \times Y \to \mathbb{R}$ is as follows:
B_1(J) For every $u \in Y$, the function $t \mapsto J(t,u)$ is measurable.
B_2(J) For any $t \in I$, the function $t \mapsto J(t,u)$ is locally Lipshitz.
B_3(J) There exists the function $a \in L^2(I, \mathbb{R})$, for $\forall\, t \in I, u \in Y$, such that

$$\|\partial J(t,u)\|_Y := \sup\{\|f\|_Y : f \in \partial J(t,u)\} \le a(t).$$

Corollary 4.1. *If the conditions* $(B1)$–$(B4)$, $(C1)$–$(C2)$, *and* $B_1(J)$–$B_3(J)$ *are satisfied, then inequality* (1.1) *is approximately controllable on J.*

Proof. Conditions B_1(J)–B_3(J) imply conditions H_1(K), H_2(K), and $H_4(K)$ for $K(t,u) = \partial J(t,u)$. Hence, the proof of this inference can be completed by Theorem 4.1 and Definitions 2.3. □

5. Application

Byszewski studied the existence and uniqueness of moderate solutions for evolutionary equations with non-local initial conditions in [6]. The author pointed out that evolutionary equations with non-local initial conditions can more realistically depict the diffusion phenomena of gases in a heavy tube. In the context of practical problems, non-local initial conditions have a better application effect compared to conventional initial conditions. Therefore, it is necessary to discuss the non-local condition. To demonstrate that our main results can solve practical problems, we consider the following non local condition problem:

$$\begin{cases} \langle -\frac{\partial}{\partial t}w(y,t) + \frac{\partial^2}{\partial y^2}w(y,t) - b_1(t)w(y,t) + k\vartheta(y,t), \rho\rangle_H + J^0(w(y,t),t;\rho) \geq 0, \quad \forall\, \rho \in Y, \\ w(0,t) = w(\pi,t) = 0, \quad \forall\, t \in P, \\ w(y,0) = \sum\limits_{k=1}^{n} d_k w(t_k), \quad \forall\, y \in [0,\pi], \end{cases} \tag{5.1}$$

where $w(y,t)$ represents temperature, t represents time, and $\frac{\partial}{\partial t}w(y,t)$ represents the rate of change of temperature versus time. The function $b_1(t)$ is a continuously differentiable, Y is a Hilbert space,

$$b_{\min} = \min_{t\in P} b_1(t) > -1,$$

a_1 is a constants, $P = [0, a_1]$, $\vartheta \in L^2(J, L^2(0,\pi;\mathbb{R}))$, and $d_k \in \mathbb{R}, k = 1, 2, \cdots, m$.

Inequality (5.1) can be transformed into the following evolutionary inclusion problem with non-local conditions.

$$\begin{cases} \frac{\partial}{\partial t}w(y,t) \in \frac{\partial^2}{\partial y^2}w(y,t) - b_1(t)w(y,t) + k\vartheta(y,t) + \partial J(w(y,t),t), \quad \forall\, t \in P = [0,a_1], \\ w(0,t) = w(\pi,t) = 0, \quad \forall\, t \in P, \\ w(y,0) = \sum\limits_{k=1}^{n} d_k w(t_k), \quad \forall\, y \in [0,\pi]. \end{cases} \tag{5.2}$$

Let $Y = L^2((0,\pi);\mathbb{R})$. We define the operator Q

$$Qw := \frac{\partial^2}{\partial y^2}w, \quad w \in D(Q), \tag{5.3}$$

where

$$D(Q) := \{w \in Y, w'' \in Y, w(0) = w(\pi) = 0\}.$$

According to Pazy [20], we can conclude that Q generates a compact and analytic C_0-semigroup in Y, with eigenvalues $-n^2$, and eigenvectors $v_n = \sqrt{\frac{2}{\pi}}\sin(ny), n \in \mathbb{N}^+$.

We define the operator $\mathcal{N}$ on Y as

$$\mathcal{N}(t)w = Qw - b_1(t)w.$$

The domain

$$D(\mathcal{N}(t)) = D(Q), \quad \forall\, t \in P.$$

It is known from [24] that there is a constants $C_1 \geq 0$ and there exists $\lambda \in \mathbb{C}$ such that $\mathcal{N}(t)$ satisfies the condition ($B2$). Additionally, using the coefficient b_1 and results from [20], we know that there exists constants $C_2 > 0$ and $\beta \in (0,1]$ such that condition ($B3$) is satisfied. Because the operator Q generates a compact semigroup, it is easy to verify, and condition ($B4$) holds. Hence, the $\mathcal{N}(t)$ generates a strongly continuous evolution family

$$\mathcal{T}(t,s)w = \sum_{n=1}^{\infty} e^{-(\int_s^t b(\tau)d\tau + n^2(t-s))}\langle w, v_n\rangle v_n, \quad 0 \leq s \leq t \leq 1,\ w \in Y.$$

It is easy to know that

$$\|\mathcal{T}(t,s)\| \leq e^{-(1+b_{\min})(t-s)}, \tag{5.4}$$

and we have

$$M := \sup_{0\leq s\leq t\leq a_1} \|\mathcal{T}(t,s)\| = 1.$$

From [20], we know that $\mathcal{T}(t,s)$ is compact. The bounded linear operator

$$\mathcal{B}\vartheta(t) = k\vartheta(\cdot,t).$$

At this point, we can easily see that the partial differential and variational inequality problem (5.1) is transformed into an abstract that includes problem (2.6).

Theorem 5.1. *If condition $B_3(J)$ and $\sum_{k=1}^{m} |d_k| < 1$ are satisfied, then the hemivariational evolution inequality with nonlocal condition* (5.1) *is approximately controllable on P.*

Proof. We can see that conditions $(B1)$, $B_1(J)$, and $B_2(J)$ are satisfied. Because $M := \sup_{0\leq s\leq t\leq a_1} \|\mathcal{T}(t,s)\| = 1$, then $\frac{1}{M} = 1$. From condition $\sum_{k=1}^{m} |d_k| < 1$ we can know that

$$\sum_{k=1}^{m} |d_k| < \frac{1}{M}.$$

This means that condition $(C1)$ holds. From the boundedness of linear operator $\mathcal{B}$, we can verify that condition $(C2)$ is satisfied. Applying Corollary 4.1, we can see that problem (5.1) is approximately controllable on P. □

6. Summary and outlook

The main work of this article is to discuss the near sighted controllability of the hemivariational evolution inequality under a non-local condition in Hilbert space. Due to the limitations of Hilbert space, we will consider extending the relevant results to ordinary Banach spaces in the future, which is a challenging task.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. X. Li, Z. Liu, N. S. Papageorgiou, Solvability and pullback attractor for a class of differential hemivariational inequalities with its applications, *Nonlinearity*, **36** (2023), 1323–1348. https://doi.org/10.1088/1361-6544/acb191
2. Y. Liu, Z. Liu, N. S. Papageorgiou, Sensitivity analysis of optimal control problems driven by dynamic history-dependent variational-hemivariational inequalities, *J. Differ. Equations*, **342** (2023), 559–595. https://doi.org/10.1016/j.jde.2022.10.009
3. Z. Liu, On boundary variational-hemivariational inequalities of elliptic type, *Proc. Roy. Soc. Edinb. A*, **140** (2010), 419–434. https://doi.org/10.1017/S0308210509000808
4. K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, **179** (1993), 630–637. https://doi.org/10.1006/jmaa.1993.1373
5. A. Boucherif, Semilinear evolution inclutions with nonlocal conditions, *Appl. Math. Lett.*, **22** (2009), 419–434. https://doi.org/10.1016/j.aml.2008.10.004
6. L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.*, **162** (1991), 494–505. https://doi.org/10.1016/0022-247X(91)90164-U
7. P. Chen, Y. Li, Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, *Results Math.*, **63** (2013), 731–744. https://doi.org/10.1007/s00025-012-0230-5
8. S. Migorski, A. Ochal, Optimal control of parabolic hemivariational inequalities, *J. Global Optim.*, **17** (2000), 285–300. https://doi.org/10.1023/A:1026555014562
9. H. Yuan, Q. Zhu, Stabilities of delay stochastic McKean-Vlasov equations in the G-framework, *Sci. China Inf. Sci.*, **68** (2025), 112203. https://doi.org/10.1007/s11432-024-4075-2
10. G. Zhang, Y. Song, X. Liu, Exponential synchronization of coupled neural networks with hybrid delays and stochastic distributed delayed impulses, *Mathematics*, **12** (2024), 1995. https://doi.org/10.3390/math12131995
11. Q. Zhu, Event-triggered sampling problem for exponential stability of stochastic nonlinear delay systems driven by Levy processes, *IEEE Trans. Automat. Contr.*, **70** (2025), 1176–1183. https://doi.org/10.1109/TAC.2024.3448128
12. Z. Liu, J. Lv, R. Sakthivel, Approximate controllability of fractional functional evolution inclusions with delay in Hilbert spaces, *IMA J. Math. Control I.*, **31** (2014), 363–383. https://doi.org/10.1093/imamci/dnt015
13. Z. Ma, H. Jiang, C. Li, D. Zhang, X. Liu, Stochastic intermittent control with uncertainty, *Mathematics*, **12** (2024), 1947. https://doi.org/10.3390/math12131947
14. X. Yang, Q. Zhu, H. Wang, Exponential stabilization of stochastic systems via novel event-triggered switching controls, *IEEE Trans. Automat. Contr.*, **69** (2024), 7948–7955. https://doi.org/10.1109/TAC.2024.3406668
15. Z. Liu, X. Li, Approximate controllability for a class of hemivariational inequalities, *Nonlinear Anal.-Real*, **22** (2015), 581–591. https://doi.org/10.1016/j.nonrwa.2014.08.010

16. Z. Liu, X. Li, D. Motreanu, Approximate controllability for nonlinear evolution hemivariational inequalities in Hilbert spaces, *SIAM J. Control Optim.*, **53** (2015), 3228–3244. https://doi.org/10.1137/140994058
17. S. Hu, N. S. Papageorgiou, *Handbook of multivalued analysis, volume I: theory*, New York: Springer, 1997.
18. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003. https://doi.org/10.1007/978-0-387-21593-8
19. F. H. Clarke, *Optimization and nonSmooth analysis*, Philadelphia: Society for Industrial and Applied Mathematics, 1990. https://doi.org/10.1137/1.9781611971309
20. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New York: Springer, 1983. https://doi.org/10.1007/978-1-4612-5561-1
21. W. Fitzgibbon, Semilinear functional differential equations in Banach spaces, *J. Differ. Equations*, **29** (1978), 1–14. https://doi.org/10.1016/0022-0396(78)90037-2
22. S. Migorski, A. Ochal, Quasi-Static hemivariational inequality via vanishing acceleration approach, *SIAM J. Math. Anal.*, **41** (2009), 1415–1435. https://doi.org/10.1137/080733231
23. S. Migorski, A. Ochal, M. Sofonea, *Nonlinear inclusions and hemivariational inequalities: models and analysis of contact problems*, New York: Springer, 2013. https://doi.org/10.1007/978-1-4614-4232-5
24. J. Prüss, *Evolutionary integral equations and applications*, Basel: Birkhäuser, 2012. https://doi.org/10.1007/978-3-0348-0499-8