Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 3428–3456.
DOI: 10.3934/math.2025159
Received: 31 October 2024
Revised: 04 February 2025
Accepted: 12 February 2025
Published: 24 February 2025

Research article

The $\mathbb{C}^*$-action and stratifications of the moduli space of semi-stable Higgs bundles of rank 5

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Abstract: Let X be a compact Riemann surface of genus $g \geq 2$. The moduli space $\mathcal{M}(r, d)$ of rank r and degree d semi-stable Higgs bundles over X admitted a stratification, called Shatz stratification, which was defined by the Harder-Narasimhan type of the Higgs bundles. There was also a $\mathbb{C}^*$-action on $\mathcal{M}(r, d)$ given by the product on the Higgs field, which provided the Białynicki-Birula stratification by considering the Hodge limit bundles $\lim_{z\to 0}(E, z \cdot \varphi)$. In this paper, these limit bundles were computed for all possible Harder-Narasimhan types when the rank of the Higgs bundles was $r = 5$, explicit vector forms were provided for the Hodge limit bundles, and necessary and sufficient conditions were given for them to be stable. In addition, it was proved that, in rank 5, the Shatz strata traversed the Białynicki-Birula strata. Specifically, it was checked that there existed different semi-stable rank 5 Higgs bundles with the same Harder-Narasimhan type such that their associated Hodge limit bundles were not S-equivalent, and explicit constructions of those Higgs bundles were also provided.

Keywords: Higgs bundle; stratification; action; Harder-Narasimhan type; semi-stable; moduli space
Mathematics Subject Classification: 14H60, 14H10, 57R57

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. Higgs bundles over X were first introduced by Hitchin [21] in the context of the resolution of self-duality equations. He also proved that the moduli space of Higgs bundles forms a hyperkähler manifold. After that, Nitsure [24] provided a construction of the moduli space of Higgs bundles over X by using techniques from geometric invariant theory. Simpson [26–28] extended the framework provided by Hitchin and linked Higgs bundles to non-Abelian Hodge theory and flat connections on varieties.

The study of the geometry and topology of the moduli space of Higgs bundles is an interesting field of study, not only for its importance in mathematics, but also for its connections with theoretical physics, particularly mirror symmetry, Langlands duality, and string theory [1, 18]. The lines of research on Higgs bundles are rich and diverse, including the computation of cohomology of moduli spaces of Higgs bundles [13, 19], the classification of its automorphisms [7], the study of certain subvarieties of the moduli space, such as the subvariety of fixed point of an automorphism of it [3,5,14] or automorphisms of the underlying vector bundles [4], the relation of Higgs bundles with the non-Abelian Hodge correspondence [29], Hitchin systems [10], or the relation of Higgs bundles with Donaldson invariants [12,23] or other topological properties such as Gopakumar-Vafa invariants [22].

The line in which the present work is framed is that of the study of stratifications of the moduli space of Higgs bundles over X. Specifically, there are two stratifications of interest for the purposes intended here. First, the Shatz stratification, defined from the vector of slopes (called Harder-Narasimhan type) associated with the Harder-Narasimhan filtration of the underlying vector bundle of strictly semi-stable Higgs bundles [24, 25]. This is a stratification of great interest in geometry and has been intensively studied even in the field of bundles with structure groups different from the linear group, such as G_2 [6]. The second is the Białynicki-Birula stratification, constructed from the limits $\lim_{z\to 0}(E, z \cdot \varphi)$ of semi-stable Higgs bundles, where $\cdot$ denotes the action of the multiplicative group $\mathbb{C}^*$ on the Higgs field [8, 11, 26]. The case of rank 2 Higgs bundles was studied by Hausel and Thaddeus [17, 19, 20]. They proved that the two mentioned stratifications coincide in rank 2. Later, Gothen and Zúñiga-Rojas proved that this is not the case for rank 3. The mentioned works were extended to rank 4 in [2], where it was proved that Shatz and Białynicki-Birula stratifications do not coincide in rank 4, but both stratifications have some common strata, defined by certain conditions of the Harder-Narasimhan types.

In the present work, the previous papers are extended to the rank 5 situation. In particular, it is proved that the Shatz and Białynicki-Birula stratifications do not coincide even in rank 5. The strategy followed is analogous to that of the previous paper [2] and the article by Gothen and Zúñiga-Rojas [15]. Specifically, a family of intermediate bundles of any strictly semi-stable Higgs bundle is constructed (Definition 1), and a set of bounds for the slopes of these Higgs bundles is provided (Proposition 1). From this, a family of cases for the Harder-Narasimhan type are distinguished that need to be differentiated to conduct the analysis. Finally, the limit bundles, which are always Hodge bundles, are explicitly computed in each of the differentiated cases (Theorems 1 to 6). Indeed, explicit vector forms of all possible Hodge limit bundles are provided, which constitutes an original contribution of the paper. Although the strategy is similar to that of previous works [2,15], the situation in rank 5 is notably more difficult, due to the much more complicated topology of the moduli space. This means that it is necessary to distinguish up to 6 different situations, as opposed to 4 in [2] and 2 in [15]. Moreover, as far as it has been possible to explore, the extension made here is not directly generalizable to any rank, so it is only possible to address the specific rank.

In addition, it is proved here that the strata of the two mentioned stratifications in rank 5 are transversal, in the sense that each Shatz stratum crosses several Białynicki-Birula strata (Theorem 7). This has been proven by analyzing the S-equivalence condition of the Hodge limit bundles in each of the differentiated situations, which could be grouped into four results (Propositions 2 to 5). This contribution is an original novelty and also an essential difference with respect to what happens in rank 4. Moreover, explicit constructions of strictly semi-stable Higgs bundles of rank 5 of the same Harder-Narasimhan type whose associated Hodge limit bundles are not S-equivalent are provided for each of the differentiated situations in the analysis made. Therefore, in addition to extending the results of [2, 15] to rank 5, the present work contributes with the novelty of the explicit description of the Shatz strata in rank 5 and with the verification of the transversality of the Shatz and Białynicki-Birula stratifications in rank 5, an aspect that does not occur in all the previously studied ranks.

The structure of the article is as follows. In Section 2, the notions of stability and S-equivalence of Higgs bundles are presented. From this, the Harder-Narasimhan types, the $\mathbb{C}^*$-action, and the Shatz and Białynicki-Birula stratifications are introduced. The bounds on the slopes of the intermediate vector bundles are checked in Section 3, where the main 6 situations of the study are distinguished and the explicit computation of the Hodge limit bundles for the $\mathbb{C}^*$-action are provided. In Section 4 it is proved that, regardless of the situation considered, it is always possible to find semi-stable Higgs bundles with a fixed Harder-Narasimhan type whose Hodge limit bundles are not S-equivalent, from which the announced transversality of the Shatz and Białynicki-Birula stratifications follows. Finally, the main conclusions of the work are drawn.

2. Shatz and Białynicki-Birula stratifications of the moduli space of Higgs bundles

Given a compact Riemann surface of genus $g \geq 2$, a Higgs bundle over X is a pair (E, φ) where E is a holomorphic vector bundle over X and $\varphi : E \to E \otimes K$ is a homomorphism of vector bundles, K being the canonical line bundle over X. The rank and the degree of the Higgs bundle are the rank and the degree, respectively, of E, and the slope of the Higgs bundle is also the slope of E, $\mu = \mu(E) = \frac{\deg E}{\operatorname{rk} E}$. The Higgs bundle (E, φ) is semi-stable if for every φ-invariant proper sub-bundle F of E it is satisfied that $\mu(F) \leq \mu(E)$. It is stable if the inequality is strict for every φ-invariant proper sub-bundle F. Finally, it is poly-stable if E is isomorphic to the direct sum of proper sub-bundles whose slopes are $\mu = \mu(E)$ [21, 26–28].

Any strictly semi-stable Higgs bundle (E, φ) over X admits a filtration $0 \subset E_1 \subset \cdots \subset E_n = E$ into φ-invariant proper sub-bundles E_k such that the pair (E_k, φ_k) is stable as a Higgs bundle, where $\varphi_k : E_k \to E_k \otimes K$ denotes the restriction of φ. This filtration provides a graded object associated with (E, φ), which is a poly-stable Higgs bundle. Two semi-stable Higgs bundles are said to be S-equivalent if their associated graded objects are isomorphic. Then the moduli space of semi-stable Higgs bundles over X is a complex algebraic variety that parametrizes S-equivalence classes of semi-stable Higgs bundles or, equivalently, isomorphism classes of poly-stable Higgs bundles over X, since every S-equivalence class admits a unique isomorphism class of a poly-stable representative.

Given any vector bundle E over X, it admits a unique Harder-Narasimhan filtration, which is a filtration of the form $0 \subset E_1 \subset \cdots \subset E_k = E$, where E_j/E_{j-1} is semi-stable as a vector bundle for every $j = 1, \ldots, k$, being $E_0 = 0$, and $\mu(E_{j+1}/E_j) < \mu(E_j/E_{j-1})$ for $j = 1, \ldots, k-1$. Here, E_1 is the maximal destabilizing sub-bundle of E and the graded vector bundle is defined by $\oplus_{j=1}^{k} E_j/E_{j-1}$. Let (E, φ) be a strictly semi-stable Higgs bundle over X. Then, the underlying vector bundle E must be unstable. Consider the Harder-Narasimhan filtration $0 \subset E_1 \subset \cdots \subset E_k = E$ of E and let, for each $j = 1, \ldots, k$, $\mu_j = \mu(E_j/E_{j-1})$. Then, a vector of slopes $(\mu_1, \ldots, \mu_r)$, where $r = \operatorname{rk} E$, is defined, where each μ_j is repeated $\operatorname{rk}(E_j) - \operatorname{rk}(E_{j-1})$ times in the vector for $j = 1, \ldots, k$. This vector of slopes is called *Harder-Narasimhan type* of the underlying vector bundle of the Higgs bundle [16, 23]. Notice that it must be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$, with at least one of these inequalities being strict. Also, to make the presentation clearer in the following, when appealing to the Harder-Narasimhan filtration of the underlying vector bundle E of a strictly semi-stable Higgs bundle, a filtration of the form $0 \subset E_1 \subseteq \cdots \subseteq E_r = E$, the inclusions being non-strict, will be taken, where $r = \operatorname{rk} E$ and each E_j is repeated $\operatorname{rk}(E_j) - \operatorname{rk}(E_{j-1})$ times in the filtration. Then, $\mu_1 = \mu(E_1)$ and $\mu_k = \mu_{k+1}$ whenever $E_k = E_{k+1}$.

Remark 1. *To fix notation, notice that the sub-bundle of the Harder-Narasimhan filtration of the underlying vector bundle E of a strictly semi-stable Higgs bundle (E, φ) of certain rank k will be denoted by E_k. Then, all sub-bundles within the mentioned Harder-Narasimhan filtration with the same rank k will be denoted as E_k. That is, if $E_k = E_{k+1}$, both of rank k, then E_{k+1} will be called E_k. Of course, in this case, it will be $\mu_k = \mu_{k+1}$. For example, in rank* 5*, which is the case of interest in this work, if the Harder-Narasimhan filtration of E is of the form* $0 \subset E_1 \subset E_3 \subset E_5$ *(with strict inclusions), this means that* $\operatorname{rk} E_1 = 1$, $\operatorname{rk} E_3 = 3$*, and the Harder-Narasimhan type of E satisfies* $\mu_1 = \mu_2 > \mu_3 = \mu_4 > \mu_5$*. This notation is consistent with that of [2] but rather different from that of [15]. The slope of E will be denoted by μ.*

With the above notation, the moduli space $\mathcal{M}(r, d)$ of semi-stable Higgs bundles of given rank r and degree d admits a stratification whose strata are defined by the equality of the Harder-Narasimhan type of the underlying vector bundle of the Higgs bundles. This is called *Shatz stratification* of $\mathcal{M}(r, d)$ [24, 25].

The moduli space $\mathcal{M}(r, d)$ also admits the *Białynicki-Birula stratification*, which arises from the action of the multiplicative group $\mathbb{C}^*$ on Higgs bundles defined by $z \cdot (E, \varphi) \mapsto (E, z \cdot \varphi)$ [8, 26]. Although this limit bundle may not be unique in the configuration space of semi-stable bundles, as the orbit space is not Hausdorff, the action is well-defined on the configuration space of poly-stable Higgs bundles, in which the orbit space is indeed Hausdorff [28, Corollary 9.20]. Two semi-stable Higgs bundles over X are in the same stratum of the Białynicki-Birula stratification if their associated limit bundles for the action of $\mathbb{C}^*$ are S-equivalent. In addition, the limit bundle will be a Hodge bundle, since it is in turn fixed by the action of $\mathbb{C}^*$ on the Higgs fields.

3. Hodge limit bundles for the $\mathbb{C}^*$-action

Let X be a compact Riemann surface of genus $g \geq 2$. In this section, all possible Harder-Narasimhan types of the underlying vector bundle of semi-stable rank 5 Higgs bundles with a given degree are considered. Specifically, for a given semi-stable Higgs bundle (E, φ) of rank 5, the Hodge limit bundle $\lim_{z\to 0}(E, z \cdot \varphi)$ is computed, and explicit forms of these limit bundles are provided depending on the different possibilities for the Harder-Narasimhan type of the underlying vector bundle of a strictly semi-stable Higgs bundle. Notice that the case when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ cannot occur for strictly semi-stable Higgs bundles. This case would correspond to a stable Higgs bundle (E, φ), for which obviously $\lim_{z\to 0}(E, z \cdot \varphi) = (E, 0)$. To do all the above, some intermediate bundles are first introduced, for which certain bounding properties of their slopes will be proved in the following [2, 15].

Given a strictly semi-stable rank 5 Higgs bundle (E, φ) with slope $\mu = \mu(E)$, its Harder-Narasimhan type will be denoted by $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$, where $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq \mu_5$ and $\frac{1}{5}(\mu_1+\mu_2+\mu_3+\mu_4+\mu_5) = \mu$. As in the previous section, if the Harder-Narasimhan filtration of the underlying vector bundle E of (E, φ) is $0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$, which is preserved by φ, then $\mu_1 = \mu(E_1)$ and $\mu_k = \mu_{k+1}$ whenever $E_k = E_{k+1}$, and the same notation convention will be followed.

Definition 1. *Let (E, φ) be a strictly semi-stable rank 5 Higgs bundle over X such that the Harder-Narasimhan filtration of the underlying vector bundle E is $0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$ and the Harder-Narasimhan type of E is $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$. Then the following intermediate bundles are defined:*

(1) For $k = 1, 2, 3, 4$, I_k is the sub-bundle of E/E_k obtained by saturating the sheaf $\varphi_{k+1,k}(E_k) \otimes K^{-1}$, where $\varphi_{k+1,k} : E_k \to E/E_k \otimes K$ is the morphism induced by φ.

(2) For $k = 1, 2, 3, 4$, $N_k = \ker(\varphi_{5-k+1,5-k})$, where $\varphi_{5-k+1,5-k}$ is the morphism, $E_{5-k} \to E/E_{5-k} \otimes K$ is induced by φ.

Remark 2. *In Definition 1, the sub-bundles I_4 and N_4 have been included because they make sense, although these two sub-bundles will not play a role in the following description of the different Shatz strata.*

Remark 3. *The bundles I_k in Definition 1 are defined as the saturation of the images of certain morphisms induced by φ. Saturation guarantees that it is a vector bundle. On their part, the bundles N_k are defined as kernels of certain homomorphisms of vector bundles induced by φ, which are always homomorphisms by semi-stability of (E, φ). The ranks of these bundles depend on the rank of the morphisms $\varphi_{k+1,k}$. Specifically, the following cases can be differentiated, which are discussed in the result below:*

(1) $\mu_1 > \mu_2$.

(2) $\mu_2 > \mu_3$ *and* $\mathrm{rk}(\varphi_{3,2}) = 1$.

(3) $\mu_2 > \mu_3$ *and* $\mathrm{rk}(\varphi_{3,2}) = 2$.

(4) $\mu_4 > \mu_5$.

(5) $\mu_3 > \mu_4$ *and* $\mathrm{rk}(\varphi_{4,3}) = 1$.

(6) $\mu_3 > \mu_4$ *and* $\mathrm{rk}(\varphi_{4,3}) = 2$.

Proposition 1. *Let (E, φ) be a strictly semi-stable rank 5 Higgs bundle over X such that the Harder-Narasimhan filtration of the underlying vector bundle E is $0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$ and the Harder-Narasimhan type of E is $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$, and I_1, I_2, I_3, N_1, N_2, and N_3 be the vector bundles given in Definition 1. Then, the following is satisfied:*

(1) If $\mu_1 > \mu_2$, then I_1 is a line bundle and $\mu_1 - (2g - 2) \le \mu(I_1) \le \mu_2$.

(2) If $\mu_2 > \mu_3$ and the morphism $\varphi_{3,2} : E_2 \to E/E_2 \otimes K$ induced by φ has rank 1, then I_2 and N_3 are line bundles, $\mu_2 - (2g - 2) \le \mu(I_2) \le \mu_3$, and $\mu_1 + \mu_2 - \mu_3 - (2g - 2) \le \mu(N_3) \le \mu_1$.

(3) If $\mu_2 > \mu_3$ and the rank of the morphism $\varphi_{3,2} : E_2 \to E/E_2 \otimes K$ induced by φ is 2, then I_2 is a rank 2 vector bundle and $\mu_2 - (2g - 2) \le \mu(I_2) \le \mu_3$.

(4) If $\mu_4 > \mu_5$, then N_1 is a rank 2 vector bundle and $\frac{1}{2}(\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5 - (2g - 2)) \le \mu(N_1) \le \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$.

(5) *If* $\mu_3 > \mu_4$ *and the morphism* $\varphi_{4,3} : E_3 \to E/E_3 \otimes K$ *induced by* φ *has rank* 1*, then* I_3 *is a line bundle,* N_2 *is a rank* 2 *vector bundle,* $\mu_3 - (2g-2) \leq \mu(I_3) \leq \mu_4$*, and* $\frac{1}{2}(\mu_1 + \mu_2 + \mu_3 - \mu_4 - \mu_5 - (2g-2)) \leq \mu(N_2) \leq \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$.

(6) *If* $\mu_3 > \mu_4$ *and the morphism* $\varphi_{4,3} : E_3 \to E/E_3 \otimes K$ *induced by* φ *has rank* 2*, then* N_2 *is a rank* 2 *vector bundle and* $\frac{1}{2}(\mu_1 + \mu_2 + \mu_3 - \mu_4 - \mu_5 - (2g-2)) \leq \mu(N_2) \leq \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$.

Proof. For case (1), suppose that $\mu_1 > \mu_2$. Then, E_1 is a line bundle by hypothesis and $\varphi_{2,1} \neq 0$ by semi-stability of (E, φ). Then, I_1 is also a line bundle and, since the morphism $E_1 \to I_1 \otimes K$ induced by $\varphi_{2,1}$ is nonzero, it follows that $\mu(I_1) + \mu(K) \geq \mu_1$, so $\mu(I_1) \geq \mu_1 - (2g-2)$. In addition, since E_2/E_1 is a maximal destabilizer with rank 1 in E/E_1, then it must be $\mu(I_1) \leq \mu(E_2/E_1) = \mu_2$.

The condition on I_2 of case (2) is similar to case (1), by considering the morphism $\varphi_{3,2}$. For the condition on N_3, notice that the morphism $\varphi_{3,2} : E_2 \to E/E_2 \otimes K$ has rank 1 and, since $\mu_2 > \mu_3$, E_2 has rank 2, so there is an exact sequence of sheaves

$$0 \to N_3 \to E_2 \to \mathrm{Im}(\varphi_{3,2}) \to 0,$$

where $\mathrm{Im}(\varphi_{3,2})$ has rank 1 (it may be considered as a vector bundle by saturating it). Then, N_3 is a line bundle. Now, since (E, φ) is semi-stable, the slope of N_3 must be $\mu(N_3) \leq \mu_1$. Moreover, by the behavior of the slopes in exact sequences applied to the short exact sequence above,

$$\mu(E_2) = \frac{\mathrm{rk}(N_3)\mu(N_3) + \mathrm{rk}(\mathrm{Im}(\varphi_{3,2}))\mu(\mathrm{Im}(\varphi_{3,2}))}{\mathrm{rk}(E_2)}.$$

Since $\mathrm{rk}(N_3) = \mathrm{rk}(\mathrm{Im}(\varphi_{3,2})) = 1$, $\mathrm{rk}(E_2) = 2$, and $\mu(E_2) = \frac{\mu_1+\mu_2}{2}$, it follows that

$$\frac{\mu_1 + \mu_2}{2} = \frac{\mu(N_3) + \mu(\mathrm{Im}(\varphi_{3,2}))}{2}.$$

On the other hand, since $\mathrm{Im}(\varphi_{3,2})$ is a sub-sheaf of $E/E_2 \otimes K$, it is obtained that

$$\mu(\mathrm{Im}(\varphi_{3,2})) \leq \mu(E/E_2 \otimes K) = \mu(E/E_2) + (2g-2) \leq \mu_3 + (2g-2).$$

By substituting in the above expression,

$$\frac{\mu_1 + \mu_2}{2} \leq \frac{\mu(N_3) + \mu_3 + (2g-2)}{2},$$

thus, by clearing $\mu(N_3)$, it follows that $\mu(N_3) \geq \mu_1 + \mu_2 - \mu_3 - (2g-2)$.

For the conditions on I_2 of the case (3), notice that I_2 has rank 2 because the morphism $\varphi_{3,2} : E_2 \to E/E_2 \otimes K$ has rank 2. There is an exact sequence of the form

$$0 \to I_2 \to E/E_2 \to Q \to 0,$$

where Q is the quotient bundle, which has rank 1, since E/E_2 has rank 3 and I_2 has rank 2. It is known that $\mu(E/E_2) = \frac{\mu_3+\mu_4+\mu_5}{3}$ and $\mu(Q) \geq \mu(E/E_2)$. To check the last inequality, consider the exact sequence defined by

$$0 \to I_2 \to E/E_2 \to Q \to 0.$$

If $\mu(Q) < \mu(E/E_2)$, then $\mu(I_2) > \mu(E/E_2)$. This would contradict the semi-stability of (E, φ), as $I_2 \otimes K^{-1}$ would be a φ-invariant sub-bundle of E/E_2 with slope higher than $\mu(E/E_2)$. Thus, $\mu(Q) \geq \mu(E/E_2)$. Therefore,

$$\frac{\mu_3 + \mu_4 + \mu_5}{3} \geq \frac{2\mu(I_2) + \frac{\mu_3+\mu_4+\mu_5}{3}}{3},$$

and then $\mu(I_2) \leq \frac{\mu_3+\mu_4+\mu_5}{3} \leq \mu_3$, as stated. On the other hand, the morphism $\varphi_{3,2} : E_2 \to I_2 \otimes K$ is surjective, so the sequence

$$0 \to \ker(\varphi_{3,2}) \to E_2 \to I_2 \otimes K \to 0$$

is exact. Therefore,

$$\mu(E_2) = \frac{\mathrm{rk}(\ker(\varphi_{3,2}))\mu(\ker(\varphi_{3,2})) + \mathrm{rk}(I_2)\mu(I_2 \otimes K)}{\mathrm{rk}(E_2)}.$$

Since $\mu(I_2) = \mu_2 - (2g-2)$ and $\mu(\ker(\varphi_{3,2})) \leq \mu(E_2) = \mu_2$ (by semi-stability), it follows that

$$\mu_2 \leq \frac{\mathrm{rk}(\ker(\varphi_{3,2}))\mu_2 + \mathrm{rk}(I_2)(\mu(I_2) + (2g-2))}{\mathrm{rk}(E_2)},$$

so, finally, $\mu(I_2) \geq \mu_2 - (2g-2)$.

In the case (4), consider $N_1 = \ker(\varphi_{5,4})$. Since $\mu_4 > \mu_5$, E_4 has rank 4, and $\varphi_{5,4} : E_4 \to E/E_4 \otimes K$ is nonzero (by semi-stability of (E, φ)). Thus, N_1 has rank 3 or 2. However, if N_1 had rank 3, then $\varphi_{5,4}$ would factor through E_4/N_1, which is a line bundle. This would imply that $\varphi_{4,3}$ has rank at most 1, contradicting the assumption. Therefore, N_1 has rank 2. Moreover, it is known that $\mu(N_1) \leq \mu(E_3) = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$, because $N_1 \subset E_4$ and E_3 is the maximal destabilizing sub-bundle of rank 3. For the lower bound, consider the exact sequence given by

$$0 \to N_1 \to E_4 \to E_4/N_1 \to 0.$$

It is known that $E_4/N_1 \to E/E_4 \otimes K$ is nonzero, so $\mu(E_4/N_1) \leq \mu_5 + (2g-2)$. By the properties of slopes in exact sequences, it is clear that

$$\mu(E_4) = \frac{2\mu(N_1) + \mu(E_4/N_1)}{3}$$

and

$$\frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{4} = \frac{2\mu(N_1) + (\mu_5 + (2g-2))}{3}.$$

Thus, $\mu(N_1) \geq \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5 - (2g-2))$, as stated.

For case (5), the condition on I_3 is similar as that of I_1 of case (1), by taking the morphism $\varphi_{4,3}$. For the condition on N_2, take the exact sequence

$$0 \to N_2 \to E_3 \to \mathrm{Im}(\varphi_{4,3}) \to 0,$$

where the rank of $\mathrm{Im}(\varphi_{4,3})$ is 1, as $\varphi_{4,3} : E_3 \to E/E_3 \otimes K$ has rank 1, and E_3 has rank 3, since $\mu_3 > \mu_4$. This proves that the rank of N_2 is 2. Moreover, as in case (2),

$$\mu(E_3) = \frac{\mathrm{rk}(N_2)\mu(N_2) + \mathrm{rk}(\mathrm{Im}(\varphi_{4,3}))\mu(\mathrm{Im}(\varphi_{4,3}))}{\mathrm{rk}(E_3)},$$

hence,

$$\frac{\mu_1+\mu_2+\mu_3}{3}=\frac{2\mu(N_2)+\mu(\mathrm{Im}(\varphi_{4,3}))}{3},$$

since $\mu(E_3)=\frac{\mu_1+\mu_2+\mu_3}{3}$. By semi-stability, $\mu(\mathrm{Im}(\varphi_{4,3}))\leq\mu(E/E_3\otimes K)=\mu(E/E_3)+(2g-2)=\frac{\mu_4+\mu_5}{2}+(2g-2)$, so, by substituting in the equation above, it is obtained that

$$\frac{\mu_1+\mu_2+\mu_3}{3}\leq\frac{2\mu(N_2)+\frac{\mu_4+\mu_5}{2}+(2g-2)}{3},$$

thus, $\mu(N_2)\geq\frac{1}{2}(\mu_1+\mu_2+\mu_3-\mu_4-\mu_5-(2g-2))$.

The proof of the conditions on N_2 of case (6) is similar to that of case (4), by considering E_3 instead of E_4 and N_2 instead of N_1. □

Remark 4. *For a better understanding of the results to be given below where the Hodge limit bundles are computed, it is convenient to briefly discuss the comparison between $\mu=\mu(E)$ and the bounds of the slope of the sub-bundles N_k given in Proposition 1. To fix ideas, the conditions of the case (2) of the proposition will be assumed. The rest of the cases are analogous. Of course, under the assumption that $\mu_2>\mu_3$, it is satisfied that $\mu<\mu_1$, so μ is always less or equal to the upper bound of $\mu(N_3)$. However, it is not always true that, given a valid Harder-Narasimhan type of the underlying vector bundle of a strictly semi-stable Higgs bundle with rank 5, it must be $\mu\geq\mu_1+\mu_2-\mu_3-(2g-2)$. This inequality is satisfied depending on the value of the genus g. For example, if the Harder-Narasimhan type is $(2,2,1,1,1)$, then $\mu=\frac{\mu_1+\mu_2+\mu_3+\mu_4+\mu_5}{5}=\frac{7}{5}$ and $\mu_1+\mu_2-\mu_3-(2g-2)=3-(2g-2)$. Notice that $\frac{7}{5}\geq 3-(2g-2)$ if, and only if, $g\geq 1+\frac{4}{5}$, which is always true since, under our assumptions, $g\geq 2$. However, if the Harder-Narasimhan type is $(10,2,1,1,1)$, then $\mu=3$ and $\mu_1+\mu_2-\mu_3-(2g-2)=11-(2g-2)$, so $3\geq 11-(2g-2)$ if, and only if, $g\geq 5$. This explains that, when $\mu(N_k)=\mu$ or $\mu(N_k)<\mu$ is supposed in the results below, an additional assumption is required, such as $\mu\geq\mu_1+\mu_2-\mu_3-(2g-2)$.*

In Theorems 1 to 6 below, all possible Hodge limit bundles $\lim_{z\to 0}(E,z\varphi)$ for a strictly semi-stable Higgs bundle of rank 5 (E,φ) are computed, according to the cases described in the Remark after Definition 1. This extends to rank 5 the results given in [15] for rank 3 and in [2] for rank 4. The above possibilities for the Hodge limit bundles depend on the various Harder-Narasimhan types of the underlying vector bundles (notice that $\mu_1\geq\mu_2\geq\mu_3\geq\mu_4\geq\mu_5$ is always satisfied) and ranks of the different components of the Higgs fields, and, within each case, different limit Hodge bundles arise depending on several conditions assumed on the intermediate bundles I_1, I_2, I_3, N_1, N_2, and N_3 given in Definition 1. In all theorems, the conditions on the rank and slope of these intermediate bundles follow from Proposition 1.

Theorem 1. *Let (E,φ) be a rank 5 semi-stable Higgs bundle over X such that the Harder-Narasimhan type $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$ of E satisfies $\mu_1>\mu_2=\mu_3=\mu_4=\mu_5$, $\mu=\mu(E)$, and let I_1 be the line bundle given by Definition 1. Let (E_0,φ_0) be the Hodge limit bundle $\lim_{z\to 0}(E,z\cdot\varphi)$. Then, the explicit form of (E_0,φ_0) is the following:*

(1) If $\mu(I_1)<2\mu-\mu_1$, then

$$(E_0,\varphi_0)=\left(E_1\oplus E/E_1,\begin{pmatrix}0&0\\ \psi_{2,1}&0\end{pmatrix}\right),$$

where $\psi_{2,1}:E_1\to E/E_1\otimes K$ is nonzero and induced by φ.

(2) If $\mu(I_1) > 2\mu - \mu_1$, then

$$(E_0, \varphi_0) = \left(E_1 \oplus I_1 \oplus (E/E_1)/I_1, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right),$$

where $\psi_{2,1} : E_1 \to I_1 \otimes K$ and $\psi_{3,2} : I_1 \to (E/E_1)/I_1 \otimes K$ are the nonzero morphisms induced by φ.

(3) If $\mu(I_1) = 2\mu - \mu_1$, then

$$(E_0, \varphi_0) = \left(E_1 \oplus I_1, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix} \right) \oplus ((E/E_1)/I_1, 0),$$

where $\psi_{2,1} : E_1 \to I_1 \otimes K$ is nonzero and induced by φ.

Proof. Notice that, under the assumptions, I_1 is a line bundle by Proposition 1 and the morphisms $\psi_{i,j}$ should be nonzero by semi-stability of (E, φ). Notice also that the choice of the symbol ψ is due to that $\psi_{i,j}$ may not coincide with the morphisms $\varphi_{i,j}$ of Definition 1.

A family $g(z)$ of gauge transformations parametrized by $z \in \mathbb{C}^*$ will be found such that the limit

$$\lim_{z \to 0} (g(z) \cdot \bar{\partial}_E, g(z)^{-1} z \varphi g(z)),$$

is semi-stable, $\bar{\partial}_E$ being the operator $\bar{\partial}$ corresponding to the holomorphic structure of E. This is then the Hodge limit bundle.

For the first case, let $\mathbb{E}$, $\mathbb{E}_1$, and $\mathbb{E}'_1$ be, respectively, the underlying C^∞ vector bundles of E, E_1, and E/E_1. Then, $\mathbb{E} \cong \mathbb{E}_1 \oplus \mathbb{E}'_1$, and the holomorphic structure of E is given by the operator $\bar{\partial}_E$ defined by

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta \\ 0 & \bar{\partial}'_1 \end{pmatrix},$$

where $\bar{\partial}_1$ and $\bar{\partial}'_1$ are the $\bar{\partial}$-operators associated to the holomorphic structures of E_1 and E/E_1, respectively, and $\beta \in A^{0,1}(\mathrm{Hom}(\mathbb{E}'_1, \mathbb{E}_1))$ [9, 21]. The Higgs field φ, expressed with respect to the decomposition above, is of the form

$$\varphi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{pmatrix}.$$

Define the gauge transformation $g(z)$, for $z \in \mathbb{C}^*$, as

$$g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \cdot I \end{pmatrix}.$$

Then,

$$g(z) \cdot \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & z\beta \\ 0 & \bar{\partial}'_1 \end{pmatrix}$$

and

$$g(z)^{-1} (z\varphi) g(z) = \begin{pmatrix} z\varphi_{1,1} & z^2 \varphi_{1,2} \\ \varphi_{2,1} & z\varphi_{2,2} \end{pmatrix},$$

so

$$\lim_{z\to 0}(g(z)\cdot\bar{\partial}_E, g(z)^{-1}z\varphi g(z)) = \left(E_1\oplus E/E_1, \begin{pmatrix} 0 & 0\\ \psi_{2,1} & 0\end{pmatrix}\right),$$

which is the Hodge bundle (E_0,φ_0) of the first case. In addition, (E_0,φ_0) is stable. To check it, notice that the only φ_0-invariant proper sub-bundles of the vector bundle E_0 are $E_1\oplus I_1$, E/E_1, or a φ_0-invariant proper vector sub-bundle F of E/E_1. In the first situation, one has that $\mu(E_1\oplus I_1) = \frac{1}{2}(\mu_1+\mu(I_1)) < \mu$, since $\mu(I_1) < 2\mu-\mu_1$, by assumption. For the second possibility, note that $\mu(E/E_1) = \frac{5\mu-\mu_1}{4} = \frac{1}{4}(\mu_2+\mu_3+\mu_4+\mu_5) < \mu$, as $\mu_1 > \mu_2$. For the third, $\mu(F)\leq\mu(E_2/E_1) = \mu_2 < \mu$, due again to the assumption that $\mu_1>\mu_2$. This concludes the first case of the statement.

For the second case, suppose that $\mu(I_1) > 2\mu-\mu_1$, let $\mathbb{E}$, $\mathbb{E}_1$, $\mathbb{I}_1$ and $\mathbb{E}_1''$ be the underlying C^∞ vector bundles corresponding to E, E_1, I_1 y $(E/E_1)/I_1$, which satisfy $\mathbb{E}=\mathbb{E}_1\oplus\mathbb{I}_1\oplus\mathbb{E}_1''$, $\bar{\partial}_E$, $\bar{\partial}_1$, $\bar{\partial}_{I_1}$; and let $\bar{\partial}_1''$ be the associated $\bar{\partial}$-operators. Let β_{ij} be the forms such that

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \beta_{12} & \beta_{13}\\ 0 & \bar{\partial}_{I_1} & \beta_{23}\\ 0 & 0 & \bar{\partial}_1''\end{pmatrix}.$$

Then, the $(3,1)$-component $\varphi_{3,1}$ of φ with respect to this decomposition is 0. Let

$$g(z) = \begin{pmatrix} 1 & 0 & 0\\ 0 & z & 0\\ 0 & 0 & z^2\end{pmatrix}.$$

The following is satisfied:

$$g(z)\cdot\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & z\beta_{1,2} & z^2\beta_{1,3}\\ 0 & \bar{\partial}_{I_1} & z\beta_{23}\\ 0 & 0 & \bar{\partial}_1''\end{pmatrix}$$

and

$$g(z)^{-1}\varphi g(z) = \begin{pmatrix} z\varphi_{1,1} & z^2\varphi_{1,2} & z^3\varphi_{1,3}\\ \varphi_{2,1} & z\varphi_{2,2} & z^2\varphi_{2,3}\\ 0 & \varphi_{3,2} & z\varphi_{3,3}\end{pmatrix},$$

so

$$\lim_{z\to 0}(g(z)\cdot\bar{\partial}_E, g(z)^{-1}z\varphi g(z)) = \left(E_1\oplus I_1\oplus(E/E_1)/I_1, \begin{pmatrix} 0 & 0 & 0\\ \psi_{2,1} & 0 & 0\\ 0 & \psi_{3,2} & 0\end{pmatrix}\right).$$

To check the stability of this limit Hodge bundle, notice again that the only possibilities for a proper φ_0-invariant sub-bundle of E_0 are $(E/E_1)/I_1$, a φ_0-invariant sub-bundle F of $(E/E_1)/I_1$, and $I_1\oplus(E/E_1)/I_1$. For the first, $\mu((E/E_1)/I_1) = \frac{1}{3}(\mu_2+\mu_3+\mu_4+\mu_5-\mu(I_1)) < \frac{1}{5}(\mu_1+\mu_2+\mu_3+\mu_4+\mu_5) = \mu$, since $\mu(I_1) > 2\mu-\mu_1$; for the second, $\mu(F)\leq\mu((E/E_1)/I_1) < \mu$; and for the third,

$$\begin{aligned}\mu(I_1\oplus(E/E_1)/I_1) &= \frac{1}{4}\mu(I_1) + \frac{3}{4}\mu((E/E_1)/I_1)\\ &= \frac{1}{4}\mu(I_1) + \frac{1}{4}(\mu_2+\mu_3+\mu_4+\mu_5-\mu(I_1))\end{aligned}$$

$$= \frac{1}{4}(\mu_2 + \mu_3 + \mu_4 + \mu_5) < \mu.$$

This concludes the second part of the theorem.

In the case that $\mu(I_1) = 2\mu - \mu_1$, then $(E/E_1)/I_1$ is the only proper φ_0-invariant sub-bundle of E_0 and $\mu((E/E_1)/I_1) = \mu = \mu(E_0)$, so in this case, the limit Hodge bundle is strictly semi-stable. To check this, notice that

$$\begin{aligned}\mu((E/E_1)/I_1) &= \frac{\deg((E/E_1)/I_1)}{\mathrm{rk}((E/E_1)/I_1)} = \frac{\deg(E) - \deg(E_1) - \deg(I_1)}{3} \\ &= \frac{5\mu - \mu_1 - \mu(I_1)}{3} = \frac{\mu_2 + \mu_3 + \mu_4 + \mu_5 - \mu(I_1)}{3},\end{aligned}$$

thus, by substituting the given condition $\mu(I_1) = 2\mu - \mu_1 = \frac{-3\mu_1+2\mu_2+2\mu_3+2\mu_4+2\mu_5}{5}$, it follows that

$$\begin{aligned}\mu((E/E_1)/I_1) &= \frac{5\mu - \mu_1 - \frac{-3\mu_1+2\mu_2+2\mu_3+2\mu_4+2\mu_5}{5}}{3} \\ &= \frac{5\mu - 2\mu}{3} = \frac{3\mu}{3} = \mu.\end{aligned}$$

□

Theorem 2. *Let (E, φ) be a rank 5 semi-stable Higgs bundle over X such that the Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ of E satisfies $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$, $\mu = \mu(E)$, let $\varphi_{3,2}$ be the homomorphism $E_2 \to E/E_2 \otimes K$ induced by φ, and let I_2 and N_3 be the vector bundles given in Definition 1. Assume that $\varphi_{3,2}$ has rank 1. Let (E_0, φ_0) be the limit $\lim_{z\to 0}(E, z\cdot\varphi)$. Then, (E_0, φ_0) has the following explicit form:*

(1) If $\mu(I_2) < 3\mu - \mu_1 - \mu_2$, $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g-2)$, and $\mu(N_3) < \mu$, then

$$(E_0, \varphi_0) = \left(E_2 \oplus E/E_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_2 \to E/E_2 \otimes K$ is the nonzero homomorphism induced by φ.

(2) If $\mu(I_2) > 3\mu - \mu_1 - \mu_2$, $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g-2)$, and $\mu(N_3) < \mu$, then

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2 \oplus (E/E_2)/I_2, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_2 \to I_2 \otimes K$ and $\psi_{3,2} : I_2 \to (E/E_2)/I_2 \otimes K$ are the nonzero homomorphisms induced by φ.

(3) If $\mu(I_2) = 3\mu - \mu_1 - \mu_2$, $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g-2)$, and $\mu(N_3) < \mu$, then

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right) \oplus ((E/E_2)/I_2, 0),$$

where $\psi_{2,1} : E_2 \to I_2 \otimes K$ is the nonzero homomorphism induced by φ.

(4) *If* $\mu(I_2) < 3\mu - \mu_1 - \mu_2$ *and* $\mu(N_3) > \mu$, *then*

$$(E_0, \varphi_0) = \left(N_3 \oplus E_2/N_3 \oplus E/E_2, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right),$$

where $\psi_{2,1} : N_3 \to E_2/N_3 \otimes K$ *and* $\psi_{3,2} : E_2/N_3 \to E/E_2 \otimes K$ *are the nonzero homomorphisms induced by* φ.

(5) *If* $\mu(I_2) > 3\mu - \mu_1 - \mu_2$ *and* $\mu(N_3) > \mu$, *then*

$$(E_0, \varphi_0) = \left(N_3 \oplus E_2/N_3 \oplus I_2 \oplus (E/E_2)/I_2, \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 & 0 \\ 0 & \psi_{3,2} & 0 & 0 \\ 0 & 0 & \psi_{4,3} & 0 \end{pmatrix} \right),$$

where $\psi_{2,1} : N_3 \to E_2/N_3 \otimes K$, $\psi_{3,2} : E_2/N_3 \to I_2 \otimes K$ *and* $\psi_{4,3} : I_2 \to (E/E_2)/I_2 \otimes K$ *are the nonzero homomorphisms induced by* φ.

(6) *If* $\mu(I_2) = 3\mu - \mu_1 - \mu_2$ *and* $\mu(N_3) > \mu$, *then*

$$(E_0, \varphi_0) = \left(N_3 \oplus E_2/N_3 \oplus I_2, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right) \oplus ((E/E_2)/I_2, 0),$$

where $\psi_{2,1} : N_3 \to E_2/N_3 \otimes K$ *and* $\psi_{3,2} : E_2/N_3 \to I_2 \otimes K$ *are the nonzero homomorphisms induced by* φ.

(7) *If* $\mu(I_2) < 3\mu - \mu_1 - \mu_2$, $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g - 2)$, *and* $\mu(N_3) = \mu$, *then*

$$(E_0, \varphi_0) = (N_3, 0) \oplus \left(E_2/N_3 \oplus E/E_2, \begin{pmatrix} 0 & 0 \\ \psi_{3,2} & 0 \end{pmatrix} \right),$$

where $\psi_{3,2} : E_2/N_3 \to E/E_2 \otimes K$ *is the nonzero homomorphism induced by* φ.

(8) *If* $\mu(I_2) > 3\mu - \mu_1 - \mu_2$, $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g - 2)$, *and* $\mu(N_3) = \mu$, *then*

$$(E_0, \varphi_0) = (N_3, 0) \oplus \left(E_2/N_3 \oplus I_2 \oplus (E/E_2)/I_2, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{3,2} & 0 & 0 \\ 0 & \psi_{4,3} & 0 \end{pmatrix} \right),$$

where $\psi_{3,2} : E_2/N_3 \to I_2 \otimes K$ *and* $\psi_{4,3} : I_2 \to (E/E_2)/I_2 \otimes K$ *are the nonzero homomorphisms induced by* φ.

(9) *If* $\mu(I_2) = 3\mu - \mu_1 - \mu_2$, $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g - 2)$, *and* $\mu(N_3) = \mu$, *then*

$$(E_0, \varphi_0) = (N_3, 0) \oplus \left(E_2/N_3 \oplus I_2, \begin{pmatrix} 0 & 0 \\ \psi_{3,2} & 0 \end{pmatrix} \right) \oplus ((E/E_2)/I_2, 0),$$

where $\psi_{3,2} : E_2/N_3 \to I_2 \otimes K$ *is the nonzero homomorphism induced by* φ.

Proof. As in the previous result, the theorem is proved by constructing appropriate gauge transformations for each case and showing how they lead to the stated limit bundles. Suppose first that $\mu(I_2) < \mu_3$ and $\mu(N_3) < \mu$. Let $\mathbb{E}$, $\mathbb{E}_2$, and $\mathbb{E}_2'$ be the underlying C^∞ vector bundles of E, E_2, and E/E_2, respectively. Then, $\mathbb{E} \cong \mathbb{E}_2 \oplus \mathbb{E}_2'$. The holomorphic structure of the vector bundle E is given by the operator $\bar{\partial}_E$ defined by

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_2 & \beta \\ 0 & \bar{\partial}_2' \end{pmatrix},$$

where $\bar{\partial}_2$ and $\bar{\partial}_2'$ are the $\bar{\partial}$-operators defining the holomorphic structures of E_2 and E/E_2, respectively, and $\beta \in A^{0,1}(\mathrm{Hom}(E_2', E_2))$. The Higgs field φ is then of the form

$$\varphi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{pmatrix}$$

expressed concerning the above decomposition. Similar computations as those made in the case (1) of Theorem 1 show that

$$\lim_{z\to 0}(g(z) \cdot \bar{\partial}_E, g(z)^{-1} z\varphi g(z)) = \left(E_2 \oplus E/E_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \cdot I \end{pmatrix}$ for $z \in \mathbb{C}^*$; thus, $g(z) \cdot \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_2 & z\beta \\ 0 & \bar{\partial}_2' \end{pmatrix}$ and $g(z)^{-1}(z\varphi)g(z) = \begin{pmatrix} z\varphi_{1,1} & z^2\varphi_{1,2} \\ \varphi_{2,1} & z\varphi_{2,2} \end{pmatrix}$, and $\psi_{2,1} = \varphi_{2,1}$ is the nonzero homomorphism induced by φ.

Notice now that the only proper φ_0-invariant sub-bundles of E_0 are N_3, E/E_2, any proper φ_0-invariant subbunsle of E/E_2, and $E_2 \oplus I_2$. First, $\mu(N_3) < \mu$ by hypothesis. Second, $\mu(E/E_2) = \frac{1}{3}(\mu_3 + \mu_4 + \mu_5) < \mu$, since $\mu_2 > \mu_3$. Third, $\mu(F) \le \mu(E/E_2) < \mu$. Finally,

$$\mu(E_2 \oplus I_2) = \frac{2}{3}\mu(E_2) + \frac{1}{3}\mu(I_2) < \frac{2}{3}\frac{\mu_1 + \mu_2}{2} + \frac{3\mu - \mu_1 - \mu_2}{3} = \mu,$$

since $\mu(I_2) < 3\mu - \mu_1 - \mu_2$ by hypothesis. Then, the slope of every proper φ_0-invariant sub-bundle of E_0 is less than $\mu = \mu(E_0)$, so the Hodge limit bundle is stable in this case.

For the case (2), suppose that $\mu(I_2) > \mu_3$ and $\mu(N_3) < \mu$, and let $\mathbb{E}$, $\mathbb{E}_2$, $\mathbb{I}_2$, and $\mathbb{E}_2''$ be the C^∞ vector bundles of E, E_2, I_2, and $(E/E_2)/I_2$, respectively, so $\mathbb{E} \cong \mathbb{E}_2 \oplus \mathbb{I}_2 \oplus \mathbb{E}_2''$. The holomorphic structure of E is given by:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_2 & \beta_{1,2} & \beta_{1,3} \\ 0 & \bar{\partial}_{I_2} & \beta_{2,3} \\ 0 & 0 & \bar{\partial}_2'' \end{pmatrix},$$

thus, by using the following induced decomposition of φ

$$\varphi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} \end{pmatrix}$$

and the gauge transformation

$$g(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix},$$

the stated form of (E_0, φ_0) holds by taking the limit as $z \to 0$ in the expression of $\left(g(z)\cdot\bar{\partial}_E, g(z)^{-1}(z\varphi)g(z)\right)$, as in the case (2) of Theorem 1.

The stability of this Hodge limit bundle follows since the proper φ_0-invariant sub-bundles are $(E/E_2)/I_2$, $I_2 \oplus (E/E_2)/I_2$, and $F \oplus I_2 \oplus (E/E_2)/I_2$ for a line sub-bundle F of E_2. The slopes of these bundles are the following:

$$\begin{aligned}
\mu((E/E_2)/I_2) &= \frac{1}{2}(5\mu - \mu_1 - \mu_2 - \mu(I_2)) < \frac{1}{2}\left(\mu_3 + \mu_4 + \mu_5 - (3\mu - \mu_1 - \mu_2)\right)\\
&= \frac{5\mu - 3\mu}{2} = \mu,\\
\mu(I_2 \oplus (E/E_2)/I_2) &= \frac{1}{3}(\mu(I_2) + 2\mu((E/E_2)/I_2)) < \frac{1}{3}(\mu_3 + \mu_4 + \mu_5) < \mu,\\
\mu(F \oplus I_2 \oplus ((E/E_2)/I_2)) &= \frac{\mu(F) + 3\cdot\frac{\mu_3+\mu_4+\mu_5}{3}}{4}\\
&< \frac{\frac{\mu_1+\mu_2}{2} + 3\cdot\frac{\mu_3+\mu_4+\mu_5}{3}}{4}\\
&\leq \frac{\mu_2 + \mu_3 + \mu_4 + \mu_5}{4} < \mu.
\end{aligned}$$

To check case (3), suppose $\mu(I_2) = \frac{1}{3}(\mu_3 + \mu_4 + \mu_5)$ and $\mu(N_3) < \mu$. This case is similar to case (2), but the Hodge limit bundle

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right) \oplus ((E/E_2)/I_2, 0)$$

is now strictly semi-stable. Indeed, notice that,

$$\begin{aligned}
\mu((E/E_2)/I_2) &= \frac{1}{2}(5\mu - \mu_1 - \mu_2 - \mu(I_2))\\
&= \frac{1}{2}(5\mu - \mu_1 - \mu_2 - (3\mu - \mu_1 - \mu_2))\\
&= \frac{1}{2}\cdot 2\mu = \mu,
\end{aligned}$$

where it has been used that $\mu(I_2) = 3\mu - \mu_1 - \mu_2$. From this, it follows that $\mu((E/E_2)/I_2) = \mu = \mu(E_0)$, as announced. Then, $(E/E_2)/I_2$ is a φ_0-invariant destabilizing sub-bundle of E_0 and the expression of the poly-stable representative of (E_0, φ_0) is as stated.

The remaining cases can be proved similarly to the previous cases, adjusting the gauge transformations as needed based on the slopes of I_2 and N_3, and choosing the gauge transformation to isolate the components that will remain nonzero in the limit. In cases (4) and (5), the Hodge limit bundles are stable (it can be checked by using similar arguments as the above cases). However, in cases (7) to (9), the limit bundles are strictly semi-stable, since N_3 is a φ_0-invariant sub-bundle of E_0 with $\mu(N_3) = \mu = \mu(E_0)$. In case (6), the φ_0-invariant sub-bundle of E_0 with slope μ is I_2, which is easily checked as done in case (3). □

Theorem 3. *Let (E, φ) be a rank 5 semi-stable Higgs bundle over X such that the Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ of E satisfies $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$, $\mu = \mu(E)$, let $\varphi_{3,2}$ be the homomorphism $E_2 \to E/E_2 \otimes K$ induced by φ, and let I_2 be the vector bundle given in Definition 1. Assume that $\varphi_{3,2}$ has rank 2. Let (E_0, φ_0) be the limit $\lim_{z\to 0}(E, z\cdot\varphi)$. Then (E_0, φ_0) has the following explicit form:*

(1) If $\mu(I_2) < 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$, then

$$(E_0, \varphi_0) = \left(E_2 \oplus E/E_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_2 \to E/E_2 \otimes K$ is the nonzero homomorphism induced by φ.

(2) If $\mu(I_2) > 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$, then

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2 \oplus (E/E_2)/I_2, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_2 \to I_2 \otimes K$ and $\psi_{3,2} : I_2 \to (E/E_2)/I_2 \otimes K$ are the nonzero homomorphisms induced by φ.

(3) If $\mu(I_2) = 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$, then

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right) \oplus ((E/E_2)/I_2, 0),$$

where $\psi_{2,1} : E_2 \to I_2 \otimes K$ is the nonzero homomorphism induced by φ.

Proof. Suppose, for the first case, that $\mu(I_2) < \frac{1}{3}(\mu_3 + \mu_4 + \mu_5)$, and let, as in the preceding theorems, $\mathbb{E}$, $\mathbb{E}_2$, and $\mathbb{E}'_2$ be the underlying C^∞ vector bundles of E, E_2, and E/E_2. Then, the holomorphic structure of E is given by the $\bar{\partial}_E$-operator

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_2 & \beta \\ 0 & \bar{\partial}'_2 \end{pmatrix},$$

$\bar{\partial}_2$ and $\bar{\partial}'_2$ being the $\bar{\partial}$-operators of the holomorphic structures of E_2 and E/E_2, respectively, and $\beta \in A^{0,1}(\mathrm{Hom}(E'_2, E_2))$. Reasoning as in case (1) of Theorem 1 or Theorem 2, but with E_2 instead of E_1, it follows that

$$\lim_{z\to 0}(g(z) \cdot \bar{\partial}E, g(z)^{-1}z\varphi g(z)) = \left(E_2 \oplus E/E_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} = \varphi_{2,1}$ is the nonzero homomorphism induced by φ and $g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \cdot I \end{pmatrix}$. Then, the Hodge limit bundle is of the announced form. Moreover, (E_0, φ_0) is stable, as it is easily checked by reasoning as in Theorem 2 making the appropriate formal adaptations. Specifically, $N_2 = 0$ since the rank of $\varphi_{3,2}$ is 2 in this case; $\mu(E/E_2) = \frac{\mu_3+\mu_4+\mu_5}{3} < \mu$ since $\mu_2 > \mu_3$; and

$$\mu(E_2 \oplus I_2) = \frac{2\mu(E_2) + 2\mu(I_2)}{4} < \frac{\frac{\mu_1+\mu_2}{2} + 2\mu - \frac{\mu_1+\mu_2}{2}}{2} = \mu,$$

where it has been used that $\mu(I_2) < 2\mu + \frac{\mu_1+\mu_2}{2}$. This completes the analysis of the slopes of the φ_0-invariant sub-bundles of E_0.

For case (2) ($\mu(I_2) > \frac{1}{3}(\mu_3 + \mu_4 + \mu_5)$), the holomorphic structure of E is given by

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_2 & \beta_{1,2} & \beta_{1,3} \\ 0 & \bar{\partial}_{I_2} & \beta_{2,3} \\ 0 & 0 & \bar{\partial}''_2 \end{pmatrix},$$

computed through the decomposition $\mathbb{E} \cong \mathbb{E}_2 \oplus \mathbb{I}_2 \oplus \mathbb{E}_2''$, where $\mathbb{E}$, $\mathbb{E}_2$, $\mathbb{I}_2$, and $\mathbb{E}_2''$ are the C^∞ vector bundles of E, E_2, I_2, and $(E/E_2)/I_2$, respectively. This case is similar to the case (2) of Theorem 2, so the same program leads to the stated form of the Hodge limit bundle (E_0, φ_0) follows. By reasoning again as in Theorem 2, it is easily checked that (E_0, φ_0) is stable. The only computation that requires some attention is the slope of the φ_0-invariant sub-bundle $(E/E_2)/I_2$:

$$\mu((E/E_2)/I_2) = 5\mu - \mu_1 - \mu_2 - 2\mu(I_2) < 5\mu - \mu_1 - \mu_2 - 4\mu + 2\frac{\mu_1 + \mu_2}{2} = \mu,$$

where it has been used that $\mu(I_2) > 2\mu - \frac{\mu_1+\mu_2}{2}$.

For case (3), the same construction of case (2) works but, in this case, the Hodge limit bundle

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right) \oplus ((E/E_2)/I_2, 0)$$

is strictly semi-stable, since $(E/E_2)/I_2$ is a φ_0-invariant sub-bundle satisfying that $\mu((E/E_2)/I_2) = \mu$, as it is easily checked:

$$\mu((E/E_2)/I_2) = 5\mu - \mu_1 - \mu_2 - 2\mu(I_2) < 5\mu - \mu_1 - \mu_2 - 4\mu + 2 \cdot \frac{\mu_1 + \mu_2}{2} = \mu.$$

□

Remark 5. *Suppose that $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$ and that the rank of $\varphi_{3,2}$ is 2 (that is, the conditions of Theorem 3 are satisfied). Notice that in this case, the only possibility for N_3 is $N_3 = 0$.*

Theorem 4. *Let (E, φ) be a rank 5 semi-stable Higgs bundle over X such that the Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ of E satisfies $\mu_1 \geq \mu_2 = \mu_3 = \mu_4 > \mu_5$, $\mu = \mu(E)$, let $\varphi_{4,3}$ be the homomorphism $E_3 \to E/E_3 \otimes K$ induced by φ, and let N_1 be the rank 2 vector bundle given in Definition 1. Let (E_0, φ_0) be the limit $\lim_{z\to 0}(E, z \cdot \varphi)$. Then, (E_0, φ_0) has the following explicit form:*

(1) If $\mu \geq \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5 - (2g - 2))$ and $\mu(N_1) < \mu$, then

$$(E_0, \varphi_0) = \left(E_4 \oplus E/E_4, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_4 \to E/E_4 \otimes K$ is the nonzero homomorphism induced by φ.

(2) If $\mu \geq \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 + \mu_4 - \mu_5 - (2g - 2))$ and $\mu(N_1) = \mu$, then

$$(E_0, \varphi_0) = (N_1, 0) \oplus \left(E_4/N_1 \oplus E/E_4, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_4/N_1 \to E/E_4 \otimes K$ is the nonzero homomorphism induced by φ.

(3) If $\mu(N_1) > \mu$, then

$$(E_0, \varphi_0) = \left(N_1 \oplus E_4/N_1 \oplus E/E_4, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : N_1 \to E_4/N_1 \otimes K$ and $\psi_{3,2} : E_4/N_1 \to E/E_4 \otimes K$ are the nonzero homomorphisms induced by φ.

Proof. First, suppose that $\mu(N_1) < \mu$, under the conditions of the statement. As in the case (1) of Theorem 1, the holomorphic structure of E is given by the operator $\bar{\partial}_E$ defined by

$$\partial_E = \begin{pmatrix} \partial_4 & \beta \\ 0 & \partial_4' \end{pmatrix},$$

where $\mathbb{E}$, $\mathbb{E}_4$, and $\mathbb{E}_4'$ are the differentiable bundles of E, E_4, and E/E_4, respectively, and $\bar{\partial}_4$ and $\bar{\partial}_4'$ are the $\bar{\partial}$-operators associated to the holomorphic structures of E_4 and E/E_4 and $\beta \in A^{0,1}(\mathrm{Hom}(\mathbb{E}_4', \mathbb{E}_4))$. Similar computations as those of the case (1) of Theorem 1 allow us to compute

$$\lim_{z\to 0}(g(z)\cdot\partial_E, g(z)^{-1}z\varphi g(z)) = \left(E_4 \oplus E/E_4, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

which is the Hodge bundle (E_0, φ_0) of the first case, being $g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z\cdot I \end{pmatrix}$. Notice also that the only φ_0-invariant proper sub-bundles of E_0 are E/E_4 and $F \oplus (E/E_4)$ for any φ_0-invariant proper sub-bundle F of E_4. For E/E_4, it is clear that $\mu(E/E_4) = \mu_5 < \mu$. If $F \subset E_4$ is φ_0-invariant of rank r, then

$$\mu(F \oplus (E/E_4)) = \frac{r\mu(F) + \mu_5}{r+1} < \frac{r\mu + \mu_5}{r+1} < \mu,$$

since $\mu_5 < \mu$, as $\mu_4 > \mu_5$, and $\mu(F) \le \mu(E_4) = \frac{1}{4}(\mu_1 + \mu_2 + \mu_3 + \mu_4) < \mu$, since $\mu_5 < \mu$, by semi-stability of (E_4, φ_4). This proves that the Hodge limit bundle (E_0, φ_0) is stable.

For the second case, suppose that, under the assumptions of the statement, $\mu(N_1) = \mu$, and let $\mathbb{E}$, $\mathbb{N}_1$, $\mathbb{E}_4/\mathbb{N}_1$, and $\mathbb{E}_4''$ be the differentiable bundles corresponding to E, N_1, E_4/N_1; let E/E_4, $\bar{\partial}_E$, $\bar{\partial}_{N_1}$, $\bar{\partial}_{E_4/N_1}$, and $\bar{\partial}_4''$ be the associated $\bar{\partial}$-operators; and let β_{ij} be the forms such that

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{N_1} & \beta_{12} & \beta_{13} \\ 0 & \bar{\partial}_{E_4/N_1} & \beta_{23} \\ 0 & 0 & \bar{\partial}_4'' \end{pmatrix}.$$

Then, the (3, 1)-component $\varphi_{3,1}$ of φ with respect to the decomposition above is 0. Let

$$g(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}.$$

Then, it is satisfied that

$$\lim_{z\to 0}(g(z)\cdot\bar{\partial}_E, g(z)^{-1}z\varphi g(z)) = \left(N_1 \oplus E_4/N_1 \oplus E/E_4, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix}\right),$$

and the announced Hodge limit bundle (E_0, φ_0) is obtained (this case is similar to the case (7) of Theorem 2 with N_1 instead of N_3 and E_4 instead of E_2). Moreover, it is strictly semi-stable, since N_1 is a φ_0-invariant sub-bundle of E_0 with $\mu(N_1) = \mu = \mu(E_0)$. The S-equivalence class of this Hodge limit bundle can be represented by the poly-stable bundle

$$(E_0, \varphi_0) = (N_1, 0) \oplus \left(E_4/N_1 \oplus E/E_4, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

as stated.

The third case is analogous to the second one, but noticing that N_1 is not a φ_0-invariant sub-bundle here, which makes the Hodge limit bundle stable as a Higgs bundle. □

Theorem 5. *Let (E,φ) be a rank 5 semi-stable Higgs bundle over X such that the Harder-Narasimhan type $(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$ of E satisfies $\mu_1 \geq \mu_2 \geq \mu_3 > \mu_4 \geq \mu_5$, $\mu = \mu(E)$, let $\varphi_{4,3}$ be the homomorphism $E_3 \to E/E_3 \otimes K$ induced by φ, and let I_3 and N_2 be the vector bundles given in Definition 1. Assume that $\varphi_{4,3}$ has rank 1. Let (E_0,φ_0) be the limit $\lim_{z\to 0}(E, z\cdot\varphi)$. Then, (E_0,φ_0) has the following explicit form:*

(1) If $\mu(I_3) < \mu_4+\mu_5-\mu$ and $\mu(N_2) < \mu$, then

$$(E_0,\varphi_0) = \left(E_3 \oplus E/E_3, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_3 \to E/E_3 \otimes K$ is the nonzero homomorphism induced by φ.

(2) If $\mu(I_3) > \mu_4+\mu_5-\mu$ and $\mu(N_2) < \mu$, then

$$(E_0,\varphi_0) = \left(E_3 \oplus I_3 \oplus (E/E_3)/I_3, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_3 \to I_3 \otimes K$ and $\psi_{3,2} : I_3 \to (E/E_3)/I_3 \otimes K$ are the nonzero homomorphisms induced by φ.

(3) If $\mu(I_3) = \mu_4+\mu_5-\mu$ and $\mu(N_2) < \mu$, then

$$(E_0,\varphi_0) = \left(E_3 \oplus I_3, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right) \oplus ((E/E_3)/I_3, 0),$$

where $\psi_{2,1} : E_3 \to I_3 \otimes K$ is the nonzero homomorphism induced by φ.

(4) If $\mu(I_3) < \mu_4+\mu_5-\mu$, $\mu \geq \frac{1}{2}(\mu_1+\mu_2+\mu_3-\mu_4-\mu_5-(2g-2))$, and $\mu(N_2) = \mu$, then

$$(E_0,\varphi_0) = (N_2, 0) \oplus \left(E_3/N_2 \oplus E/E_3, \begin{pmatrix} 0 & 0 \\ \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{3,2} : E_3/N_2 \to E/E_3 \otimes K$ is the nonzero homomorphism induced by φ.

(5) If $\mu(I_3) > \mu_4+\mu_5-\mu$, $\mu \geq \frac{1}{2}(\mu_1+\mu_2+\mu_3-\mu_4-\mu_5-(2g-2))$, and $\mu(N_2) = \mu$, then

$$(E_0,\varphi_0) = (N_2, 0) \oplus \left(E_3/N_2 \oplus I_3 \oplus (E/E_3)/I_3, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{3,2} & 0 & 0 \\ 0 & \psi_{4,3} & 0 \end{pmatrix}\right),$$

where $\psi_{3,2} : E_3/N_2 \to I_3 \otimes K$ and $\psi_{4,3} : I_3 \to (E/E_3)/I_3 \otimes K$ are the nonzero homomorphisms induced by φ.

(6) If $\mu(I_3) = \mu_4 + \mu_5 - \mu$, $\mu \geq \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 - \mu_4 - \mu_5 - (2g-2))$, *and* $\mu(N_2) = \mu$, *then*

$$(E_0, \varphi_0) = (N_2, 0) \oplus \left(E_3/N_2 \oplus I_3, \begin{pmatrix} 0 & 0 \\ \psi_{3,2} & 0 \end{pmatrix} \right) \oplus ((E/E_3)/I_3, 0),$$

where $\psi_{3,2} : E_3/N_2 \to I_3 \otimes K$ *is the nonzero homomorphism induced by* φ.

(7) If $\mu(I_3) < \mu_4 + \mu_5 - \mu$ *and* $\mu(N_2) > \mu$, *then*

$$(E_0, \varphi_0) = \left(N_2 \oplus E_3/N_2 \oplus E/E_3, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right),$$

where $\psi_{2,1} : N_2 \to E_3/N_2 \otimes K$ *and* $\psi_{3,2} : E_3/N_2 \to E/E_3 \otimes K$ *are the nonzero homomorphisms induced by* φ.

(8) If $\mu(I_3) > \mu_4 + \mu_5 - \mu$ *and* $\mu(N_2) > \mu$, *then*

$$(E_0, \varphi_0) = \left(N_2 \oplus E_3/N_2 \oplus I_3 \oplus (E/E_3)/I_3, \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 & 0 \\ 0 & \psi_{3,2} & 0 & 0 \\ 0 & 0 & \psi_{4,3} & 0 \end{pmatrix} \right),$$

where $\psi_{2,1} : N_2 \to E_3/N_2 \otimes K$, $\psi_{3,2} : E_3/N_2 \to I_3 \otimes K$ *and* $\psi_{4,3} : I_3 \to (E/E_3)/I_3 \otimes K$ *are the nonzero homomorphisms induced by* φ.

(9) If $\mu(I_3) = \mu_4 + \mu_5 - \mu$ *and* $\mu(N_2) > \mu$, *then*

$$(E_0, \varphi_0) = \left(N_2 \oplus E_3/N_2 \oplus I_3, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right) \oplus ((E/E_3)/I_3, 0),$$

where $\psi_{2,1} : N_2 \to E_3/N_2 \otimes K$ *and* $\psi_{3,2} : E_3/N_2 \to I_3 \otimes K$ *are the nonzero homomorphisms induced by* φ.

Proof. For the first case, assume $\mu(I_3) < \frac{1}{2}(\mu_4 + \mu_5)$ and $\mu(N_2) < \mu$. Let $\mathbb{E} = \mathbb{E}_3 \oplus \mathbb{E}_3''$, where $\mathbb{E}_3'' = \mathbb{E}/\mathbb{E}_3$, and the gauge transformation

$$g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

for $z \in \mathbb{C}^*$. Then,

$$g(z) \cdot \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_3 & z\beta \\ 0 & \bar{\partial}_3'' \end{pmatrix} \quad \text{and} \quad g(z)^{-1}(z\varphi)g(z) = \begin{pmatrix} z\varphi_{1,1} & z^2\varphi_{1,2} \\ \varphi_{2,1} & z\varphi_{2,2} \end{pmatrix}.$$

Taking the limit as $z \to 0$, the announced form for the Hodge limit bundle is obtained as in the case (1) of Theorem 1 with E_3 instead of E_1.

For the second case, assume $\mu(I_3) > \frac{1}{2}(\mu_4 + \mu_5)$ and $\mu(N_2) < \mu$, let $\mathbb{E} = \mathbb{E}_3 \oplus \mathbb{I}_3 \oplus \mathbb{E}_3''$, and define

$$g(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}.$$

Applying this transformation and taking the limit provides the announced form for the Hodge limit bundle as in the case (2) of Theorem 1 or Theorem 3, but considering I_3 instead of I_1 or I_2.

For the third case, suppose that $\mu(I_3) = \frac{1}{2}(\mu_4 + \mu_5)$ and $\mu(N_2) < \mu$. Then, the same gauge transformations as in the preceding case work and the limit bundle is the one stated.

For the fourth case, assume $\mu(I_3) < \frac{1}{2}(\mu_4 + \mu_5)$ and $\mu(N_2) = \mu$, and let $\mathbb{E} = \mathbb{N}_2 \oplus (\mathbb{E}_3/\mathbb{N}_2) \oplus \mathbb{E}_3''$. The stated form for the Hodge limit bundle arises from applying the gauge transformation $g(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2 \end{pmatrix}$, with respect to the decomposition above, and taking the limit as $z \to 0$.

For the fifth case, it is supposed that $\mu(I_3) > \frac{1}{2}(\mu_4+\mu_5)$ and $\mu(N_2) = \mu$. Take $\mathbb{E} = \mathbb{N}_2 \oplus (\mathbb{E}_3/\mathbb{N}_2) \oplus \mathbb{I}_3 \oplus \mathbb{E}_3''$, and define

$$g(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z^2 & 0 \\ 0 & 0 & 0 & z^3 \end{pmatrix}.$$

The limit bundle of the statement arises from this by computing the limit as done in the preceding theorems.

For the sixth case, the same gauge transformation as in the case above works. Similarly, the cases (7) to (9) follow by taking suitable gauge transformations.

By reasoning as in Theorems 1 and 2 with the appropriate formal adaptations, it follows that in the cases (1), (2), (7), and (8), the Hodge limit bundle is stable, and in the remaining cases it is strictly semi-stable, so the graded object gives a strictly poly-stable representative of those Hodge limit bundles. The first two cases will be computed (the remaining cases are similar). Specifically, for the first case, notice that $\mu(E/E_3) = \frac{\mu_4+\mu_5}{2} < \mu$, as $\mu_3 > \mu_4$, and that

$$\begin{aligned} \mu(I_3 \oplus E_3) = \frac{3\mu(E_3) + \mu(I_3)}{4} &< \frac{\mu_1 + \mu_2 + \mu_3 - \mu + \mu_4 + \mu_5}{4} \\ &= \frac{5\mu - \mu}{4} = \mu, \end{aligned}$$

since $\mu(I_3) < \mu_4 + \mu_5 - \mu$ in this case. For the second case, the only φ_0-invariant vector sub-bundle that requires some computation is

$$\mu((E/E_3)/I_3) = \mu_4 + \mu_5 - \mu(I_3) < \mu_4 + \mu_5 - \mu_4 - \mu_5 + \mu = \mu,$$

since, in this case, $\mu(I_3) > \mu_4 + \mu_5 - \mu$ by hypothesis. In cases (4) to (6), $\mu(N_2) = \mu$ by hypothesis, so N_4 is a φ_0-invariant sub-bundle with the same slope as E_0, and in cases (3), (6), and (9),

$$\mu((E/E_3)/I_3) = \mu_4 + \mu_5 - \mu(I_3) = \mu_4 + \mu_5 - \mu_4 - \mu_5 + \mu = \mu,$$

because, in these cases, $\mu(I_3) = \mu_4 + \mu_5 - \mu$. □

Theorem 6. *Let (E, φ) be a rank 5 semi-stable Higgs bundle over Xsuch that the Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ of E satisfies $\mu_1 \geq \mu_2 = \mu_3 > \mu_4 \geq \mu_5$, $\mu = \mu(E)$, let $\varphi_{4,3}$ be the homomorphism $E_3 \to E/E_3 \otimes K$ induced by φ, and let N_2 be the vector bundle given in Definition 1. Assume that $\varphi_{4,3}$ has rank 2. Let (E_0, φ_0) be the limit $\lim_{z\to 0}(E, z \cdot \varphi)$. Then, (E_0, φ_0) has the following explicit form:*

(1) If $\mu \geq \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 - \mu_4 - \mu_5 - (2g-2))$ *and* $\mu(N_2) < \mu$*, then*

$$(E_0, \varphi_0) = \left(E_3 \oplus E/E_3, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : E_3 \to E/E_3 \otimes K$ *is the nonzero homomorphism induced by* φ*.*

(2) If $\mu \geq \frac{1}{2}(\mu_1 + \mu_2 + \mu_3 - \mu_4 - \mu_5 - (2g-2))$ *and* $\mu(N_2) = \mu$*, then*

$$(E_0, \varphi_0) = (N_2, 0) \oplus \left(E_3/N_2 \oplus E/E_3, \begin{pmatrix} 0 & 0 \\ \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{3,2} : E_3/N_2 \to E/E_3 \otimes K$ *is the nonzero homomorphism induced by* φ*.*

(3) If $\mu(N_2) > \mu$*, then*

$$(E_0, \varphi_0) = \left(N_2 \oplus E_3/N_2 \oplus E/E_3, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix}\right),$$

where $\psi_{2,1} : N_2 \to E_3/N_2 \otimes K$ *and* $\psi_{3,2} : E_3/N_2 \to E/E_3 \otimes K$ *are the nonzero homomorphisms induced by* φ*.*

Proof. Let $\mathbb{E}$, $\mathbb{E}_3$, $\mathbb{N}_2$, $\mathbb{I}_3$, and $\mathbb{E}_3''$ be the underlying C^∞ vector bundles corresponding to E, E_3, N_2, I_3, and $(E/E_3)/I_3$, respectively. Let $\bar{\partial}_E$, $\bar{\partial}_3$, $\bar{\partial}N_2$, $\bar{\partial}_{I_3}$, and $\bar{\partial}_3''$ be their associated $\bar{\partial}$-operators.

Suppose, for the first case, that $\mu(I_3) < \frac{1}{2}(\mu_4 + \mu_5)$ and $\mu(N_2) < \mu$. In this case, $\mathbb{E} = \mathbb{E}_3 \oplus \mathbb{E}/\mathbb{E}_3$. The holomorphic structure of E is given by

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_3 & \beta \\ 0 & \bar{\partial}_3' \end{pmatrix},$$

where $\beta \in A^{0,1}(\mathrm{Hom}(E/E_3, E_3))$. Being the Higgs field φ of the form $\varphi = \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{pmatrix}$ with respect to the decomposition above and $g(z)$, the gauge transformation $g(z) = \begin{pmatrix} 1 & 0 \\ 0 & z \cdot I \end{pmatrix}$ for $z \in \mathbb{C}^*$, it follows that

$$g(z) \cdot \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_3 & z\beta \\ 0 & \bar{\partial}_3' \end{pmatrix} \quad \text{and} \quad g(z)^{-1}(z\varphi)g(z) = \begin{pmatrix} z\varphi_{1,1} & z^2\varphi_{1,2} \\ \varphi_{2,1} & z\varphi_{2,2} \end{pmatrix},$$

thus, by taking the limit as $z \to 0$, it follows that

$$\lim_{z \to 0}(g(z) \cdot \bar{\partial}E, g(z)^{-1}z\varphi g(z)) = \left(E_3 \oplus E/E_3, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix}\right),$$

similarly to the case (1) of Theorem 1 or Theorem 2. Then, the stated Hodge limit bundle is set. This limit bundle is stable because $\mu(N_2) < \mu$ by hypothesis and $\mu(E/E_3) = \frac{1}{3}(\mu_3 + \mu_4 + \mu_5) < \mu$ due to the assumption $\mu_1 \geq \mu_2 = \mu_3 > \mu_4 \geq \mu_5$.

For cases (2) to (9), the proofs follow a similar pattern, using suitable gauge transformations based on the decomposition of E and the conditions on $\mu(I_3)$ and $\mu(N_2)$ in each case.

For the cases where $\mu(I_3) = \frac{1}{2}(\mu - \mu_1 - \mu_2 - \mu_3)$ or $\mu(N_2) = \mu$ (that is, cases (3) to (6) and (9)), the resulting Hodge limit bundles are strictly semi-stable, having $(E/E_3)/I_3$ (for the cases (3), (6), and (9)) or N_2 (for the cases (4) to (6)) as destabilizing sub-bundles, which is easily checked as in Theorem 2. Indeed, notice that in cases (4) to (6), $\mu(N_2) = \mu = \mu(E_0)$, so it is the destabilizing vector sub-bundle, and in the cases (3), (6), and (9), since

$$\mu(E/E_2) = \frac{2\mu(I_3) + \mu((E/E_2)/I_3))}{3},$$

it easily follows that $\mu((E/E_2)/I_3) = 3\frac{\mu_3+\mu_4+\mu_5}{3} - 2\mu(I_3) = \mu$, under the assumption that $\mu(I_3) = \frac{1}{2}(\mu - \mu_3 - \mu_4 - \mu_5)$. □

Remark 6. *Suppose that $\mu_1 \geq \mu_2 \geq \mu_3 > \mu_4 \geq \mu_5$ and that the rank of $\varphi_{4,3}$ is 2 (that is, the conditions of Theorem 6 are satisfied). Then, $I_3 = E/E_3$, thus, $\mu(I_3) = \frac{\mu_4+\mu_5}{2}$ in this case, and it does not play a role in the description of the Hodge limit bundles given in Theorem 6.*

Remark 7. *It is worth noting that the bounds used for the intermediate bundles I_k and N_k in Theorems 1 to 6 (the notation is always that of Definition 1) are coherent with those proved for the same bundles in Proposition 1. To avoid being repetitive with very similar computations, this issue will be exemplified with the bounds found in Theorems 1 and 2, which will be confronted with the bounds of sections (1) and (2) of Proposition 1.*

In Theorem 1, $\mu(I_1)$ is compared with $2\mu - \mu_1$, and in part (1) of Proposition 1 it is proved that $\mu_1 - (2g-2) \leq \mu(I_1) \leq \mu_2$ whenever $\mu_1 > \mu_2$. Notice that $2\mu - \mu_1 = \frac{-3\mu_1+2\mu_2+2\mu_3+2\mu_4+2\mu_5}{5}$, so $2\mu - \mu_1 < \mu_2$ if, and only if, $-3\mu_1 + 2\mu_2 + 2\mu_3 + 2\mu_4 + 2\mu_{\leq}5\mu_2$, that is, $-3(\mu_1 + \mu_2) + 2(\mu_3 + \mu_4 + \mu_5) \leq 0$. This is equivalent to stating that

$$\frac{\mu_3 + \mu_4 + \mu_5}{3} \leq \frac{\mu_1 + \mu_2}{2},$$

which is true under the assumption that $\mu_1 > \mu_2$. This proves that, certainly, $2\mu - \mu_1 \leq \mu_2$. Moreover, $2\mu - \mu_1 \geq \mu_1 - (2g-2)$ if, and only if, $g - 1 \geq \mu_1 - \mu > 0$, and this is again true since, under our assumptions, $g \geq 2$. Therefore, $\mu_1 - (2g-2) \leq 2\mu - \mu_1 \leq \mu_2$.

For the case of the slopes used in Theorem 2, $\mu(I_2)$ is compared with $3\mu - \mu_1 - \mu_2$ and $\mu(N_3)$ is compared with μ. Take the bounds proved in part (2) of Proposition 1 under the assumption that $\mu_2 > \mu_3$. Notice first that, since $\frac{\mu_1+\mu_2+\mu_3}{3} \geq \mu$ (since $\mu_2 > \mu_3$), it follows that $3\mu - \mu_1 - \mu_2 \leq \mu_3$. In addition, since that morphism $E_2/E_1 \to E_3/E_2 \otimes K$ induced by the Higgs field is nonzero, it must be $\mu_2 - (2g-2) \geq \mu_3 \geq 3\mu - \mu_1 - \mu_2$. It has been then proved that $\mu_2 - (2g-2) \leq 3\mu - \mu_1 - \mu_2 \leq \mu_3$. For $\mu(N_3)$, notice that $\mu \leq \mu_1$ as $\mu_2 > \mu_3$. Now, the inequality $\mu \geq \mu_1 + \mu_2 - \mu_3 - (2g-2)$ can occur depending on the value of the genus g, as discussed in the Remark after Proposition 1.

In Theorems 1 to 6, not only are Hodge limit bundles computed, but it is discussed whether they are stable or strictly semi-stable. In the case of strict semi-stability, the statements of the theorems provide the expression of the strictly poly-stable representative of the respective S-equivalence class. Specifically, Hodge limit bundles are stable in the situations described in the following result, which follows immediately from the proofs of Theorems 1 to 6.

Corolary 1. *Let (E, φ) be a semi-stable rank 5 Higgs bundle over X such that the Harder-Narasimhan type of E is $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ and $\mu = \mu(E)$ be the slope of E. Then, the Hodge limit bundle for the $\mathbb{C}^*$-action $(E_0, \varphi_0) = \lim_{z\to 0}(E, z \cdot \varphi)$ described in Theorems 1 to 6 is stable if, and only if, one of the following conditions, expressed with the notation of Definition 1, holds:*

(1) $\mu_1 > \mu_2 = \mu_3 = \mu_4 = \mu_5$ *and* $\mu(I_1) \neq 2\mu - \mu_1$.

(2) $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$, $\mathrm{rk}(\varphi_{3,2}) = 1$, $\mu(I_2) \neq 3\mu - \mu_1 - \mu_2$, *and* $\mu(N_3) \neq \mu$.

(3) $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$, $\mathrm{rk}(\varphi_{3,2}) = 2$, *and* $\mu(I_2) \neq 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$.

(4) $\mu_1 \geq \mu_2 \geq \mu_3 = \mu_4 > \mu_5$, *and* $\mu(N_1) \neq \mu$.

(5) $\mu_1 \geq \mu_2 \geq \mu_3 > \mu_4 \geq \mu_5$, $\mathrm{rk}(\varphi_{4,3}) = 1$, $\mu(I_3) \neq \mu_4 + \mu_5 - \mu$, *and* $\mu(N_2) \neq \mu$.

(6) $\mu_1 \geq \mu_2 \geq \mu_3 > \mu_4 \geq \mu_5$, $\mathrm{rk}(\varphi_{4,3}) = 2$, *and* $\mu(N_2) \neq \mu$.

4. Transversality of the stratifications

In this section, it will be proved that the Shatz and the Bialynycki-Birula stratifications of the moduli space of rank 5 and degree d Higgs bundles over X are transversal, in the sense that no stratum of one stratification coincides with a stratum of the other. Specifically, it will be checked that, for every possible Harder-Narasimhan type of the underlying vector bundle of a strictly semi-stable rank 5 Higgs bundle, there exist semi-stable Higgs bundles whose underlying vector bundle admits the given Harder-Narasimhan type such that the associated Hodge limit bundles are not S-equivalent. For this purpose, the following four cases will be distinguished:

(1) $\mu_1 > \mu_2 = \mu_3 = \mu_4 = \mu_5$,

(2) $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$,

(3) $\mu_1 \geq \mu_2 = \mu_3 = \mu_4 > \mu_5$,

(4) $\mu_1 \geq \mu_2 = \mu_3 > \mu_4 \geq \mu_5$,

and the descriptions of the Hodge limit bundles made in the Theorems 1 to 6 will be used. The key is that, for the same Harder-Narasimhan type, it will be possible to find different semi-stable Higgs bundles whose underlying vector bundles admit different intermediate bundles I_1, I_2, I_3, N_1, N_2, or N_3, introduced in Definition 1, so that, depending on the particular description of the possible Hodge limit bundles given in the mentioned theorems, the associated limits will have non-isomorphic graded objects.

Proposition 2. *Let X be a compact Riemann surface of genus $g \geq 2$. For any Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ satisfying $\mu_1 > \mu_2 = \mu_3 = \mu_4 = \mu_5$, there exist semi-stable rank 5 Higgs bundles (E, φ) and (E', φ') such that E admits this Harder-Narasimhan type and whose Hodge limit bundles under the $\mathbb{C}^*$-action are not S-equivalent.*

Proof. Suppose that a Harder-Narasimhan type $(\mu_1, \mu_2, \mu_2, \mu_2, \mu_2)$ with $\mu_1 > \mu_2$ is taken. Let E_1 be a line bundle of degree $d_1 = \mu_1$ and F be a stable rank 4 vector bundle of degree $4\mu_2$. Let $E = E_1 \oplus F$ and choose the Higgs field $\varphi : E \to E \otimes K$ to be

$$\varphi = \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix},$$

for a nonzero morphism $\psi : E_1 \to F \otimes K$. Then, (E, φ) is semi-stable by construction, since E_1 and F are stable and $\varphi(E_1) \not\subset E_1 \otimes K$. Let I_1 be the saturation of $\psi(E_1) \otimes K^{-1}$ in $E/E_1 = F$. Since F is stable as a vector bundle, $\mu(I_1) \leq \mu(F) = \mu_2$, and, by definition of I_1, $\mu(I_1) \geq \mu(E_1) - (2g-2) = \mu_1 - (2g-2)$. Choose ψ such that $\mu(I_1) < 2\mu - \mu_1$, which is always possible, as $\mu_1 > \mu_2$.

Let now E_1' be a line bundle of degree $d_1 = \mu_1$, L be a line bundle of certain degree l such that $2\mu - \mu_1 < l \leq \mu_2$ (this is always possible, since $\mu < \frac{\mu_1+\mu_2}{2}$, thus, $2\mu - \mu_1 < \mu_2$), and G be a stable rank 3 vector bundle of degree $4\mu_2 - l$. Define $E' = E_1' \oplus L \oplus G$ and $\varphi' : E' \to E' \otimes K$ by

$$\varphi' = \begin{pmatrix} 0 & 0 & 0 \\ \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \end{pmatrix},$$

where $\psi_1 : E_1' \to L \otimes K$ and $\psi_2 : L \to G \otimes K$ are nonzero morphisms. Define $I_1' = L$. By the condition on l, (E', φ') is semi-stable and E' has the same Harder-Narasimhan type that E.

Since $\mu(I_1) < 2\mu - \mu_1$, the Hodge limit bundle (E_0, φ_0) of (E, φ) is computed in case (1) of Theorem 1:

$$(E_0, \varphi_0) = \left(E_1 \oplus F, \begin{pmatrix} 0 & 0 \\ \psi_{2,1} & 0 \end{pmatrix} \right).$$

Similarly, since $\mu(I_1) > 2\mu - \mu_1$, the Hodge limit bundle of (E', φ') is given in case (2) of Theorem 1, so it is expressed by

$$(E_0', \varphi_0') = \left(E_1' \oplus L \oplus G, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right).$$

The Hodge limit bundles (E_0, φ_0) and (E_0', φ_0') are not S-equivalent, since they have different decompositions into stable sub-bundles, so their graded objects cannot be isomorphic. Therefore, (E, φ) and (E', φ') are semi-stable Higgs bundles whose underlying vector bundles have the same Harder-Narasimhan type $(\mu_1, \mu_2, \mu_2, \mu_2, \mu_2)$ and whose respective Hodge limit bundles are not S-equivalent. □

Proposition 3. *Let X be a compact Riemann surface of genus $g \geq 2$. For any Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ satisfying $\mu_1 \geq \mu_2 > \mu_3 \geq \mu_4 \geq \mu_5$, there exist semi-stable rank 5 Higgs bundles (E, φ) and (E', φ') whose underlying vector bundles admit this Harder-Narasimhan type and whose Hodge limit bundles under the $\mathbb{C}^*$-action are not S-equivalent.*

Proof. Let E_2 be a stable vector bundle of rank 2 and slope $\frac{\mu_1+\mu_2}{2}$, and F be a stable vector bundle of rank 3 and slope $\frac{\mu_3+\mu_4+\mu_5}{3}$. Let $E = E_2 \oplus F$. Choose a rank 2 morphism $\psi : E_2 \to F \otimes K$. Define φ as

$$\varphi = \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix} : E_2 \oplus F \to (E_2 \oplus F) \otimes K.$$

Then, (E, φ) is semi-stable and E has the Harder-Narasimhan type fixed in the statement, by construction.

Similarly, let E_2' be a stable vector bundle of rank 2 and slope $\frac{\mu_1+\mu_2}{2}$, chosen to be non-isomorphic to E_2, and F' be a stable vector bundle of rank 3 and slope $\frac{\mu_3+\mu_4+\mu_5}{3}$, chosen to be non-isomorphic to F. Define $E' = E_2' \oplus F'$, choose a nonzero morphism $\psi' : E_2' \to F' \otimes K$, and let

$$\varphi' = \begin{pmatrix} 0 & 0 \\ \psi' & 0 \end{pmatrix} : E_2' \oplus F' \to (E_2' \oplus F') \otimes K.$$

Again, (E', φ') is semi-stable, E' has the same Harder-Narasimhan type, and ψ' can be chosen such that (E, φ) and (E', φ') are not S-equivalent. In both (E, φ) and (E', φ'), the rank of ψ determines the rank of $\varphi_{3,2}$, which is 2.

Let I_2 and I_2' be the saturations of $\mathrm{Im}(\psi)\otimes K^{-1}$ and $\mathrm{Im}(\psi')\otimes K^{-1}$, respectively. For the computation of the limit Hodge bundles, let I_2 and I_2' be the saturations of $\mathrm{Im}(\psi) \otimes K^{-1}$ and $\mathrm{Im}(\psi') \otimes K^{-1}$, respectively, as in Definition 1. Three cases can now be distinguished, depending on the slope of I_2. First, if $\mu(I_2) < 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$, then the limit Hodge bundles are:

$$(E_0, \varphi_0) = \left(E_2 \oplus F, \begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix}\right)$$

and

$$(E_0', \varphi_0') = \left(E_2' \oplus F', \begin{pmatrix} 0 & 0 \\ \psi' & 0 \end{pmatrix}\right),$$

by Theorem 3. The Harder-Narasimhan filtrations of E_0 and E_0' are

$$0 \subset E_2 \subset E_0 \quad \text{and} \quad 0 \subset E_2' \subset E_0',$$

so the associated graded objects are $\mathrm{gr}(E_0) = E_2 \oplus F$ and $\mathrm{gr}(E_0') = E_2' \oplus F'$. Since $E_2 \ncong E_2'$ and $F \ncong F'$, these graded objects are non-isomorphic, proving that the limit Hodge bundles are not S-equivalent.

Second, if $\mu(I_2) > 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$, the limit Hodge bundles are:

$$(E_0, \varphi_0) = \left(E_2 \oplus I_2 \oplus F/I_2, \begin{pmatrix} 0 & 0 & 0 \\ \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \end{pmatrix}\right)$$

and

$$(E_0', \varphi_0') = \left(E_2' \oplus I_2' \oplus F'/I_2', \begin{pmatrix} 0 & 0 & 0 \\ \psi_1' & 0 & 0 \\ 0 & \psi_2' & 0 \end{pmatrix}\right),$$

where $\psi_1 : E_2 \to I_2 \otimes K$, $\psi_2 : I_2 \to (F/I_2) \otimes K$, and similarly for ψ_1' and ψ_2'.

Third, if $\mu(I_2) = 2\mu - \frac{1}{2}(\mu_1 + \mu_2)$, then the limit Hodge bundles are

$$(E_0, \varphi_0) = \left(\left(E_2 \oplus I_2, \begin{pmatrix} 0 & 0 \\ \psi_1 & 0 \end{pmatrix}\right) \oplus (F/I_2, 0)\right)$$

and

$$(E_0', \varphi_0') = \left(\left(E_2' \oplus I_2', \begin{pmatrix} 0 & 0 \\ \psi_1' & 0 \end{pmatrix}\right) \oplus (F'/I_2', 0)\right),$$

so, in this case, the Harder-Narasimhan filtrations of the underlying vector bundles are

$$0 \subset E_2 \subset E_2 \oplus I_2 \subset E_0 \quad \text{and} \quad 0 \subset E_2' \subset E_2' \oplus I_2' \subset E_0'.$$

The associated graded objects are then $\mathrm{gr}(E_0) = E_2 \oplus I_2 \oplus F/I_2$ and $\mathrm{gr}(E_0') = E_2' \oplus I_2' \oplus F'/I_2'$. As in the preceding case, since $E_2 \ncong E_2'$ and $I_2 \ncong I_2'$, these graded objects are non-isomorphic, so the limit Hodge bundles are not S-equivalent.

Therefore, in all cases, the semi-stable Higgs bundles (E, φ) and (E', φ') satisfy that the underlying vector bundles admit the given Harder-Narasimhan type satisfying the conditions of the statement and they admit limit Hodge bundles which are not S-equivalent. □

Proposition 4. *Let X be a compact Riemann surface of genus $g \geq 2$. For any Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ satisfying $\mu_1 \geq \mu_2 = \mu_3 = \mu_4 > \mu_5$, there exist semi-stable Higgs bundles (E, φ) and (E', φ') of rank 5 whose underlying vector bundles admit this Harder-Narasimhan type, and whose respective Hodge limit bundles (E_0, φ_0) and (E'_0, φ'_0) are not S-equivalent.*

Proof. Under the assumptions of the statement, let L_1 be a line bundle of degree $d_1 = \mu_1(2g-1)$ and L_5 be a line bundle of degree $d_5 = \mu_5(2g-1)$. Take $E = L_1 \oplus F_2 \oplus L_5$, where F_2 is a rank 3 vector bundle fitting in the exact sequence:

$$0 \to O_X \to F_2 \to O_X^{\oplus 2} \to 0,$$

such that $\deg(F_2) = 3\mu_2(2g-1)$, O_X being the trivial line bundle over X. Choose the Higgs field ϕ such that $N_1 = L_1 \oplus O_X$, which can be written in block form for this decomposition as

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix},$$

where $\psi_{2,1} : L_1 \oplus O_X \to F_2/O_X \otimes K$ and $\psi_{3,2} : F_2 \to L_5 \otimes K$ are nonzero morphisms.

For the construction of the other Higgs bundle (E', φ'), let L'_1 be another line bundle of degree $d_1 = \mu_1(2g-1)$ not isomorphic to L_1, and define

$$E' = L'_1 \oplus F_2 \oplus L_5.$$

Choose ϕ' with the same block structure as ϕ but ensuring that $N'_1 = L'_1 \oplus O_X$.

Both Higgs bundles are clearly semi-stable, the respective underlying vector bundles have the same fixed Harder-Narasimhan type and with $\mu(N_1) > \mu$ and $\mu(N'_1) > \mu$; thus, by Theorem 4, the corresponding Hodge limit bundles are

$$(E_0, \varphi_0) = \left(N_1 \oplus E_4/N_1 \oplus E/E_4, \begin{pmatrix} 0 & 0 & 0 \\ \psi_{2,1} & 0 & 0 \\ 0 & \psi_{3,2} & 0 \end{pmatrix} \right)$$

and

$$(E'_0, \varphi'_0) = \left(N'_1 \oplus E'_4/N'_1 \oplus E'/E'_4, \begin{pmatrix} 0 & 0 & 0 \\ \psi'_{2,1} & 0 & 0 \\ 0 & \psi'_{3,2} & 0 \end{pmatrix} \right),$$

where $\psi_{2,1}, \psi_{3,2}, \psi'_{2,1}, \psi'_{3,2}$ are the nonzero morphisms induced by ϕ and ϕ', respectively. These Hodge limit bundles are not S-equivalent, since N_1 contains L_1 while N'_1 contains L'_1 as direct summands and $L_1 \not\cong L'_1$ by construction.

Therefore, for any valid Harder-Narasimhan type under the conditions of the statement, there exist semi-stable Higgs bundles whose underlying vector bundles admit this Harder-Narasimhan type and whose Hodge limits are not S-equivalent. □

Proposition 5. *Let X be a compact Riemann surface of genus $g \geq 2$. For any Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ satisfying $\mu_1 \geq \mu_2 = \mu_3 > \mu_4 \geq \mu_5$, there exist semi-stable rank 5 Higgs bundles (E, φ) and (E', φ') whose underlying vector bundles admit this Harder-Narasimhan type, and whose respective limit Hodge bundles under the $\mathbb{C}^*$-action are not S-equivalent.*

Proof. Fix a Harder-Narasimhan type $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ satisfying $\mu_1 \geq \mu_2 = \mu_3 > \mu_4 \geq \mu_5$. For the construction of (E, φ), take N_2 to be a stable bundle of rank 2 with slope $\mu(N_2) > \mu$ (this is always possible), E_3/N_2 to be a stable line bundle such that $\mu(E_3/N_2) = 3\mu_3 - 2\mu(N_2)$, and E/E_3 to be a stable bundle of rank 2 with $\mu(E/E_3) = \frac{5\mu - 3\mu_3}{2}$. Choose $\psi_{2,1} : N_2 \to E_3/N_2 \otimes K$ and $\psi_{3,2} : E_3/N_2 \to E/E_3 \otimes K$ to be nonzero morphisms.

For (E', φ'), choose N'_2 to be a stable bundle of rank 2 with $\mu(N'_2) = \mu(N_2)$ but not isomorphic to N_2 and keep the same construction for E'_3/N'_2, E'/E'_3, $\psi'_{2,1}$, and $\psi'_{3,2}$ as above.

These Higgs bundles satisfy that the underlying vector bundles admit the same fixed Harder-Narasimhan type and also verify the conditions of Theorem 6, specifically of the case (3), but the corresponding Hodge limit bundles are not S-equivalent because, by construction, $N_2 \ncong N'_2$, being these vector sub-bundles summands of the respective graded objects of (E, φ) and (E', φ').

Therefore, there exist semi-stable Higgs bundles whose underlying vector bundles have the same Harder-Narasimhan type of the given form whose Hodge limits are not S-equivalent. □

Theorem 7. *Let X be a compact Riemann surface of genus $g \geq 2$ and $\mathcal{M}(5, d)$ be the moduli space of semi-stable rank 5 and degree d Higgs bundles over X. Then, the Shatz strata traverse the Białynicki-Birula strata of $\mathcal{M}(5, d)$, in the sense that each Shatz stratum contains semi-stable Higgs bundles with rank 5 of several Białynicki-Birula strata.*

Proof. The result holds from Propositions 2 to 5 by noticing that, if (E, φ) is stable, then $\lim_{z\to 0}(E, z \cdot \varphi) = (E, 0)$. In this case, it suffices to take two stable vector bundles E and E' with the same degree (hence, with the same Harder-Narasimhan type of the considered form) to ensure that the associated Hodge limit bundles are not S-equivalent, because $(E, 0)$ and $(E', 0)$ are stable as Higgs bundles but they are not isomorphic. □

5. Conclusions

Let X be a compact Riemann surface of genus $g \geq 2$ and $\mathcal{M}(5, d)$ be the moduli space of rank 5 semi-stable holomorphic Higgs bundles over X of a given degree d. The multiplicative group $\mathbb{C}^*$ acts on $\mathcal{M}(5, d)$ by the product on the Higgs field. One of the main contributions of the paper is the computation of all possible limits $\lim_{z\to 0}(E, z \cdot \varphi)$, where (E, φ) is a rank 5 semi-stable Higgs bundle over X. Explicit vector forms for these limit bundles are provided, which give a description of the Białynicki-Birula strata in $\mathcal{M}(5, d)$. The possibilities for the limit bundles are in accordance with a certain classification of the Harder-Narasimhan type associated with the underlying vector bundle of a semi-stable Higgs bundle that has been given, and with some bounds for the slope of certain intermediate vector bundles that have also been provided. In addition, necessary and sufficient conditions are given on the Harder-Narasimhan types and the slopes of the intermediate bundles for the Hodge limit bundle to be stable. The other main contribution of the paper is proving that, whatever the Harder-Narasimhan type, there exist semi-stable Higgs bundles with rank 5 whose associated limit bundles are not S-equivalent but whose underlying vector bundles have the same Harder-Narasimhan type, which is equal to the given. Indeed, in each of the situations differentiated in the paper to analyze this question, explicit constructions of these Higgs bundles are provided. As a consequence, it is proved that the Shatz strata of $\mathcal{M}(5, d)$, defined by the Harder-Narasimhan type of the underlying vector bundles, traverse the Białynicki-Birula strata, in the sense that every Shatz stratum contains Higgs bundles of different

Białynicki-Birula strata. This behavior that has been proved in rank 5 contrasts with that of rank 4, where there are overlapping strata.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declare there is no conflict of interest.

References

1. L. B. Anderson, L. Fredrickson, M. Esole, L. P. Shaposnick, Singular geometry and Higgs bundles in string theory, *SIGMA*, **14** (2018), 1–27. https://doi.org/10.3842/SIGMA.2018.037
2. Á. Antón-Sancho, Shatz and Białynicki-Birula stratifications of the moduli space of Higgs bundles, *Hokkaido Math. J.*, **51** (2022), 25–56. https://doi.org/10.14492/hokmj/2019-202
3. Á. Antón-Sancho, F_4 and PSp$(8, \mathbb{C})$-Higgs pairs understood as fixed points of the moduli space of E_6-Higgs bundles over a compact Riemann surface, *Open Math.*, **20** (2022), 1723–1733. https://doi.org/10.1515/math-2022-0543
4. Á. Antón-Sancho, Fixed points of automorphisms of the vector bundle moduli space over a compact Riemann surface, *Mediterr. J. Math.*, **21** (2024), 20. https://doi.org/10.1007/s00009-023-02559-z
5. Á. Antón-Sancho, Fixed points of involutions of G-Higgs bundle moduli spaces over a compact Riemann surface with classical complex structure group, *Front. Math.*, **19** (2024), 1025–1039. https://doi.org/10.1007/s11464-023-0014-0
6. Á. Antón-Sancho, A construction of Shatz strata in the polystable G_2-bundles moduli space using Hecke curves, *Electron. Res. Arch.*, **32** (2024), 6109–6119. https://doi.org/10.3934/era.2024283
7. D. Baraglia, Classification of the automorphism and isometry groups of Higgs bundle moduli spaces, *Proc. London Math. Soc.*, **112** (2016), 827–854. https://doi.org/10.1112/plms/pdw014
8. A. Białynicki-Birula, Some theorems on actions of algebraic groups, *Ann. Math.*, **98** (1973), 480–497. https://doi.org/10.2307/1970915
9. S. B. Bradlow, O. García-Prada, I. Mundet-Riera, Relative Hitchin-Kobayashi correspondences for principal pairs, *Quart. J. Math.*, **54** (2003), 171–208. https://doi.org/10.1093/qmath/hag013
10. O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase, A. Neitzke, From the Hitchin section to opers through nonabelian Hodge, *J. Differential Geom.*, **117** (2021), 223–253. https://doi.org/10.4310/jdg/1612975016
11. B. Collier, R. Wentworth, Conformal limits and the Białynicki-Birula stratification of the space of λ-connections, *Adv. Math.*, **350** (2019), 1193–1225. https://doi.org/10.1016/j.aim.2019.04.034
12. R. Fedorov, A. Soibelman, Y. Soibelman, Motivic Donaldson-Thomas invariants of parabolic Higgs bundles and parabolic connections on a curve, *SIGMA*, **16** (2020), 1–49. https://doi.org/10.3842/SIGMA.2020.070

13. O. García-Prada, J. Heinloth, A. H. W. Schmitt, On the motives of moduli of chains and Higgs bundles, *J. Eur. Math. Soc.*, **16** (2014), 2617–2668. https://doi.org/10.4171/JEMS/494
14. O. García-Prada, S. Ramanan, Involutions and higher order automorphisms of Higgs bundle moduli spaces, *Proc. London Math. Soc.*, **119** (2019), 681–732. https://doi.org/10.1112/plms.12242
15. P. Gothen, R. A. Zúñiga-Rojas, Stratifications on the moduli space of Higgs bundles, *Portugal. Math.*, **74** (2017), 127–148. https://doi.org/10.4171/PM/1996
16. G. Harder, M. S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, *Math. Ann.*, **212** (1975), 215–248. https://doi.org/10.1007/BF01357141
17. T. Hausel, *Geometry of Higgs bundles*, PhD thesis, Cambridge University, United Kingdom, 1998.
18. T. Hausel, M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, *Invent. Math.*, **153** (2003), 197–229. https://doi.org/10.1007/s00222-003-0286-7
19. T. Hausel, M. Thaddeus, Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles, *J. Amer. Math. Soc.*, **16** (2003), 303–327.
20. T. Hausel, M. Thaddeus, Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles, *Proc. London Math. Soc.*, **88** (2004), 632–658. https://doi.org/10.1112/S0024611503014618
21. N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.*, **55** (1987), 59–126. https://doi.org/10.1112/plms/s3-55.1.59
22. T. Kinjo, N. Koseki, Cohomological χ-independence for Higgs bundles and Gopakumar–Vafa invariants, *J. Eur. Math. Soc.*, 2024. https://doi.org/10.4171/JEMS/1487
23. T. Mochizuki, *Donaldson type invariants for algebraic surfaces*, Lecture Notes in Mathematics, Vol. 1972, Springer Berlin, Heidelberg, 2009. https://doi.org/10.1007/978-3-540-93913-9
24. N. Nitsure, Moduli space of semistable pairs on a curve, *Proc. London Math. Soc.*, **62** (1991), 275–300.
25. S. S. Shatz, The decomposition and specialization of algebraic families of vector bundles, *Compos. Math.*, **35** (1977), 163–187.
26. C. T. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.*, **75** (1992), 5–95.
27. C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety I, *Inst. Hautes Études Sci. Publ. Math.*, **79** (1994), 47–129.
28. C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety II, *Inst. Hautes Études Sci. Publ. Math.*, **80** (1994), 5–79.
29. A. Thomas, A gentle introduction to the non-abelian Hodge correspondence, *Enseign. Math.*, 2024. https://doi.org/10.4171/LEM/1072