Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 3392–3407.
DOI: 10.3934/math.2025157
Received: 24 December 2024
Revised: 26 January 2025
Accepted: 12 February 2025
Published: 21 February 2025

Research article

Construction optical solitons of generalized nonlinear Schrödinger equation with quintuple power-law nonlinearity using Exp-function, projective Riccati, and new generalized methods

Islam Samir[1,*], **Hamdy M. Ahmed**[2], **Wafaa Rabie**[3], **W. Abbas**[4] **and Ola Mostafa**[4]

[1] Department of Physics and Mathematics Engineering, Faculty of Engineering, Ain Shams University, Cairo, Egypt

[2] Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El Shorouk Academy, Cairo, Egypt

[3] Department of Basic Sciences, Higher Institute of Engineering and Technology, Menoufia, Egypt

[4] Basic and Applied Science Department, College of Engineering and Technology, Arab Academy for Science, Technology and Maritime Transport, Cairo, Egypt

* **Correspondence:** Email: islamsamir@eng.asu.edu.eg.

Abstract: This work investigates the generalized nonlinear Schrödinger equation (NLSE), which imitates the wave transmission along optical fibers. This model incorporates a quintuple power-law of non-linearity and nonlinear chromatic dispersion. To demonstrate the significance and motivation for this work, a review of the prior research is presented in the literature. Three integration strategies are applied during the study process in order to produce a variety of novel solutions. These techniques include the modified exp-function approach, the general projective Riccati method (GPRM), and the new generalized method. The extracted solutions include bright solitons, singular solitons, dark solitons, and trigonometric solutions.

Keywords: generalized nonlinear Schrödinger equation; optical solitons; chromatic dispersion; analytic methods
Mathematics Subject Classification: 35C05, 35C07, 35C08, 47J35

1. Introduction

The field of telecommunications engineering has benefited greatly from optical soliton dynamics in recent decades [1–3]. There are two types of telecommunications engineering: Wireless communications, which transmit data over long distances using radio waves rather than wires, and wired communications, which use subterranean communications cables. Fiber optics elucidates nonlinear responses of incident light characteristics, including phase and polarization [4]. Numerous optical phenomena are produced by these nonlinear interactions. To enhance the performance of telecommunications engineering, many novel ideas have been proposed. Propagation dynamics of solitons is one of them. These consist of the nonlinear refractive index and chromatic dispersion (CD) of the optical fiber. The nonlinear Schrödinger model is widely applied to mimic many physical phenomena in different fields such as plasma physics, condensed matter, nonlinear optics, fluid dynamics, solid-state physics, biochemistry, and many more [5–9]. The soliton theory is essential to many models, including those in fiber optics and other fields. These models such as the Gerdjikov–Ivanov equation [10], Radhakrishnan—Kundu-–Lakshmanan equation [11], Chen–Lee–Liu equation [12, 13], complex Ginzburg–Landau model [14, 15], Gilson–Pickering equation [16, 17] and Lakshmanan–Porsezian–Daniel model [18]. One of the basic models for nonlinear waves is NLSE, which describes nonlinear phenomena in optical fibers and is also used to describe spatial solitons [19]. It is necessary to solve nonlinear partial differential equations (NLPDEs) analytically in order to understand these nonlinear phenomena [20–22]. Many academics are interested in the extraction of solitons in NLPDEs. There are efficient techniques that have been put forth to get exact solutions to NLPDEs including extended hyperbolic function technique [23], the Sardar sub-equation method [24], the extended Tanh–Coth scheme [25], the Jacobi elliptic function method [26] and many more.

This work examines generalized NLSE with nonlinear chromatic dispersion and quintuple power law non-linearity. This model is as follows: [27]:

$$i\psi_t + \alpha(|\psi|^p\psi)_{xx} + \left(k_1|\psi|^{2m} + k_2|\psi|^{2m+n} + k_3|\psi|^{2m+2n} + k_4|\psi|^{2m+n+r} + k_5|\psi|^{2m+2n+r} + (|\psi|^r)_{xx}\right)\psi = 0, \quad (1.1)$$

where $\psi_{x,t}$ represents the wave profile. α is the coefficient of the nonlinear CD. k_1, k_2, k_3, k_4, and k_5 are the coefficients of nonlinear refractive index. $p \geq 0$ and r, m, and n are positive constants. When $p = 0$, it becomes the case of linear chromatic dispersion in the exceptional case. The possible values of the parameters remain unknown, though. Benjamin–Feir stability analysis could be used to aid with this issue. This project is distinct and should be handled later. This work focuses on soliton solutions with non-negative parameters. Optical solitons have become a popular topic in fibre optics during the past decade due to their potential to block pulse transmission over intercontinental distance transmission [28, 29]. Pre-existing and current publications serve as models for studying the genesis and existence of optical solitons. In order to obtain stationary soliton solutions, the proposed model was examined using the G'/G expansion approach [30]. In addition, the modified extended mapping approach is implemented to derive dark and singular solitons [27].

Three integration strategies are used in this study to find novel and different solutions for Eq (1.1). These techniques including the modified exp-function method, the general projective Riccati method (GPRM), and new generalized method. With the aid of these techniques, different solutions can be offered, including trigonometric solutions, singular solitons, dark solitons, and bright solitons.

This paper's structure will be as follows. Section 2 provides a brief introduction to the proposed techniques. Analytical solutions for Eq (1.1) are then provided in Section 3 using the proposed techniques. Section 4 presents both 2D and 3D graphs to show the properties of the propagating waves. In the last section, the manuscript is concluded.

2. Mathematical methods

This section contains concise summaries of the modified exp-function approach [31], GPRM [32] and new generalized method [4]

Let us consider NLPDE as follows:

$$F(\psi, \psi_t, \psi_x, \psi_{xx}, ...) = 0, \tag{2.1}$$

where F is polynomial in $\psi(x, t)$.

Suppose $\psi(x, t) = \psi(\xi)$, $\xi = k(x - c\ t)$. Then, Eq (2.1) is transformed to a nonlinear ordinary differential equation (NLODE) as follows:

$$Q(\psi, \psi\prime, \psi\prime\prime, \psi\prime\prime\prime, ...) = 0. \tag{2.2}$$

2.1. The modified exp-function method

Step 1. The solution to the generated NLODE is considered as follows:

$$\psi(\xi) = \sum_{j=-N}^{N} a_j(exp(-\phi(\xi)))^j, \tag{2.3}$$

where a_j are constants to be determined and ϕ meets the following auxiliary equation

$$\phi\prime(\xi) = -\sqrt{\lambda_1 + \lambda_2(exp(-\phi(\xi)))^2},\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.4}$$

Step 2. Applying the balance rule in Eq (2.2), N can be raised.
Step 3. A set of nonlinear equations is raised by putting (2.3) and (2.4) into (2.2), then gathering and equating all the terms of $(exp(-\phi(\xi)))^j$ to zero. The extracted system can be handled by Mathematica or Maple software tools to obtain all unknowns.
Step 4. Finally, we obtain the new exact solutions for NLPDE by combining Eq (2.3) with the general solutions of Eq (2.4).
Step 5. To obtain more exact solutions, the steps from (1) to (4) will be repeated again by using the following auxiliary equation:

$$\phi\prime(\xi) = -\lambda_1\ exp(\phi(\xi)) - \lambda_2\ exp(-\phi(\xi)),\ \lambda_1,\ \lambda_2 \in \mathbb{R}. \tag{2.5}$$

The general solutions of Eq (2.4) are:

$$\Psi(\xi) = -ln\left(-\sqrt{\frac{\lambda_1}{\lambda_2}}\text{csch}[\sqrt{\lambda_1}\xi]\right),\ \lambda_1 > 0,\ \lambda_2 > 0, \tag{2.6}$$

$$\Psi(\xi) = -ln\left(\sqrt{\frac{-\lambda_1}{\lambda_2}}\sec[\sqrt{-\lambda_1}\xi]\right),\ \lambda_1 < 0,\ \lambda_2 > 0, \tag{2.7}$$

$$\Psi(\xi) = -ln\left(\sqrt{\frac{\lambda_1}{-\lambda_2}}\text{sech}[\sqrt{\lambda_1}\xi]\right),\ \lambda_1 > 0,\ \lambda_2 < 0, \tag{2.8}$$

$$\Psi(\xi) = -ln\bigg(\sqrt{\frac{-\lambda_1}{\lambda_2}}\csc[\sqrt{-\lambda_1}\xi]\bigg),\ \lambda_1 < 0,\ \lambda_2 > 0, \tag{2.9}$$

while the general solutions of Eq (2.5) are

$$\Psi(\xi) = -ln\bigg(\sqrt{\frac{\lambda_1}{\lambda_2}}\tan[\sqrt{\lambda_1\lambda_2}\xi]\bigg),\ \lambda_1\ \lambda_2 > 0, \tag{2.10}$$

$$\Psi(\xi) = -ln\bigg(-\sqrt{\frac{\lambda_1}{\lambda_2}}\cot[\sqrt{\lambda_1\lambda_2}\xi]\bigg),\ \lambda_1\ \lambda_2 > 0, \tag{2.11}$$

$$\Psi(\xi) = -ln\bigg(\sqrt{\frac{\lambda_1}{-\lambda_2}}\tanh[\sqrt{-\lambda_1\lambda_2}\xi]\bigg),\ \lambda_1\ \lambda_2 < 0, \tag{2.12}$$

$$\Psi(\xi) = -ln\bigg(\sqrt{\frac{\lambda_1}{-\lambda_2}}\coth[\sqrt{-\lambda_1\lambda_2}\xi]\bigg),\ \lambda_1\ \lambda_2 < 0. \tag{2.13}$$

2.2. GPRM method

Step 1. Assuming that Eq (2.2) has a solution of the form

$$\psi(\xi) = s_0 + \sum_{i=1}^{N}\phi^{i-1}(\xi)\Big(s_i\phi(\xi) + \beta_i\varphi(\xi)\Big), \tag{2.14}$$

where the functions $\phi(\xi)$ and $\varphi(\xi)$ fulfill the next equation:

$$\begin{aligned}&\phi'(\xi) = \epsilon\phi(\xi)\varphi(\xi),\\ &\varphi'(\xi) = \sigma + \epsilon\varphi^2(\xi) - \delta\phi(\xi),\end{aligned} \tag{2.15}$$

where

$$\varphi^2(\xi) = -\epsilon\Big(\sigma - 2\delta\phi(\xi) + \frac{\delta^2+\tau}{\sigma}\phi^2(\xi)\Big). \tag{2.16}$$

The following are the general solutions to Eq (2.15):
Set (1): $\epsilon = \tau = -1$:

$$\phi(\xi) = \frac{\sigma\,\mathrm{sech}\big[\sqrt{\sigma}\xi\big]}{1+\delta\,\mathrm{sech}\big[\sqrt{\sigma}\xi\big]},\quad \varphi(\xi) = \frac{\sqrt{\sigma}\tanh\big[\sqrt{\sigma}\xi\big]}{1+\delta\,\mathrm{sech}\big[\sqrt{\sigma}\xi\big]}, \tag{2.17}$$

Set (2): $\epsilon = -1,\ \tau = 1$:

$$\phi(\xi) = \frac{\sigma\,\mathrm{csch}\big[\sqrt{\sigma}\xi\big]}{1+\delta\,\mathrm{csch}\big[\sqrt{\sigma}\xi\big]},\quad \varphi(\xi) = \frac{\sqrt{\sigma}\coth\big[\sqrt{\sigma}\xi\big]}{1+\delta\,\mathrm{csch}\big[\sqrt{\sigma}\xi\big]}, \tag{2.18}$$

Set (3): $\epsilon = 1,\ \tau = -1$:

$$\phi(\xi) = \frac{\sigma\sec\big[\sqrt{\sigma}\xi\big]}{1+\delta\sec\big[\sqrt{\sigma}\xi\big]},\quad \varphi(\xi) = \frac{\sqrt{\sigma}\tan\big[\sqrt{\sigma}\xi\big]}{1+\delta\sec\big[\sqrt{\sigma}\xi\big]}, \tag{2.19}$$

Set (4): $\epsilon = \tau = 1$:

$$\phi(\xi) = \frac{\sigma \csc\left[\sqrt{\sigma}\xi\right]}{1 + \delta \csc\left[\sqrt{\sigma}\xi\right]}, \quad \varphi(\xi) = -\frac{\sqrt{\sigma} \cot\left[\sqrt{\sigma}\xi\right]}{1 + \delta \csc\left[\sqrt{\sigma}\xi\right]}. \tag{2.20}$$

Step 2. Along with (2.15) and (2.16), (2.14) is substituted into (2.2). A polynomial of $\varphi(\xi)$ and $\phi(\xi)$ is thus obtained. A collection of nonlinear equations is generated via equating terms of the identical power to zero. The extracted system can be handled by Mathematica or Maple software tools to assess all unknowns. Consequently, we obtain exact solutions for Eq (2.1).

2.3. The new generalized method

Step 1. The solution of the obtained NLODE is considered as follows:

$$\psi(\xi) = \alpha_0 + \sum_{i=1}^{N} \frac{\alpha_i + \beta_i \phi'(\xi)^i}{\phi(\xi)^i}, \tag{2.21}$$

and $\phi(\xi)$ fulfills the next equation:

$$\phi'(\xi) = \sqrt{-\sigma + \phi(\xi)^2}, \tag{2.22}$$

and

$$\begin{aligned} &\phi^{(n)}(\xi) = \phi(\xi), \quad n \geq 2 \textit{ and } n \text{ is even}, \\ &\phi^{(n)}(\xi) = \phi'(\xi), \quad n \geq 2 \textit{ and } n \text{ is odd}. \end{aligned} \tag{2.23}$$

Equation (2.22) has the following solution:

$$\Phi(\xi) = ae^{\xi} + \frac{\sigma}{4ae^{\xi}}, \tag{2.24}$$

where σ and a are constants.

Step 2. Substituting (2.21) into (2.2) along with (2.22) and its derivatives (2.23). This substitution yields a polynomial of the form $\frac{1}{\Phi(\xi)}\left(\frac{\Phi'(\xi)}{\Phi(\xi)}\right)$. Once all of the terms in this polynomial are combined and their values are set to zero, an overdetermined system of algebraic equations is produced. All of the unknowns in this system can be found by solving it with Mathematica. Consequently, we have analytical solutions for Eq (2.1).

3. Mathematical analysis

Our goal is to achieve analytic solutions in the following form for Eq (1.1):

$$\psi(x,t) = V(kx)e^{i(wt+\theta)}, \tag{3.1}$$

where θ represents the phase constant and w is the wave number. k is a constant. Inserting Eq (3.1) into Eq (1.1) yields

$$\begin{aligned} &\alpha k^2(p+1)V^{p+1}V'' + \alpha k^2 p(p+1)V^p V'^2 + k_4 V^{2m+n+r+2} + k_5 V^{2m+2n+r+2} + k_3 V^{2m+2n+2} + k_2 V^{2m+n+2} \\ &+k_1 V^{2m+2} + k^2 r V^{r+1} V'' + k^2(r-1)rV^r V'^2 - wV^2 = 0. \end{aligned} \tag{3.2}$$

Taking $p = 2m + r$ and using the transformation

$$V = Q^{\frac{1}{n}}, \tag{3.3}$$

Eq (3.2) becomes

$$\begin{aligned}&n^2Q^2\left(-w + k_1Q^{\frac{2m}{n}} + k_2Q^{\frac{2m+n}{n}} + k_3Q^{\frac{2m+2n}{n}} + k_4Q^{\frac{2m+n+r}{n}} + k_5Q^{\frac{2m+2n+r}{n}}\right)\\&+k^2r(r-1)Q^{\frac{r}{n}}Q'^2 + \alpha k^2(r+2m)(r+2m+1)Q'^2Q^{\frac{2m+r}{n}} + k^2rQ^{\frac{r}{n}}\left(nQQ'' + (1-n)Q'^2\right)\\&\alpha k^2(2m+r+1)Q^{\frac{2m+r}{n}}\left(nQQ'' + (1-n)Q'^2\right) = 0.\end{aligned} \tag{3.4}$$

Setting $n = r = 2m$, Eq (3.4) becomes

$$\begin{aligned}&2\alpha mk^2(1+4m)QQ'' + \alpha k^2(1+2m)(1+4m)Q'^2 + 4k_5m^2Q^4 + 4m^2(k_3+k_4)Q^3 + 4k_2m^2Q^2\\&+4k_1m^2Q + 4k^2m^2Q'' - 4m^2w = 0.\end{aligned} \tag{3.5}$$

Balancing QQ'' with Q^4 in Eq (3.5) results in $N = 1$.

3.1. Application of the modified exp-function scheme

By applying the modified exponential method, the following form is the solution to Eq (3.5)

$$Q(x) = a_0 + a_1exp(-\phi(x)) + a_{-1}exp(\phi(x)). \tag{3.6}$$

Applying the steps discussed in Section 2.1, we get the next results:

$$\begin{aligned}&a_0 = \frac{-\alpha k_3 - \alpha k_4 - 24\alpha k_3m^2 - 24\alpha k_4m^2 + 8k_5m^2 - 10\alpha k_3m - 10\alpha k_4m}{4\alpha k_5\left(20m^2+9m+1\right)},\ a_{-1} = 0,\\&w = \frac{a_0k^2\left(a_0^2\lambda_2\left(\alpha a_0\left(8m^2+6m+1\right) - 8m^2\right) + a_1^2\lambda_1\left(\alpha a_0\left(8m^2+6m+1\right) - 4m^2\right)\right)}{4a_1^2m^2},\\&\lambda_2 = -\frac{4a_1^2k_5m^2}{\alpha k^2\left(24m^2+10m+1\right)},\ \lambda_1 = \frac{2a_1^2k_1m^2 - 2a_0^2k^2\lambda_2\left(\alpha a_0(3m+1)(4m+1) - 6m^2\right)}{a_1^2k^2\left(\alpha a_0(3m+1)(4m+1) - 2m^2\right)}.\end{aligned}$$

Afterwards, we have

$$\begin{aligned}\psi(x,t) = &\left\{a_0 - \sqrt{-\frac{4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right) - 6m^2\right) + \alpha k_1\left(24m^2+10m+1\right)}{2k_5\left(\alpha a_0\left(12m^2+7m+1\right) - 2m^2\right)}}\ \times\right.\\&\left.\operatorname{csch}\left(\sqrt{2}x\sqrt{\frac{m^2\left(4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right) - 6m^2\right) + \alpha k_1\left(24m^2+10m+1\right)\right)}{\alpha\left(24m^2+10m+1\right)\left(\alpha a_0\left(12m^2+7m+1\right) - 2m^2\right)}}\right)\right\}^{1/n} e^{i(wt+\theta)},\end{aligned} \tag{3.7}$$

$$\psi(x,t) = \left\{a_0 + \sqrt{\frac{4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right) - 6m^2\right) + \alpha k_1\left(24m^2+10m+1\right)}{2k_5\left(\alpha a_0\left(12m^2+7m+1\right) - 2m^2\right)}}\ \times\right.$$

$$\operatorname{sech}\left(\sqrt{2}x\sqrt{\frac{m^2\left(4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-6m^2\right)+\alpha k_1\left(24m^2+10m+1\right)\right)}{\alpha\left(24m^2+10m+1\right)\left(\alpha a_0\left(12m^2+7m+1\right)-2m^2\right)}}\right)\Bigg\}^{1/n} e^{i(wt+\theta)}, \tag{3.8}$$

$$\psi(x,t)=\Bigg\{a_0+\sqrt{\frac{4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-6m^2\right)+\alpha k_1\left(24m^2+10m+1\right)}{2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-2m^2\right)}}\times$$
$$\csc\left(\sqrt{2}x\sqrt{-\frac{m^2\left(4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-6m^2\right)+\alpha k_1\left(24m^2+10m+1\right)\right)}{\alpha\left(24m^2+10m+1\right)\left(\alpha a_0\left(12m^2+7m+1\right)-2m^2\right)}}\right)\Bigg\}^{1/n} e^{i(wt+\theta)}, \tag{3.9}$$

$$\psi(x,t)=\Bigg\{a_0+\sqrt{\frac{4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-6m^2\right)+\alpha k_1\left(24m^2+10m+1\right)}{2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-2m^2\right)}}\times$$
$$\sec\left(\sqrt{2}x\sqrt{-\frac{m^2\left(4a_0^2k_5\left(\alpha a_0\left(12m^2+7m+1\right)-6m^2\right)+\alpha k_1\left(24m^2+10m+1\right)\right)}{\alpha\left(24m^2+10m+1\right)\left(\alpha a_0\left(12m^2+7m+1\right)-2m^2\right)}}\right)\Bigg\}^{1/n} e^{i(wt+\theta)}. \tag{3.10}$$

A singular solitary is provided by Eq (3.7), while a bright solitary is provided by Eq (3.8). Equations (3.9) and (3.10) represent singular periodic solutions.

Substituting by the auxiliary equation of (2.5) instead of Eq (2.4), we obtain the next results for Eq (1.1).

Result (1):

$$a_0=a_{-1}=0,\ \lambda_1=-\frac{k_1}{2k^2\lambda_2},\ a_1=\frac{\sqrt{2}k\lambda_2}{\sqrt{-k_3-k_4}},\ w=\frac{\alpha a_1^2k^2\lambda_1^2\left(8m^2+6m+1\right)}{4m^2},$$
$$k_5=-\frac{\alpha k^2\lambda_2^2\left(24m^2+10m+1\right)}{4a_1^2m^2},\ k_2=-\frac{\alpha k^2\lambda_1\lambda_2(4m+1)^2}{2m^2}.$$

The following solutions are then provided

$$\psi(x,t)=\left\{\frac{\sqrt{k_1}\tanh\left(\sqrt{\frac{k_1}{2}}x\right)}{\sqrt{-k_3-k_4}}\right\}^{1/n} e^{i(wt+\theta)},\ k_1>0,\ k_3+k_4<0, \tag{3.11}$$

$$\psi(x,t)=\left\{\frac{\sqrt{k_1}\coth\left(\sqrt{\frac{k_1}{2}}x\right)}{\sqrt{-k_3-k_4}}\right\}^{1/n} e^{i(wt+\theta)},\ k_1>0,\ k_3+k_4<0, \tag{3.12}$$

$$\psi(x,t)=\left\{\frac{\sqrt{-k_1}\tan\left(\sqrt{-\frac{k_1}{2}}x\right)}{\sqrt{-k_3-k_4}}\right\}^{1/n} e^{i(wt+\theta)},\ k_1<0,\ k_3+k_4<0, \tag{3.13}$$

$$\psi(x,t)=\left\{-\frac{\sqrt{-k_1}\cot\left(\sqrt{-\frac{k_1}{2}}x\right)}{\sqrt{-k_3-k_4}}\right\}^{1/n}e^{i(wt+\theta)},\ k_1<0,\ k_3+k_4<0. \tag{3.14}$$

The dark soliton is provided by Eq (3.11), whereas the singular soliton is provided by Eq (3.12). Equations (3.13) and (3.14) represent singular periodic solutions.

Result (2):

$$a_0=0,\ a_{-1}=-\frac{a_1\lambda_1}{\lambda_2},\ \lambda_1=-\frac{k_1}{8k^2\lambda_2},\ a_1=\frac{\sqrt{2}k\lambda_2}{\sqrt{-k_3-k_4}},\ w=\frac{4\alpha a_1^2k^2\lambda_1^2\left(8m^2+6m+1\right)}{m^2},$$
$$k_5=-\frac{\alpha k^2\lambda_2^2\left(24m^2+10m+1\right)}{4a_1^2m^2},\ k_2=-\frac{2\alpha k^2\lambda_1\lambda_2(4m+1)^2}{m^2}.$$

Afterwards, we have

$$\psi(x,t)=\left\{\frac{\sqrt{k_1}\tanh\left(\frac{\sqrt{k_1}x}{2\sqrt{2}}\right)\left(\coth^2\left(\frac{\sqrt{k_1}x}{2\sqrt{2}}\right)+1\right)}{2\sqrt{-k_3-k_4}}\right\}^{1/n}e^{i(wt+\theta)},\ k_1>0,\ k_3+k_4<0, \tag{3.15}$$

$$\psi(x,t)=\left\{\frac{\sqrt{-k_1}\csc\left(\frac{\sqrt{-k_1}x}{\sqrt{2}}\right)}{\sqrt{-k_3-k_4}}\right\}^{1/n}e^{i(wt+\theta)},\ k_1<0,\ k_3+k_4<0. \tag{3.16}$$

A singular solitary is provided by Eq (3.15) while a singular periodic solution is provided by Eq (3.16).

3.2. Application of GPRM

By applying GPRM, solution of Eq (3.5) is represented as:

$$Q(x)=s_0+s_1\phi(x)+\beta_1\varphi(x). \tag{3.17}$$

Applying Step-2 of the proposed method, we get the following results:
Case (1). When $\tau=-1$ and $\epsilon=-1$

$$s_0=0,\ w=\frac{\alpha\beta_1^6\left(\delta^2-1\right)^2k^2\left(8m^2+6m+1\right)}{16m^2s_1^4},\ \beta_1=\frac{k}{\sqrt{2}\sqrt{-\left(k_3+k_4\right)}},\ s_1=\frac{\sqrt{\alpha}\sqrt{1-\delta^2}k^2(4m+1)}{\sqrt{16k_2k_3m^2+16k_2k_4m^2}},$$
$$\sigma=\frac{2k_1}{k^2},\ k_5=-\frac{\alpha k^2\left(24m^2+10m+1\right)}{16\beta_1^2m^2}.$$

Afterwards, we have

$$\psi(x,t)=\left\{\frac{\sqrt{\alpha}\sqrt{\delta^2-1}k_1(4m+1)\mathrm{sech}\left(\sqrt{2}\sqrt{k_1}x\right)+2\sqrt{k_1}\sqrt{k_2}m\tanh\left(\sqrt{2}\sqrt{k_1}x\right)}{2\sqrt{k_2}\sqrt{-k_3-k_4}\left(\delta m\ \mathrm{sech}\left(\sqrt{2}\sqrt{k_1}x\right)+m\right)}\right\}^{1/n}e^{i(wt+\theta)},$$
$$k_1>0,\ k_3+k_4<0,\ k_2>0,\ \alpha(\delta^2-1)>0. \tag{3.18}$$

Case (2). When $\tau = 1$ and $\epsilon = -1$

$$s_0 = 0,\ w = \frac{\alpha\beta_1^6\left(\delta^2+1\right)^2 k^2\left(8m^2+6m+1\right)}{16m^2 s_1^4},\ s_1 = \frac{\sqrt{\alpha}\sqrt{-\delta^2-1}k^2(4m+1)}{\sqrt{16k_2k_3m^2+16k_2k_4m^2}},\ \beta_1 = \frac{k}{\sqrt{2}\sqrt{-k_3-k_4}},$$
$$\sigma = \frac{2k_1}{k^2},\ k_5 = -\frac{\alpha k^2\left(24m^2+10m+1\right)}{16\beta_1^2 m^2}.$$

Afterwards, we have

$$\psi(x,t) = \left\{\frac{\sqrt{\alpha}\sqrt{\delta^2+1}k_1(4m+1)\operatorname{csch}\left(\sqrt{2}\sqrt{k_1}x\right)+2\sqrt{k_1}\sqrt{k_2}m\coth\left(\sqrt{2}\sqrt{k_1}x\right)}{2\sqrt{k_2}\sqrt{-k_3-k_4}\left(\delta m\operatorname{csch}\left(\sqrt{2}\sqrt{k_1}x\right)+m\right)}\right\}^{1/n} e^{i(wt+\theta)},$$
$$k_1 > 0,\ k_3+k_4 < 0,\ k_2 > 0,\ \alpha > 0. \tag{3.19}$$

Case (3). When $\tau = -1$ and $\epsilon = 1$

$$s_0 = 0,\ w = \frac{\alpha\beta_1^6\left(\delta^2-1\right)^2 k^2\left(8m^2+6m+1\right)}{16m^2 s_1^4},\ s_1 = \frac{\sqrt{\alpha}\sqrt{1-\delta^2}k^2(4m+1)}{\sqrt{16k_2k_3m^2+16k_2k_4m^2}},\ \beta_1 = \frac{k}{\sqrt{2}\sqrt{-k_3-k_4}},$$
$$\sigma = -\frac{2k_1}{k^2},\ k_5 = -\frac{\alpha k^2\left(24m^2+10m+1\right)}{16\beta_1^2 m^2}.$$

Afterwards, we have

$$\psi(x,t) = \left\{\frac{\sqrt{\alpha}\sqrt{\delta^2-1}k_1(-4m-1)+2\sqrt{-k_1}\sqrt{k_2}m\sin\left(\sqrt{2}\sqrt{-k_1}x\right)}{2\sqrt{k_2}\sqrt{-k_3-k_4}m\left(\delta+\cos\left(\sqrt{2}\sqrt{-k_1}x\right)\right)}\right\}^{1/n} e^{i(wt+\theta)},$$
$$k_1 < 0,\ k_3+k_4 < 0,\ k_2 > 0,\ \alpha(\delta^2-1) > 0. \tag{3.20}$$

Case (4). When $\tau = 1$ and $\epsilon = 1$

$$s_0 = 0,\ w = \frac{\alpha\beta_1^6\left(\delta^2+1\right)^2 k^2\left(8m^2+6m+1\right)}{16m^2 s_1^4},\ s_1 = \frac{\sqrt{\alpha}\sqrt{-\delta^2-1}k^2(4m+1)}{\sqrt{16k_2k_3m^2+16k_2k_4m^2}},\ \beta_1 = \frac{k}{\sqrt{2}\sqrt{-k_3-k_4}},$$
$$\sigma = -\frac{2k_1}{k^2},\ k_5 = -\frac{\alpha k^2\left(24m^2+10m+1\right)}{16\beta_1^2 m^2}.$$

Afterwards, we have

$$\psi(x,t) = \left\{\frac{-\sqrt{\alpha}\sqrt{\delta^2+1}k_1(4m+1)-2\sqrt{-k_1}\sqrt{k_2}m\cos\left(\sqrt{2}\sqrt{-k_1}x\right)}{2\sqrt{k_2}\sqrt{-k_3-k_4}m\left(\delta+\sin\left(\sqrt{2}\sqrt{-k_1}x\right)\right)}\right\}^{1/n} e^{i(wt+\theta)},$$
$$k_1 < 0,\ k_3+k_4 < 0,\ k_2 > 0,\ \alpha > 0. \tag{3.21}$$

3.3. Application of the new generalized method

By applying the new generalized method, the following form is the solution to Eq (3.5)

$$Q(x) = \alpha_0 + \frac{\beta_1 \phi'(x) + \alpha_1}{\phi(x)}, \tag{3.22}$$

Applying Step-2 of the proposed method, We obtain the findings listed below:

Result (1):

$$\alpha_0 = \frac{m^2\left(2k^2\sigma - \alpha_1^2 k_3 - \alpha_1^2 k_4\right)}{\alpha k^2\left(20m^2 + 9m + 1\right)\sigma},\ \beta_1 = 0,\ \alpha_1 = \frac{\sqrt{\alpha}k\sqrt{24m^2 + 10m + 1}\sqrt{\sigma}}{2\sqrt{k_5}m},$$

$$w = \frac{\alpha_0 k^2\left(\alpha_0^2\sigma\left(8m^2 - \alpha\alpha_0\left(8m^2 + 6m + 1\right)\right) + \alpha_1^2\left(\alpha\alpha_0\left(8m^2 + 6m + 1\right) - 4m^2\right)\right)}{4\alpha_1^2 m^2},$$

$$k_2 = \frac{k^2\left(-24\alpha_0 m^2\sigma + 6\alpha\alpha_0^2(4m+1)^2\sigma - \alpha\alpha_1^2(4m+1)^2\right)}{4\alpha_1^2 m^2},$$

$$k_1 = \frac{k^2\left(-2\alpha\alpha_0^3\left(12m^2 + 7m + 1\right)\sigma + 12\alpha_0^2 m^2\sigma + \alpha\alpha_1^2\alpha_0\left(12m^2 + 7m + 1\right) - 2\alpha_1^2 m^2\right)}{2\alpha_1^2 m^2}.$$

Then, we have the following solutions

$$\psi(x,t) = \left\{\frac{1}{4}\left(\frac{8a\sqrt{\alpha}k\sqrt{24m^2 + 10m + 1}\sqrt{\sigma}\,e^x}{\sqrt{k_5}\left(4a^2 m e^{2x} + m\sigma\right)} - \frac{(k_3 + k_4)(6m+1)}{k_5(5m+1)} + \frac{8m^2}{\alpha + 20\alpha m^2 + 9\alpha m}\right)\right\}^{1/n} e^{i(wt+\theta)}. \tag{3.23}$$

When $\sigma = \pm 4a^2$, Eq (3.23) gives a bright soliton solution

$$\psi(x,t) = \left\{\frac{1}{4}\left(\frac{2\sqrt{\alpha}k\sqrt{(4m+1)(6m+1)}\,\mathrm{sech}(x)}{\sqrt{k_5}m} - \frac{(k_3 + k_4)(6m+1)}{k_5(5m+1)} + \frac{8m^2}{\alpha + 20\alpha m^2 + 9\alpha m}\right)\right\}^{1/n} e^{i(wt+\theta)}, \tag{3.24}$$

and a singular soliton solution

$$\psi(x,t) = \left\{\frac{1}{4}\left(\frac{2\sqrt{-\alpha}k\sqrt{(4m+1)(6m+1)}\,\mathrm{csch}(x)}{\sqrt{k_5}m} - \frac{(k_3 + k_4)(6m+1)}{k_5(5m+1)} + \frac{8m^2}{\alpha + 20\alpha m^2 + 9\alpha m}\right)\right\}^{1/n} e^{i(wt+\theta)}. \tag{3.25}$$

Result (2):

$$\alpha_0 = \frac{m^2\left(2k^2 + \beta_1^2 k_3 + \beta_1^2 k_4\right)}{\alpha k^2\left(20m^2 + 9m + 1\right)},\ \alpha_1 = 0,\ \beta_1 = \frac{\sqrt{-\alpha}k\sqrt{24m^2 + 10m + 1}}{2\sqrt{k_5}m},$$

$$w = \frac{k^2\left(\beta_1^2 - \alpha_0^2\right)\left(\alpha\beta_1^2\left(8m^2 + 6m + 1\right) + 8\alpha_0 m^2 - \alpha\alpha_0^2\left(8m^2 + 6m + 1\right)\right)}{4\beta_1^2 m^2},$$

$$k_2 = \frac{k^2\left(12\alpha_0 m^2 + \alpha\beta_1^2(4m+1)^2 - 3\alpha\alpha_0^2(4m+1)^2\right)}{2\beta_1^2 m^2},$$

$$k_1 = \frac{k^2\left(-\alpha\alpha_0\beta_1^2\left(12m^2+7m+1\right) + \alpha\alpha_0^3\left(12m^2+7m+1\right) - 6\alpha_0^2 m^2 + 2\beta_1^2 m^2\right)}{\beta_1^2 m^2}.$$

Afterwards, we have

$$\psi(x,t) = \left\{\frac{\sqrt{-\alpha}k\sqrt{24m^2+10m+1}\left(ae^x - \frac{\sigma e^{-x}}{4a}\right)}{2\sqrt{k_5}m\left(\frac{\sigma e^{-x}}{4a} + ae^x\right)} + \frac{2m^2}{\alpha\left(20m^2+9m+1\right)}\right\}^{1/n} e^{i(wt+\theta)}. \quad (3.26)$$

When $\sigma = \pm 4a^2$, Eq (3.26) gives a dark soliton solution

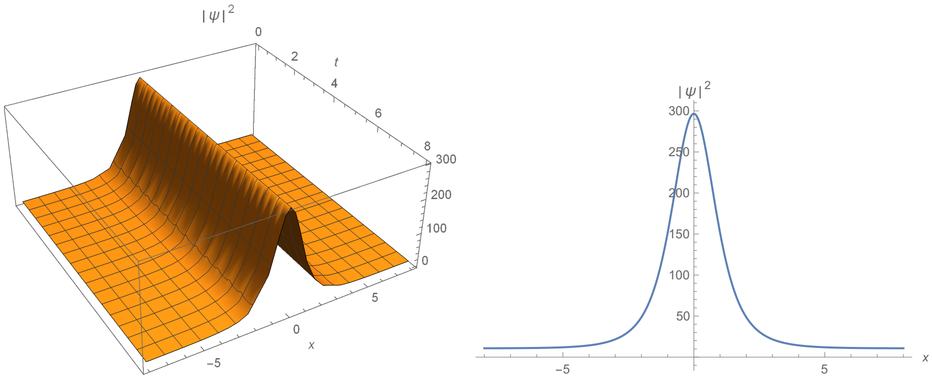
$$\psi(x,t) = \left\{\frac{1}{4}\left(\frac{2\sqrt{-\alpha}k\sqrt{(4m+1)(6m+1)}\tanh(x)}{\sqrt{k_5}m} - \frac{(k_3+k_4)(6m+1)}{k_5(5m+1)} + \frac{8m^2}{\alpha+20\alpha m^2+9\alpha m}\right)\right\}^{1/n} e^{i(wt+\theta)}, \quad (3.27)$$

and a singular soliton solution

$$\psi(x,t) = \left\{\frac{1}{4}\left(\frac{2\sqrt{-\alpha}k\sqrt{(4m+1)(6m+1)}\coth(x)}{\sqrt{k_5}m} - \frac{(k_3+k_4)(6m+1)}{k_5(5m+1)} + \frac{8m^2}{\alpha+20\alpha m^2+9\alpha m}\right)\right\}^{1/n} e^{i(wt+\theta)}. \quad (3.28)$$

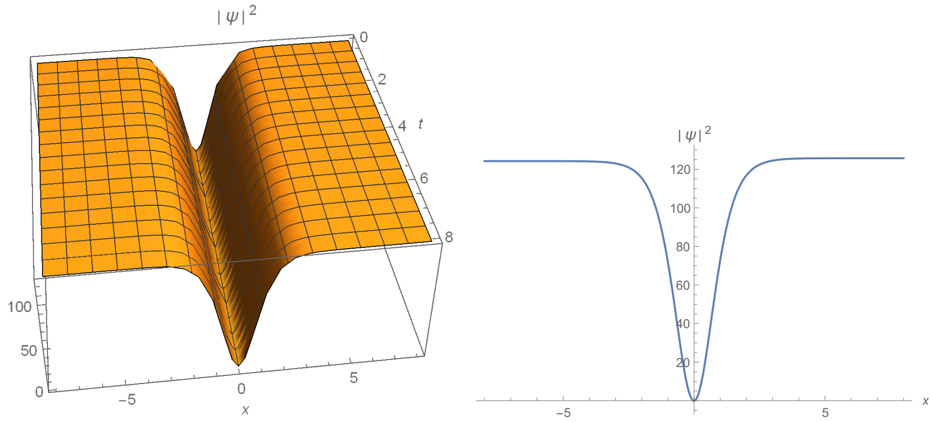
4. Graphical illustrations

The nature of some extracted solutions is demonstrated by the presentation of graphic simulations. A bright solitary solution of Eq (3.24) with $m = n = 1,\ k = k_3 = k_4 = \alpha = -2,\ k_5 = -0.36$ is shown in Figure 1. A dark solitary solution of Eq (3.27) with $n = m = 1,\ k = k_3 = \alpha = -2,\ k_4 = 2,\ k_5 = 0.56$ is graphically illustrated in Figure 2. The extracted solutions demonstrate a certain balance between dispersion and nonlinearity for the investigated model. This results in solitons that have the advantage of being highly stable, as they can travel very long distances while still maintaining their velocity and shape. Figure 3 depicts the graphical illustration of a singular solitary solution of Eq (3.28) with $n = m = 1,\ k = k_3 = \alpha = -2,\ k_4 = 1.75,\ k_5 = 2$. This solution represents a rare phenomenon in nonlinear physics, characterized by a point of singularity or divergence in intensity. It captures the abrupt change at the point, offering insight into the interplay of nonlinearity and dispersion in forming exotic solitary waves. Figure 4 depicts the graphical illustration of a singular periodic solution of Eq (3.16) with $k_1 = k_3 = k_4 = -2$. This solution displays a periodically repeating wave with a point of singularity, showing valuable insights into the behavior of nonlinear systems with recurring singularities.



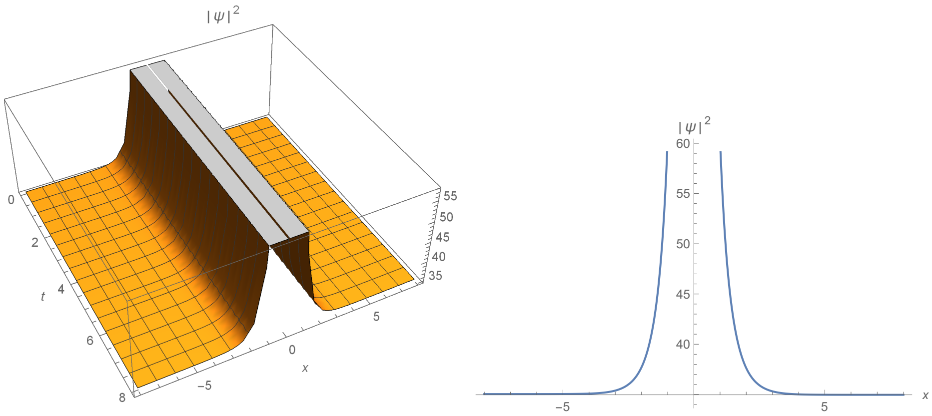
(a) **(b)**

Figure 1. Graphical representation of Eq (3.24) using $n = m = 1,\ k = k_3 = k_4 = \alpha = -2,\ k_5 = -0.36$.



(a) **(b)**

Figure 2. Graphical representation of Eq (3.27) using $n = m = 1,\ k = k_3 = \alpha = -2,\ k_4 = 2,\ k_5 = 0.56$.



(a) **(b)**

Figure 3. Graphical representation of Eq (3.28) using $n = m = 1,\ k = k_3 = \alpha = -2,\ k_4 = 1.75,\ k_5 = 2$.

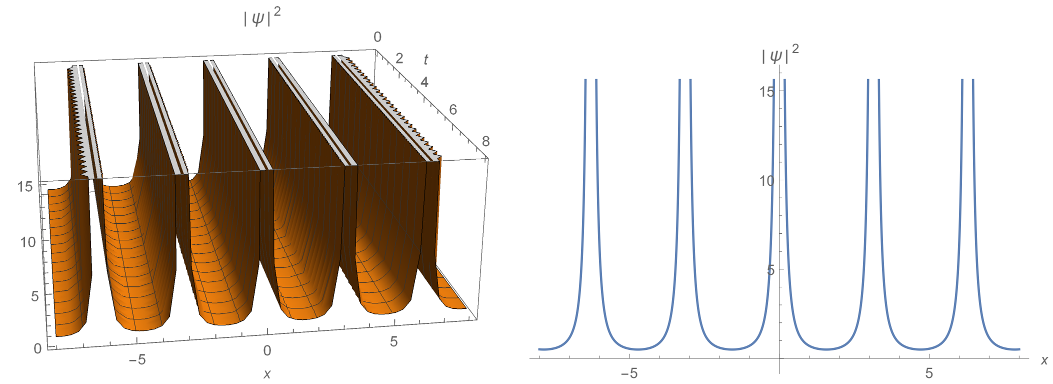


Figure 4. Graphical representation of Eq (3.16) using $k_1 = k_3 = k_4 = -2$.

5. Conclusions

The generalized NLSE, which simulates wave transmission via optical fibers, was examined in this work. This model incorporates a quintuple power-law of non-linearity and nonlinear chromatic dispersion. Three integration techniques are implemented to conduct this study. These techniques are the modified exp-function method, the general projective Riccati method, and the new generalized method. Different solutions are derived for the studied model, including singular solitons, bright solitons, dark solitons, and singular periodic solutions. These findings may be useful for optical communication and for a deeper comprehension of many phenomena that arise in various physical systems that are governed by the current model.

Author contributions

Islam Samir: Formal analysis, Software; Hamdy M. Ahmed: Validation, Methodology; Wafaa B. Rabie: Resources, Writing–review & editing; W. Abbas: Writing–review & editing, Investigation; Ola Mostafa: Software, Investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. Y. Sun, Z. Hu, H. Triki, M. Mirzazadeh, W. Liu, A. Biswas, et al., Analytical study of three-soliton interactions with different phases in nonlinear optics, *Nonlinear Dyn.*, **111** (2023), 18391–18400. https://doi.org/10.1007/s11071-023-08786-z

2. Y. Zhong, K. Yu, Y. Sun, H. Triki, Q. Zhou, Stability of solitons in Bose–Einstein condensates with cubic–quintic–septic nonlinearity and non-PT-symmetric complex potentials, *Eur. Phys. J. Plus*, **139** (2024), 119. https://doi.org/10.1140/epjp/s13360-024-04930-9
3. Q. Li, W. Zou, Normalized ground states for Sobolev critical nonlinear Schrödinger equation in the L^2-supercritical case, *DCDS*, **44** (2024), 205–227. https://doi.org/10.3934/dcds.2023101
4. S. Malik, S. Kumar, Pure-cubic optical soliton perturbation with full nonlinearity by a new generalized approach, *Optik*, **258** (2022), 168865. https://doi.org/10.1016/j.ijleo.2022.168865
5. Q. Li, J. Nie, W. Wang, J. Zhou, Normalized solutions for Sobolev critical fractional Schrödinger equation, *Adv. Nonlinear Anal.*, **13** (2024), 20240027. https://doi.org/10.1515/anona-2024-0027
6. W. Zhang, J. Zhang, V. Rădulescu, Semiclassical states for the pseudo-relativistic Schrödinger equation with competing potentials, *Commun. Math. Sci.*, **23** (2025), 465–507.
7. S. Shen, Z. J. Yang, Z. G. Pang, Y. R. Ge, The complex-valued astigmatic cosine-Gaussian soliton solution of the nonlocal nonlinear Schrödinger equation and its transmission characteristics, *Appl. Math. Lett.*, **125** (2022), 107755. https://doi.org/10.1016/j.aml.2021.107755
8. M. A. E. Abdelrahman, A. Alharbi, M. B. Almatrafi, Fundamental solutions for the generalised third-order nonlinear Schrödinger equation, *Int. J. Appl. Comput. Math.*, **6** (2020), 160. https://doi.org/10.1007/s40819-020-00906-2
9. W. B. Rabie, H. M. Ahmed, Construction cubic-quartic solitons in optical metamaterials for the perturbed twin-core couplers with Kudryashov's sextic power law using extended F-expansion method, *Chaos Soliton Fract.*, **160** (2022), 112289. https://doi.org/10.1016/j.chaos.2022.112289
10. I. Onder, A. Secer, M. Ozisik, M. Bayram, Investigation of optical soliton solutions for the perturbed Gerdjikov-Ivanov equation with full-nonlinearity, *Heliyon*, **9** (2023), e13519. https://doi.org/10.1016/j.heliyon.2023.e13519
11. I. Samir, H. M. Ahmed, S. Alkhatib, E. M. Mohamed, Construction of wave solutions for stochastic Radhakrishnan–Kundu–Lakshmanan equation using modified extended direct algebraic technique, *Results Phys.*, **55** (2023), 107191. https://doi.org/10.1016/j.rinp.2023.107191
12. R. Kumar, R. Kumar, A. Bansal, A. Biswas, Y. Yildirim, S. P. Moshokoa, et al., Optical solitons and group invariants for Chen-Lee-Liu equation with time-dependent chromatic dispersion and nonlinearity by Lie symmetry, *Ukr. J. Phys. Opt.*, 2023.
13. M. M. Khatun, M. A. Akbar, New optical soliton solutions to the space-time fractional perturbed Chen-Lee-Liu equation, *Results Phys.*, **46** (2023), 106306. https://doi.org/10.1016/j.rinp.2023.106306
14. M. Sadaf, G. Akram, S. Arshed, K. Farooq, A study of fractional complex Ginzburg–Landau model with three kinds of fractional operators, *Chaos Soliton. Fract.*, **166** (2023), 112976. https://doi.org/10.1016/j.chaos.2022.112976
15. K. J. Wang, J. Si, Diverse optical solitons to the complex Ginzburg–Landau equation with Kerr law nonlinearity in the nonlinear optical fiber, *Eur. Phys. J. Plus*, **138** (2023), 187. https://doi.org/10.1140/epjp/s13360-023-03804-w

16. W. Chen, J. Manafian, K. H. Mahmoud, A. S. Alsubaie, A. Aldurayhim, A. Alkader, Cutting-edge analytical and numerical approaches to the Gilson–Pickering equation with plenty of soliton solutions, *Mathematics*, **11** (2023), 3454. https://doi.org/10.3390/math11163454

17. A. Yokuş, H. Durur, K. A. Abro, D. Kaya, Role of Gilson–Pickering equation for the different types of soliton solutions: a nonlinear analysis, *Eur. Phys. J. Plus*, **135** (2020), 657. https://doi.org/10.1140/epjp/s13360-020-00646-8

18. N. A. Kudryashov, The Lakshmanan–Porsezian–Daniel model with arbitrary refractive index and its solution, *Optik*, **241** (2021), 167043. https://doi.org/10.1016/j.ijleo.2021.167043

19. G. Liang, H. Zhang, L. Fang, Q. Shou, W. Hu, Q. Guo, Influence of transverse cross-phases on propagations of optical beams in linear and nonlinear regimes, *Laser Photonics Rev.*, **14** (2020), 2000141. https://doi.org/10.1002/lpor.202000141

20. I. Samir, H. M. Ahmed, A. Darwish, H. H. Hussein, Dynamical behaviors of solitons for NLSE with Kudryashov's sextic power-law of nonlinear refractive index using improved modified extended tanh-function method, *Ain Shams Eng. J.*, **15** (2024), 102267. https://doi.org/10.1016/j.asej.2023.102267

21. O. El-shamy, R. El-barkoki, H. M. Ahmed, W. Abbas, I. Samir, Exploration of new solitons in optical medium with higher-order dispersive and nonlinear effects via improved modified extended tanh function method, *Alex. Eng. J.*, **68** (2023), 611–618. https://doi.org/10.1016/j.aej.2023.01.053

22. I. Samir, A. Abd-Elmonem, H. M. Ahmed, General solitons for eighth-order dispersive nonlinear Schrödinger equation with ninth-power law nonlinearity using improved modified extended tanh method, *Opt. Quant. Electron.*, **55** (2023), 470. https://doi.org/10.1007/s11082-023-04753-5

23. Y. Shang, The extended hyperbolic function method and exact solutions of the long–short wave resonance equations, *Chaos Soliton. Fract.*, **36** (2008), 762–771. https://doi.org/10.1016/j.chaos.2006.07.007

24. L. Akinyemi, Two improved techniques for the perturbed nonlinear Biswas–Milovic equation and its optical solitons, *Optik*, **243** (2021), 167477. https://doi.org/10.1016/j.ijleo.2021.167477

25. N. Savaissou, B. Gambo, H. Rezazadeh, A. Bekir, S. Y. Doka, Exact optical solitons to the perturbed nonlinear Schrödinger equation with dual-power law of nonlinearity, *Opt. Quant. Electron.*, **52** (2020), 318. https://doi.org/10.1007/s11082-020-02412-7

26. E. A. Az-Zo'bi, W. A. Alzoubi, L. Akinyemi, M. Şenol, B. S. Masaedeh, A variety of wave amplitudes for the conformable fractional (2+1)-dimensional Ito equation, *Mod. Phys. Lett. B*, **35** (2021), 2150254. https://doi.org/10.1142/S0217984921502547

27. W. B. Rabie, H. M. Ahmed, I. Samir, M. Alnahhass, Optical solitons and stability analysis for NLSE with nonlocal nonlinearity, nonlinear chromatic dispersion and Kudryashov's generalized quintuple-power nonlinearity, *Results Phys.*, **59** (2024), 107589. https://doi.org/10.1016/j.rinp.2024.107589

28. A. Biswas, M. Ekici, A. Sonmezoglu, Stationary optical solitons with Kudryashov's quintuple power–law of refractive index having nonlinear chromatic dispersion, *Phys. Lett. A*, **426** (2022), 127885. https://doi.org/10.1016/j.physleta.2021.127885

29. M. Ekici, Stationary optical solitons with complex Ginzburg–Landau equation having nonlinear chromatic dispersion and Kudryashov's refractive index structures, *Phys. Lett. A*, **440** (2022), 128146. https://doi.org/10.1016/j.physleta.2022.128146

30. A. Sonmezoglu, Stationary optical solitons having Kudryashov's quintuple power law nonlinearity by extended G'/G–expansion, *Optik*, **253** (2022), 168521. https://doi.org/10.1016/j.ijleo.2021.168521

31. M. G. Hafez, M. A. Akbar, An exponential expansion method and its application to the strain wave equation in microstructured solids, *Ain Shams Eng. J.*, **6** (2015), 683–690. https://doi.org/10.1016/j.asej.2014.11.011

32. G. Akram, M. Sadaf, S. Arshed, F. Sameen, Bright, dark, kink, singular and periodic soliton solutions of Lakshmanan–Porsezian–Daniel model by generalized projective Riccati equations method, *Optik*, **241** (2021), 167051. https://doi.org/10.1016/j.ijleo.2021.167051

AIMS Press