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Research article

Stabilization in distribution of hybrid stochastic differential delay equations with Lévy noise by discrete-time state feedback controls

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Abstract: This paper was concerned with stabilization in distribution by feedback controls based on discrete-time state observations for a class of nonlinear stochastic differential delay equations with Markovian switching and Lévy noise (SDDEs-MS-LN). Compared with previous literature, we employed Lévy noise in the discussion about stabilization in distribution for hybrid stochastic delay systems and we considered using a discrete-time linear feedback control which is more realistic and costs less. In addition, by constructing a new Lyapunov functional, stabilization in distribution of controlled systems can be achieved with the coefficients satisfying globally Lipschitz conditions. In particular, we discussed the design of feedback controls in two structure cases: state feedback and output injection. At the same time, the lower bound for the duration between two consecutive observations τ (τ^*) was obtained as well. Finally, a numerical experiment with some computer simulations was given to illustrate the new results.

Keywords: stochastic differential equations; Markov chain; Lévy noise; stability in distribution; discrete-time feedback controls
Mathematics Subject Classification: 93C55, 93D15, 93E03

1. Introduction

Stochastic differential equations (SDEs) with Markovian switching (also known as hybrid SDEs) has been widely used to model many systems in biological systems, financial systems, and other fields. A field of common interest in the study of hybrid SDEs is automatic control, taking subsequent emphasis on the stability analysis [1, 2]. Most of the literature, such as [3–7], only consider Brown motions. However, Brown motions are continuous and cannot describe discontinuous noises like jump-type noises. Compared with Brown motions, Lévy noise, which contains both continuous Brown motions and discontinuous Poisson jumps, can model the extreme sudden events, such as earthquakes, storms, floods, wars, and so on. For example, in [8] some sufficient conditions were put forward to achieve almost surely exponential stability of neural networks with Markovian switching and Lévy noise. Therefore, with the development of stochastic analysis, stochastic differential equations with Markovian switching and Lévy noise are considered by many researchers, see [9–12].

It is well known that time delays are often and inevitably encountered for various reasons in many fields such as population systems, manufacturing, chemistry and chemical engineering, finance, etc. Meanwhile, a time delay is often one of the main causes of poor performance in systems, see [13, 14]. Hence, taking time delays into account is reasonable and necessary when studying the stability of SDDEs-MS-LN. Nowadays, stability and stabilization of such SDDEs have been studied, see [15–17]. For example, Yuan et al. in [16] investigated sufficient conditions for stability of delay jump diffusion processes. Li in [17] focused on the mean square stability of stochastic differential equations with Lévy noise.

A common feature in these papers is that most of the research is focused on the stability of the trivial solutions. However, many hybrid systems do not have an equilibrium state or their solutions do not converge to zero, see [18, 19]. It is not sufficient to only study the stability of trivial solutions in the real world. For example, for many population systems under realistic backdrops, stochastic permanence is a more suitable control goal than extinction (see [20–22]). In this case, it is of great significance to know whether the solution will converge to some distribution or not (but not necessarily to zero). This property is known as asymptotic stability in distribution. Stability in distribution of SDEs-MS with Brownian motion has attracted some attention of scholars recently, for example, Yuan et al. [19] and You et al. [23]. In 2010, Yuan et al. [16] studied stability in distribution of hybrid delay systems with jumps. As a classical area of stability of hybrid systems, Li et al. [24] recently considered to employ delay feedback controls to stabilize a given SDEs-MS-LN in distribution. But for the stabilization in distribution of SDDEs-MS-LN, the discussion is still open. In addition, to reduce the practical cost of control design, feedback controls based on discrite-time state observations [1, 14] are considered in this paper.

Mathematically speaking, let us consider an unstable SDDE-MS-LN

$$\begin{aligned} dX(t) = \quad & f(X(t), X(t-h), r(t))dt + g(X(t), X(t-h), r(t))dB(t) \\ & + \int_{\mathbb{R}_0^n} H\left(X(t^-), X((t-h)^-), r(t), z\right) \tilde{N}(\, dt,\ dz), \end{aligned} \tag{1.1}$$

where $X(t) \in \mathbb{R}^d$, h is a time delay of the system, $r(t)$ is a Markov chain, $B(t)$ is a Brownian motion, $\tilde{N}(\, dt,\ dz)$ is a compensated Poisson random measure, and $\mathbb{R}_0^n = \mathbb{R}^n - \{0\}$ (For formal definitions, see Section 2.) Such a regular feedback control requires the continuous observations of the state $X(t)$ for all $t \geq 0$. This is of course expensive and sometimes not possible as the observations are often of discrete time. Now we can design a feedback control $u(X([t/\tau]\tau), r(t))$ based on the discrete-time observations of the state $X(t)$ at times 0, τ, 2τ,..., so that the controlled system

$$\begin{aligned} dX(t) = & [f(X(t), X(t-h), r(t)) + u(X([t/\tau]\tau), r(t))]dt \\ & + g(X(t), X(t-h), r(t))dB(t) \\ & + \int_{\mathbb{R}_0^n} H\left(X(t^-), X((t-h)^-), r\left(t\right), z\right) \tilde{N}(\, dt,\ dz), \end{aligned} \tag{1.2}$$

becomes stable in distribution.

The main aim of this paper is to explore how to use feedback control $u(X([t/\tau]\tau), r(t))$ to stabilize a given unstable SDDE-MS-LN in distribution. The key points of this paper are as follows.

- We introduce Lévy noise to remodel hybrid stochastic delay systems and study the stability in distribution for controlled SDDEs-MS-LN.
- Due to the discontinuity of Lévy noise, we need to study the stability in distribution for SDDEs-MS-LN in functional space $\mathcal{D}_h$ (for formal definitions, see Section 2) rather than C_h in [23].
- Making use of the generalized Itô formula for Lévy-type stochastic integrals [25], we construct a special Lyapunov functional based on Lévy noise, the property of stability and discrete-time feedback control to achieve the asymptotic stability in distribution for controlled SDDEs-MS-LN.
- In order to reduce the cost of the continuous working time of the controller, feedback control $u(X([t/\tau]\tau), r(t))$ based on the discrete-time observations is an efficient strategy to stabilize the unstable systems. Moreover, we show that there is a positive number τ^* such that the feedback control $u(X([t/\tau]\tau), r(t))$ will make the controlled system (1.2) asymptotically stable in distribution provided $\tau \leq \tau^*$. We will also give a lower bound on τ^* which is computable numerically.

The structure of this paper is organized as follows. In Section 2, we present some notations, definitions, and assumptions related to Eq (1.1). In Section 3, we study the stability in distribution of the solution to Eq (1.2) based on the Lyapunov functionl and Itô formula. In Section 4, the method for designing the control function is discussed. In Section 5, we provide a numerical example to verify the effectiveness of the new results. Finally, this article is concluded in Section 6.

2. Notations and assumptions

Throughout this paper, unless otherwise specified, we use the following notations. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\mathcal{B}\left(\mathbb{R}^d\right)$ denote the family of all Borel measurable sets in $\mathbb{R}^d$. Let $|\cdot|$ denote the Euclidean norm or the matric trace norm, respectively. For a matrix A, $|A| = \sqrt{\text{trace}\,(A^T A)}$ denotes its trace norm and $\|A\| = max\{|Ax| : |x| = 1\}$ denotes its operator norm. If A is a symmetric matrix, the largest and smallest eigenvalue are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. In general, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ signifies a complete probability space whose filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. Denote by $\mathcal{D}_h$ (or $D\left([-h, 0]; \mathbb{R}^d\right)$) the family of all càdlàg (i.e., right continuous with left limits) functions $\xi : [-h, 0] \to \mathbb{R}^d$ in the Skorokhod topology. For any $\xi_1, \xi_2 \in \mathcal{D}_h$, define the Skorohod metric $d_S(\xi_1, \xi_2) = \inf_{\lambda\in\Lambda} \{\|\lambda\|^\circ \vee \|\xi_1 - \xi_2 \circ \lambda\|_h\}$, where Λ denotes the class of strictly increasing, continuous mappings of $[-h, 0]$ onto itself, $\xi_2 \circ \lambda$ denotes the composition of two functions ξ_2 and λ, $\|\lambda\|^\circ = \sup_{-h\leq s<t\leq 0} \left|\log \frac{\lambda(t)-\lambda(s)}{t-s}\right|$, and $\|\xi\|_h = \sup_{-h\leq s\leq 0} |\xi(s)|$. Under the Skorohod metric $d_S, D\left([-h, 0]; \mathbb{R}^d\right)$ is complete and separable ([26], Theorem 12.2, p. 128). In addition, $\mathcal{B}(\mathcal{D}_h)$ denotes the family of all Borel measurable sets in $\mathcal{D}_h$. Let $B(t) = (B_1, \ldots, B_m)$ be an m-dimensional Brownian motion. Denote by $N(t, z)$ an n-dimensional Poisson process, and denote the compensated Poisson random measure by

$$\begin{aligned}\tilde{N}(\mathrm{d}t, \mathrm{d}z)^T &= N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t \\ &= (N_1(\mathrm{d}t, \mathrm{d}z_1) - \nu_1(\mathrm{d}z_1)\,\mathrm{d}t, \ldots, N_n(\mathrm{d}t, \mathrm{d}z_n) - \nu_n(\mathrm{d}z_n)\,\mathrm{d}t),\end{aligned}$$

where $\{N_k, k = 1, \ldots, n\}$ are independent 1-dimensional Poisson random measures with characteristic measure $\{\nu_k, k = 1, \ldots, n\}$ coming from n independent 1-dimensional Poisson point processes.

Let $r(t)$, $t \geq 0$, be a right-continuous irreducible Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with the generator $\Gamma = \left(\gamma_{ij}\right)_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$ satisfies $\lim\limits_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$ and γ_{ij} is the transition rate from i to j satisfying $\gamma_{ij} > 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that $r(t)$, $B(t)$, and $N(t, z)$ are independent of each other.

Let us consider a d-dimension SDDE-MS-LN (1.1), where $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}^{d\times m}$, and $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_0^n \to \mathbb{R}^{d\times n}$ are Borel measurable functions, $X(t^-) = \lim_{s\uparrow t} X(s)$. We note that each column $H^{(k)}$ of the $d \times n$ matrix $H = \left[H_{lj}\right]$ depends on z only through the kth coordinate z_k, that is

$$H^{(k)}(X, i, z) = H^{(k)}(X, i, z_k); \quad z = (z_1, \ldots, z_n) \in \mathbb{R}_0^n.$$

We refer to [16, 27] where this type of dependence is discussed and investigated for SDDEs-MS-LN. We can rewrite out in detail component $X_l(t)$, $1 \leq l \leq d$, in (1.1), that is

$$\begin{aligned} \mathrm{d}X_l(t) = \quad & f_l(X(t), X(t-h), r(t))\mathrm{d}t + \sum_{j=1}^{m} g_{lj}(X(t), X(t-h), r(t))\mathrm{d}B_j(t) \\ & + \sum_{k=1}^{n} \int_{\mathbb{R}\setminus\{0\}} H_{lk}\left(X(t^-), X((t-h)^-), r(t), z_k\right) \tilde{N}_k(\mathrm{d}t, \mathrm{d}z_k). \end{aligned}$$

Next we will state an assumption about the coefficients of SDDE-MS-LN (1.1).

Assumption 2.1. *There exist positive constants a_1, a_2, and a_3 such that*

$$|f(x, \bar{x}, i) - f(y, \bar{y}, i)|^2 \leq a_1(|x-y|^2 + |\bar{x}-\bar{y}|^2), \quad |g(x, \bar{x}, i) - g(y, \bar{y}, i)|^2 \leq a_2(|x-y|^2 + |\bar{x}-\bar{y}|^2),$$

and

$$\sum_{k=1}^{n} \int_{\mathbb{R}\setminus\{0\}} \left|H^{(k)}(x, \bar{x}, i, z_k) - H^{(k)}(y, \bar{y}, i, z_k)\right|^2 \nu_k(\mathrm{d}z_k) \leq a_3(|x-y|^2 + |\bar{x}-\bar{y}|^2),$$

for all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and $i \in \mathbb{S}$.

It is easy to see from Assumption (2.1) that

$$|f(x, \bar{x}, i)|^2 \leq 2a_1(|x|^2 + |\bar{x}|^2) + a_0, \quad |g(x, \bar{x}, i)|^2 \leq 2a_2(|x|^2 + |\bar{x}|^2) + a_0 \tag{2.1}$$

and

$$\sum_{k=1}^{n} \int_{\mathbb{R}\setminus\{0\}} \left|H^{(k)}(x, \bar{x}, i, z_k)\right|^2 \nu_k(\mathrm{d}z_k) \leq 2a_3(|x|^2 + |\bar{x}|^2) + a_0 \tag{2.2}$$

for all $(x, \bar{x}, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$ and $i \in \mathbb{S}$, where $a_0 = 2\max_{i\in\mathbb{S}}\Big(|f(0,0,i)|^2 \vee |g(0,0,i)|^2\Big) \vee \sum_{k=1}^{n} \int_{\mathbb{R}\setminus\{0\}} \left|H^{(k)}(0, 0, i, z_k)\right|^2 \nu_k(dz_k)$.

By Assumption 2.1, it is known (see [16]) that the SDDE-MS-LN (1.1) has a unique global solution $X(t)$ for all $t \geq 0$. Assume that the original SDDE-MS-LN (1.1) does not have the desired property of stability in distribution. Therefore we need to design a feedback control to stabilize the system (1.1). To make the design more concise and simple, we use the linear form of feedback control $u(X(\delta(t)), r(t)) = A(r(t))X(\delta(t))$, where $A(i) \equiv A_i \in \mathbb{R}^{d\times d}(1 \leq i \leq N)$, $\delta(t) = [t/\tau]\tau$. In addition, throughout this paper, we will set $a_4 = \max_{i\in\mathbb{S}} \|A_i\|^2$. The controlled system (1.2) therefore becomes

$$\begin{aligned} \mathrm{d}X(t) = \quad & [f(X(t), X(t-h), r(t)) + A(r(t))X(\delta(t))]\mathrm{d}t \\ & + g(X(t), X(t-h), r(t))dB(t) \\ & + \int_{\mathbb{R}_0^n} H\left(X(t^-), X((t-h)^-), r(t), z\right) \tilde{N}(\mathrm{~d}t, \mathrm{~d}z) \end{aligned} \tag{2.3}$$

with the initial data as

$$\begin{cases} \{X(s) : -h \leq s \leq 0\} = \xi \in \mathcal{D}_h, \\ r(0) = i \in \mathbb{S}. \end{cases} \tag{2.4}$$

It is well known to all (see [28]), under Assumption 2.1, SDDE-MS-LN (2.3) has a unique global solution for any initial data (2.4). Define $X_t = \{X(t+s) : -h \leq s \leq 0\}$ for $t \geq 0$, which is a $\mathcal{D}_h$-valued process. $X^{\xi,i}(t)$ denotes the solution of SDDE-MS-LN (2.3) with initial data (2.4). $r^i(t)$ denotes the Markov chain starting from i. It is also known that (see [29])

$$\mathbb{E}\left[\left\|X_t^{\xi,i}\right\|_h^2\right] < C_t\left(1 + \|\xi\|_h^2\right) \quad \forall t \geq 0, \tag{2.5}$$

where C_t is a positive constant that depends on t but is independent of the initial data (ξ, i).

We notice that the joint process $(X_t, r(t))$ is not a time-homogeneous Markov process. But when h can be divisible by τ, for $k \geq 0$, we can easily get that the joint process $(X_{k\tau}, r(k\tau))$ is a time-homogeneous Markov process with transition probability $p(k, \xi, i; d\zeta \times \{j\})$, where $p(k, \xi, i; d\zeta \times \{j\})$ denotes the transition probability measure on $\mathcal{D}_h \times \mathbb{S}$, that is

$$\mathbb{P}\left(\left(X_{k\tau}^{\xi,i}, r^i(k\tau)\right) \in E \times J\right) = \sum_{j\in J} \int_E p(k, \xi, i; d\zeta \times \{j\}) \tag{2.6}$$

for any $E \in \mathcal{B}(\mathcal{D}_h)$ and $J \in \mathbb{S}$.

Denote by $\mathcal{P}(\mathcal{D}_h)$ the family of probability measures on the measurable space $(\mathcal{D}_h, \mathcal{B}(\mathcal{D}_h))$. For $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\mathcal{D}_h)$, metric $d_{\mathbb{L}}$ is given by

$$d_{\mathbb{L}}(\mathbb{P}_1, \mathbb{P}_2) = \sup_{\phi\in\mathbb{L}} \left| \int_{\mathcal{D}_h} \phi(\xi)\mathbb{P}_1(\mathrm{~d}\xi) - \int_{\mathcal{D}_h} \phi(\xi)\mathbb{P}_2(\mathrm{~d}\xi) \right|,$$

where $\mathbb{L} = \{\phi : \mathcal{D}_h \to \mathbb{R}$ satisfying $|\phi(\xi) - \phi(\zeta)| \leq d_{\mathbb{S}}(\xi, \zeta)$ and $|\phi(\xi)| \leq 1$ for $\xi, \zeta \in D_h\}$. In addition, let $\mathcal{L}(X_t)$ denote the probability measure generated by X_t on $(\mathcal{D}_h, \mathcal{B}(\mathcal{D}_h))$.

Definition 2.1. *The SDDE-MS-LN* (2.3) *is said to be asymptotically stable in distribution if there exists a probability measure* $\mu_h \in \mathcal{P}(\mathcal{D}_h)$ *such that*

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(\mathcal{L}\left(X_{k\tau}^{\xi,i}\right), \mu_h\right) = 0$$

for all $(\xi, i) \in \mathcal{D}_h \times \mathbb{S}$.

3. Stabilization in distribution

Let $C^2\left(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+\right)$ denote the family of all non-negative continuous functions $\Psi(x,i)$ defined on $\mathbb{R}^d \times \mathbb{S}$ which are twice continuously differentiable in x for all $i \in \mathbb{S}$. Assume that there exists one $\Psi \in C^2\left(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+\right)$, and define an operator $L\Psi$ from $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$ to $\mathbb{R}$ by:

$$\begin{aligned} & L\Psi(x,\bar{x},i) \\ = \;& \Psi_x(x,i)\left[f(x,\bar{x},i) + A_i x\right] \\ & + \frac{1}{2}\operatorname{trace}\left[g(x,\bar{x},i)^T \Psi_{xx}(x,i) g(x,\bar{x},i)\right] \\ & + \int_{\mathbb{R}\backslash\{0\}} \sum_{k=1}^{n} \left[\Psi\left(x + H^{(k)}(x,\bar{x},i,z_k), i\right) - \Psi(x,i)\right. \\ & \left. - \Psi_x(x,i) H^{(k)}(x,\bar{x},i,z_k)\right] \nu_k(\,\mathrm{d}z_k) + \sum_{j=1}^{N} \gamma_{ij} \Psi(x,j), \end{aligned} \tag{3.1}$$

where $\Psi_x(x,i) = \left(\frac{\partial \Psi(x,i)}{\partial x_1}, \frac{\partial \Psi(x,i)}{\partial x_2}, \ldots, \frac{\partial \Psi(x,i)}{\partial x_d}\right)$, $\Psi_{xx}(x,i) = \left(\frac{\partial^2 \Psi(x,i)}{\partial x_i \partial x_j}\right)_{d\times d}$.

The difference between two solutions of the system (2.3) with different initial values is as follows:

$$\begin{aligned} & X^{\xi,i}(t) - X^{\zeta,i}(t) \\ = \;& \xi - \zeta + \int_0^t \left[f\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r^i(s)\right) - f\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r^i(s)\right)\right. \\ & + \left. A\left(r^i(s)\right)\left(X^{\xi,i}(\delta(t)) - X^{\zeta,i}(\delta(t))\right)\right] \mathrm{d}s \\ & + \int_0^t \left[g\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r^i(s)\right) - g\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r^i(s)\right)\right] \mathrm{d}B(s) \\ & + \int_0^t \int_{\mathbb{R}_0^n} \left[H\left(X^{\xi,i}(s^-), X^{\xi,i}((s-h)^-), r^i(s), z\right)\right. \\ & \left. - H\left(X^{\zeta,i}(s^-), X^{\zeta,i}((s-h)^-), r^i((s-h)), z\right)\right] \tilde{N}(\,\mathrm{d}s,\ \mathrm{d}z). \end{aligned} \tag{3.2}$$

Let $\Phi \in C^2\left(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+\right)$, and define an operator $\mathbb{L}\Phi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}$ concerning Eq (3.2) by

$$\begin{aligned} & \mathbb{L}\Phi(x,y,\bar{x},\bar{y},i) \\ = \;& \Phi_x(x-y,i)\left[f(x,\bar{x},i) - f(y,\bar{y},i) + A_i(x-y)\right] \\ & + \frac{1}{2}\operatorname{trace}\left[(g(x,\bar{x},i) - g(y,\bar{y},i))^T \Phi_{xx}(x-y,i)(g(x,\bar{x},i) - g(y,\bar{y},i))\right] \\ & + \int_{\mathbb{R}\backslash\{0\}} \sum_{k=1}^{n} \left[\Phi(x - y + H^{(k)}(x,\bar{x},i,z_k) - H^{(k)}(y,\bar{y},i,z_k), i)\right. \\ & \left. - \Phi(x-y,i) - \Phi_x(x-y,i)\left(H^{(k)}(x,\bar{x},i,z_k) - H^{(k)}(y,\bar{y},i,z_k)\right)\right] \nu_k(\,\mathrm{d}z_k) \\ & + \sum_{j=1}^{N} \gamma_{ij} \Phi(x-y,j). \end{aligned} \tag{3.3}$$

To study stabilization in distribution of system (2.3), we need the following assumptions.

Assumption 3.1. *There exist positive constants* c_1, θ_1, b_2, *and* $b_0 > b_1 \geq 0$, *function* $\Psi(x,i) \in C^2\left(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+\right)$, *and* $Q_1(x) \in C\left(\mathbb{R}^d; \mathbb{R}_+\right)$ *such that*

$$\begin{aligned} & c_1|x|^2 \leq \Psi(x,i) \leq Q_1(x), \\ & L\Psi(x,\bar{x},i) + \theta_1 \left|\Psi_x(x,i)\right|^2 \leq -b_0 Q_1(x) + b_1 Q_1(\bar{x}) + b_2 \end{aligned} \tag{3.4}$$

for all $(x,\bar{x},i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$.

Assumption 3.2. *There exist positive constants* c_2, θ_2, *and* $b_3 > b_4 \geq 0$, *function* $\Phi(x,i) \in C^2\left(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+\right)$, *and* $Q_2(x) \in C\left(\mathbb{R}^d; \mathbb{R}_+\right)$ *such that*

$$\begin{aligned} & c_2|x-y|^2 \leq \Phi(x,y,i) \leq Q_2(x-y), \\ & \mathbb{L}\Phi(x,y,\bar{x},\bar{y},i) + \theta_2 \left|\Phi_x(x-y,i)\right|^2 \leq -b_3 Q_2(x-y) + b_4 Q_2(\bar{x}-\bar{y}) \end{aligned} \tag{3.5}$$

for all $(x,y,\bar{x},\bar{y},i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$.

3.1. Lyapunov functionals

To obtain our results, we need to establish the Lyapunov functional on the segments $\hat{X}_t := \{X(t+s) : -\tau - h \leq s \leq 0\}$ and $\hat{r}_t = \{r(t+s) : -\tau - h \leq s \leq 0\}$ for $t \geq \tau$. Let $r(s) = r(0)$ for $-\tau - h \leq s \leq 0$. Evidently, $\hat{X}_t$ is $D\left([-\tau-h,0];\mathbb{R}^d\right)$-valued which is different with X_t. The Lyapunov functional will be of the form

$$V\left(\hat{X}_t,\hat{r}_t,t\right) := \Psi(X(t),r(t)) + \hat{V}\left(\hat{X}_t,\hat{r}_t,t\right), \quad \text{for } t \geq h, \tag{3.6}$$

where

$$\begin{aligned} \hat{V}\left(\hat{X}_t,\hat{r}_t,t\right) = \; & \alpha \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau \left|f(X(v), f(X(v-h), r(v)) + A_{r(v)}X(\delta(v))\right|^2 \\ & + |g(X(v), g(X(v-h), r(v))|^2 \\ & + \int_{\mathbb{R}_0^n} \left|H\left(X\left(v^-\right), X\left((v-h)^-\right), r\left(v\right), z\right)\right|^2 \nu(\mathrm{d}z)\Big]\mathrm{d}v\,\mathrm{d}s \end{aligned}$$

and α is a positive constant selected later.

Remark 3.1. *Since our feedback control is based on discrete-time state observations, the Lyapunov functional in [23, 24] is not appropriate which is employed for the stabilization in distribution problem by delay feedback control. Therefore, we consider to employ a new Lyapunov functional motivated by [14, 24] to prove the stability in distribution of controlled system (2.3).*

We can observe that

$$c_1|X(t)|^2 \leq V\left(\hat{X}_t,\hat{r}_t,t\right) \leq Q_1(X(t)) + \hat{V}\left(\hat{X}_t,\hat{r}_t,t\right). \tag{3.7}$$

For convenience, $X(t)$ denotes $X^{\xi,i}(t)$ and we fix the initial data (ξ,i) arbitrarily. Applying the generalized functional Itô formula to the Lyapunov functional defined by (3.6) yields

$$\mathrm{d}V\left(\hat{X}_t,\hat{r}_t,t\right) = \mathrm{LV}\left(\hat{X}_t,\hat{r}_t,t\right)\mathrm{d}t + \mathrm{d}M(t)$$

for $t \geq \tau$, where $M(t)$ is a martingale with $M(0) = 0$, and

$$
\begin{aligned}
\mathrm{LV}\left(\hat{X}_t, \hat{r}_t, t\right) = \quad & L\Psi(X(t), X(t-h), r(t)) - \Psi_X(X(t), r(t))A_{r(t)}(X(t) - X(\delta(t))) \\
& + \alpha\tau\Big[\tau\left|f(X(t), X(t-h), r(t)) + A_{r(t)}X(\delta(t))\right|^2 + |g(X(t), X(t-h), r(t))|^2 \\
& + \int_{\mathbb{R}_0^n}\left|H\left(X(t^-), X((t-h)^-), r\left(t\right), z\right)\right|^2 \nu(\,\mathrm{d}z)\Big] \\
& - \alpha\int_{t-\tau}^{t}\Big[\tau\left|f(X(s), X(s-h), r(s)) + A_{r(s)}X(\delta(s))\right|^2 \\
& + |g(X(s), X(s-h), r(s))|^2 + \int_{\mathbb{R}_0^n}\left|H\left(X(s^-), X((s-h)^-), r\left(s\right), z\right)\right|^2 \nu(\mathrm{d}z)\Big]ds \\
\leq \quad & L\Psi(X(t), X(t-h), r(t)) + \theta_1\left|\Psi_X(X(t), r(t))\right|^2 \\
& + \frac{1}{4\theta_1}\left\|A_{r(t)}\right\|^2|X(t) - X(\delta(t))|^2 \\
& + \alpha\tau\Big[\tau\left|f(X(t), X(t-h), r(t)) + A_{r(t)}X(\delta(t))\right|^2 + |g(X(t), X(t-h), r(t))|^2 \\
& + \int_{\mathbb{R}_0^n}\left|H\left(X(t^-), X((t-h)^-), r\left(t\right), z\right)\right|^2 \nu(\,\mathrm{d}z)\Big] \\
& - \alpha\int_{t-\tau}^{t}\Big[\tau\left|f(X(s), X(s-h), r(s)) + A_{r(s)}X(\delta(s))\right|^2 \\
& + |g(X(s), X(s-h), r(s))|^2 + \int_{\mathbb{R}_0^n}\left|H\left(X(s^-), X((s-h)^-), r\left(s\right), z\right)\right|^2 \nu(\mathrm{d}z)\Big]ds.
\end{aligned} \tag{3.8}
$$

By Assumption 2.1, we can derive

$$
\begin{aligned}
& \alpha\tau\Big[\tau\left|f(X(t), X(t-h), r(t)) + A_{r(t)}X(\delta(t))\right|^2 + |g(X(t), X(t-h), r(t))|^2 \\
& + \int_{\mathbb{R}_0^n}\left|H\left(X(t^-), X((t-h)^-), r\left(t\right), z\right)\right|^2 \nu(dx)\Big] \\
\leq \quad & \alpha\tau\Big[4a_1\tau(|X(t)|^2 + |X(t-h)|^2) + 2a_0\tau + 2a_4\tau|X(\delta(t))|^2 \\
+ \quad & 2a_2(|X(t)|^2 + |X(t-h)|^2) + a_0 + 2a_3(|X(t)|^2 + |X(t-h)|^2) + a_0\Big] \\
\leq \quad & \alpha\tau\Big[2\left(2a_1\tau + a_2 + a_3\right)|X(t)|^2 \\
+ \quad & a_0(2\tau + 1) + 2\left(2a_1\tau + a_2 + a_3\right)|X(t-h)|^2 + 2a_4\tau|X(\delta(t))|^2\Big] \\
\leq \quad & \alpha\tau\Big[2\left(2a_1\tau + a_2 + a_3\right)|X(t)|^2 \\
+ \quad & a_0(2\tau + 1) + 2\left(2a_1\tau + a_2 + a_3\right)|X(t-h)|^2 + 4a_4\tau|X(t)|^2 + 4a_4\tau|X(t) - X(\delta(t))|^2\Big] \\
\leq \quad & \alpha\tau\Big[2\left(2a_1\tau + a_2 + a_3 + 2a_4\tau\right)|X(t)|^2 \\
+ \quad & a_0(2\tau + 1) + 2\left(2a_1\tau + a_2 + a_3\right)|X(t-h)|^2 + 4a_4\tau|X(t) - X(\delta(t))|^2\Big].
\end{aligned} \tag{3.9}
$$

Under Assumption 3.1, we get from Eqs (3.8) and (3.9) that

$$
\begin{aligned}
&\mathrm{LV}\left(\hat{X}_t, \hat{r}_t, t\right)\\
\leq& -b_0 Q_1(X(t)) + b_1 Q_1(X(t-h)) + b_2 + (\frac{a_4}{4\theta} + 4a_4\alpha\tau^2)|X(t) - X(\delta(t))|^2\\
&+ \alpha\tau\Big[2\left(2a_1\tau + a_2 + a_3 + 2a_4\tau\right)|X(t)|^2\\
&+ \quad a_0(2\tau+1) + 2\left(2a_1\tau + a_2 + a_3\right)|X(t-h)|^2\Big]\\
&- \alpha\int_{t-\tau}^{t}\Big[\tau\left|f(X(s), X(s-h), r(s)) + A_{r(s)}X(\delta(s))\right|^2\\
&+ |g(X(s), X(s-h), r(s))|^2 + \int_{\mathbb{R}_0^n}\left|H\left(X(s^-), X((s-h)^-), r\left(s\right), z\right)\right|^2 \nu(\mathrm{d}z)\Big]ds \qquad (3.10)\\
\leq& -bQ_1(X(t)) + b_1 Q_1(X(t-h)) + b_2 + (\frac{a_4}{4\theta_1} + 4a_4\alpha\tau^2)|X(t) - X(\delta(t))|^2\\
&+ \alpha\tau a_0(2\tau+1) + 2\alpha\tau\left(2a_1\tau + a_2 + a_3\right)|X(t-h)|^2\\
&- \alpha\int_{t-\tau}^{t}\Big[\tau\left|f(X(s), X(s-h), r(s)) + A_{r(s)}X(\delta(s))\right|^2\\
&+ |g(X(s), X(s-h), r(s))|^2 + \int_{\mathbb{R}_0^n}\left|H\left(X(s^-), X((s-h)^-), r\left(s\right), z\right)\right|^2 \nu(\mathrm{d}z)\Big]ds
\end{aligned}
$$

for $t \geq \tau$, where $b = b_0 - 2\alpha\tau\left(2a_1\tau + a_2 + a_3 + 2a_4\tau\right)/c_1$.

3.2. Lemmas

Before proving the key theorem, we need to prove two lemmas, where Lemma 3.1 will prove the uniform boundedness and Lemma 3.2 will prove the exponential convergence.

Lemma 3.1. *Let Assumptions 2.1 and 3.1 hold. If $\tau > 0$ is sufficiently small for*

$$b = b_0 - 2\alpha\tau\left(2a_1\tau + a_2 + a_3 + 2a_4\tau\right)/c_1 > 0 \quad \text{and} \quad \tau < \frac{1}{\sqrt{15a_4}}, \tag{3.11}$$

then the solution of Eq (2.3) *with initial data* (2.4) *satisfies*

$$\mathbb{E}\left\|X_t^{\xi,i}\right\|^2 \leq C\left(1 + \|\xi\|^2\right) \tag{3.12}$$

for all $t \geq 0$, where C is a positive constant.

Proof. Applying the functional Itô formula to $e^{\beta_0 t}(V(\hat{X}_t, \hat{r}_t, t))$, we can show that

$$
\begin{aligned}
&e^{\beta_0 t}\mathbb{E}(V(\hat{X}_t, \hat{r}_t, t)) - e^{\beta_0\tau}\mathbb{E}(V(\hat{X}_\tau, \hat{r}_\tau, \tau))\\
=\;& \mathbb{E}\int_{\tau}^{t} e^{\beta_0 s}(\beta_0 V(\hat{X}_s, \hat{r}_s, s) + \mathrm{LV}(\hat{X}_s, \hat{r}_s, s))ds,
\end{aligned}
$$

for $t \geq \tau$, where β_0 is a positive number to be chosen later. Using Eqs (2.5) and (3.7), one can see that

$$
\begin{aligned}
&c_1 e^{\beta_0 t}\mathbb{E}|X(t)|^2 - \beta_1\left(1 + \|\xi\|^2\right)\\
\leq& \mathbb{E}\int_{\tau}^{t} e^{\beta_0 s}\left[\beta_0(Q_1(X(s)) + \hat{V}(\hat{X}_s, \hat{r}_s, s)) + \mathrm{LV}(\hat{X}_s, \hat{r}_s, s)\right]\mathrm{d}s,
\end{aligned} \tag{3.13}
$$

where β_1 is a positive number. Moreover, we note that

$$\begin{aligned} &\mathbb{E}\left(\hat{V}\left(\hat{X}_s,\hat{r}_s,s\right)\right)\\ \le\ &\alpha\tau\mathbb{E}\int_{s-\tau}^{s}\Big[\tau\left|f(X(v),X(v-h),r(v))+A_{r(v)}X(\delta(v))\right|^2\\ &+\ |g(X(v),X(v-h),r(v))|^2+\int_{\mathbb{R}_0^n}\left|H\left(X(v^-),X((v-h)^-),r\left(v\right),z\right)\right|^2\nu(\mathrm{d}z)\Big]\mathrm{d}v. \end{aligned} \tag{3.14}$$

Substituting Eqs (3.10) and (3.14) into Eq (3.13), we can obtain

$$\begin{aligned} &c_1e^{\beta_0 t}\mathbb{E}|X(t)|^2-\beta_1\left(1+\|\xi\|^2\right)\\ \le\ &\int_{\tau}^{t}e^{\beta_0 s}\beta_0\mathbb{E}\left(Q_1(X(s)\right)ds+\int_{\tau}^{t}e^{\beta_0 s}\beta_0\alpha\tau\int_{s-\tau}^{s}\mathbb{E}\Big(\tau\left|f(X(v),X(v-h),r(v))+A_{r(v)}X(\delta(v))\right|^2\\ &+\ |g(X(v),X(v-h),r(v))|^2+\int_{\mathbb{R}_0^n}\left|H\left(X(v^-),X((v-h)^-),r\left(v\right),z\right)\right|^2\nu(\mathrm{d}z)\Big)\mathrm{d}v ds\\ &+\int_{\tau}^{t}e^{\beta_0 s}\mathbb{E}\,\mathrm{LV}(\hat{X}_s,\hat{r}_s,s)\mathrm{d}s\\ \le\ &\int_{\tau}^{t}e^{\beta_0 s}\beta_0\mathbb{E}\left(Q_1(X(s)\right)ds+\int_{\tau}^{t}e^{\beta_0 s}\beta_0\alpha\tau\int_{s-\tau}^{s}\mathbb{E}\Big(\tau\left|f(X(v),X(v-h),r(v))+A_{r(v)}X(\delta(v))\right|^2\\ &+\ |g(X(v),X(v-h),r(v))|^2+\int_{\mathbb{R}_0^n}\left|H\left(X(v^-),X((v-h)^-),r\left(v\right),z\right)\right|^2\nu(\mathrm{d}z)\Big)\mathrm{d}v ds\\ &+\int_{\tau}^{t}e^{\beta_0 s}\Big[-b\mathbb{E}Q_1(X(s))+b_1\mathbb{E}Q_1(X(s-h))+b_2+(\frac{a_4}{4\theta_1}+4a_4\alpha\tau^2)\mathbb{E}|X(s)-X(\delta(s))|^2\\ &+\ \alpha\tau a_0(2\tau+1)+2\alpha\tau\left(2a_1\tau+a_2+a_3\right)\mathbb{E}|X(s-h)|^2\\ &-\ \alpha\int_{s-\tau}^{s}\mathbb{E}(\tau\left|f(X(v),X(v-h),r(v))+A_{r(v)}X(\delta(v))\right|^2\\ &+\ |g(X(v),X(v-h),r(v))|^2+\int_{\mathbb{R}_0^n}\left|H\left(X(v^-),X((v-h)^-),r\left(v\right),z\right)\right|^2\nu(\mathrm{d}z))dv\Big]ds. \end{aligned} \tag{3.15}$$

It follows from Eq (2.3) that

$$\begin{aligned} &\mathbb{E}|X(t)-X(\delta(t))|^2\\ =\ &\mathbb{E}\Big|\int_{\delta(t)}^{t}[f(X(s),X(s-h),r(s))+A(r(s))X(\delta(s))]\mathrm{d}s\\ &+\int_{\delta(t)}^{t}g(X(s),X(s-h),r(s))\mathrm{d}B(s)\\ &+\int_{\delta(t)}^{t}\int_{\mathbb{R}_0^n}H\left(X\left(s^-\right),X\left((s-h)^-\right),r\left(s\right),z\right)\tilde{N}(\,\mathrm{d}s,\ \mathrm{d}z)\Big|^2\\ \le\ &3\tau\mathbb{E}\int_{t-\tau}^{t}|f(X(s),X(s-h),r(s))+A(r(s))X(\delta(s))|^2\,\mathrm{d}s\\ &+3\mathbb{E}\Big|\int_{t-\tau}^{t}g(X(s),X(s-h),r(s))\mathrm{d}B(s)\Big|^2\\ &+3\mathbb{E}\Big|\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n}H\left(X(s^-),X((s-h)^-),r\left(s\right),z\right)\tilde{N}(\,\mathrm{d}s,\ \mathrm{d}z)\Big|^2. \end{aligned} \tag{3.16}$$

By Itô isometry,

$$\begin{aligned}
&\mathbb{E}|X(t)-X(\delta(t))|^2 \\
\leq\ & 3\tau\mathbb{E}\int_{t-\tau}^{t} |f(X(s),X(s-h),r(s))+A(r(s))X(\delta(s))|^2\, \mathrm{d}s \\
&+3\mathbb{E}\int_{t-\tau}^{t} |g(X(s),X(s-h),r(s))|^2\, ds \\
&+3\mathbb{E}\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n} \left|H\left(X(s^-),X((s-h)^-),r\left(s\right),z\right)\right|^2 \nu_(dx)ds.
\end{aligned} \tag{3.17}$$

Set $\alpha=\frac{15a_4}{4\theta_1}$ and $\tau<\frac{1}{\sqrt{15a_4}}$, and we have that $3(\frac{a_4}{4\theta_1}+4a_4\alpha\tau^2)-\alpha<0$. Then we can find a sufficiently small β_0 which satisfies the following condition:

$$\begin{aligned}
&\beta_0\alpha\tau+3(\frac{a_4}{4\theta_1}+4a_4\alpha\tau^2)-\alpha<0, \\
&\beta_0-b+e^{\beta_0 h}b_1+2\alpha\tau e^{\beta_0 h}\left(2a_1\tau+a_2+a_3\right)/c_1<0.
\end{aligned} \tag{3.18}$$

Using Eqs (3.17) and (3.18), we derive that

$$\begin{aligned}
&\int_{\tau}^{t} e^{\beta_0 s}\beta_0\alpha\tau\int_{s-\tau}^{s}\mathbb{E}\Big(\tau\left|f(X(v),X(v-h),r(v))+A_{r(v)}X(\delta(v))\right|^2 \\
&+\quad |g(X(v),X(v-h),r(v))|^2+\int_{\mathbb{R}_0^n}\left|H\left(X(v^-),X((v-h)^-),r\left(v\right),z\right)\right|^2\nu(\mathrm{d}z)\Big)\mathrm{d}v ds \\
&+\int_{\tau}^{t} e^{\beta_0 s}(\frac{a_4}{4\theta_1}+4a_4\alpha\tau^2)\mathbb{E}|X(s)-X(\delta(s))|^2 ds \\
&-\int_{\tau}^{t} e^{\beta_0 s}\Bigg[\alpha\int_{s-\tau}^{s}\mathbb{E}(\tau\left|f(X(v),X(v-h),r(v))+A_{r(v)}X(\delta(v))\right|^2 \\
&+\quad |g(X(v),X(v-h),r(v))|^2+\int_{\mathbb{R}_0^n}\left|H\left(X(v^-),X((v-h)^-),r\left(v\right),z\right)\right|^2\nu(\mathrm{d}z))dv\Bigg]ds \\
&<0.
\end{aligned} \tag{3.19}$$

It follows from Eq (3.15) that one gains

$$\begin{aligned}
&c_1e^{\beta_0 t}\mathbb{E}|X(t)|^2-\beta_1\left(1+\|\xi\|^2\right) \\
\leq &\int_{\tau}^{t} e^{\beta_0 s}\beta_0\mathbb{E}\left(Q_1(X(s)\right)ds+\int_{\tau}^{t} e^{\beta_0 s}\left[-b\mathbb{E}Q_1(X(s))+b_1\mathbb{E}Q_1(X(s-h))+b_2\right. \\
&+\quad \left.\alpha\tau a_0(2\tau+1)+2\alpha\tau\left(2a_1\tau+a_2+a_3\right)\mathbb{E}|X(s-h)|^2\right]ds.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\tau}^{t} e^{\beta_0 s}\left[b_1\mathbb{E}Q_1(X(s-h))+2\alpha\tau\left(2a_1\tau+a_2+a_3\right)\mathbb{E}|X(s-h)|^2\right]ds, \\
\leq\ & e^{\beta_0 h}\int_{\tau-h}^{t-h} e^{\beta_0 s}\left[b_1\mathbb{E}Q_1(X(s))+2\alpha\tau\left(2a_1\tau+a_2+a_3\right)\mathbb{E}|X(s)|^2\right]ds, \\
\leq\ & e^{\beta_0 h}\int_{\tau}^{t} e^{\beta_0 s}\left[(b_1+2\alpha\tau\left(2a_1\tau+a_2+a_3\right)/c_1)\mathbb{E}Q_1(X(s))\right]ds, \\
&+e^{\beta_0 h}\int_{\tau-h}^{\tau} e^{\beta_0 s}\left[(b_1+2\alpha\tau\left(2a_1\tau+a_2+a_3\right)/c_1)\mathbb{E}Q_1(X(s))\right]ds.
\end{aligned}$$

By condition (3.18), we derive that

$$
\begin{aligned}
& c_1 e^{\beta_0 t}\mathbb{E}|X(t)|^2 - \beta_1\left(1+\|\xi\|^2\right) \\
\leq & \int_\tau^t e^{\beta_0 s}(\beta_0 - b + e^{\beta_0 h} b_1 + 2\alpha\tau e^{\beta_0 h}\left(2a_1\tau + a_2 + a_3\right)/c_1)\mathbb{E}\left(Q_1(X(s)\right) ds \\
& + \int_\tau^t e^{\beta_0 s}\left[b_2 + \alpha\tau a_0(2\tau+1)\right] ds + e^{\beta_0 h}\int_{\tau-h}^{\tau} e^{\beta_0 s}\left[(b_1 + 2\alpha\tau\left(2a_1\tau + a_2 + a_3\right)/c_1)\mathbb{E}Q_1(X(s))\right] ds \\
\leq & \beta_2 e^{\beta_0 t},
\end{aligned}
$$

where β_2 is a positive number. Hence

$$
\mathbb{E}|X(t)|^2 \leq \beta_3\left(1+\|\xi\|^2\right), \quad t \geq \tau. \tag{3.20}
$$

After that, we can make an estimate of the segment process X_t. Let $t \geq \tau + h$ and $\theta \in [0, \tau]$. According to the Itô formula and Eq (2.3), we obtain that

$$
\begin{aligned}
& |X(t-\theta)|^2 \\
= & |X(t-\tau)|^2 + 2\int_{t-\tau}^{t-\theta} X^T(s)\left[f(X(s), X(s-h), r(s)) + A_{r(s)}X(\delta(s))\right] \mathrm{d}s \\
& + 2\int_{t-\tau}^{t-\theta} X^T(s) g(X(s), X(s-h), r(s)) \mathrm{d}B(s) + \int_{t-\tau}^{t-\theta} |g(X(s), X(s-h), r(s))|^2 \mathrm{d}s \\
& + \int_{t-\tau}^{t-\theta}\int_{\mathbb{R}\backslash\{0\}}\sum_{k=1}^n \Big[\left|X(s) + H^{(k)}\left(X(s^-), X((s-h)^-), r(s), z_k\right)\right|^2 \\
& - |X(s)|^2 - 2X^T(s)H^{(k)}\left(X(s^-), X((s-h)^-), r(s), z_k\right)\Big]\nu_k(\mathrm{d}z_k)\mathrm{d}s \\
& + \sum_{k=1}^n\int_{t-\tau}^{t-\theta}\int_{\mathbb{R}\backslash\{0\}}\Big[\left|X(s^-) + H^{(k)}\left(X(s^-), X((s-h)^-), r(s), z_k\right)\right|^2 \\
& - \left|X(s^-)\right|^2\Big]\tilde{N}(\mathrm{d}s, \mathrm{d}z_k).
\end{aligned}
$$

According to Kunita's inequality ([30], Corollary 2.12, p. 332),

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq\theta\leq\tau}|X(t-\theta)|^2 \\
\leq & c_3\Bigg\{\mathbb{E}\int_{t-\tau}^t\left[|f(X(s), X(s-h), r(s))|^2 + \left|A_{r(s)}X(\delta(s))\right|^2\right]\mathrm{d}s \\
& + \mathbb{E}|X(t-\tau)|^2 + \mathbb{E}\int_{t-\tau}^t |g(X(s), X(s-h), r(s))|^2 \mathrm{d}s \\
& + \mathbb{E}\int_{t-\tau}^t\int_{\mathbb{R}\backslash\{0\}}\sum_{k=1}^n\left|H^{(k)}\left(X(s^-), X((s-h)^-), r(s), z_k\right)\right|^2\nu_k(\mathrm{d}z_k)\mathrm{d}s\Bigg\},
\end{aligned}
$$

where c_3 is a positive constant. It follows from Eqs (2.1) and (2.2) that

$$
\begin{aligned}
& \mathbb{E}\sup_{0\leq\theta\leq\tau}|X(t-\theta)|^2 \\
\leq & c_4\Bigg(\mathbb{E}|X(t-\tau)|^2 + \int_{t-\tau}^t\mathbb{E}|X(s)|^2\mathrm{d}s + \int_{t-\tau}^t\mathbb{E}|X(\delta(s))|^2\mathrm{d}s \\
& + \int_{t-\tau}^t\mathbb{E}|X(s-h)|^2\mathrm{d}s + c_5\Bigg)
\end{aligned}
$$

$$\leq \quad c_4\Bigg(\mathbb{E}|X(t-\tau)|^2 + \int_{t-\tau}^{t} \mathbb{E}|X(s)|^2\, \mathrm{d}s + \int_{t-\tau}^{t} \mathbb{E}|X(\delta(s))|^2\, \mathrm{d}s$$
$$+ \int_{t-\tau-h}^{t-h} \mathbb{E}|X(s)|^2\, \mathrm{d}s + c_5\Bigg), \tag{3.21}$$

where c_4 and c_5 are positive numbers. By Eqs (3.20) and (3.21), it is easy to show

$$\mathbb{E}\,\|X_t\|^2 \leq \beta_4\left(1 + \|\xi\|^2\right)$$

where β_4 is a positive number. Together with Eq (2.5), the assertion (3.12) holds. The proof is hence complete. □

Lemma 3.2. *Let Assumptions 2.1 and 3.2 hold. If $\tau > 0$ is sufficiently small enough for*

$$\bar{b} = b_3 - \alpha\tau\left(2a_1\tau + a_2 + a_3 + 2a_4\tau\right)/c_2 > 0 \quad \textit{and} \quad \tau < \sqrt{\frac{2}{15a_4}}, \tag{3.22}$$

then for any $(\xi, \zeta, i) \in \mathcal{D}_h \times \mathcal{D}_h \times \mathbb{S}$,

$$\mathbb{E}\left\|X_t^{\xi,i} - X_t^{\zeta,i}\right\|^2 \leq \alpha_1\|\xi - \zeta\|^2 e^{-\alpha_2 t} \tag{3.23}$$

for all $t \geq \tau + h$, where α_1 and α_2 are positive constants.

Proof. Denote by $O(t) = X^{\xi,i}(t) - X^{\zeta,i}(t)$ for any $(\xi, \zeta, i) \in \mathcal{D}_h \times \mathcal{D}_h \times \mathbb{S}$. Moreover, $O_t = \{O(t+s) : -\tau \leq s \leq 0\}$ for $t \geq 0$ and $\hat{O}_t = \{O(t+s) : -\tau - h \leq s \leq 0\}$ for $t \geq \tau + h$. Design a new Lyapunov functional $\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)$:

$$\begin{aligned}
&\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right) \\
:= \quad &\Phi\left(X^{\xi,i}(t) - X^{\zeta,i}(t), r(t)\right) + \alpha \int_{t-\tau}^{t}\int_{s}^{t} \Big[\tau \mid f\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) \\
&- f\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right) + A_{r(v)}O(\delta(v))\Big|^2 \\
&+ \left|g\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) - g\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right)\right|^2 \\
&+ \int_{\mathbb{R}_0^n} \mid H\left(X^{\xi,i}(v^-), X^{\xi,i}((v-h)^-), , r(v), z\right) \\
&- H\left(X^{\zeta,i}(v^-), X^{\zeta,i}((v-h)^-), r(v), z\right)\Big|^2 \nu(\mathrm{d}z)\Big]\, \mathrm{d}v\, \mathrm{d}s
\end{aligned} \tag{3.24}$$

for $t \geq \tau$. Applying the functional Itô formula, we have

$$\mathrm{d}\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right) = \mathrm{L}\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)\mathrm{d}t + \mathrm{d}\tilde{M}(t)$$

for $t \geq \tau$, where $\tilde{M}(t)$ is a martingale with $\tilde{M}(0) = 0$, and

$$
\begin{aligned}
& \mathrm{L}\,\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right) \\
= & \ \mathbb{L}\Phi\left(X^{\xi,i}(t), X^{\zeta,i}(t), X^{\xi,i}(t-h), X^{\zeta,i}(t-h), r(t)\right) \\
- & \quad \Phi_X\left(X^{\xi,i}(t), X^{\zeta,i}(t), r(t)\right) A_{r(t)}(O(t)-O(\delta(t))) \\
+ & \quad \alpha\tau\Big[\tau \mid f\left(X^{\xi,i}(t), X^{\xi,i}(t-h), r(t)\right) - f\left(X^{\zeta,i}(t), X^{\zeta,i}(t-h), r(t)\right) \\
& + A_{r(t)}O(\delta(t))\Big|^2 + \left|g\left(X^{\xi,i}(t), X^{\xi,i}(t-h), r(t)\right) - g\left(X^{\zeta,i}(t), X^{\zeta,i}(t-h), r(t)\right)\right|^2 \\
& + \int_{\mathbb{R}_0^n} \left|H\left(X^{\xi,i}(t^-), X^{\xi,i}((t-h)^-), r(t), z\right) - H\left(X^{\zeta,i}(t^-), X^{\zeta,i}((t-h)^-), r(t), z\right)\right|^2 \nu(\mathrm{d}z)\Big] \\
- & \quad \alpha \int_{t-\tau}^{t} \Big[\tau \mid f\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - f\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right) \\
& + A_{r(s)}O(\delta(s))\Big|^2 + \left|g\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - g\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right)\right|^2 \\
& + \int_{\mathbb{R}_0^n} \mid H\left(X^{\xi,i}(s^-), X^{\xi,i}((s-h)^-), r(s), z\right) \\
& - H\left(X^{\zeta,i}(s^-), X^{\zeta,i}((s-h)^-), r(s), z\right)\Big|^2 \nu(\mathrm{d}z)\Big]\mathrm{d}s.
\end{aligned}
$$

By Assumptions (2.1) and (3.2), we have

$$
\begin{aligned}
& \mathrm{L}\,\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right) \\
\leq & \quad \bar{b}Q_2(O(t)) + b_4 Q_2(O(t-h)) + \left(\frac{a_4}{4\theta_2} + 2a_4\alpha\tau^2\right)|O(t)-O(\delta(t))|^2 \\
& + \alpha\tau\left(2a_1\tau + a_2 + a_3\right)|O(t-h)|^2 \\
& - \alpha \int_{t-\tau}^{t} \Big[\tau\Big|f\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - f\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right) \\
& + A_{r(s)}O(\delta(s))\Big|^2 + \left|g\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - g\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right)\right|^2 \\
& + \int_{\mathbb{R}_0^n} \Big|H\left(X^{\xi,i}(s^-), X^{\xi,i}((s-h)^-), r(s), z\right) \\
& - H\left(X^{\zeta,i}(s^-), X^{\zeta,i}((s-h)^-), r(s), z\right)\Big|^2 \nu(\mathrm{d}z)\Big]\mathrm{d}s
\end{aligned}
\tag{3.25}
$$

for $t \geq \tau$, where $\bar{b} = b_3 - \alpha\tau\left(2a_1\tau + a_2 + a_3 + 2a_4\tau\right)/c_2$.

Applying the functional It$\hat{o}$ formula to $e^{\alpha_2 t}\mathbb{E}\left(\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)\right)$, we have

$$
\begin{aligned}
& e^{\alpha_2 t}\mathbb{E}|O(t)|^2 - \alpha_4\|\xi - \zeta\|_\tau^2 \\
= & \quad e^{\alpha_2 t}\mathbb{E}\left(\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)\right) - e^{\alpha_2 \tau}\mathbb{E}\left(\tilde{V}\left(\hat{O}_\tau, \hat{r}_\tau, \tau\right)\right) \\
= & \quad \mathbb{E}\int_\tau^t e^{\alpha_2 s}\left(\alpha_2\tilde{V}\left(\hat{O}_s, \hat{r}_s, s\right) + \mathrm{L}\tilde{\mathrm{V}}\left(\hat{O}_s, \hat{r}_s, s\right)\right)\mathrm{d}s,
\end{aligned}
\tag{3.26}
$$

where α_4 is a positive number and α_2 is a positive number to be determined. Substituting Eq (3.25) into Eq (3.26) yields

$$
\mathbb{E}\int_\tau^t e^{\alpha_2 s}\left(\alpha_2\tilde{V}\left(\hat{O}_s, \hat{r}_s, s\right)\right)ds + \mathbb{E}\int_\tau^t e^{\alpha_2 s}\,\mathrm{L}\,\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)ds
$$

$$
\begin{aligned}
\leq\quad & \mathbb{E}\int_{\tau}^{t} e^{\alpha_2 s}\left(\alpha_2\tilde{V}\left(\hat{O}_s,\hat{r}_s,s\right)\right)ds \\
&+\mathbb{E}\int_{\tau}^{t} e^{\alpha_2 s}\Big[-\bar{b}Q_2(O(s))+b_4Q_2(O(s-h))+(\frac{a_4}{4\theta_2}+2a_4\alpha\tau^2)|O(s)-O(\delta(s))|^2 \\
&+\alpha\tau\left(2a_1\tau+a_2+a_3\right)|O(s-h)|^2 \\
&-\alpha\int_{t-\tau}^{t}(\tau\Big|f\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right)-f\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right) \\
&+A_{r(v)}O(\delta(v))\Big|^2+\Big|g\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right)-g\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right)\Big|^2 \\
&+\int_{\mathbb{R}_0^n}\Big|H\left(X^{\xi,i}\left(v^-\right),X^{\xi,i}\left((v-h)^-\right),r\left(v\right),z\right) \\
&-H\left(X^{\zeta,i}\left(v^-\right),X^{\zeta,i}\left((v-h)^-\right),r\left(v\right),z\right)\Big|^2\nu(\mathrm{d}z))dv\Big]\mathrm{d}s.
\end{aligned}
\tag{3.27}
$$

Moreover, by Assumption 3.2, one can see that

$$
\begin{aligned}
&\mathbb{E}\left(\tilde{V}\left(\hat{O}_s,\hat{r}_s,s\right)\right) \\
\leq\ &\mathbb{E}Q_2(O(s))+\mathbb{E}\alpha\tau\int_{s-\tau}^{s}\Big[\tau\Big|f\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right) \\
&-f\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right)+A_{r(v)}O(\delta(v))\Big|^2 \\
&+\quad\Big|g\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right)-g\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right)\Big|^2 \\
&+\int_{\mathbb{R}_0^n}\Big|H\left(X^{\xi,i}(v^-),X^{\xi,i}((v-h)^-),r\left(v\right),z\right) \\
&-\quad H\left(X^{\zeta,i}(v^-),X^{\zeta,i}((v-h)^-),r\left(v\right),z\right)\Big|^2\nu(\mathrm{d}z)\Big]\mathrm{d}v.
\end{aligned}
\tag{3.28}
$$

Substituting Eq (3.28) into Eq (3.27), we can get

$$
\begin{aligned}
&\mathbb{E}\int_{\tau}^{t} e^{\alpha_2 s}\left(\alpha_2\tilde{V}\left(\hat{O}_s,\hat{r}_s,s\right)\right)ds+\mathbb{E}\int_{\tau}^{t} e^{\alpha_2 s}\,\mathrm{L}\,\tilde{V}\left(\hat{O}_t,\hat{r}_t,t\right)ds \\
\leq&\int_{\tau}^{t} e^{\alpha_2 s}\Big[\alpha_2\mathbb{E}Q_2(O(s)) \\
&+\mathbb{E}\alpha\tau\alpha_2\int_{s-\tau}^{s}\left(\tau\Big|f\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right)-f\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right)+A_{r(v)}O(\delta(v))\Big|^2\right. \\
&+\Big|g\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right)-g\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right)\Big|^2 \\
&+\int_{\mathbb{R}_0^n}\Big|H\left(X^{\xi,i}(v^-),X^{\xi,i}((v-h)^-),r\left(v\right),z\right) \\
&\left.-H\left(X^{\zeta,i}(v^-),X^{\zeta,i}((v-h)^-),r\left(v\right),z\right)\Big|^2\nu(\mathrm{d}z)\right)\mathrm{d}v\Big]ds \\
&+\mathbb{E}\int_{\tau}^{t} e^{\alpha_2 s}\Big[-\bar{b}Q_2(O(s))+b_4Q_2(O(s-h))+(\frac{a_4}{4\theta_2}+2a_4\alpha\tau^2)|O(s)-O(\delta(s))|^2 \\
&+\alpha\tau\left(2a_1\tau+a_2+a_3\right)|O(s-h)|^2 \\
&-\alpha\int_{s-\tau}^{s}\left(\tau\Big|f\left(X^{\xi,i}(v),X^{\xi,i}(v-h),r(v)\right)-f\left(X^{\zeta,i}(v),X^{\zeta,i}(v-h),r(v)\right)\right.
\end{aligned}
$$

$$+ A_{r(v)}O(\delta(v))\Big|^2 + \Big|g\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) - g\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right)\Big|^2$$
$$+ \int_{\mathbb{R}_0^n} \Big|H\left(X^{\xi,i}\left(v^-\right), X^{\xi,i}\left((v-h)^-\right), r\left(v\right), z\right)$$
$$- H\left(X^{\zeta,i}\left(v^-\right), X^{\zeta,i}\left((v-h)^-\right), r\left(v\right), z\right)\Big|^2 \nu(\mathrm{d}z)\Big)dv\Big]\mathrm{d}s.$$

Moreover, we can obtain that

$$\begin{aligned}
&\mathbb{E}|O(t) - O(\delta(t))|^2 \\
\leq\ & \mathbb{E}\Big|\int_{\delta(t)}^{t}\left[f\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - f\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right) + A(r(s))O(\delta(s))\right]\mathrm{d}s \\
&+ \int_{\delta(t)}^{t}\left[g\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - g\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right)\right]\mathrm{d}B(s) \\
&+ \int_{\delta(t)}^{t}\int_{\mathbb{R}_0^n}\left[H\left(X^{\xi,i}(s^-), X^{\xi,i}((s-h)^-), r\left(s\right), z\right) - H\left(X^{\zeta,i}(s^-), X^{\zeta,i}((s-h)^-), r\left(s\right), z\right)\right]\tilde{N}(\,\mathrm{d}s,\ \mathrm{d}z)\Big|^2 \\
\leq\ & 3\tau\mathbb{E}\int_{t-\tau}^{t}\Big|f\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - f\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right) + A(r(s))O(\delta(s))\Big|^2\,\mathrm{d}s \\
&+3\mathbb{E}\int_{t-\tau}^{t}\Big|g\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - g\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right)\Big|^2\,\mathrm{d}s \\
&+3\mathbb{E}\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n}\Big|H\left(X^{\xi,i}(s^-), X^{\xi,i}((s-h)^-), r\left(s\right), z\right) - H\left(X^{\zeta,i}(s^-), X^{\zeta,i}((s-h)^-), r\left(s\right), z\right)\Big|^2\nu(\mathrm{d}z)\mathrm{d}s.
\end{aligned} \tag{3.29}$$

Set $\alpha = \frac{15a_4}{4\theta_2}$ and $\tau < \sqrt{\frac{2}{15a_4}}$, and we have that $3(\frac{a_4}{4\theta_2} + 2a_4\alpha\tau^2) - \alpha < 0$. Then we can find a sufficiently small α_2 which satisfies the following condition:

$$\begin{aligned}
&\alpha_2\alpha\tau + 3(\frac{a_4}{4\theta_2} + 2a_4\alpha\tau^2) - \alpha < 0, \\
&-\bar{b} + \alpha_2 + e^{\alpha_2 h}b_4 + \alpha\tau e^{\alpha_2 h}\left(2a_1\tau + a_2 + a_3\right)/c_2 < 0.
\end{aligned} \tag{3.30}$$

By Eq (3.30), we can show that

$$\begin{aligned}
&\int_{\tau}^{t} e^{\alpha_2 s}\Big[\alpha\tau\alpha_2\mathbb{E}\int_{s-\tau}^{s}\Big[\tau\Big|f\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) \\
&- f\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right) + A_{r(v)}O(\delta(v))\Big|^2 \\
&+ \Big|g\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) - g\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right)\Big|^2 \\
&+ \int_{\mathbb{R}_0^n}\Big|H\left(X^{\xi,i}\left(v^-\right), X^{\xi,i}\left((v-h)^-\right), r\left(v\right), z\right) \\
&- H\left(X^{\zeta,i}\left(v^-\right), X^{\zeta,i}\left((v-h)^-\right), r\left(v\right), z\right)\Big|^2\nu(\mathrm{d}z)\Big]\mathrm{d}v \\
&+ (\frac{a_4}{4\theta_2} + 2a_4\alpha\tau^2)\mathbb{E}|O(s) - O(\delta(s))|^2 \\
&- \alpha\mathbb{E}\int_{s-\tau}^{s}\Big(\tau\Big|f\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) - f\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right) \\
&+ A_{r(v)}O(\delta(v))\Big|^2 + \Big|g\left(X^{\xi,i}(v), X^{\xi,i}(v-h), r(v)\right) - g\left(X^{\zeta,i}(v), X^{\zeta,i}(v-h), r(v)\right)\Big|^2
\end{aligned}$$

$$
\begin{aligned}
& + \int_{\mathbb{R}_0^n} \Big| H\left(X^{\xi,i}\left(v^-\right), X^{\xi,i}\left((v-h)^-\right), r\left(v\right), z\right) \\
& - H\left(X^{\zeta,i}\left(v^-\right), X^{\zeta,i}\left((v-h)^-\right), r\left(v\right), z\right)\Big|^2 \nu(\mathrm{d}z))dv\Big]\mathrm{d}s \\
< \quad & 0.
\end{aligned}
$$

Following the condition (3.30), we derive that

$$
\begin{aligned}
& e^{\alpha_2 t}\mathbb{E}|O(t)|^2 - \alpha_4\|\xi-\zeta\|_\tau^2 \\
\leq \quad & e^{\alpha_2 t}\mathbb{E}\left(\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)\right) - e^{\alpha_2 \tau}\mathbb{E}\left(\tilde{V}\left(\hat{O}_\tau, \hat{r}_\tau, \tau\right)\right) \\
= \quad & \mathbb{E}\int_\tau^t e^{\alpha_2 s}\left(\alpha_2 \tilde{V}\left(\hat{O}_s, \hat{r}_s, s\right)\right)ds + \mathbb{E}\int_\tau^t e^{\alpha_2 s}\,\mathrm{L}\,\tilde{V}\left(\hat{O}_t, \hat{r}_t, t\right)ds \\
\leq & \int_\tau^t e^{\alpha_2 s}\left(\alpha_2 \mathbb{E}Q_2(O(s))\right)ds + \mathbb{E}\int_\tau^t e^{\alpha_2 s}[-\bar{b}Q_2(O(s)) \\
& + \quad (b_4 + \alpha\tau\left(2a_1\tau + a_2 + a_3\right)/c_2)Q_2(O(s-h)]\,\mathrm{d}s \\
\leq \quad & \mathbb{E}\int_\tau^t e^{\alpha_2 s}[-\bar{b} + \alpha_2 + e^{\alpha_2 h}b_4 + \alpha\tau e^{\alpha_2 h}\left(2a_1\tau + a_2 + a_3\right)/c_2)Q_2(O(s)]ds \\
& + e^{\alpha_2 h}\mathbb{E}\int_{\tau-h}^{\tau} e^{\alpha_2 s}[b_4 + \alpha\tau\left(2a_1\tau + a_2 + a_3\right)/c_2]Q_2 O(s)ds \\
\leq \quad & \alpha_5.
\end{aligned}
\tag{3.31}
$$

This implies that

$$
\mathbb{E}|O(t)|^2 \leq \alpha_3\|\xi-\zeta\|_\tau^2 e^{-\alpha_2 t} \tag{3.32}
$$

for $t \geq \tau$, where α_5, α_3, and α_2 are positive constants. However, for $t \geq \tau + h$, we have that

$$
\begin{aligned}
& \sup_{0\leq\theta\leq\tau} |O(t-\theta)|^2 \\
\leq \quad & c_6\Bigg\{\mathbb{E}|O(t-\tau)|^2 + \mathbb{E}\int_{t-\tau}^t \Big[\Big|f\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) \\
& -f\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right)\Big|^2 + \left|A_{r(s)}O(\delta(s))\right|^2\Big]\mathrm{d}s \\
& + \mathbb{E}\int_{t-\tau}^t \left|g\left(X^{\xi,i}(s), X^{\xi,i}(s-h), r(s)\right) - g\left(X^{\zeta,i}(s), X^{\zeta,i}(s-h), r(s)\right)\right|^2 \mathrm{d}s \\
& + \mathbb{E}\int_{t-\tau}^t \int_{\mathbb{R}\backslash\{0\}} \sum_{k=1}^n \Big|H^{(k)}\left(X^{\xi,i}(s^-), X^{\xi,i}((s-h)^-), r\left(s\right), z_k\right) \\
& - \quad H^{(k)}\left(X^{\zeta,i}(s^-), X^{\zeta,i}((s-h)^-), r\left(s\right), z_k\right)\Big|^2 \nu_k\left(\mathrm{d}z_k\right)\mathrm{d}s\Bigg\}.
\end{aligned}
\tag{3.33}
$$

From Assumption 2.1, one can see that

$$
\sup_{0\leq\theta\leq\tau} |O(t-\theta)|^2 \leq c_7\left(\mathbb{E}|O(t-\tau)|^2 + \int_{t-\tau}^t \mathbb{E}|O(s)|^2\,\mathrm{d}s + \int_{t-\tau}^t \mathbb{E}|O(\delta(s))|^2\,\mathrm{d}s + \int_{t-\tau}^t \mathbb{E}|O(s-h)|^2\,\mathrm{d}s\right),
$$

where c_6 and c_7 are all positive numbers. By Eq (3.32), we have

$$
\mathbb{E}\left\|O_t\right\|^2 \leq \alpha_1\|\xi-\zeta\|^2 e^{-\alpha_2 t}, \quad \forall t \geq \tau + h,
$$

where α_1 is a positive number. Therefore, the required assertion (3.23) holds. This completes the proof. □

3.3. Key theorem

Next, let us prove that the SDDE-MS-LN (2.3) is stable in distribution by Lemmas 3.1 and 3.2.

Theorem 3.1. *Let Assumptions* 2.1 *and* 3.2 *hold. Let* $\tau_1^*, \tau_2^*, \tau_3^*$, *and* τ_4^* *be the unique positive roots to the following equations*

$$\begin{aligned} b_0 c &= \frac{15a_4}{2\theta}\tau_1^*\left(2a_1\tau_1^* + a_2 + a_3 + 2a_4\tau_1^*\right), \tau_2^* = \frac{1}{\sqrt{15a_4}}, \\ b_3 c &= \frac{15a_4}{4\theta}\tau_3^*\left(2a_1\tau_3^* + a_2 + a_3 + 2a_4\tau_3^*\right), \tau_4^* = \sqrt{\frac{2}{15a_4}}, \end{aligned} \tag{3.34}$$

respectively, and set $\tau^* = \tau_1^* \wedge \tau_2^* \wedge \tau_3^* \wedge \tau_4^*$. *Then for each* $\tau < \tau^*$ *and where* h *can be divisible by* τ, *there exists a unique probability measure* $\mu_h \in \mathcal{P}(\mathcal{D}_h)$ *such that*

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(\mathcal{L}\left(X_{k\tau}^{\xi,i}\right), \mu_h\right) = 0 \tag{3.35}$$

for all $(\xi, i) \in \mathcal{D}_h \times \mathbb{S}$.

Proof. Step 1: We first claim that for any compact set $\mathcal{K} \subset \mathcal{D}_h$,

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(\mathcal{L}\left(X_{k\tau}^{\xi,i}\right), \mathcal{L}\left(X_{k\tau}^{\zeta,j}\right)\right) = 0 \tag{3.36}$$

uniformly in $(\xi, \zeta, i, j) \in \mathcal{K} \times \mathcal{K} \times \mathbb{S} \times \mathbb{S}$. Define the sequence of the stopping time $\kappa_{ij} = \inf\left\{k\tau : r^i(k\tau) = r^j(k\tau), k \geq 0\right\}$. Using the ergodicity of the Markov chain, we can obtain that $\kappa_{ij} < \infty$ a.s. Consequently, for any $\varepsilon \in (0, 1)$, there exists a number $T_1 > 0$ such that

$$\mathbb{P}\left(\kappa_{ij} \leq T_1\right) > 1 - \frac{\varepsilon}{6}.$$

Recalling a known result that

$$\sup_{(\xi,i)\in\mathcal{K}\times\mathbb{S}} \mathbb{E}\left(\sup_{-\tau\leq t\leq T_1}\left|X^{\xi,i}(t)\right|\right) < \infty,$$

we can find enough large $T_2 > 0$ such that

$$\mathbb{P}\left(\Omega_{\xi,i}\right) > 1 - \frac{\varepsilon}{12} \quad \forall(\xi, i) \in \mathcal{K} \times \mathbb{S},$$

where $\Omega_{\xi,i} = \left\{\omega \in \Omega : \sup_{-\tau\leq t\leq T_1}\left|X^{\xi,i}(t, \omega)\right| \leq q\right\}$. For any $\phi \in \mathbb{L}$ and $k\tau \geq T_1$, we obtain

$$\left|\mathbb{E}\phi\left(X_{k\tau}^{\xi,i}\right) - \mathbb{E}\phi\left(X_{k\tau}^{\zeta,j}\right)\right| \leq \frac{\varepsilon}{3} + \rho(t),$$

where $\rho(t) := \mathbb{E}\left(I_{\{\kappa_{ij}\le T_1\}}\left|\phi\left(X_{k\tau}^{\xi,i}\right)-\phi\left(X_{k\tau}^{\zeta,j}\right)\right|\right)$. Set $\Omega_1=\Omega_{\xi,i}\cap\Omega_{\zeta,i}\cap\left\{\kappa_{ij}\le T_1\right\}$. By the Markov property of joint process $\left(X_{k\tau}^{\xi,i}, r^i(k\tau)\right)$ and the property of conditional expectation, we derive

$$\begin{aligned}
&\rho(t)\\
=\ & \mathbb{E}\left(I_{\{\kappa_{ij}\le T_1\}}\mathbb{E}(|\ \phi(X_{k\tau}^{\xi,i})-\phi(X_{k\tau}^{\zeta,j})|\big|\phi_{\kappa_{ij}})\right)\\
=\ & \mathbb{E}\Big(I_{\{\kappa_{ij}\le T_1\}}\times\mathbb{E}(|\phi(X_{k\tau-\kappa_{ij}}^{\tilde{\xi},l})-\phi(X_{k\tau-\kappa_{ij}}^{\tilde{\zeta},l})|)\big|_{\tilde{\xi}=X_{\kappa_{ij}}^{\xi,i},\tilde{\zeta}=X_{\kappa_{ij}}^{\zeta,j},l=q_{\kappa_{ij}}^{i}=q_{\kappa_{ij}}^{j}}\Big)\\
\le\ & \frac{\varepsilon}{3}+\mathbb{E}\Big(I_{\Omega_1}\times\mathbb{E}(|\phi(X_{k\tau-\kappa_{ij}}^{\tilde{\xi},l})-\phi(X_{k\tau-\kappa_{ij}}^{\tilde{\zeta},l})|)\big|_{\tilde{\xi}=X_{\kappa_{ij}}^{\xi,i},\tilde{\zeta}=X_{\kappa_{ij}}^{\zeta,j},l=q_{\kappa_{ij}}^{i}=q_{\kappa_{ij}}^{j}}\Big)\\
\le\ & \frac{\varepsilon}{3}+\mathbb{E}\Big(\ I_{\Omega_1}\mathbb{E}d_S(X_{k\tau-\kappa_{ij}}^{\tilde{\xi},l},X_{k\tau-\kappa_{ij}}^{\tilde{\zeta},l})\Big|_{\tilde{\xi}=X_{\kappa_{ij}}^{\xi,i},\tilde{\zeta}=X_{\kappa_{ij}}^{\zeta,j},l=q_{\kappa_{ij}}^{i}=q_{\kappa_{ij}}^{j}}\ \Big).
\end{aligned}$$

It is known (see [31]) that $d_S(\xi_1,\xi_2)\le\|\xi_1-\xi_2\|$, for any $\xi_1,\xi_2\in\mathcal{D}_h$. Using this and the Proposition 1.17 in [32], we derive that

$$\rho(t)\le\frac{\varepsilon}{3}+\mathbb{E}\left(I_{\Omega_1}\mathbb{E}\left(\left\|X_{k\tau-\kappa_{ij}}^{\tilde{\xi},l}-X_{k\tau-\kappa_{ij}}^{\tilde{\zeta},l}\right\|\right)\right).$$

It is easy to observe that $\|\tilde{\xi}\|\vee\|\tilde{\zeta}\|\le q$, for any $\omega\in\Omega_1$. By using Lemma 3.2, we are able to find positive number T_2 such that

$$\mathbb{E}\left(\left\|X_{k\tau-\kappa_{ij}}^{\tilde{\xi},l}-X_{k\tau-\kappa_{ij}}^{\tilde{\zeta},l}\right\|_{\tau}\right)\le\frac{\varepsilon}{3},\quad\forall k\tau\ge T_1+T_2$$

for any given $\omega\in\Omega_1$. Then we have that

$$\left|\mathbb{E}\phi\left(X_{k\tau-\kappa_{ij}}^{\xi,l}\right)-\mathbb{E}\phi\left(X_{k\tau-\kappa_{ij}}^{\zeta,l}\right)\right|\le\varepsilon,\quad\forall k\tau\ge T_1+T_2.$$

Due to the arbitrariness of ϕ, for all $(\xi,\zeta,i,j)\in\mathcal{K}\times\mathcal{K}\times\mathbb{S}\times\mathbb{S}$, we get

$$d_{\mathbb{L}}\left(\mathcal{L}\left(X_{k\tau}^{\xi,i}\right),\mathcal{L}\left(X_{k\tau}^{\zeta,j}\right)\right)\le\varepsilon,\quad\forall k\tau\ge T_1+T_2.$$

Our claim is proved.

Step 2: Next, we claim that for any $(\xi,i)\in\mathcal{D}_h\times\mathbb{S}$, $\left\{\mathcal{L}\left(X_{k\tau}^{\xi,i}\right)\right\}_{k\in\mathbb{N}_+}$ is a Cauchy sequence in $\mathcal{P}(\mathcal{D}_h)$ with metric $d_{\mathbb{L}}$. That is, we need to show that for any $\varepsilon>0$, there is a positive number k_0 such that

$$d_{\mathbb{L}}\left(\mathcal{L}\left(X_{(s+v)\tau}^{\xi,i}\right),\mathcal{L}\left(X_{s\tau}^{\xi,i}\right)\right)\le\varepsilon\tag{3.37}$$

for all integers $s\ge k_0$ and $v\ge1$. Let $\varepsilon\in(0,1)$ be arbitrary. By Lemma 3.1, there is a $\bar{q}>0$ such that

$$\mathbb{P}\left\{\omega\in\Omega:\left\|X_{v\tau}^{\xi,i}(\omega)\right\|\le\bar{q}\right\}>1-\varepsilon/4\quad\forall v\ge1.\tag{3.38}$$

For any $\phi \in \mathbb{L}$ and integers $s \geq 1$, we can then derive, using (2.6) and (3.38), that

$$\begin{aligned}
&\left|\mathbb{E}\phi\left(X^{\xi,i}_{(s+v)\tau}\right) - \mathbb{E}\phi\left(X^{\xi,i}_{s\tau}\right)\right| \\
=&\left|\sum_{j\in\mathbb{S}} \int \mathbb{E}\phi\left(X^{\zeta,j}_{s\tau}\right) p(v,\xi,i;d\zeta\times\{j\}) - \mathbb{E}\phi\left(X^{\xi,i}_{s\tau}\right)\right| \\
\leq&\sum_{j\in\mathbb{S}} \int \left|\mathbb{E}\phi\left(X^{\zeta,j}_{s\tau}\right) - \mathbb{E}\phi\left(X^{\xi,i}_{s\tau}\right)\right| p(v,\xi,i;d\zeta\times\{j\}) \\
\leq&\quad \frac{\varepsilon}{2} + \sum_{j\in\mathbb{S}} \int_{Z_{\bar{q}}} d_{\mathbb{L}}\Big(\mathcal{L}\left(X^{\zeta,j}_{s\tau}\right), \mathcal{L}\left(X^{\xi,i}_{s\tau}\right)\Big) p(v,\xi,i;d\zeta\times\{j\})
\end{aligned}$$

where $Z_{\bar{q}} = \{\zeta \in \mathcal{D}_h : \|\zeta\| \leq \bar{q}\}$. But, by (3.36), there is a positive integer k_0 such that

$$d_{\mathbb{L}}\left(\mathcal{L}\left(X^{\zeta,j}_{s\tau}\right)\right), \mathcal{L}\left(X^{\xi,i}_{s\tau}\right)\Big) \leq \frac{\varepsilon}{2} \quad \forall s \geq k_0$$

whenever $(\zeta, j) \in Z_{\bar{q}} \times \mathbb{S}$. We, therefore, obtain

$$\left|\mathbb{E}\phi\left(X^{\xi,i}_{(s+v)\tau}\right) - \mathbb{E}\phi\left(X^{\xi,i}_{s\tau}\right)\right| \leq \varepsilon$$

for $s \geq k_0$ and $v \geq 1$. As this holds for any $\phi \in \mathbb{L}$, we must have (3.37) as claimed.
Step 3: By Eq (3.37), there exists a unique $\mu_h \in \mathcal{P}(\mathcal{D}_h)$ such that

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(\mathcal{L}\left(X^{0,1}_{k\tau}\right), \mu_h\right) = 0,$$

which together with Eq (3.36) gains

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(\mathcal{L}\left(X^{\xi,i}_{k\tau}\right), \mu_h\right) \leq \lim_{k\to\infty} d_{\mathbb{L}}\left(\left(X^{\xi,i}_{k\tau}\right), \mathcal{L}\left(X^{0,1}_{k\tau}\right)\right) + \lim_{k\to\infty} d_{\mathbb{L}}\left(\mathcal{L}\left(X^{0,1}_{k\tau}\right), \mu_h\right) = 0$$

for all $(\xi, i) \in \mathcal{D}_h \times \mathbb{S}$. That is assertion (3.35). The proof is, hence, complete. □

4. Design of matrices $A_i's$

To simplify the calculation and design of matrices, we choose the forms of the function as follows:

$$\Psi(x,i) = \Phi(x,i) = x^T W_i x$$

for some N symmetric positive definite matrices $W_i (i \in \mathbb{S})$. It follows easily from Eqs (3.1) and (3.3) that

$$\begin{aligned}
&L\Psi(x,\bar{x},i) \\
=&\quad 2x^T W_i \left[f(x,\bar{x},i) + A_i(x)\right] + \operatorname{trace}\left[g(x,\bar{x},i)^T W_i g(x,\bar{x},i)\right] \\
&+ \int_{\mathbb{R}\setminus\{0\}} \sum_{k=1}^{n} \Big[\left(H^{(k)}(x,\bar{x},i,z_k)\right)^T W_i H^{(k)}(x,\bar{x},i,z_k) \\
&+ \left(H^{(k)}(x,\bar{x},i,z_k)\right)^T W_i x - x^T W_i H^{(k)}(x,\bar{x},i,z_k)\Big] \nu_k(\,\mathrm{d}z_k) \\
&+ \sum_{j=1}^{N} \gamma_{ij} x^T W_j x,
\end{aligned}$$

and

$$
\begin{aligned}
& \mathbb{L}\Phi(x, y, \bar{x}, \bar{y}, i) \\
= \quad & 2(x-y)^T W_i \left[f(x, \bar{x}, i) - f(y, \bar{y}, i) + A_i(x-y) \right] \\
& + \operatorname{trace}\left[(g(x, \bar{x}, i) - g(y, \bar{y}, i))^T W_i (g(x, \bar{x}, i) - g(y, \bar{y}, i)) \right] \\
& + \int_{\mathbb{R}\setminus\{0\}} \sum_{k=1}^{n} \left[\left(H^{(k)}(x, \bar{x}, i, z_k) - H^{(k)}(y, \bar{y}, i, z_k) \right)^T \right. \\
& \times W_i \left(H^{(k)}(x, \bar{x}, i, z_k) - H^{(k)}(y, \bar{y}, i, z_k) \right) \\
& + \left(H^{(k)}(x, \bar{x}, i, z_k) - H^{(k)}(y, \bar{y}, i, z_k) \right)^T W_i (x-y) \\
& \left. - (x-y)^T W_i \left(H^{(k)}(x, \bar{x}, i, z_k) - H^{(k)}(y, \bar{y}, i, z_k) \right) \right] \nu_k(\,\mathrm{d}z_k) \\
& + \sum_{j=1}^{N} \gamma_{ij} (x-y)^T W_j (x-y).
\end{aligned}
$$

Assumption 4.1. *Let Assumption* 2.1 *hold. There exist positive numbers* j_0*,* b_1*,* j_3*,* b_2*,* b_4*,* $j_0 \geq b_1$*,* $j_3 \geq b_4$*, and positive definite matrices* W_i *such that*

$$
\begin{aligned}
& L\Psi(x, \bar{x}, i) \leq -j_0|x|^2 + b_1|\bar{x}|^2 + b_2, \\
& \mathbb{L}\Phi(x, y, \bar{x}, \bar{y}, i) \leq -j_3|x-y|^2 + b_4|\bar{x}-\bar{y}|^2
\end{aligned}
$$

for all x*,* $y \in \mathbb{R}^d$ *and* $i \in \mathbb{S}$*.*

If we set $b_0 = j_0 - 4\check{c}^2\theta_1$, $b_3 = j_3 - 4\check{c}^2\theta_2$, it reaches the desired conditions (3.4) and (3.5). That is to say, we have shown that Assumption 4.1 implies Assumptions 3.1 and 3.2.

By Lemmas 3.1, 3.2, and Theorem 3.1, the following corollary therefore follows.

Corollary 4.1. *Let Assumptions* 3.1 *and* 3.2 *be replaced by Assumption 4.1. If* h *can be divisible by* τ *and is small enough for*

$$
\begin{aligned}
& j_0 c - 4\check{c}^2\bar{\theta}c - 2\alpha\tau\,(2a_1\tau + a_2 + a_3 + 2a_4\tau) > 0, \quad \tau < \tfrac{1}{\sqrt{15a_4}}, \\
& j_3 c - 4\check{c}^2\bar{\theta}c - \alpha\tau\,(2a_1\tau + a_2 + a_3 + 2a_4\tau) > 0, \quad \tau < \sqrt{\tfrac{2}{15a_4}},
\end{aligned}
$$

where $\bar{\theta} \in \left(0, \frac{j_0 - b_1}{4\check{c}^2} \wedge \frac{j_3 - b_4}{4\check{c}^2}\right)$*,* $c = \min_{i\in\mathbb{S}} \lambda_{\min} W_i$*,* $\check{c} = \max_{i\in\mathbb{S}} \|W_i\|$*, and* $\alpha = \dfrac{15a_4}{4\bar{\theta}}$*. Then the SDDE-MS-LN* (2.3) *is stable in distribution.*

The key to the problem of stabilization in distribution lies in the design of the matrices $A_i (i \in \mathbb{S})$. Therefore we need to find the matrices of the form $A_i = F_i G_i$ with $F_i \in R^{n\times l}$ and $G_i \in R^{n\times l}$ for some positive integer l. The two cases of state feedback and output injection are discussed below.

(i) State feedback

When G_i's are given, we need to seek suitable F_i's to make SDDE-MS-LN (2.3) stable in distribution. The matrices are designed in two steps.

Step 1: Seek N couples of positive-definite symmetric matrices $\left(W_i, \hat{R}_i, \hat{S}_i\right)$ such that

$$\begin{aligned}
& 2(x-y)^T W_i[f(x,\bar{x},i) - f(y,\bar{y},i)] \\
& + \operatorname{trace}\left[(g(x,\bar{x},i) - g(y,\bar{y},i))^T W_i (g(x,\bar{x},i) - g(y,\bar{y},i))\right] \\
& + \int_{\mathbb{R}\backslash\{0\}} \sum_{k=1}^{n} \Big[\left(H^{(k)}(x,\bar{x},i,z_k) - H^{(k)}(y,\bar{y},i,z_k)\right)^T \\
& \times W_i \left(H^{(k)}(x,\bar{x},i,z_k) - H^{(k)}(y,\bar{y},i,z_k)\right) \\
& + \left(H^{(k)}(x,\bar{x},i,z_k) - H^{(k)}(y,\bar{y},i,z_k)\right)^T W_i(x-y) \\
& - (x-y)^T W_i \left(H^{(k)}(x,\bar{x},i,z_k) - H^{(k)}(y,\bar{y},i,z_k)\right] \nu_k(\,\mathrm{d}z_k) \\
\leq\ & (x-y)^T \hat{R}_i (x-y) + (\bar{x}-\bar{y})^T \hat{S}_i (\bar{x}-\bar{y}),
\end{aligned} \tag{4.1}$$

and then

$$\begin{aligned}
& \mathbb{L}\Phi(x,y,\bar{x},\bar{y},i) \\
\leq\ & (x-y)^T \hat{R}_i (x-y) + (\bar{x}-\bar{y})^T \hat{S}_i (\bar{x}-\bar{y}) + 2(x-y)^T W_i A_i (x-y) + \sum_{j=1}^{N} \gamma_{ij} (x-y)^T W_j (x-y) \\
\leq\ & (x-y)^T (\hat{R}_i + W_i F_i G_i + G_i^T F_i^T W_i + \sum_{j=1}^{N} \gamma_{ij} W_j)(x-y) + (\bar{x}-\bar{y})^T \hat{S}_i (\bar{x}-\bar{y}).
\end{aligned} \tag{4.2}$$

Step 2: Seek a solution of matrices F_i to the following linear matrix inequalities ensuring that $j_3 > b_4$:

$$\hat{R}_i + W_i F_i G_i + G_i^T F_i^T W_i + \sum_{j=1}^{N} \gamma_{ij} W_j + \hat{S}_i < 0, \quad i \in \mathbb{S}. \tag{4.3}$$

Corollary 4.2. *Under Assumption* 2.1*, seek matrices $F_i (i \in \mathbb{S})$ with the above Steps 1 and 2. Then Corollary* 4.1 *holds with $A_i = F_i G_i$ and*

$$\begin{aligned}
j_3 &= -\max_{i\in\mathbb{S}} \lambda_{\max}\left(\hat{R}_i + W_i F_i G_i + G_i^T F_i^T W_i + \textstyle\sum_{j=1}^{N} \gamma_{ij} W_j\right), \\
b_4 &= \max_{i\in\mathbb{S}} \lambda_{\max} \hat{S}_i.
\end{aligned}$$

(ii) Output injection

When F_i's are given, we need to seek G_i's. This case is similar to the case of state feedback, and therefore we can get another corollary.

Corollary 4.3. *Under Assumption* 2.1*, find matrices W_i, $\hat{R}_i$, $\hat{S}_i$, and $G_i (i \in \mathbb{S})$ with the above Steps 1 and 2. Then Corollary* 4.1 *holds with $A_i = F_i G_i$, and moreover j_3 and b_4 are the same as in Corollary* 4.2*.*

$$\begin{aligned}
j_3 &= -\max_{i\in\mathbb{S}} \lambda_{\max}\left(\hat{R}_i + W_i F_i G_i + G_i^T F_i^T W_i + \textstyle\sum_{j=1}^{N} \gamma_{ij} W_j\right), \\
b_4 &= \max_{i\in\mathbb{S}} \lambda_{\max} \hat{S}_i.
\end{aligned}$$

5. Example

In this section, we will give an example to illustrate our results.

Example 5.1. *Let us consider the following unstable SDDE-MS-LN:*

$$\begin{aligned} \mathrm{d}X(t) = \ & f(X(t), X(t-h), r(t))\mathrm{d}t + g(X(t), X(t-h), r(t))\mathrm{d}B(t) \\ & + H\left(X(t^-), X((t-h)^-), r(t)\right)\ \mathrm{d}\tilde{N}(t) \end{aligned} \tag{5.1}$$

with initial value $X(0) = 1$*,* $r(0) = 1$*, and* $N(0) = 0$*, where the coefficients f, g, and H are defined by*

$$\begin{aligned} f(x, \bar{x}, 1) &= 0.4 + 0.2x - 0.1\bar{x}, \quad & f(x, \bar{x}, 2) &= 0.3 + 0.1x - 0.3\bar{x}, \\ g(x, \bar{x}, 1) &= 0.3 + 0.2x + 0.1\bar{x}, \quad & g(x, \bar{x}, 2) &= 0.4 + 0.1x + 0.2\bar{x}, \\ H(x, \bar{x}, 1) &= 0.5x + \bar{x}, \quad & H(x, \bar{x}, 2) &= x + 0.5\bar{x}, \end{aligned}$$

for all $x, \bar{x} \in \mathbb{R}$*,* $B(t)$ *is a scalar Brownian motion,* $N(t)$ *is a scalar Poisson process with intensity* λ*,* $\tilde{N}(t)$ *denotes its compensated Poisson random measure,* $r(t)$ *is a Markov chain on the state space* $\mathbb{S} = \{1, 2\}$ *with its generator*

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix},$$

and the time delay $h = 0.01$*.*

From Figure 1, we know the SDDE-MS-LN (5.1) *is unstable. Let us now apply our new theory to design a linear feedback control to make the SDDE-MS-LN* (5.1) *stable in distribution. Suppose that the type of linear feedback control is* $-k(i)X(\delta(t))$*, where* $k(i)$ *will be computed later. Then the controlled system becomes*

$$\begin{aligned} dX(t) = \ & [f(X(t), X(t-h), r(t)) - k(r(t))X(\delta(t))]dt + g(X(t), X(t-h), r(t))dB(t) \\ & + H\left(X(t^-), X((t-h)^-), r(t)\right) d\tilde{N}(t). \end{aligned} \tag{5.2}$$

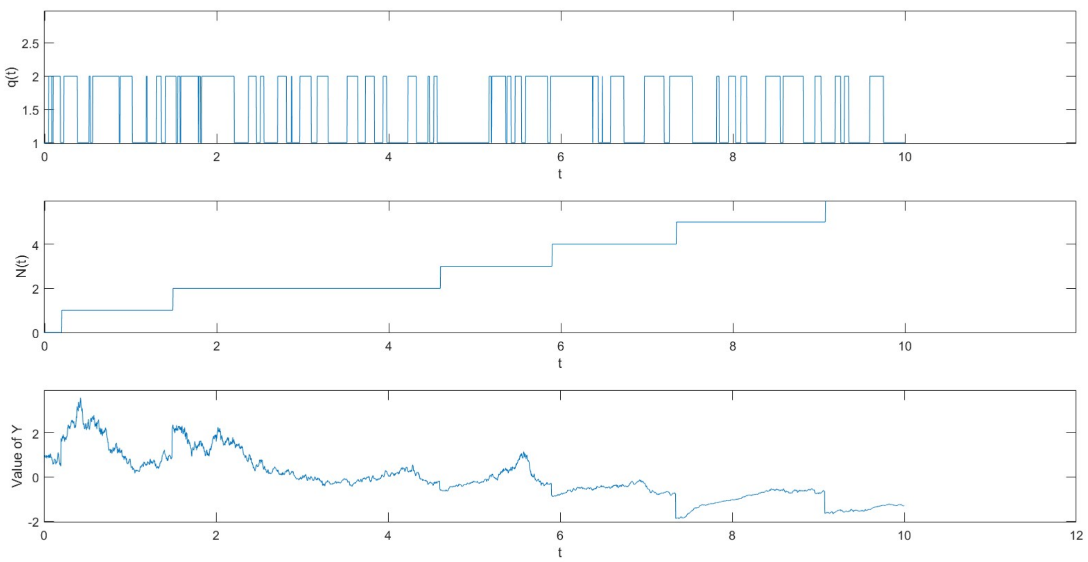

Figure 1. The sample path of SDDE-MS-LN (5.1) with the initial data $X(0) = 1$.

Let $W_i(i \in 1, 2)$ be the identity matrix. Set $\lambda = 1$. After the calculation, we can get that $b_4 = 2.12$. Next, we need to choose a number j_3 which satisfies $j_3 > b_4$ and $-j_3 = -\max_{i\in\mathbb{S}}(1.08 - 2k(i)) < 0$ for $i \in \mathbb{S}$. Hence, Corollary 4.1 holds for $j_3 = 6$. Then we can deduce that $k(1) = 3.54$ and $k(2) = 4.26$. It is easy to verify that Assumption 2.1 holds for $a_1 = 0.18$, $a_2 = 0.08$, and $a_3 = 2$. Furthermore, by $a_4 = \max_{i\in\mathbb{S}} \|k(i)\|^2$, we can get $a_4 = 18.1476$ and $\check{c} = 1$. Thus, setting $c = 1$, $\bar{\theta} = 0.1$, we can derive

$$\tau_1^* = 0.00158, \quad \tau_2^* = 0.0606, \quad \tau_3^* = 0.00371, \quad \tau_4^* = 0.0857.$$

Consequently, $\tau^* = 0.00158$. By Corollary 4.1, the controlled system (5.2) is stable in distribution when $\tau < 0.00158$ and h can be divisible by τ.

In addition, we plot the marginal density function of $X(t)$ by using of the Euler-Maruyama method with step size 0.001 in Figure 2. From the figure, as t increases, the change in distribution becomes smaller and smaller. This show that the controlled system (5.2) is stable in distribution.

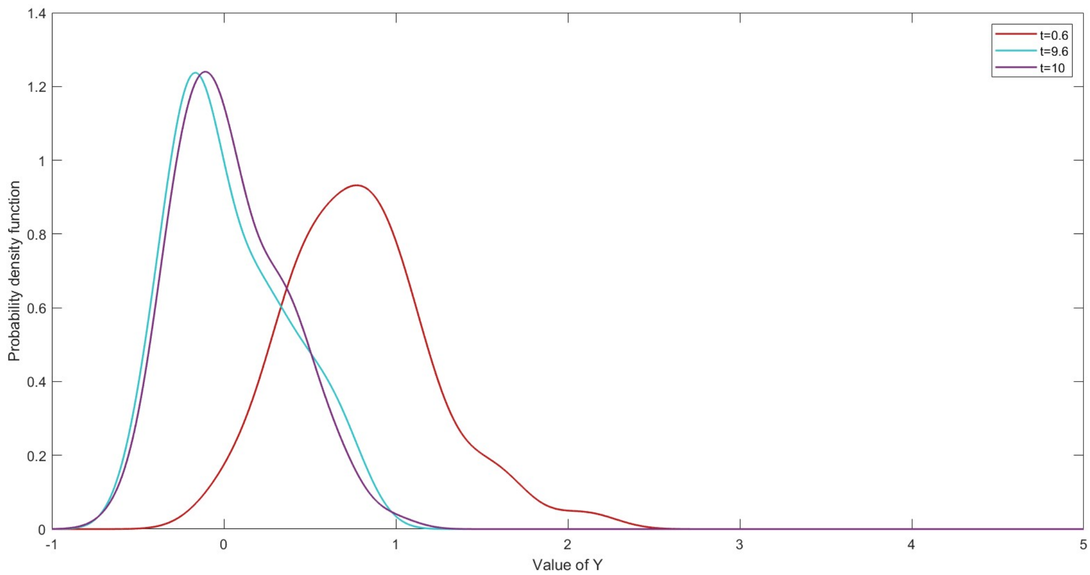

Figure 2. Distribution numerical solution of the controlled SDDE-MS-LN (5.2).

6. Conclusions

In this paper, stabilization in distribution for given unstable SDDEs-MS-LN whose drift and diffusion coefficients are globally Lipshitz continuous has been investigated. We successfully showed that the stability in distribution of controlled SDDE-MS-LN can be achieved by linear feedback controls based on discrete-time state observations. A lower bound on duration τ^* is given so that the controlled SDDEs-MS-LN is stable in distribution as long as $\tau < \tau^*$ and h can be divisible by τ. We specifically discussed how to design the linear feedback control in two structure cases. Finally, a numerical example is illustrated to support our theory. But the system coefficients are still under a linear growth condition, and the stochastic systems are driven by Markovian switching. Hence we will devote our future work to releasing the linear growth condition on f, g [5] and investigating stabilization in distribution for nonlinear stochastic differential delay equations with semi-Markovian switching and Lévy noise (SDDEs-SMS-LN) controlled by discrete feedback controls [33, 34].

Author contributions

Jingjing Yang: Formal analysis, methodology, investigation, resources, software, writing-original draft. Jianqiu Lu: Conceptualization, methodology, supervision, writing-review and editing, funding acquisition.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare that there are no conflicts of interest.

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