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Research article

Controller design for chain-type wave network models with boundary delays

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Abstract: In this paper, we consider the design method of full-state integral feedback controllers, selecting appropriate target systems, establishing equivalent transformations, and completing the stability analysis of the original system while avoiding a large amount of complex calculations and proofs. This research attempts to establish a universally applicable controller design strategy that can not only eliminate the negative effects of system delays, but also ensure the closed-loop stability. Therefore, this paper further explores the applicability of this controller design strategy in complex PDE-coupled systems, seeks general patterns, and has further theoretical and practical value for the stability study of infinite-dimensional time-delay systems.

Keywords: chain wave network; input delay; controller design; exponential stabilization; PDE-coupled system
Mathematics Subject Classification: 31B10, 35R02, 93C35, 93C43, 93D15

1. Introduction

Wave networks are an important class of partial differential systems, with a wide range of applications in engineering domains such as transportation and aerospace. These systems have the capacity to describe dynamic behaviors, including fluid flow and neural networks in [1,2]. A notable example is the elastic string network, which is employed to model the dynamic behavior of elastic systems where the cross-sectional area of the material is negligible in comparison to its length. Elastic beam networks are employed to model the dynamic behavior of elastic systems where the cross-sectional dimensions are significant relative to the length of the material. Among these, the Euler-Bernoulli beam network represents the simplest model for describing the behavior of elastic beams in [3,4].

Due to the infinite dimensionality of the state of the partially differential coupled system itself, the control methods that involve infinite-dimensional distributions are characterized by control inputs that are functions of both space and time, making the study of control problems quite challenging.

In recent years, the methods for studying the stability of delay-free network systems are mostly as follows. The Lyapunov function method establishes the stability of the closed-loop system through the construction of an appropriate Lyapunov function in [5]. The frequency domain analysis method evaluates system stability through analysis of the system's frequency response characteristics in [6]. The Riesz basis method, as demonstrated in [7, 8], establishes system stability by verifying that the spectrum of the system operator meets certain stability criteria. The Riemann Geometry Method, as illustrated in [9], is particularly effective for analyzing the stability of semi-linear systems with memory-type boundary coupling or variable coefficient wave equations. By employing this method, it can be shown that the energy of the weak solution of the system exhibits exponential decay, which in turn confirms the stability of the system.

In the context of wave network systems characterized by time delays, the development of suitable control strategies that are designed to counteract the impact of these delays on system stability is imperative. To illustrate this point, consider dynamic systems with differential-type input delays; in such cases, the implementation of dynamic feedback control strategies can be a fruitful approach. A methodology that has been proven effective in achieving this objective involves the design of a partial state estimator in conjunction with the designated feedback controller, a strategy that has been shown to facilitate the attainment of exponential stability for the system under consideration. It is worth mentioning that the pioneering contributions in this field can be credited to Xu and his colleagues, who were the first to investigate the dynamics of differential-type boundary input delays in [10].

In their seminal work, Xu and Shang designed a dynamic feedback controller that improved the coefficient constraint to achieve exponential stability or asymptotic stability for the original time-delay system in the Euler-Bernoulli beam model in [11, 12]. Many researchers have applied different models and some complex forms of memory to this dynamic feedback controller. For instance, the wave equation in [13, 14], the Timoshenko beam model in [15, 16], the design of output-based dynamic feedback controllers in [17], and the stability analysis of the Schrödinger equation with time delays in boundary inputs in [18].

However, the use of partial state estimators to analyze the stability of the control system is challenging. Moreover, traditional methods such as the multiplier method in [19] and spectral analysis in [20] are no longer applicable, except in certain special cases in [21], and these methods are difficult to generalize to high-dimensional systems.

In their analysis and summary, Xu in [22] proposed a new method of controller design. This method is based on the concept of system feedback equivalence, whereby the controller is manifested as an integral full-state feedback controller. Although it also studied similar issues in [23–25], the design methods are different. The parameterized controller can counteract any control delays and is easier to analyze the stability of the closed-loop system, which is conducive to fully utilizing existing results. In [26–28], it discussed constant coefficient one-dimensional systems.

In this paper, we first proposed an exponentially stable coupled system, which serves as the target system. Subsequently, we discovered a bounded reversible transformation that effectively maps the original coupled system into this target system. Based on this, a state feedback controller is meticulously designed. Ultimately, by leveraging the equivalence between the original and target systems, we rigorously proved the exponential stability of the closed-loop system. In this way, we have not only eliminated the time delay, but also demonstrated the stability of the closed-loop system. We aspire to extend the application of this integral-type controller via the controller design presented in this model. Our innovation is to develop a versatile integral-type controller capable of effectively tackling the challenges of time delays and stability in infinite-dimensional systems.

We discuss controller design for chain-type wave network models with input delay, which behave according to the following equations:

$$\begin{cases} u_{i,tt}(y,t) = C_i^2 u_{i,yy}(y,t),\ y \in (0,1),\ t > 0,\ i = 1,2,\ldots,n, \\ u_1(0,t) = 0,\ t > 0, \\ u_i(1,t) = u_{i+1}(0,t),\ t > 0,\ i = 1,2,\ldots,n-1, \\ m_i u_{i,tt}(1,t) + C_i^2 u_{i,y}(1,t) = C_{i+1}^2 u_{i+1,y}(0,t),\ t > 0,\ i = 1,2,\ldots,n-1, \\ m_n u_{n,tt}(1,t) + C_n^2 u_{n,y}(1,t) = \mu(t-\tau),\ t > 0, \\ u_i(y,0) = u_{i,0}(y), \\ u_{i,t}(y,0) = u_{i,1}(y),\ y \in (0,1),\ i = 1,2,\ldots,n, \end{cases} \tag{1.1}$$

where $u_i(x,t)$ represents the amplitude of the wave at position x and time t. $u_{i,tt}(y,t)$ is for the wave acceleration, and $u_{i,yy}(y,t)$ is for the second-order spatial variation rate in the x-direction. C_i is a positive constant that represents wave speed. $u_{i,0}(y)$ and $u_{i,1}(y)$ are the initial states of waves, $i = 1,2,\cdots,n$. $\mu(t)$ is a control function, and τ is an arbitrary positive constant representing the storage time.

In the following, we will introduce the controller design approach. First of all, with reference to the idea in [12], we introduce an auxiliary function

$$\phi(s,t) = \mu(s+t-\tau),\ s \in (0,\tau),$$

hence, $\phi(0,t) = \mu(t-\tau)$. System (1.1) is reformulated to

$$\begin{cases} \phi_t(s,t) = \phi_s(s,t),\ s \in (0,\tau),\ t > 0, \\ \phi(\tau,t) = \mu(t),\ t > 0, \\ u_{i,tt}(y,t) = C_i^2 u_{i,yy}(y,t),\ y \in (0,1),\ t > 0,\ i = 1,2,\ldots,n, \\ u_1(0,t) = 0,\ t > 0, \\ u_i(1,t) = u_{i+1}(0,t),\ i = 1,2,\ldots,n-1, \\ m_i u_{i,tt}(1,t) + C_i^2 u_{i,y}(1,t) = C_{i+1}^2 u_{i+1,y}(0,t),\ t > 0,\ i = 1,2,\ldots,n-1, \\ m_n u_{n,tt}(1,t) + C_n^2 u_{n,y}(1,t) = \phi(0,t),\ t > 0, \\ u_i(y,0) = u_{i,0}(y), \\ u_{i,t}(y,0) = u_{i,1}(y),\ y \in (0,1)\ i = 1,2,\ldots,n. \end{cases} \tag{1.2}$$

It is evident that the integral-type feedback controller in [12] is the primary factor in this matter, and $\mu(t)$ should be like

$$\begin{aligned} \mu(t) = &\int_0^\tau p(\tau-r)\phi(r,t)dr + \sum_{i=1}^n \int_0^1 \gamma_i(\tau,x)u_i(x,t)dx + \sum_{i=1}^n \int_0^1 \eta_i(\tau,x)u_{i,t}(x,t)dx \\ &+ \sum_{i=1}^n \int_0^1 \theta_i(\tau,x)u_{i,x}(x,t)dx + \sum_{i=1}^n \int_0^1 \rho_i(\tau,x)u_{i,xt}(x,t)dx, \end{aligned} \tag{1.3}$$

where $p(s)$, $\gamma_i(s,y)$, $\eta_i(s,y)$, $\theta_i(s,y)$, and $\rho_i(s,y)$ are the parameter functions. The right parameter function is vital for the closed-loop system's stability.

In order to resolve the issue of instability in the closed-loop system stability analysis, the following system is selected for study, where a and b are positive constant

$$\begin{cases} \psi_t(s,t) = \psi_s(s,t), \ s \in (0,\tau), \ t > 0, \\ \psi(\tau,t) = 0, \ t > 0, \\ u_{i,tt}(y,t) = C_i^2 u_{i,yy}(y,t), \ y \in (0,1), \ t > 0, \ i = 1,2,\ldots,n, \\ u_1(0,t) = 0, \ t > 0, \\ u_i(1,t) = u_{i+1}(0,t), \ t > 0, \ i = 1,2,\ldots,n-1, \\ m_i u_{i,tt}(1,t) + C_i^2 u_{i,y}(1,t) = C_{i+1}^2 u_{i+1,y}(0,t), \ t > 0, \ i = 1,2,\ldots,n-1, \\ m_n u_{n,tt}(1,t) + C_n^2 u_{n,y}(1,t) = \psi(0,t) - a u_{n,yt}(1,t) - b u_{n,t}(1,t), \ t > 0, \\ u_i(y,0) = u_{i,0}(y), u_{i,t}(y,0) = u_{i,1}(y), \ y \in (0,1), \ i = 1,2,\ldots,n. \end{cases} \tag{1.4}$$

The rest of the paper is structured as follows. In section 2, we prove that the original system is stable, where we make an equivalent transformation between (1.2) and (1.4). In section 3, the solvability of kernel function equations and the boundedness of transformations are demonstrated. Finally, in section 4, we give a summary of this article.

2. Proofs of stability of the closed-loop system

2.1. *The transformation from system* (1.2) *to system* (1.4)

In order to determine the boundary condition of ϕ, we give a transformation as follows:

$$\begin{cases} \psi(s,t) = \phi(s,t) - \int_0^s p(s-r)\phi(r,t)dr - \sum_{i=1}^n \int_0^1 \gamma_i(s,x)u_i(x,t)dx - \sum_{i=1}^n \int_0^1 \eta_i(s,x)u_{i,t}(x,t)dx \\ \qquad - \sum_{i=1}^n \int_0^1 \theta_i(s,x)u_{i,x}(x,t)dx - \sum_{i=1}^n \int_0^1 \rho_i(s,x)u_{i,xt}(x,t)dx, \\ u_i(y,t) = u_i(y,t), \ i = 1,2,\ldots,n. \end{cases} \tag{2.1}$$

It is important to select these appropriate parameter functions $p(s)$, $\gamma_i(s,x)$, $\eta_i(s,y)$, $\theta_i(s,y)$, and $\rho_i(s,y)$ to make $(\psi(s,t), u_i(x,t))$ satisfy (1.4). The following theorem presents a selection for these parameters.

Theorem 2.1. *Suppose that $\gamma_i(s,x)$, $\eta_i(s,x)$, $\theta_i(s,x)$, and $\rho_i(s,x)$ satisfy the equations*

$$\begin{cases} \gamma_i(s,x) = \eta_{i,s}(s,x), \\ \gamma_{i,s}(s,x) = C_i^2 \eta_{i,xx}(s,x), \ s \in (0,\tau), \ x \in (0,1), \ i = 1,2,\ldots,n, \\ \theta_i(s,x) = \rho_{i,s}(s,x), \\ \theta_{i,sx}(s,x) = C_i^2 \rho_{i,xxx}(s,x), \ s \in (0,\tau), \ x \in (0,1), \ i = 1,2,\ldots,n, \\ \eta_i(s,1) = \eta_{i+1}(s,0), \ \rho_{i,x}(s,1) = \rho_{i,x+1}(s,0), \ s \in (0,\tau), \ i = 1,2,\ldots,n-1, \\ m_i\eta_i(s,1) - m_i\rho_{i,x}(s,1) = \rho_i(s,1), \\ C_i^2 \eta_{i,x}(s,1) = C_{i+1}^2 \eta_{i+1,x}(s,0), \\ C_i^2 \rho_{i,xx}(s,1) = C_{i+1}^2 \rho_{i+1,xx}(s,0), \ s \in (0,\tau), \ i = 1,2,\ldots,n-1, \\ \theta_{i,s}(s,1) = \theta_{i+1,s}(s,0), \\ \eta_1(s,0) = 0, \ \rho_{1,x}(s,0) = 0, \ \rho_i(s,0) = 0, \ s \in (0,\tau), \\ \theta_{n,s}(s,1) = 0, \ \eta_{n,x}(s,1) = 0, \ \rho_{n,xx}(s,1) = 0, \ s \in (0,\tau), \\ \gamma_i(0,x) = 0, \ x \in (0,1), \ i = 1,2,\ldots,n, \\ \eta_i(0,x) = 0, \ x \in (0,1), \ i = 1,2,\ldots,n-1, \\ \eta_n(0,x) = -b\delta(x-1), \ x \in (0,1), \\ \rho_{i,x}(0,x) = 0, \ i = 1,2,\ldots,n-1, \rho_{n,x}(0,x) = -a\delta'(x-1), \ x \in (0,1), \end{cases} \tag{2.2}$$

and $p(s)$ is given by $p(s) = \eta_n(s,1) - \rho_{n,x}(s,1)$, and $\delta(x-1)$ is an impulse function. Then, $\psi(s,t)$ and $u_i(y,t)$ satisfy (1.4).

Proof. Let $(\phi(s,t), u_i(y,t))$ be the solution of system (1.2). According to (2.1), we have

$$\begin{aligned}\psi_t(s,t) &= \phi_t(s,t) - \int_0^s p(s-r)\phi_t(r,t)dr - \sum_{i=1}^n \int_0^1 \gamma_i(s,x)u_{i,t}(x,t)dx - \sum_{i=1}^n \int_0^1 \eta_i(s,x)u_{i,tt}(x,t)dx \\ &\quad - \sum_{i=1}^n \int_0^1 \theta_i(s,x)u_{i,xt}(x,t)dx - \sum_{i=1}^n \int_0^1 \rho_i(s,x)u_{i,xtt}(x,t)dx,\end{aligned}$$

cause

$$\begin{aligned}&\sum_{i=1}^n \int_0^1 \eta_i(s,x)u_{i,tt}(x,t)dx + \sum_{i=1}^n \int_0^1 \rho_i(s,x)u_{i,xtt}(x,t)dx \\ &= \sum_{i=1}^n C_i^2 \int_0^1 \eta_i(s,x)u_{i,xx}(x,t)dx + \sum_{i=1}^n \int_0^1 \rho_i(s,x)du_{i,tt}(x,t) \\ &= \sum_{i=1}^n C_i^2 \left(\eta_i(s,1)u_{i,x}(1,t) - \eta_i(s,0)u_{i,x}(0,t) - \eta_{i,x}(s,1)u_i(1,t) + \eta_{i,x}(s,0)u_i(1,0)\right. \\ &\left.+ \int_0^1 \eta_{i,xx}(s,x)u_i(x,t)dx\right) + \sum_{i=1}^n \left(\rho_i(s,1)u_{i,tt}(1,t) - \rho_i(s,0)u_{i,tt}(0,t) - C_i^2 \int_0^1 \rho_{i,x}(s,x)u_{i,xx}(x,t)dx\right) \\ &= \eta_n(s,1)\phi(0,t) - \sum_{i=1}^n m_i u_{i,tt}(1,t)\eta_i(s,1) + \sum_{i=1}^n C_i^2 \int_0^1 u_i(x,t)\eta_{i,xx}(s,x)dx - \rho_{n,x}(s,1)\phi(0,t) \\ &+ \sum_{i=1}^n (\rho_i(s,1)u_{i,tt}(1,t) - \rho_i(s,0)u_{i,tt}(0,t)) + \sum_{i=1}^n m_i u_{i,tt}(1,t)\rho_{i,x}(s,1) - \sum_{i=1}^n C_i^2 \int_0^1 u_i(x,t)\rho_{i,xxx}(s,x)dx \\ &= \phi(0,t)(\eta_n(s,1) - \rho_{n,x}(s,1)) + \sum_{i=1}^n C_i^2 \int_0^1 u_i(x,t)\left(\eta_{i,xx}(s,x) - \rho_{i,xxx}(s,x)\right)dx,\end{aligned}$$

therefore,

$$\begin{aligned}\psi_t(s,t) =&\phi_t(s,t) - \int_0^s p(s-r)\phi_t(r,t)dr - \sum_{i=1}^n \int_0^1 \gamma_i(s,x)u_{i,t}(x,t)dx - \phi(0,t)(\eta_n(s,1) - \rho_{n,x}(s,1)) \\ &- \sum_{i=1}^n C_i^2 \int_0^1 u_i(x,t)\left(\eta_{i,xx}(s,x) - \rho_{i,xxx}(s,x)\right)dx - \sum_{i=1}^n \int_0^1 \theta_i(s,x)u_{i,xt}(x,t)dx,\end{aligned}$$

and

$$\begin{aligned}\psi_s(s,t) =&\phi_s(s,t) - \int_0^s p(r)\phi_s(s-r,t)dr - p(s)\phi(0,t) - \sum_{i=1}^n \int_0^1 u_i(x,t)\left(\gamma_{i,s}(s,x) - \theta_{i,sx}(s,x)\right)dx \\ &- \sum_{i=1}^n \int_0^1 \eta_{i,s}(s,x)u_{i,t}(x,t)dx - \sum_{i=1}^n \int_0^1 \rho_i(s,x)u_{i,xt}(x,t)dx,\end{aligned}$$

where we have used the differential equations in (1.2). From this, we conclude that $\psi_t(s,t) = \psi_s(s,t)$ for any $s \in (0,\tau)$ and $t > 0$. Obviously,

$$\psi(\tau,t)=\phi(\tau,t)-\int_0^\tau \phi(r,t)(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1))dr-\sum_{i=1}^n\int_0^1 \eta_{i,s}(\tau,x)u_i(x,t)dx$$
$$-\sum_{i=1}^n\int_0^1 \eta_i(\tau,x)u_{i,x}(x,t)dx-\sum_{i=1}^n\int_0^1 \rho_{i,s}(\tau,x)u_{i,t}(x,t)dx-\sum_{i=1}^n\int_0^1 \rho_i(\tau,x)u_{i,xt}(x,t)dx=0,$$

and

$$\psi(0,t)=\phi(0,t)-\sum_{i=1}^n\int_0^1 \gamma_i(0,x)u_i(x,t)dx-\sum_{i=1}^n\int_0^1 \eta_i(0,x)u_{i,t}(x,t)dx-\sum_{i=1}^n\int_0^1 \rho_i(0,x)u_{i,xt}(x,t)dx$$
$$-\sum_{i=1}^n\int_0^1 \theta_i(0,x)u_{i,x}(x,t)dx=\phi(0,t)+au_{n,xt}(1,t)+bu_{n,t}(1,t).$$

This ends the proof. □

2.2. The inverse transformation from system (1.4) *to system* (1.2)

This subsection will construct the inverse transformation.

$$\begin{cases} \phi(s,t)=\psi(s,t)-\int_0^s \tilde{p}(s-r)\phi(r,t)dr-\sum_{i=1}^n\int_0^1 \tilde{\gamma}_i(s,x)u_i(x,t)dx-\sum_{i=1}^n\int_0^1 \tilde{\eta}_i(s,x)u_{i,t}(x,t)dx \\ \qquad -\sum_{i=1}^n\int_0^1 \tilde{\theta}_i(s,x)u_{i,x}(x,t)dx-\sum_{i=1}^n\int_0^1 \tilde{\rho}_i(s,x)u_{i,xt}(x,t)dx, \\ u_i(y,t)=u_i(y,t),\ i=1,2,\ldots,n. \end{cases} \tag{2.3}$$

Theorem 2.2. *Let* $(\psi(s,t),\ u_i(x,t))$ *be solutions of* (1.4)*; let* $(\phi(s,t),\ u_i(y,t))$ *be defined as in* (2.3)*. Suppose that* $\tilde{\gamma}(s,x)$, $\tilde{\eta}(s,x)$ *and* $\tilde{\theta}(s,x)$, $\tilde{\rho}(s,x)$ *satisfy the following equations*

$$\begin{cases} \tilde{\gamma}_i(s,x)=\tilde{\eta}_{i,s}(s,x)-\eta_i(0,x)(\tilde{\eta}_i(s,x)-\tilde{\rho}_{i,x}(s,x)), \\ \tilde{\gamma}_{i,s}(s,x)=C_i^2\tilde{\eta}_{i,xx}(s,x),\ s\in(0,\tau),\ x\in(0,1),\ i=1,2,\ldots,n, \\ \tilde{\theta}_i(s,x)=\tilde{\rho}_{i,s}(s,x)-\rho_i(0,x)(\tilde{\eta}_i(s,x)-\tilde{\rho}_{i,x}(s,x)), \\ \tilde{\theta}_{i,s}(s,x)=C_i^2\tilde{\rho}_{i,xxx}(s,x),\ s\in(0,\tau),\ x\in(0,1),\ i=1,2,\ldots,n, \\ \tilde{\eta}_i(s,1)=\tilde{\eta}_{i+1}(s,0),\ \tilde{\rho}_{i,x}(s,1)=\tilde{\rho}_{i,x+1}(s,0)\ s\in(0,\tau),\ i=1,2,\ldots,n-1, \\ m_i\tilde{\eta}_i(s,1)-m_i\tilde{\rho}_{i,x}(s,1)=\rho_i(s,1), \\ C_i^2\tilde{\eta}_{i,x}(s,1)=C_{i+1}^2\tilde{\eta}_{i+1,x}(s,0), \\ C_i^2\tilde{\rho}_{i,xx}(s,1)=C_{i+1}^2\tilde{\rho}_{i+1,xx}(s,0),\ s\in(0,\tau),\ i=1,2,\ldots,n-1, \\ \tilde{\theta}_{i,s}(s,1)=\tilde{\theta}_{i+1,s}(s,0), \\ \tilde{\eta}_1(s,0)=0,\ \tilde{\rho}_1(s,0)=0,\ \tilde{\rho}_{i,x}(s,0)=0,\ s\in(0,\tau), \\ \tilde{\theta}_{n,s}(s,1)=0,\ \tilde{\eta}_{n,x}(s,1)=0,\ \tilde{\rho}_{n,xx}(s,1)=0,\ s\in(0,\tau), \\ \tilde{\gamma}_i(0,x)=0,\ x\in(0,1),\ i=1,2,\ldots,n, \\ \tilde{\eta}_i(0,x)=0,\ x\in(0,1),\ i=1,2,\ldots,n-1, \\ \tilde{\eta}_n(0,x)=b\delta(x-1),\ x\in(0,1), \\ \tilde{\rho}_{i,x}(0,x)=0,\ i=1,2,\ldots,n-1, \\ \tilde{\rho}_{n,x}(0,x)=a\delta'(x-1),\ x\in(0,1), \end{cases} \tag{2.4}$$

then, $(\phi(s,t),\ u_i(y,t))$ *is the solution of* (1.2)*. In addition,* $\tilde{p}(s)$ *is given by* $\tilde{p}(s)=\tilde{\eta}_n(s,1)-\tilde{\rho}_{n,x}(s,1)$*, and we have*

$$\begin{cases} \tilde{\eta}_i(\tau,x)=-\eta_i(\tau,x)+\int_0^\tau \tilde{\eta}_i(r,x)(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1))dr, \\ \tilde{\rho}_i(\tau,x)=-\rho_i(\tau,x)+\int_0^\tau \tilde{\rho}_i(r,x)(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1))dr. \end{cases} \tag{2.5}$$

Proof. Let $(\psi(s,t), \tilde{u}_i(y,t))$ be the solution of system (1.4); it can be directly calculated that

$$\begin{aligned}
\phi_t(s,t) - \phi_s(s,t) \quad &= \psi_t(s,t) - \psi_s(s,t) + \int_0^s \tilde{p}(r)\psi_s(s-r,t)dr \\
&- \int_0^s \tilde{p}(s-r)\psi_t(r,t)dr + (\tilde{p}(s) - \tilde{\eta}_n(s,1) + \tilde{\rho}_{n,x}(s,1))\psi(0,t) \\
&+ \sum_{i=1}^n \int_0^1 \left(\tilde{\gamma}_{i,s}(s,x) - \tilde{\theta}_{i,sx}(s,x) - C_i^2 \left(\tilde{\eta}_{i,xx}(s,x) - \tilde{\rho}_{i,xxx}(s,x)\right)\right) u_i(x,t)dx \\
&+ \sum_{i=1}^n \int_0^1 \left(\tilde{\eta}_{i,s}(s,x) - \tilde{\gamma}_i(s,x) - \eta_i(0,x)\left(\tilde{\eta}_i(s,x) - \tilde{\rho}_{i,x}(s,x)\right)\right) u_{i,t}(x,t)dx \\
&+ \sum_{i=1}^n \int_0^1 \left(\tilde{\rho}_{i,s}(s,x) - \tilde{\theta}_i(s,x) - \rho_i(0,x)\left(\tilde{\eta}_i(s,x) - \tilde{\rho}_{i,x}(s,x)\right)\right) u_{i,xt}(x,t)dx \\
&= 0,
\end{aligned}$$

where we have used the differential equations and the boundary conditions in (2.4). From the expression, it holds that $\phi_t(s,t) = \phi_s(s,t)$ for any $s \in (0,\tau)$ and $t > 0$. Here, we provide an explanation of the sifting property of the impulse function,

$$\begin{aligned}
\sum_{i=1}^n \int_0^1 \eta_i(0,x)u_{i,t}(x,t)\left(\eta_i(s,x) - \rho_{i,x}(s,x)\right)dx &= \int_0^1 \eta_n(0,x)u_{n,t}(x,t)\left(\eta_n(s,x) - \rho_{n,x}(s,x)\right)dx \\
&= -bu_{n,t}(1,t)\left(\eta_n(s,1) - \rho_{n,x}(s,1)\right),
\end{aligned}$$

similarly, we can obtain

$$\begin{aligned}
\sum_{i=1}^n \int_0^1 \rho_i(0,x)u_{i,xt}(x,t)\left(\eta_i(s,x) - \rho_{i,x}(s,x)\right)dx &= \int_0^1 \rho_n(0,x)u_{n,xt}(x,t)\left(\eta_n(s,x) - \rho_{n,x}(s,x)\right)dx \\
&= -au_{n,xt}(1,t)\left(\eta_n(s,1) - \rho_{n,x}(s,1)\right).
\end{aligned}$$

In the end, we testify the boundary conditions of $\phi(s,t)$. By the definition in (2.3), we have

$$\begin{aligned}
\phi(\tau,t) \;\; &= \;\; \psi(\tau,t) - \int_0^\tau \tilde{p}(\tau-r)\psi(r,t)dr - \sum_{i=1}^n \int_0^1 \tilde{\gamma}_i(\tau,x)u_i(x,t)dx \\
&\quad - \sum_{i=1}^n \int_0^1 \tilde{\eta}_i(\tau,x)u_{i,t}(x,t)dx - \sum_{i=1}^n \int_0^1 \tilde{\theta}_i(\tau,x)u_{i,x}(x,t)dx - \sum_{i=1}^n \int_0^1 \tilde{\rho}_i(\tau,x)u_{i,xt}(x,t)dx \\
&= \;\; \int_0^\tau \left(\tilde{\rho}_{n,x}(\tau-r,1) - \tilde{\eta}_n(\tau-r,1)\right)\psi(r,t)dr - \sum_{i=1}^n \int_0^1 u_{i,x}(x,t)\tilde{\eta}_{i,s}(\tau,x)dx \\
&\quad - \sum_{i=1}^n \int_0^1 u_{i,t}(x,t)\tilde{\eta}_i(\tau,x)dx - \sum_{i=1}^n \int_0^1 u_{i,x}(x,t)\tilde{\rho}_{i,s}(\tau,x)dx - \sum_{i=1}^n \int_0^1 u_{i,xt}(x,t)\tilde{\rho}_i(\tau,x)dx \\
&\quad - \left(bu_n(1,t) + au_{n,x}(1,t)\right)\left(\tilde{\eta}_n(\tau,x) - \tilde{\rho}_{n,x}(\tau,x)\right).
\end{aligned}$$

To prove $\phi(\tau,t)$ satisfies the boundary condition in (1.2), we need to show

$$\phi(\tau,t) = \int_0^\tau p(\tau-r)\phi(r,t)dr + \sum_{i=1}^n \int_0^1 \gamma_i(\tau,x)u_i(x,t)dx + \sum_{i=1}^n \int_0^1 \eta_i(\tau,x)u_{i,t}(x,t)dx$$

$$+\sum_{i=1}^{n}\int_0^1 \theta_i(\tau,x)u_{i,x}(x,t)dx+\sum_{i=1}^{n}\int_0^1 \rho_i(\tau,x)u_{i,xt}(x,t)dx,$$

since

$$\begin{aligned}&\psi(s,t)-\int_0^s\left(\tilde{\eta}_n(s-r,1)-\tilde{\rho}_{n,x}(s-r,1)\right)\psi(r,t)dr=\phi(s,t)+\sum_{i=1}^{n}\int_0^1 u_i(x,t)\tilde{\eta}_{i,s}(s,x)dx\\&+\sum_{i=1}^{n}\int_0^1 u_{i,t}(x,t)\tilde{\eta}_i(s,x)dx+\sum_{i=1}^{n}\int_0^1 u_{i,x}(x,t)\tilde{\rho}_{i,s}(s,x)dx+\sum_{i=1}^{n}\int_0^1 u_{i,xt}(x,t)\tilde{\rho}_i(s,x)dx\\&+\left(bu_n(1,t)+au_{n,x}(1,t)\right)\left(\tilde{\eta}_n(s,x)-\tilde{\rho}_{n,x}(s,x)\right),\end{aligned}$$

we have

$$\begin{aligned}\phi(r,t)=&\psi(r,t)-\int_0^r\left(\tilde{\eta}_n(r-r',1)-\tilde{\rho}_{n,x}(r-r',1)\right)\psi(r,t)dr'-\sum_{i=1}^{n}\int_0^1 u_i(x,t)\tilde{\eta}_{i,s}(r,x)dx\\&-\sum_{i=1}^{n}\int_0^1 u_{i,t}(x,t)\tilde{\eta}_i(r,x)dx-\sum_{i=1}^{n}\int_0^1 u_{i,x}(x,t)\tilde{\rho}_{i,s}(r,x)dx-\sum_{i=1}^{n}\int_0^1 u_{i,xt}(x,t)\tilde{\rho}_i(r,x)dx\\&-\left(bu_n(1,t)+au_{n,x}(1,t)\right)\left(\tilde{\eta}_n(r,x)-\tilde{\rho}_{r,x}(s,x)\right),\end{aligned}$$

then by substituting the above equation into (1.3), we can obtain (2.5) and end the proof. In fact

$$\begin{aligned}\mu(t)=&\int_0^\tau \psi(r,t)\Big(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\\&-\int_r^\tau\left(\eta_n(\tau-\xi,1)-\rho_{n,x}(\tau-\xi,1)\right)\left(\tilde{\eta}_n(\xi-r,1)-\tilde{\rho}_{n,x}(\xi-r,1)\right)d\xi\Big)dr\\&-\sum_{i=1}^{n}\int_0^1 u_i(x,t)\left(\int_0^\tau \tilde{\eta}_{i,s}(r,x)\left(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\right)dr-\eta_{i,s}(\tau,x)\right)dx\\&-bu_n(1,t)\int_0^\tau\left(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\right)\left(\tilde{\eta}_n(r,x)-\tilde{\rho}_{n,x}(r,x)\right)dr\\&-\sum_{i=1}^{n}\int_0^1 u_{i,t}(x,t)\left(\int_0^\tau\left(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\right)\tilde{\eta}_i(r,x)dr-\eta_i(\tau,x)\right)dx\\&-\sum_{i=1}^{n}\int_0^1 u_{i,x}(x,t)\left(\int_0^\tau \tilde{\rho}_{i,s}(r,x)\left(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\right)dr-\rho_{i,s}(\tau,x)\right)dx\\&-au_{n,x}(1,t)\int_0^\tau\left(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\right)\left(\tilde{\eta}_n(r,x)-\tilde{\rho}_{n,x}(r,x)\right)dr\\&-\sum_{i=1}^{n}\int_0^1 u_{i,xt}(x,t)\left(\int_0^\tau\left(\eta_n(\tau-r,1)-\rho_{n,x}(\tau-r,1)\right)\tilde{\rho}_i(r,x)dr-\rho_i(\tau,x)\right)dx.\end{aligned}$$

In particular, we have

$$m_nu_{n,tt}(1,t)+C_n^2u_{n,y}(1,t)\;=\;\psi(0,t)-au_{n,yt}(1,t)-bu_{n,t}(1,t)$$

$$
\begin{aligned}
&= \psi(0,t) - \sum_{i=1}^{n}\int_0^1 \tilde{\gamma}_i(0,x)u_i(x,t)dx - \sum_{i=1}^{n}\int_0^1 \tilde{\eta}_i(0,x)u_{i,t}(x,t)dx \\
&\quad - \sum_{i=1}^{n}\int_0^1 \tilde{\theta}_i(0,x)u_{i,x}(x,t)dx \sum_{i=1}^{n}\int_0^1 \tilde{\rho}_i(0,x)u_{i,xt}(x,t)dx \\
&= \phi(0,t).
\end{aligned}
$$

□

3. Solvability of kernel function equations and boundedness of transformations

In this subsection, we consider the solvability of system (2.2) and system (2.4). In order to facilitate this process, the initial step involves the observation of the transmission condition in both equations. Since the continuity condition

$$
\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\eta\\ \gamma\\ \rho_x\\ \theta_x\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ C_i^2\partial xx & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & C_i^2\partial xx & 0\end{pmatrix}\begin{pmatrix}\eta\\ \gamma\\ \rho_x\\ \theta_x\end{pmatrix},\ s\in(0,\tau),\ x\in(0,1).
$$

This observation requires defining the correct state space and operators to study the solvability of (2.2) and (2.4). Denote

$$
H[0,1] = \left\{f, h \in H^1[0,1], |\ f_1(0)=0, h_1(0)=0\right\},
$$

and take the space as

$$
H = \left\{(f,g,h,k) \in H[0,1]\times L^2[0,1]\times H[0,1]\times L^2[0,1]\right\},
$$

which equipped with the norm

$$
\|(f,g,h,k)\|_H^2 = \int_0^1 (|f'(y)|^2 + |g(y)|^2 + |h'(y)|^2 + |k(y)|^2)dy,\quad (f,g,h,k)\in H,
$$

clearly, H is a Hilbert space. Then we shall define an operator A in H by

$$
Af(y) = f_s(y) = g(y), Ag(y) = g_s(y) = C^2 f_{yy}(y), Ah(y) = h_s(y) = k(y), Ak(y) = ks(y) = C^2 h_{yy}(y).
$$

So, we can rephrase it as

$$
A\begin{pmatrix}f_1\\ g_1\\ h_1\\ k_1\\ \vdots\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0 & \cdots\\ C_1^2\partial yy & 0 & 0 & 0 & \cdots\\ 0 & 0 & 1 & 0 & \cdots\\ 0 & C_2^2\partial yy & 0 & 0 & \cdots\\ & \cdots & & & \cdots\\ & \vdots & & & \vdots\end{pmatrix}\begin{pmatrix}f_1\\ g_1\\ h_1\\ k_1\\ \vdots\end{pmatrix}
$$

with domain

$$
D(A) = \left\{(f_i, g_i, h_i, k_i) \in H^2[0,1]\times H^1[0,1]\times H^2[0,1]\times H^1[0,1]\right\},
$$

where

$$\begin{cases} f_i(1) = f_{i+1}(0), \\ h_i(1) = h_{i+1}(0), i = 1, 2, \cdots, n-1, \\ C_i^2 f_i'(1) = C_{i+1}^2 f_{i+1}'(0), \\ C_i^2 h_i'(1) = C_{i+1}^2 h_{i+1}'(0), i = 1, 2, \cdots, n-1, \\ f_n'(1) = 0, h_n'(1) = 0. \end{cases}$$

Meanwhile, we define the operator B on H

$$B\begin{pmatrix} \vdots \\ f_n \\ g_n \\ h_n \\ k_n \end{pmatrix} = (f_n - h_n)\begin{pmatrix} \vdots \\ -b\delta(x-1) \\ 0 \\ -a\delta(x-1) \\ 0 \end{pmatrix}.$$

Due to A and B, we can rewrite (2.2) and (2.4) as the evolutionary equations on H,

$$\begin{cases} \frac{dW(s)}{ds} = AW(s), \ s > 0, \\ W(0) = W_0, \end{cases}$$

and

$$\begin{cases} \frac{d\widetilde{W}(s)}{ds} = A\widetilde{W}(s) + B\widetilde{W}(s), \ s > 0, \\ \widetilde{W}(0) = -W_0. \end{cases}$$

Here,

$$W(s) = \Big(\eta_1(s,y), \gamma_1(s,y), \rho_{1,y}(s,y), \theta_{1,y}(s,y), \cdots, \eta_n(s,y), \gamma_n(s,y), \rho_{n,y}(s,y), \theta_{n,y}(s,y)\Big)^T,$$

$$\widetilde{W}(s) = \Big(\tilde{\eta}_1(s,y), \tilde{\gamma}_1(s,y), \tilde{\rho}_{1,y}(s,y), \tilde{\theta}_{1,y}(s,y), \cdots, \tilde{\eta}_n(s,y), \tilde{\gamma}_n(s,y), \tilde{\rho}_{n,y}(s,y), \tilde{\theta}_{n,y}(s,y)\Big)^T,$$

$$\begin{aligned} W_0 &= \Big(\eta_{10}(0,y), \gamma_{10}(0,y), \rho_{1,y0}(0,y), \theta_{1,y0}(0,y), \cdots, \eta_{n0}(0,y), \gamma_{n0}(0,y), \rho_{n,y0}(0,y), \theta_{n,y0}(0,y)\Big)^T \\ &= \Big(0, 0, 0, 0, \cdots, -b\delta(y-1), 0, -a\delta'(y-1), 0\Big)^T. \end{aligned}$$

We can define a new inner product on H by

$$\begin{aligned} &\Big\langle(\eta_1, \gamma_1, \rho_1, \theta_1, \cdots, \eta_n, \gamma_n, \rho_n, \theta_n), (\hat{\eta}_1, \hat{\gamma}_1, \hat{\rho}_1, \hat{\theta}_1, \cdots, \hat{\eta}_n, \hat{\gamma}_n, \hat{\rho}_n, \hat{\theta}_n)\Big\rangle_H \\ =& \sum_{i=1}^n \int_0^1 C_i^2 \eta_i'(y)\bar{\hat{\eta}}_i'(y)dy + \sum_{i=1}^n \int_0^1 \gamma_i(y)\bar{\hat{\gamma}}_i(y)dy + \sum_{i=1}^n \int_0^1 C_i^2 \theta_i'(y)\bar{\hat{\theta}}_i'(y)dy + \sum_{i=1}^n \int_0^1 \rho_i(y)\bar{\hat{\rho}}_i(y)dy. \end{aligned}$$

Clearly, this is an equivalent inner product on H.

Proposition 3.1. *Define A as previously. Then, under a new inner product, A is a skew-adjoint operator in H.*

Proof. In fact, for any $(\eta_1,\gamma_1,\rho_1,\theta_1,\cdots,\eta_n,\gamma_n,\rho_n,\theta_n)\in D(A)$ and $(\hat{\eta}_1,\hat{\gamma}_1,\hat{\rho}_1,\hat{\theta}_1,\cdots,\hat{\eta}_n,\hat{\gamma}_n,\hat{\rho}_n,\hat{\theta}_n)\in H^2[0,1]\times H^1[0,1]\times H^2[0,1]\times H^1[0,1]\times\cdots\times H^2[0,1]\times H^1[0,1]\times H^2[0,1]\times H^1[0,1]$, we have

$$
\begin{aligned}
&\Big\langle A(\eta_1,\gamma_1,\rho_1',\theta_1',\cdots,\eta_n,\gamma_n,\rho_n',\theta_n'),\ (\hat{\eta}_1,\hat{\gamma}_1,\hat{\rho}'_1,\hat{\theta}'_1,\cdots,\hat{\eta}_n,\hat{\gamma}_n,\hat{\rho}'_n,\hat{\theta}'_n)\Big\rangle_H\\
=&\Big\langle(\gamma_1,C_1^2\eta_1'',\theta_1',C_1^2\rho_1'''\cdots,\gamma_n,C_n^2\eta_n'',\theta_n',C_n^2\rho_n'''),\ (\hat{\eta}_1,\hat{\gamma}_1,\hat{\rho}'_1,\hat{\theta}'_1,\cdots,\hat{\eta}_n,\hat{\gamma}_n,\hat{\rho}'_n,\hat{\theta}'_n)\Big\rangle_H\\
=&\sum_{i=1}^n\int_0^1 C_i^2\gamma_i'(y)\bar{\hat{\eta}}_i'(y)dy+\sum_{i=1}^n\int_0^1 C_i^2\eta_i''(y)\bar{\hat{\gamma}}_i(y)dy\\
&+\sum_{i=1}^n\int_0^1 C_i^2\theta_i''(y)\bar{\hat{\rho}}_i''(y)dy+\sum_{i=1}^n\int_0^1 C_i^2\rho_i'''(y)\bar{\hat{\theta}}_i'(y)dy\\
=&-\sum_{i=1}^n\int_0^1 C_i^2\left(\gamma_i(y)\bar{\hat{\eta}}_i''(y)+\eta_i'(x)\bar{\hat{\gamma}}_i'(y)+\theta_i'(y)\bar{\hat{\rho}}_i'''(y)+\rho_i''(x)\bar{\hat{\theta}}_i''(y)\right)dy\\
&+\sum_{i=1}^n C_i^2\left(\gamma_i(1)\bar{\hat{\eta}}_i'(1)+\eta_i'(1)\bar{\hat{\gamma}}_i(1)-\gamma_i(0)\bar{\hat{\eta}}_i'(0)-\eta_i'(0)\bar{\hat{\gamma}}_i(0)\right.\\
&\left.+\theta_i'(1)\bar{\hat{\rho}}_i''(1)+\rho_i''(1)\bar{\hat{\theta}}_i'(1)-\theta_i'(0)\bar{\hat{\rho}}_i''(0)-\rho_i''(0)\bar{\hat{\theta}}_i'(0)\right)\\
=&-\sum_{i=1}^n\int_0^1 C_i^2\left(\gamma_i(y)\bar{\hat{\eta}}_i''(y)+\eta_i'(x)\bar{\hat{\gamma}}_i'(y)+\theta_i'(y)\bar{\hat{\rho}}_i'''(y)+\rho_i''(x)\bar{\hat{\theta}}_i''(y)\right)dy\\
=&-\Big\langle(\eta_1,\gamma_1,\rho_1',\theta_1',\cdots,\eta_n,\gamma_n,\rho_n',\theta_n'),\ A^*(\hat{\eta}_1,\hat{\gamma}_1,\hat{\rho}'_1,\hat{\theta}'_1,\cdots,\hat{\eta}_n,\hat{\gamma}_n,\hat{\rho}'_n,\hat{\theta}'_n)\Big\rangle_H,
\end{aligned}
$$

here

$$
\begin{aligned}
&\sum_{i=1}^n C_i^2\Big(\gamma_i(1)\bar{\hat{\eta}}_i'(1)+\eta_i'(1)\bar{\hat{\gamma}}_i(1)+\theta_i'(1)\bar{\hat{\rho}}_i''(1)+\rho_i''(1)\bar{\hat{\theta}}_i'(1)\Big)\\
&-\sum_{i=1}^n C_i^2\Big(\gamma_i(0)\bar{\hat{\eta}}_i'(0)+\eta_i'(0)\bar{\hat{\gamma}}_i(0)+\theta_i'(0)\bar{\hat{\rho}}_i''(0)+\rho_i''(0)\bar{\hat{\theta}}_i'(0)\Big)=0,
\end{aligned}
$$

where we have used the condition $\eta_{n,x}(1)=0$ and $C_i^2\eta_{i,x}(1)=C_{i+1}^2\eta_{i+1,x}(0)$, $\rho_{n,x}(1)=0$ and $C_i^2\rho_{i,x}(1)=C_{i+1}^2\rho_{i+1,x}(0), i=1,2,\cdots,n-1$. And in fact

$$
\begin{aligned}
&\sum_{i=1}^n C_i^2\left(\gamma_i(1)\bar{\hat{\eta}}_i'(1)+\eta_i'(1)\bar{\hat{\gamma}}_i(1)\right)-\sum_{i=1}^n C_i^2\left(\gamma_i(0)\bar{\hat{\eta}}_i'(0)+\eta_i'(0)\bar{\hat{\gamma}}_i(0)\right)\\
=&\sum_{i=1}^{n-1}\Big(\gamma_i(1)C_{i+1}^2\bar{\hat{\eta}}_{i+1}'(0)+\bar{\hat{\gamma}}_i(1)C_{i+1}^2\eta_{i+1}'(0)-\left(\gamma_{i+1}(0)C_{i+1}^2\bar{\hat{\eta}}_{i+1}'(0)-\bar{\hat{\gamma}}_{i+1}(0)C_{i+1}^2\eta_{i+1}'(0)\right)\\
&-C_i^2\left(\gamma_1(0)\bar{\hat{\eta}}_1'(0)-\eta_1'(0)\bar{\hat{\gamma}}_1(0)\right)\Big)\\
=&\sum_{i=1}^{n-1}C_{i+1}^2\bar{\hat{\eta}}_{i+1}'(0)\left(\gamma_i(1)-\gamma_{i+1}(0)\right)+\sum_{i=1}^{n-1}C_{i+1}^2\eta_{i+1}'(0)\left(\bar{\hat{\gamma}}_i(1)-\bar{\hat{\gamma}}_{i+1}(0)\right)-0\\
=&0.
\end{aligned}
$$

Similarly, we can conclude that

$$\sum_{i=1}^{n} C_i^2 \left(\theta_i'(1)\bar{\tilde{\rho}}_i''(1) + \rho_i''(1)\bar{\tilde{\theta}}_i'(1)\right) - \sum_{i=1}^{n} C_i^2 \left(\theta_i'(0)\bar{\tilde{\rho}}_i''(0) + \rho_i''(0)\bar{\tilde{\theta}}_i'(0)\right) = 0.$$

That is, $A^* = -A$. This ends the proof. Therefore, A generates a C_0 semigroup on H. □

Lemma 3.1. *(see [29]) Let $\mathbb{X}$ be a Banach space, and A be the infinitesimal generator of a C_0 semigroup $T(t)$ on H, satisfying $\|T(t)\| \leq Me^{ut}$. If B is a bounded linear operator on H, then A + B is the infinitesimal generator of a C_0 semigroup $S(t)$, satisfying $\|S(t)\| \leq Me^{(u+M\|B\|)t}$.*

In light of the fact that A generates a C_0 semigroup on H, it follows from (3.1) that the following result is obtained.

Let $T(t)$ and $S(t)$ be the C_0 semigroups generated by A and $A + B$, respectively. It can be shown that the systems in question, namely (2.2) and (2.4), are solvable in H. The solutions to these systems are given by the following expressions: In light of the nonlocal conditions, we arrive at the equations $W_0 = T(0)W(0)$ and $\widetilde{W}(0) = -S(0)\widetilde{W}(0)$. Consequently, we can express $W(s)$ and $W(0)$ in terms of the generator matrices as follows, $W(s) = T(s)W(0)$ and $\widetilde{W}(s) = -S(s)\widetilde{W}(0)$. Similarly, we can express $W(s)$ and $W(0)$ in terms of the inverse generator matrices as follows: $W(s) = T(s)T^{-1}(0)W_0$ and $\widetilde{W}(s) = -S(s)S^{-1}(0)\widetilde{W}_0$.

In the following sections, we shall examine the boundedness of the transformations defined in (2.1) and (2.3) within an appropriate Hilbert space. For the purposes of this discussion, we shall define $H[0, 1]$ and H as previously outlined and introduce a new state space, $H_1 = L^2[0, \tau] \times H$, which we shall equip with a norm

$$\|(\phi, \eta, \gamma, \rho, \theta)\|_{H_1}^2 = \int_0^\tau |\phi(s)|^2\, ds + \|(\eta, \gamma, \rho, \theta)\|_H^2 ,$$

H is a Hilbert space. Set $p, \eta_i, \gamma_i, \rho_i, \theta_i$ be a solution to (2.1), $i = 1, 2, \cdots, n$, and define a linear operator T on H by

$$\begin{pmatrix} \psi(s) \\ f_1 \\ g_1 \\ h_1 \\ k_1 \\ \vdots \end{pmatrix} = C \begin{pmatrix} \phi(s) \\ f_1 \\ g_1 \\ h_1 \\ k_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1-p^* & -\gamma_i^* & -\eta_i^* & -\rho_i^* & -\theta_i^* & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ & \cdots & & & \cdots & \\ & \vdots & & & \vdots & \end{pmatrix} \begin{pmatrix} \phi(s) \\ f_1 \\ g_1 \\ h_1 \\ k_1 \\ \vdots \end{pmatrix},$$

and the components are defined as follows:

$$p^* : L^2[0, \tau] \longrightarrow L^2[0, \tau] : p^*\phi(s) = \int_0^s p(s-r)\phi(r)dr, \quad \phi \in L^2[0, \tau];$$

$$\gamma_i^* : H^1[0, 1] \longrightarrow L^2[0, \tau] : \gamma_i^* f_i(s) = \int_0^1 \gamma_i(s, x) f_i(x)dx, \quad f_i \in H^1[0, 1];$$

$$\eta_i^* : L^2[0, 1] \longrightarrow L^2[0, \tau] : \eta_i^* g_i(s) = \int_0^1 \eta_i(s, x) g_i(x)dx, \quad g_i \in L^2[0, 1];$$

$$\rho_i^* : H^1[0,1] \longrightarrow L^2[0,\tau] : \rho_i^* h_i(s) = \int_0^1 \rho_i(s,x) h_i(x) dx, \ \ h_i \in H^1[0,1];$$
$$\theta_i^* : L^2[0,1] \longrightarrow L^2[0,\tau] : \theta_i^* k_i(s) = \int_0^1 \theta_i(s,x) k_i(x) dx, \ \ k_i \in L^2[0,1].$$

Note that $(\eta_1, \gamma_1, \rho_1, \theta_1, \cdots, \eta_n, \gamma_n, \rho_n, \theta_n) \in H$, $p(s) = \eta_n(s,1)$. From the definition of these operators, we can conclude that $p^*, \gamma_i^*, \eta_i^*, \rho_i^*$, and θ_i^* are bounded linear operators. The analysis shows:

Proposition 3.2. *Define C as before; therefore, in space H_1, C is a bounded linear operator, and the transformation* (2.1) *is tantamount to C.*

The boundedness of transformations (2.3) is similar to (2.1); we omit it here.

Finally, this section concludes with an examination of the stability of the target system. In this context, the notation $\psi(t)$ is employed to denote the C_0 semigroup on H_1 associated with the system

$$\begin{cases} \psi_t(s,t) = \psi_s(s,t), \ s \in (0,\tau), \ t > 0, \\ \psi(\tau,t) = 0, \ t > 0, \\ u_{i,tt}(y,t) = C_i^2 u_{i,yy}(y,t), \ y \in (0,1), \ t > 0, \ i = 1,2,\ldots,n, \\ u_1(0,t) = 0, \ t > 0, \\ u_i(1,t) = u_{i+1}(0,t), \ t > 0, \ i = 1,2,\ldots,n-1, \\ m_i u_{i,tt}(1,t) + C_i^2 u_{i,y}(1,t) = C_{i+1}^2 u_{i+1,y}(0,t), \ t > 0, \ i = 1,2,\ldots,n-1, \\ m_n u_{n,tt}(1,t) + C_n^2 u_{n,y}(1,t) = -a u_{n,yt}(1,t) - b u_{n,t}(1,t), \ t > 0 \\ u_i(y,0) = u_{i,0}(y), \\ u_{i,t}(y,0) = u_{i,1}(y), \ y \in (0,1), \ i = 1,2,\ldots,n. \end{cases} \tag{3.1}$$

In accordance with the findings in [30], $\psi(t)$ is a C_0 exponential stable semigroup, which means that there exist M and ω , for $\forall t \geq 0$, we have

$$\|\psi(t)\| \leq M e^{-\omega t}.$$

Theorem 3.1. *Denote $u_0 = (\psi_0(s), u_{1,0}(y), u_{1,1}(y), u_{2,0}(y), u_{2,1}(y), \cdots, u_{n,0}(y), u_{n,1}(y)) \in H_1$, define an operate $b = (0,0,0,0,0,\cdots,0,\delta(y-1))^T$. Subsequently, b is deemed to be acceptable in relation to $\psi(t)$. The objective is to establish*

$$u_t = (\psi(s,t), \omega_1(y,t), \omega_{1,t}(y,t), \cdots, \omega_n(y,t), \omega_{n,t}(y,t)) \in H_1,$$

be solution of the target system (1.4). Then $u(t) = \psi(t)u_0 - \int_0^t \psi(t-r) b \psi_0(r) dr$, $\ \forall t \geq 0$, and $u(t) = \psi(t-\epsilon)u(\epsilon)$, $\ t > \tau$. So

$$\|u(t)\| \leq M e^{-\omega(t-\epsilon)} \|u(\epsilon)\|, \quad \forall t \geq \tau.$$

In light of Theorems (2.1) and (3.1), we consider the following assertion.

Corollary 3.1. *The system* (1.2) *is exponentially stable under the control law* (1.3).

4. Conclusions

In this paper, we are considering the exponential stabilization problem for chain wave network with a boundary delay. The right kernel functions are chosen by starting with an exponentially stable target system and then adding an auxiliary system. Then a linear transformation is constructed with the objective of establishing the equivalence of the systems under consideration and the auxiliary system. It can thus be demonstrated that a feedback control law is generated. The proof is based on the rigorous selection of kernel functions and parameter equations, which demonstrate that the system under consideration is feedback equivalent to the target system. The target system is exponentially stable, so the original system is exponentially stable as well. This approach to controller design guarantees the stability of the closed-loop system, thus obviating the necessity for complex stability analysis. The key issue is identifying functions that stabilize the system in the absence of time delay. After determining these functions, we can use (2.2) to solve the kernel functions and get the feedback controller. However, it remains unclear if this result is applicable in all cases. This topic will be pursued further in order to identify a general principle that can then be applied to more models.

Author contributions

Yaru Xie: Conceptualization, Methodology, Supervision, Funding acquisition; Ruiqing Gao: Validation, Writing-original draft, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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