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Research article

Second-order differential equations with mixed neutral terms: new oscillation theorems

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Abstract: Since differential equations play a major role in mathematics, physics, and engineering, the study of the oscillatory behavior of these equations is of great importance. In this paper, we apply the comparison method with first-order differential equations to study the oscillatory behavior of second-order differential equations. New oscillation criteria were obtained to improve some of the results of previous studies. Examples are included to illustrate the importance and novelty of the presented results.

Keywords: oscillation criteria; differential equations; non-canonical case; mixed neutral terms
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this paper, we study the oscillation of a class of second-order differential equations (DEs) with mixed neutral terms of the form

$$(a(s)\varpi'(s))' + h(s)u(\epsilon(s)) = 0, \ \ s \geq s_0, \tag{1.1}$$

where

$$\varpi(s) = u(s) + \rho_1(s)u(\delta(s)) + \rho_2(s)u(\lambda(s)).$$

Throughout this paper, we will assume that the following conditions hold:

(H1) $a \in C([s_0, \infty), (0, \infty))$ satisfies condition

$$\int_{s_0}^{\infty} \frac{1}{a(\xi)} \mathrm{d}\xi < \infty; \tag{1.2}$$

(H2) $\epsilon \in C([s_0, \infty), (0, \infty))$, $\epsilon(s) \le s$, $\epsilon'(s) > 0$, and $\lim_{s\to\infty} \epsilon(s) = \infty$;

(H3) $\rho_1, \rho_2 \in C([s_0, \infty), [0, 1))$, $h \in C([s_0, \infty), [0, \infty))$ and $h(s)$ is not identically zero in any interval of $[s_0, \infty)$;

(H4) $\delta, \lambda \in C([s_0, \infty), (0, \infty))$, $\delta(s) \le s$, $\lambda(s) \ge s$ and $\lim_{s\to\infty} \delta(s) = \lim_{s\to\infty} \lambda(s) = \infty$.

By a solution of (1.1), we mean a function $u \in C^1([s_u, \infty), \mathbb{R})$ $s_u \ge s_0$, which has the property that $a(s)(\varpi'(s))$ are continuously differentiable for $s \in [s_u, \infty)$. We only consider those solutions $u(s)$ of (1.1) satisfying $\sup\{|u(s)| : s \ge s_*\} > 0$ for all $s_* \ge s_u$, and we assume that (1.1) possesses such solutions.

A solution of (1.1) is called oscillatory if it has arbitrarily many zeros on $[s_u, \infty)$; and is called nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

The problem of the oscillation of solutions of differential equations has been widely studied by many authors and by many techniques since the pioneering work of Sturm on second-order linear differential equations. As we know, many recent studies have been interested in studying the oscillatory behavior of solutions of functional DEs of various orders. The reader can refer to the papers [1–5] for second-order equations, the papers [6–9] for third-order equations, and the papers [10–14] for higher-order equations.

One of the major branching issues of DEs is the oscillatory behavior of ordinary DEs. The oscillation problems of ordinary DEs can be used to describe the oscillatory problems in the plane's wings. There are many uses for DEs with arguments in the natural sciences and engineering (for additional information, see [15–18]).

The advancement of modern science and technology, including economics, aerospace, and modern physics, as well as social development, has led to an increasing interest in delay DEs in recent decades. As is often known, delayed DEs use the reliance on the past state to forecast the future state with accuracy and efficiency. In the meantime, many qualitative characteristics, such as periodicity, stability, and boundedness, can be explained. The delay effect will be important in expressing the time required to complete a concealed procedure if we include it in the models. Conversely, unlike genetic systems, advanced DEs can be used in practically every field of the actual world. Applications of such DEs can be found in fields such as population dynamics in mathematical biology, mechanical control in engineering, or economic difficulties [19].

The oscillatory behavior of DEs, particularly those of the neutral type, is a topic of growing interest. The fact that these equations may replicate a wide range of situations, such as electrical networks, a vibrating mass connected to an elastic rod, etc., makes them practically significant [20].

It is known that some studies have been interested in studying the oscillatory behavior of second-order neutral DEs, and we mention some of them, for example:

Tunc et al. [21] considered the second-order neutral DE

$$\left(a(s)\left((u(s) + \rho_1(s)u(\delta(s)) + \rho_2(s)u(\lambda(s)))'\right)^{\alpha}\right)' + h(s)u^{\alpha}(\epsilon(s)) = 0, \ s \ge s_0, \tag{1.3}$$

where α is the ratio of odd positive integers. They set new sufficient conditions for the oscillation of the solutions of (1.3) under the condition

$$\int_{s_0}^{\infty} \frac{1}{a^{1/\alpha}(\xi)} \mathrm{d}\xi = \infty. \tag{1.4}$$

The results they obtain improve and complete some well-known results in the relevant literature.

Grace et al. [22] established some sufficient conditions for the oscillation of the DEs

$$\left(a(s)\left(\left(u(s)+\rho_1(s)u^{\beta_1}(\delta(s))-\rho_2(s)u^{\beta_2}(\delta(s))\right)'\right)^{\alpha}\right)'+h(s)u^{\beta_3}(\epsilon(s))+c(s)u^{\beta_4}(\epsilon_1(s))=0,$$

and presented new results that extend, generalize and simplify the results found in the literature. They also analyzed the oscillatory and asymptotic behavior of solutions of the equation

$$\left(a(s)\left(\left(u(s)+\rho_1(s)u^{\beta_1}(\delta(s))-\rho_2(s)u^{\beta_2}(\delta(s))\right)'\right)^{\alpha}\right)'=h(s)u^{\beta_3}(\epsilon(s))+c(s)u^{\beta_4}(\epsilon_1(s)),$$

under the condition (1.4), where $s \geq s_0$, $\epsilon_1(s) \geq s$, $\lim_{s\to\infty}\epsilon_1(s)=\infty$, α,β_1,β_2, β_3, andβ_4 are the ratios of odd positive integers with $0<\beta_1<1$ and$\beta_1>1$.

Moaaz et al. [23] discussed the oscillation behavior of solutions of the DE

$$(a(s)((u(s)+\rho_1(s)u(\delta(s))+\rho_2(s)u(\lambda(s)))')^{\alpha})'+h(s)u^{\alpha}(\epsilon(s))=0,\ s\geq s_0, \tag{1.5}$$

where α is the ratio of odd positive integers. The authors have developed new oscillation theorems to test the oscillation of solutions of DE (1.5). These theorems aim to complement and simplify related results in the literature. They have also provided an example of the application of their results. For the convenience of the reader, we mention one of their results.

Theorem 1.1. *Assume that*

$$1-\rho_1(s)-\rho_2(s)\frac{\mu(\lambda(s))}{\mu(s)}\geq 1-\rho_1(s)\frac{\eta(\delta(s))}{\eta(s)}-\rho_2(s)>0. \tag{1.6}$$

If

$$\limsup_{s\to\infty}\eta^{\alpha}(s)\int_{s_1}^{s}h(\varrho)\left(1-\rho_1(\varrho)\frac{\eta(\delta(\varrho))}{\eta(\varrho)}-\rho_2(\varrho)\right)\mathrm{d}\varrho>1, \tag{1.7}$$

where

$$\eta(s)=\int_{s}^{\infty}\frac{1}{a(\xi)}\mathrm{d}\xi \text{ and } \mu(s)=\int_{s_0}^{s}\frac{1}{a(\xi)}\mathrm{d}\xi,$$

then (1.5) is oscillatory.

Grace et al. [24] studied the oscillatory behavior of nonlinear noncanonical neutral DEs

$$(a(s)(u(s)+\rho_1(s)u(\delta(s)))')'+h(s)u^{\alpha}(\epsilon(s))=0, \tag{1.8}$$

where $s\geq s_0>0$ and α is the ratio of odd positive integers with $0<\alpha\leq 1$. They provided sufficient conditions for all solutions to be oscillatory. For the convenience of the reader, we mention one of their results.

Theorem 1.2. *If*

$$\int_{s_0}^{\infty}\eta(\xi)h(\xi)\mathrm{d}\xi=\infty, \tag{1.9}$$

and

$$\limsup_{s\to\infty}\left(\eta(s)\int_{s_0}^{s}h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}\right)^{\alpha}\mathrm{d}\varrho\right.$$
$$\left.+\eta^{-\alpha}(\epsilon(s))\int_{s}^{\infty}\eta(\varrho)h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}\right)^{\alpha}\eta^{\alpha}(\epsilon(\varrho))\,\mathrm{d}\varrho\right)>\begin{cases}1, & \text{if } \alpha=1,\\ 0, & \text{if } 0<\alpha<1,\end{cases} \tag{1.10}$$

where

$$\eta(s)=\int_{s}^{\infty}\frac{1}{a(\xi)}\mathrm{d}\xi,$$

then (1.8) is oscillatory.

Based on the above, in this paper, we aim to establish new conditions using some relations and inequalities to obtain new oscillation criteria for the studied equation using the comparison method with first-order differential equations. We also compare our results with previous studies by providing examples to show that our results improve those studies.

2. Main results

Our first oscillation result is as follows:

Theorem 2.1. *If*

$$\int_{s_1}^{\infty}\left(\frac{1}{a(\theta)}\left(\int_{s_1}^{\theta}h(\varrho)\eta(\epsilon(\varrho))\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\right)\mathrm{d}\theta=\infty, \tag{2.1}$$

where

$$\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}+\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}<1,$$

then (1.1) is oscillatory.

Proof. Assume that (1.1) has a positive solution $u(s)$. Thus, there exists a $s_1 \geq s_0$ such that $u(\delta(s)) > 0$ and $u(\epsilon(s)) > 0$ for $s \geq s_1$. Since $u(s) > 0$ and $\rho_1, \rho_2 \in [0, 1)$, we see that $\varpi(s) > 0$, and

$$(a(s)\varpi'(s))' = -h(s)u(\epsilon(s)) \leq 0, \tag{2.2}$$

thus, we see that $a(s)\varpi'(s)$ has one sign. Therefore, we have two cases.
(i) Assume that $\varpi'(s) < 0$. Hence,

$$\begin{aligned}\varpi(s) &\geq -\int_{s}^{\infty}\frac{1}{a(\zeta)}(a(\zeta)\varpi'(\zeta))\,\mathrm{d}\xi\\ &\geq -a(s)\varpi'(s)\eta(s),\end{aligned} \tag{2.3}$$

since $a(s)\varpi'(s)$ is decreasing, we find

$$a(s)\varpi'(s) \leq a(s_1)\varpi'(s_1) := -K < 0, \tag{2.4}$$

where $K > 0$, using (2.3) and (2.4), we obtain

$$\varpi(s) \geq K\eta(s). \tag{2.5}$$

From (2.3), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\varpi(s)}{\eta(s)}\right)=\frac{\eta(s)\,a(s)\,\varpi'(s)+\varpi(s)}{\eta^2(s)\,a(s)}\geq 0. \tag{2.6}$$

From definition $\varpi(s)$, we conclude that

$$u(s)=\varpi(s)-\rho_1(s)\,u(\delta(s))-\rho_2(s)\,u(\lambda(s))\geq\varpi(s)-\rho_1(s)\,\varpi(\delta(s))-\rho_2(s)\,\varpi(\lambda(s)), \tag{2.7}$$

using (2.6) and (H4), we have

$$u(s)\geq\varpi(s)-\rho_1(s)\,\frac{\varpi(s)\,\eta(\delta(s))}{\eta(s)}-\rho_2(s)\,\varpi(s)=\varpi(s)\left(1-\rho_1(s)\,\frac{\eta(\delta(s))}{\eta(s)}-\rho_2(s)\right),$$

and so

$$u(\epsilon(s))\geq\varpi(\epsilon(s))\left(1-\rho_1(\epsilon(s))\,\frac{\eta(\delta(\epsilon(s)))}{\eta(\epsilon(s))}-\rho_2(\epsilon(s))\right), \tag{2.8}$$

using (2.2) and (2.8), we obtain:

$$(a(s)\,\varpi'(s))'\leq -h(s)\,\varpi(\epsilon(s))\left(1-\rho_1(\epsilon(s))\,\frac{\eta(\delta(\epsilon(s)))}{\eta(\epsilon(s))}-\rho_2(\epsilon(s))\right), \tag{2.9}$$

from (2.5), we obtain

$$(a(s)\,\varpi'(s))'\leq -h(s)\,K\eta(\epsilon(s))\left(1-\rho_1(\epsilon(s))\,\frac{\eta(\delta(\epsilon(s)))}{\eta(\epsilon(s))}-\rho_2(\epsilon(s))\right). \tag{2.10}$$

Since $\mu'(s)>0$, we conclude that

$$\mu(\lambda(\epsilon(s)))\geq\mu(\epsilon(s)). \tag{2.11}$$

Integrating (2.10) from s_1 to s; and using (2.11), we find

$$a(s)\,\varpi'(s)\leq -K\int_{s_1}^{s}h(\varrho)\,\eta(\epsilon(\varrho))\left(1-\rho_1(\epsilon(\varrho))\,\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\,\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho. \tag{2.12}$$

Integrating (2.12) from s_1 to s, we obtain:

$$\varpi(s)\leq\varpi(s_1)-K\int_{s_1}^{s}\left(\frac{1}{a(\theta)}\left(\int_{s_1}^{\theta}h(\varrho)\,\eta(\epsilon(\varrho))\left(1-\rho_1(\epsilon(\varrho))\,\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\,\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\right)\mathrm{d}\theta. \tag{2.13}$$

By comparing (2.1) and (2.13), we conclude that $\varpi(s)\to-\infty$ as $s\to\infty$, and this contradicts $\varpi(s)>0$.

(ii) Assume that $\varpi'(s)>0$. Thus, we see that $\varpi(s)\geq\varpi(\delta(s))\geq u(\delta(s))$, and hence,

$$\begin{aligned}\varpi(s)&=\varpi(s_1)+\int_{s_1}^{s}\frac{1}{a(\zeta)}(a(\zeta)\,\varpi'(\zeta))\,\mathrm{d}\xi\geq(a(s)\,\varpi'(s))\int_{s_1}^{s}\frac{1}{a(\zeta)}\mathrm{d}\xi\\&\geq a(s)\,\varpi'(s)\,\mu(s),\end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\varpi(s)}{\mu(s)}\right)=\frac{\mu(s)\,a(s)\,\varpi'(s)-\varpi(s)}{\mu^2(s)\,a(s)}\leq 0, \tag{2.14}$$

using (2.7) and (2.14), we see that

$$u(s) \geq \varpi(s) - \rho_1(s)\varpi(s) - \rho_2(s)\frac{\varpi(s)\mu(\lambda(s))}{\mu(s)} = \varpi(s)\left(1 - \rho_1(s) - \rho_2(s)\frac{\mu(\lambda(s))}{\mu(s)}\right),$$

and so

$$u(\epsilon(s)) \geq \varpi(\epsilon(s))\left(1 - \rho_1(\epsilon(s)) - \rho_2(\epsilon(s))\frac{\mu(\lambda(\epsilon(s)))}{\mu(\epsilon(s))}\right). \tag{2.15}$$

Using (2.2) and (2.15), we obtain:

$$(a(s)\varpi'(s))' \leq -h(s)\varpi(\epsilon(s))\left(1 - \rho_1(\epsilon(s)) - \rho_2(\epsilon(s))\frac{\mu(\lambda(\epsilon(s)))}{\mu(\epsilon(s))}\right). \tag{2.16}$$

Since $\eta'(s) < 0$, we conclude that

$$\eta(\delta(\epsilon(s))) \geq \eta(\epsilon(s)). \tag{2.17}$$

Integrating (2.16) from $s_1$to s, and using (2.17), we have

$$\begin{aligned} a(s)\varpi'(s) &\leq -\int_{s_1}^{s} h(\varrho)\varpi(\epsilon(\varrho))\left(1 - \rho_1(\epsilon(\varrho)) - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho \\ &\quad + a(s_1)\varpi'(s_1) \\ &\leq -\varpi(\epsilon(s_1))\int_{s_1}^{s} h(\varrho)\left(1 - \rho_1(\epsilon(\varrho)) - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho \\ &\quad + a(s_1)\varpi'(s_1) \\ &\leq -\varpi(\epsilon(s_1))\int_{s_1}^{s} h(\varrho)\left(1 - \rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))} - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho \\ &\quad + a^1(s_1)\varpi'(s_1). \end{aligned} \tag{2.18}$$

Since $\eta'(s) < 0$, we find

$$\begin{aligned} &\int_{s_1}^{s} \eta(\epsilon(\varrho))h(\varrho)\left(1 - \rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))} - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho \\ &\quad \leq \eta(\epsilon(s_1))\int_{s_1}^{s} h(\varrho)\left(1 - \rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))} - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho\ . \end{aligned} \tag{2.19}$$

It follow from (2.1) and (H1) that $\int_{s_1}^{s} h(\varrho)\eta(\epsilon(\varrho))\left(1 - \rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))} - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho$ must be unbounded. Hence, from (2.19), we get

$$\int_{s_1}^{s} h(\varrho)\left(1 - \rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))} - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho \to \infty \text{ as } s \to \infty. \tag{2.20}$$

Thus, and from (2.18), we conclude that $\varpi'(s) \to -\infty$ as $s \to \infty$, and this contradicts $\varpi'(s) > 0$. The proof is completed. □

Theorem 2.2. *Assume that*

$$\int_{s_0}^{\infty} h(\varrho)\left(1 - \rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))} - \rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)d\varrho = \infty, \tag{2.21}$$

and

$$\varpi'(s)+\frac{1}{a(s)}\left(\int_{s_1}^{s}h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\varpi(\epsilon(s))=0 \quad (2.22)$$

is oscillatory. Then, (1.1) is oscillatory.

Proof. As in the proof of Theorem 2.1, we find that $a(s)\varpi'(s)$ is of one sign.
(i) Assume that $\varpi'(s)<0$; therefore, we have (2.9) and (2.11) hold. Integrating (2.9) from $s_1$to s, and using (2.11), we see that

$$a(s)\varpi'(s)\leq a(s_1)\varpi'(s_1)-\int_{s_1}^{s}h(\varrho)\varpi(\epsilon(\varrho))\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho,$$

and so

$$\varpi'(s)\leq-\frac{\varpi(\epsilon(s))}{a(s)}\left(\int_{s_1}^{s}h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right).$$

Hence, we see that ϖ is a positive solution of

$$\varpi'(s)+\frac{1}{a(s)}\left(\int_{s_1}^{s}h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\varpi(\epsilon(s))\leq 0. \quad (2.23)$$

In view of [25, Lemma 1], we see that (2.22) has a positive solution, a contradiction.
(ii) Assume that $\varpi'(s)>0$, then (2.21) leads to (2.20). The rest of this proof is comparable to the proof of Theorem 2.1. The proof is completed. □

We now present a new criterion for the oscillation of (1.1) using the results of [25].

Corollary 2.1. *If (2.21) holds, and*

$$\liminf_{s\to\infty}\int_{\epsilon(s)}^{s}\frac{1}{a(\theta)}\left(\int_{s_1}^{\theta}h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\mathrm{d}\theta>\frac{1}{\mathrm{e}}, \quad (2.24)$$

then (1.1) is oscillatory.

Example 2.1. *Consider the DE*

$$\left(s^2\left[u(s)+\frac{1}{32}u\left(\frac{s}{2}\right)+\frac{1}{64}u(2s)\right]'\right)'+h_0u\left(\frac{s}{3}\right)=0, \quad (2.25)$$

where $a(s)=s^2$, $\rho_1(s)=1/32$, $\rho_2(s)=1/64$, $\delta(s)=s/2$, $\lambda(s)=2s$, $\epsilon(s)=s/3$, and $h(s)=h_0$. Now, we see that

$$\eta(s)=\frac{1}{s},\ \eta(\epsilon(s))=\frac{3}{s},\ \eta(\delta(\epsilon(s)))=\frac{6}{s},$$

$$\mu(s)=-\frac{1}{s},\ \mu(\epsilon(s))=-\frac{3}{s}\text{ and }\mu(\lambda(\epsilon(s)))=-\frac{3}{2s}.$$

Therefore, the condition (2.21) is satisfied, where

$$\begin{aligned}&\int_{s_0}^{\infty} h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\\ =\;& h_0\left(1-\frac{1}{32}(2)-\frac{1}{64}\left(\frac{1}{2}\right)\right)\int_{s_0}^{\infty}\mathrm{d}\varrho=\infty,\end{aligned}$$

and the condition (2.24); becomes

$$\begin{aligned}&\liminf_{s\to\infty}\int_{\epsilon(s)}^{s}\frac{1}{a(\theta)}\left(\int_{s_1}^{\theta} h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\mathrm{d}\theta\\ =\;& h_0\left(1-\frac{1}{16}-\frac{1}{128}\right)\ln 3>\frac{1}{\mathrm{e}},\end{aligned}$$

thus, by using Corollary 2.1, we see that (2.25) is oscillatory if $h_0 > 0.36018$.

On the other hand, we see that condition (1.6) is satisfied, where

$$0.96094 \geq 0.92188 > 0,$$

also, the condition (1.7) becomes

$$\begin{aligned}&\limsup_{s\to\infty}\eta^{\alpha}(s)\int_{s_1}^{s} h(\varrho)\left(1-\rho_1(\varrho)\frac{\eta(\delta(\varrho))}{\eta(\varrho)}-\rho_2(\varrho)\right)\mathrm{d}\varrho\\ =\;& h_0\left(1-\frac{1}{16}-\frac{1}{64}\right)>1,\end{aligned}$$

then, by using Theorem 1.1, we see that (2.25) is oscillatory if $h_0 > 1.0847$.

From the above, we notice that our results improved [23].

Example 2.2. *Let us assume the special case*

$$\left(s^2\left[u(s)+\frac{1}{8}u\left(\frac{s}{3}\right)\right]'\right)'+h_0 u\left(\frac{s}{4}\right)=0 \tag{2.26}$$

for equation (1.1), where $\rho_2(s) = 0$. Now, we see that

$$\eta(s)=\frac{1}{s},\ \eta(\epsilon(s))=\frac{4}{s}\text{ and }\eta(\delta(\epsilon(s)))=\frac{12}{s}.$$

Therefore, the condition (2.21) is satisfied, where

$$\begin{aligned}&\int_{s_0}^{\infty} h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\\ =\;& \int_{s_0}^{\infty} h_0\left(1-\frac{1}{8}(3)\right)\mathrm{d}\varrho=\infty,\end{aligned}$$

and the condition (2.24); becomes

$$\liminf_{s\to\infty}\int_{\epsilon(s)}^{s}\frac{1}{a(\theta)}\left(\int_{s_1}^{\theta} h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}-\rho_2(\epsilon(\varrho))\frac{\mu(\lambda(\epsilon(\varrho)))}{\mu(\epsilon(\varrho))}\right)\mathrm{d}\varrho\right)\mathrm{d}\theta$$

$$= \quad h_0\left(1-\frac{1}{8}(3)\right)\ln 4 > \frac{1}{\mathrm{e}},$$

thus, by using Corollary 2.1, we see that (2.26) is oscillatory if $h_0 > 0.42459$.

On the other hand, we see that condition (1.9) is satisfied, where

$$\int_{s_0}^{\infty} \eta(\xi) h(\xi) \, \mathrm{d}\xi = \int_{s_0}^{\infty} \frac{1}{\xi} h_0 \mathrm{d}\xi = \infty,$$

also, the condition (1.10) becomes

$$\begin{aligned} &\limsup_{s\to\infty}\left(\eta(s)\int_{s_0}^{s} h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}\right)^{\alpha} \mathrm{d}\varrho\right. \\ &\left.+\eta^{-\alpha}(\epsilon(s))\int_{s}^{\infty}\eta(\varrho)h(\varrho)\left(1-\rho_1(\epsilon(\varrho))\frac{\eta(\delta(\epsilon(\varrho)))}{\eta(\epsilon(\varrho))}\right)^{\alpha}\eta^{\alpha}(\epsilon(\varrho))\,\mathrm{d}\varrho\right) \\ = \quad &\left(h_0\left(1-\frac{1}{8}(3)\right)+h_0\left(1-\frac{1}{8}(3)\right)\right) > 1, \end{aligned}$$

then, by using Theorem 1.2, we see that (2.26) is oscillatory if $h_0 > 0.8$

From the above, we notice that our results improved [24].

Remark 2.1. *If $h_0 = 1/2$ in (2.26), we find that Grace et al. in [24] fail to study the oscillation of Eq (2.26) at $h_0 = 1/2$ because condition (1.10) is not satisfied. But by applying our results, we find that condition (2.24) is satisfied; thus, our results succeed in studying the oscillation Eq (2.26) at $h_0 = 1/2$. Therefore, our results improve the results of Grace et al. in [24].*

3. Conclusions

This research improves the oscillation criteria for second-order DEs with mixed neutral terms. These equations describe situations where the rate of change depends not only on the current state but also on an advanced version of it. These new criteria allow a wider range of equations to be studied. Future research may involve applying the same approach to even-order DEs with mixed neutral terms in the canonical case as well as the non-canonical case and exploring more novel criteria.

Author contributions

Fawaz Khaled Alarfaj: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-review and editing; Ali Muhib: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-original draft preparation, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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