Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 2930–2957.
DOI: 10.3934/math.2025136
Received: 22 November 2024
Revised: 31 December 2024
Accepted: 05 February 2025
Published: 17 February 2025

Research article

New expressions for certain polynomials combining Fibonacci and Lucas polynomials

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Abstract: We establish a new sequence of polynomials that combines the Fibonacci and Lucas polynomials. We will refer to these polynomials as merged Fibonacci-Lucas polynomials (MFLPs). We will show that we can represent these polynomials by combining two certain Fibonacci polynomials. This formula will be essential for determining the power form representation of these polynomials. This representation and its inversion formula for these polynomials are crucial to derive new formulas about the MFLPs. New derivative expressions for these polynomials are given as combinations of several symmetric and non-symmetric polynomials. We also provide the inverse formulas for these formulas. Some new product formulas involving the MFLPs have also been derived. We also provide some definite integral formulas that apply to the derived formulas.

Keywords: Fibonacci and Lucas polynomials; orthogonal polynomials; recursive formulas; moment formulas; generalized hypergeometric functions; definite integrals
Mathematics Subject Classification: 11B39, 11B83, 33C45

1. Introduction

Special functions are essential in fields like mathematics, engineering, and physics. The significance of these functions in various contexts is enhanced by some features they possess. Results and applications of some special functions are presented in various books. The works [1–3] might be consulted for guidance in this area. Moreover, orthogonal polynomials are classes of special functions that are widely used in many fields related to numerical analysis and approximation theory; see, for instance, [4–7].

Due to the importance of studying particular and generalized polynomial sequences, many authors were interested in investigating and utilizing different polynomial sequences in several applications. For example, the authors of [8] studied some Horadam polynomials. Some approximations were presented using modified Bernstein polynomials in [9]. A class of generalized polynomials was presented in [10]. Another class of generalized polynomials was presented in [11]. Some generalized polynomials with some operational identities and applications were given in [12]. Degenerate Bell polynomials were introduced in [13]. In [14], the authors developed some new formulas for the generalized Hermite polynomials. In [15], the authors investigated Bell polynomials of the second kind. A class of Bernoulli polynomials was introduced in [16]. A class of degenerate Bell numbers and polynomials was given in [17]. The authors of [18] studied some polynomial sequences with some applications, while the same authors studied other sequences in [19].

Further studies on other sequences of polynomials can be found in [20–24].

Among the well-known sequences in number theory are the Fibonacci and Lucas sequences. These sequences are significant in several domains, including computer science, art, coding theory, statistics, and numerical analysis. For some characteristics of these sequences and their applications, one can consult [25, 26].

The significance of Fibonacci and Lucas polynomials and their generalized and modified sequences has prompted several studies to consider them. Some theoretical results regarding the standard Fibonacci and Lucas polynomials have been obtained. In [27, 28], the authors have introduced some results of Fibonacci and Lucas polynomials and their connections with other polynomials, including orthogonal polynomials. Other formulas of Lucas polynomials were developed and used in [29] to solve the time-fractional diffusion equation. Some other contributions for these polynomials can be found in [30–33]. The studies are not restricted to the standard Fibonacci and Lucas polynomials, but many researchers are interested in introducing and investigating several modified and generalized sequences of Fibonacci and Lucas polynomials. The authors of [34] have reduced some radicals using two generalized Fibonacci and Lucas polynomial classes. Generalized bivariate Fibonacci and Lucas polynomials were studied in [35]. Some k-Fibonacci and k-Lucas polynomial formulas were introduced and investigated in [36]. Some generalizations of Fibonacci and Lucas polynomials were studied in [37]. From a numerical point of view, the different sequences of Fibonacci and Lucas polynomials and their generalized ones were employed in a variety of papers to solve different kinds of differential equations. For example, the authors of [38] have developed a numerical solution for a two-dimensional Sobolev equation using mixed Lucas and Fibonacci polynomials. The authors of [39] used the generalized Lucas polynomials and the wavelet method to treat some fractional optimal control problems.

The investigation of special functions relies heavily on hypergeometric functions (HGFs) because these functions may be used to represent almost all significant functions and polynomials, and they may be used to solve several significant problems within the domain of special functions. The necessary coefficients for solving the connection and linearization problems can be derived using various HGFs. Some references in this direction may be found in [40–45].

This paper aims to introduce new generalized polynomials of Fibonacci and Lucas polynomials. These polynomials can be written as a combination of two Fibonacci polynomials. We develop new formulas for the new merged polynomials. The objectives of this paper can be summarized as follows:

- Introducing new polynomials that generalize the standard Fibonacci and Lucas polynomials.
- Developing some new formulas, including derivative and product formulas, that are concerned

with the newly introduced set of polynomials.
- Obtaining new expressions for some new integrals.

The current article is structured as follows: Section 2 focuses on presenting some basic formulas of Fibonacci and Lucas polynomials as well as an overview of some orthogonal and non-orthogonal polynomials. Section 3 introduces new sequences of polynomials that unify Fibonacci and Lucas polynomials. Deriving new expressions of derivatives of the MFLPs as combinations of different orthogonal and non-orthogonal polynomials is the focus of Section 4. Section 5 is interested in deriving the derivatives of the MFLPs in terms of different polynomials. Section 6 derives some inverse formulas to these given in Section 5. Section 7 presents some linearization formulas for the MFLPs. An application of some new definite integrals based on the application of some of the derived formulas is given in Section 8. We end the paper with some discussions and suggest some expected future work in Section 9.

2. Fundamentals of Fibonacci, Lucas, and some other polynomials

The basic characteristics of Fibonacci and Lucas polynomials are presented in this section. In addition, an account of symmetric and non-symmetric polynomials is given.

2.1. Fibonacci and Lucas polynomials

The Fibonacci and Lucas polynomials can be generated using respectively the following two recursive formulas:

$$F_i(x) - x\,F_{i-1}(x) - F_{i-2}(x) = 0, \quad F_0(x) = 0,\ F_1(x) = 1, \tag{2.1}$$

$$L_i(x) - x\,L_{i-1}(x) - L_{i-2}(x) = 0, \quad L_0(x) = 2,\ L_1(x) = x. \tag{2.2}$$

The Fibonacci and Lucas polynomials can be represented by [46]

$$F_m(x) = \sum_{s=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-s}{s} x^{m-2s-1}, \tag{2.3}$$

$$L_m(x) = m \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{\binom{m-s}{s}}{m-s} x^{m-2s}, \tag{2.4}$$

while their inversion formulas are given by

$$x^m = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^s (m-2s+1)(m-s+2)_{s-1}}{s!} F_{m-2s+1}(x), \tag{2.5}$$

$$x^m = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} \frac{c_{m-2s}\,(-1)^s (m-s+1)_i}{s!} L_{m-2s}(x), \tag{2.6}$$

with

$$c_\ell = \begin{cases} \frac{1}{2}, & \ell = 0, \\ 1, & \ell \geq 1. \end{cases}$$

In addition, the moment formulas of these polynomials are given by [46]

$$x^m\, F_k(x) = \sum_{s=0}^{m} \binom{m}{s} F_{k+m-2s}(x), \tag{2.7}$$

$$x^m\, L_k(x) = \sum_{s=0}^{m} \binom{m}{s} L_{k+m-2s}(x). \tag{2.8}$$

2.2. An overview of well-known polynomials: Both symmetric and non-symmetric

This section presents some basic formulas related to certain symmetric and non-symmetric polynomials.

We may express the two classes of symmetric and non-symmetric polynomials, denoted respectively by $\chi_m(x)$ and $\theta_m(x)$, by the following formulas:

$$\chi_m(x) = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} G_{\ell,m}\, x^{m-2\ell}, \tag{2.9}$$

$$\theta_m(x) = \sum_{\ell=0}^{m} Y_{\ell,m}\, x^{m-\ell}, \tag{2.10}$$

with the known coefficients $G_{\ell,m}$ and $Y_{\ell,m}$.
Furthermore, suppose that the inversion formulas for (2.9) and (2.10) are as follows:

$$x^m = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \bar{G}_{\ell,m}\, \chi_{\ell-2m}(x), \tag{2.11}$$

$$x^m = \sum_{r=0}^{m} \bar{Y}_{\ell,m}\, \theta_{\ell-m}(x), \tag{2.12}$$

with the known coefficients $\bar{G}_{\ell,m}$ and $\bar{Y}_{\ell,m}$.

There are examples of non-symmetric polynomials. The most important polynomials in this regard are the normalized Jacobi polynomials that are used in a large number of articles. These polynomials are defined by [46]

$$V_\ell^{(\mu,\nu)}(x) = {}_2F_1\left(\begin{matrix} -\ell, \ell+\mu+\nu+1 \\ \mu+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right).$$

Note that for $\mu = \nu$, the polynomials are symmetric and are called ultraspherical polynomials. In fact, the ultraspherical polynomials are explicitly defined by

$$U_\ell^{(\mu)}(x) = V_\ell^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(x), \tag{2.13}$$

while, for $\mu \neq \nu$, the Jacobi polynomials are non-symmetric ones.
It is useful to define the normalized Jacobi polynomials on $[0, 1]$, which are called shifted Jacobi polynomials. They are defined by

$$\tilde{V}_\ell^{(\mu,\nu)}(x) = V_\ell^{(\mu,\nu)}(2x-1).$$

In [47], the authors investigated a type of non-symmetric polynomials called Schröder polynomials. These polynomials are defined by

$$S_i(x) = \sum_{m=0}^{i} \frac{\binom{2m}{j}\binom{i+m}{i-m}}{m+1}\, x^m.$$

For a survey of classical orthogonal polynomials and other sequences of polynomials, one can consult [48–50].

We present two categories of non-orthogonal polynomials: the generalized Fibonacci polynomials $F_r^{A,B}(x)$, and the generalized Lucas polynomials $L_r^{\bar{A},\bar{B}}(x)$, both of which were examined in [34], and can be expressed respectively as follows:

$$F_r^{A,B}(x) = A\,x\,F_{r-1}^{A,B}(x) + B\,F_{r-2}^{A,B}(x), \quad F_0^{A,B}(x) = 1,\ F_1^{A,B}(x) = A\,x, \quad r \geq 2, \tag{2.14}$$

$$L_r^{\bar{A},\bar{B}}(x) = \bar{A}\,x\,L_{r-1}^{\bar{A},\bar{B}}(x) + \bar{B}\,L_{r-2}^{\bar{A},\bar{B}}(x), \quad L_0^{\bar{A},\bar{B}}(x) = 2,\ L_1^{\bar{A},\bar{B}}(x) = \bar{A}\,x, \quad r \geq 2. \tag{2.15}$$

Remark 2.1. *The power form representations and associated inversion formulas for various polynomials are essential for deriving numerous significant formulas related to these polynomials. Taking into account the four formulas in* (2.9)–(2.12)*, the subsequent table displays the coefficients found in these formulas for some renowned symmetric and non-symmetric polynomials(see Table 1). We present the relevant formulas for shifted Jacobi polynomials* $\tilde{V}_\ell^{(\mu,\nu)}(x)$*, Schröder polynomials* $S_i(x)$*, the ultraspherical polynomials* $U_\ell^{(\lambda)}(x)$*, the generalized Fibonacci polynomials* $F_\ell^{a,b}(x)$ *that are generated by* (2.14)*, the generalized Lucas polynomials* $L_\ell^{c,d}(x)$ *that are generated by* (2.15)*, and Bernoulli polynomials:* $B_\ell(x)$*.*

Table 1. Coefficients for analytic forms and their inversion formulas.

Polynomial	$G_{\ell,m}\,(Y_{\ell,m})$	$\bar{G}_{\ell,m}\left(\bar{Y}_{\ell,m}\right)$
$\tilde{V}_\ell^{(\mu,\nu)}(x)$	$\dfrac{(-1)^\ell\, m!\,\Gamma(1+\mu)\,(1+\nu)_m\,(1+\mu+\nu)_{2m-\ell}}{(m-\ell)!\,\ell!\,\Gamma(1+m+\mu)\,(1+\nu)_{m-\ell}\,(1+\mu+\nu)_m}$	$\dfrac{\binom{m}{\ell}\,(1+\mu)_{m-\ell}\,(1+m-\ell+\nu)_\ell}{(2+2m-2\ell+\mu+\nu)_L\,(1+m-\ell+\mu+\nu)_{m-\ell}}$
$S_\ell(x)$	$\dfrac{\binom{m}{m-\ell}\binom{2m-\ell}{m-\ell}}{1+m-\ell}$	$\dfrac{i!(i+1)!\,((-1)^m(1+2i-2m))}{(2i-m+1)!m!}$
$U_\ell^{(\lambda)}(x)$	$\dfrac{(-1)^\ell\,2^{-1+m-2\ell}\,m!\,\Gamma(m-\ell+\lambda)\,\Gamma(1+2\lambda)}{(m-2\ell)!\,\ell!\,\Gamma(1+\lambda)\,\Gamma(m+2\lambda)}$	$\dfrac{2^{-m+1}\,(m-2\ell+\lambda)\,m!\,\Gamma(\lambda+1)\,\Gamma(m-2\ell+2\lambda)}{(m-2\ell)!\,\ell!\,\Gamma(2\lambda+1)\,\Gamma(1+m-\ell+\lambda)}$
$H_\ell(x)$	$\dfrac{(-1)^\ell\,2^{-2\ell+m}\,m!}{\ell!\,(-2\ell+m)!}$	$\dfrac{2^{-m}\,m!}{\ell!\,(m-2\ell)!}$
$F_\ell^{A,B}(x)$	$\dfrac{A^{m-2\ell}\,B^\ell\,(m-2\ell+1)_\ell}{\ell!}$	$\dfrac{(-1)^\ell\,(m-2\ell+1)\,(m-\ell+2)_{\ell-1}\,B^\ell}{\ell!\,A^m}$
$L_\ell^{\bar{A},\bar{B}}(x)$	$\dfrac{\bar{A}^{m-2\ell}\,\bar{B}^\ell\,m\,(1-2\ell+m)_{\ell-1}}{\ell!}$	$\dfrac{c_{m-2\ell}\,(-1)^\ell\,\bar{A}^{-m}\,\bar{B}^\ell\,(1-\ell+m)_\ell}{\ell!}$
$B_\ell(x)$	$B_\ell\dbinom{m}{m-\ell}$	$\dfrac{\binom{m+1}{m-\ell}}{m+1}$

Note that for the row before the last, c_ℓ is defined by

$$c_\ell = \begin{cases} \frac{1}{2}, & \ell = 0,\\ 1, & \ell \geq 1. \end{cases}$$

In addition, the numbers B_ℓ that appear in the last row are the well-known Bernoulli numbers.

3. Introducing a new merged sequence of Fibonacci and Lucas polynomials

This section introduces new merged Fibonacci and Lucas polynomials and gives some characteristics of these polynomials.

From the two recursive formulas in (2.1) and (2.2), it is clear that both Fibonacci and Lucas polynomials satisfy the same recursive formula but with different initials; thus, it is clear that the following recurrence relation:

$$\phi_i(x) - x\,\phi_{i-1}(x) - \phi_{i-2}(x) = 0, \quad \phi_0(x) = a, \quad \phi_1(x) = b\,x, \tag{3.1}$$

generalizes the two sequences in (2.1) and (2.2). We denote $\phi_i(x) = FL_i^{a,b}(x)$, that is

$$FL_i^{a,b}(x) - x\,FL_{i-1}^{a,b}(x) - FL_{i-2}^{a,b}(x) = 0, \quad FL_0^{a,b}(x) = a, \quad FL_1^{a,b}(x) = b\,x. \tag{3.2}$$

It is clear that both $F_i(x)$ and $L_i(x)$ are particular polynomials of $FL_i^{a,b}(x)$. In fact, we have

$$F_{i+1}(x) = FL_i^{1,1}(x), \qquad L_i(x) = FL_i^{2,1}(x). \tag{3.3}$$

Remark 3.1. *The key idea to develop the formulas related to the polynomials $FL_i^{a,b}(x)$ that satisfy* (3.1) *is the following theorem, in which we will show that $FL_i^{a,b}(x)$ can be expressed in terms of two certain Fibonacci polynomials.*

Theorem 3.1. *Consider any non-negative integer j. The MFLPs can be expressed as*

$$FL_j^{a,b}(x) = b\,F_{j+1}(x) + (a-b)\,F_{j-1}(x). \tag{3.4}$$

Proof. Consider the polynomial:

$$\xi_j(x) = b\,F_{j+1}(x) + (a-b)\,F_{j-1}(x). \tag{3.5}$$

We have

$$\xi_0(x) = b\,F_1(x) + (a-b)\,F_{-1}(x).$$

Since $F_1(x) = F_{-1}(x) = 1$, then $\xi_0(x) = a$. In addition, it is clear that $\xi_1(x) = b\,x$.
This means that

$$\xi_0(x) = FL_0^{a,b}(x), \quad \xi_1(x) = FL_1^{a,b}(x).$$

Hence, to prove that $\xi_j(x) = FL_j^{a,b}(x)$, $\forall j \geq 0$, it is sufficient to prove that $\xi_j(x)$ satisfies the same recurrence relation of $FL_j^{a,b}(x)$, for $j \geq 2$, that is we are going to prove that:

$$\xi_{j+2}(x) - x\,\xi_{j+1}(x) - \xi_j(x) = 0. \tag{3.6}$$

Now, we can write

$$\begin{aligned}\xi_{j+2}(x) - x\,\xi_{j+1}(x) - \xi_j(x) =&(a-b)F_{j+1}(x) + bF_{j+3}(x) - x\Big((a-b)F_j(x) + bF_{j+2}(x)\Big)\\ &- \Big(bF_{j+1}(x) + (a-b)F_{j-1}(x)\Big).\end{aligned} \tag{3.7}$$

By virtue of the well-known recurrence relation (2.1) in the form

$$x\,F_j(x) = F_{j+1}(x) - F_{j-1}(x),$$

it is not difficult to show that (3.7) reduces to

$$\xi_{j+2}(x) - x\,\xi_{j+1}(x) - \xi_j(x) = 0. \tag{3.8}$$

This proves Theorem 3.1. □

Now, based on the last theorem, an explicit analytic formula of $FL_k^{a,b}(x)$ can be deduced.

Theorem 3.2. *Consider a positive integer i. The power form representation of $FL_i^{a,b}(x)$ is:*

$$FL_i^{a,b}(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(i-2m+1)_{m-1}((i-2m)b + ma)}{m!}\, x^{i-2m}. \tag{3.9}$$

Proof. By virtue of the combination (3.4) together with (2.3), we get the expression in (3.9). □

The moment and inversion formulas of the $MFLPs$ are of interest. The following theorem and corollary exhibit these results.

Theorem 3.3.

$$x^m\, FL_i^{a,b}(x) = \sum_{r=0}^{m} (-1)^r \binom{m}{r} FL_{i+m-2r}^{a,b}(x). \tag{3.10}$$

Proof. The proof can be performed using induction based on the recursive formula (3.1). □

Corollary 3.1. *Consider the non-negative integer m. The following inversion formula holds:*

$$x^m = \frac{1}{a} \sum_{r=0}^{m} (-1)^r \binom{m}{r} FL_{m-2r}^{a,b}(x). \tag{3.11}$$

Proof. Substituting by $i = 0$ in (3.10) immediately yields (3.11). □

4. A new derivative expression between two families of the MFLPs

In this section, we derive the expression for the derivatives of the MFLPs in terms of other parameters of FPLs. From this expression, some important specific formulas are obtained.

We will consider the two families of merged polynomials $FL_i^{a,b}(x)$ and $FL_i^{c,d}(x)$. More definitely, we will give the derivatives of the polynomials $FL_i^{a,b}(x)$ as a combination of the polynomials $FL_i^{c,d}(x)$. First, the following lemma is useful for deriving the desired expression.

Lemma 4.1. *Consider a non-negative integer p. One has*

$$\sum_{\ell=0}^{p} \frac{(-1)^{1-\ell+p}(-bm - a\ell + 2b\ell)(m-\ell-1)!}{\ell!(p-\ell)!(m-\ell-p-s)!} = \frac{(-1)^p (s+1)_{p-1}(m-p-1)!}{p!(-p+m-s)!} \times \\ (-2bp(p+s) + ap(-m+p+s) + bm(2p+s)). \tag{4.1}$$

Proof. If we set

$$U_{p,m,s} = \sum_{\ell=0}^{p} \frac{(-1)^{1-\ell+p}(-bm - a\ell + 2b\ell)(m-\ell-1)!}{\ell!(p-\ell)!(m-\ell-p-s)!},$$

then based on Zeilberger's algorithm [51], the following recurrence relation can be generated:

$$\begin{aligned}&(1+m-p-s)(p+s-1)(2bp(p+s) - ap(-m+p+s) - bm(2p+s))U_{p-1,m,s}\\ &+ (m-p)p((a-2b)(1+m-p)(p-1) + (a - b(2+m-2p) - ap)s)U_{p,m,s} = 0,\end{aligned}$$

with the following initial value: $U_{0,m,s} = \dfrac{b\,m!}{(m-s)!}$.
The solution of the last recursive formula is given by

$$U_{p,m,s} = \frac{(-1)^p(s+1)_{p-1}(m-p-1)!}{p!(-p+m-s)!} \times (-2bp(p+s) + ap(-m+p+s) + bm(2p+s)).$$

Lemma 4.1 is now proved. □

Theorem 4.1. *Consider the two positive integers s and m with $m \geq s$. $D^sFL_m^{a,b}(x)$ has the following expression:*

$$\begin{aligned}D^sFL_m^{a,b}(x) =& \frac{1}{c}\sum_{p=0}^{m-s} \frac{(-1)^p(s+1)_{p-1}(m-p-1)!}{p!(-p+m-s)!}\times\\ &(-2bp(p+s) + ap(-m+p+s) + bm(2p+s))\ FL_{m-s-2p}^{c,d}(x).\end{aligned} \tag{4.2}$$

Proof. From the analytic form in (3.9), it is not difficult to express $D^sFL_m^{a,b}(x)$ in the following form:

$$D^sFL_m^{a,b}(x) = \sum_{\ell=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{(b(m-2\ell) + a\ell)(m-\ell-1)!}{\ell!(m-s-2\ell)!} x^{m-2\ell-s}. \tag{4.3}$$

Formula (3.11) helps to convert the last formula into

$$\begin{aligned}D^sFL_m^{a,b}(x) =& \frac{1}{c}\sum_{\ell=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{(b(m-2\ell) + a\ell)(m-\ell-1)!}{\ell!(m-s-2\ell)!}\times\\ &\sum_{t=0}^{m-2\ell-s}(-1)^t\binom{m-s-2\ell}{t} FL_{m-2\ell-s-2t}^{c,d}(x),\end{aligned} \tag{4.4}$$

which can be written again, after some lengthy algebraic calculations, into the form

$$D^sFL_m^{a,b}(x) = \frac{1}{c}\sum_{p=0}^{m-s}\left(\sum_{\ell=0}^{p} \frac{(-1)^{1-\ell+p}(-bm - a\ell + 2b\ell)(m-\ell-1)!}{\ell!(p-\ell)!(m-\ell-p-s)!}\right) FL_{m-s-2p}^{c,d}(x). \tag{4.5}$$

Inserting the result of Lemma 4.1, formula (4.2) can be proved. This proves Theorem 4.1. □

As a special case, the derivatives $D^s FL_m^{a,b}(x)$ can be expressed as combinations of their original ones. The following corollary displays this result.

Corollary 4.1. *Consider the two positive integers, s and m with $m \geq s$. $D^s FL_m^{a,b}(x)$ has the following expression:*

$$\begin{aligned} D^s FL_m^{a,b}(x) =& \frac{1}{a}\sum_{p=0}^{m-s} \frac{(-1)^p (s+1)_{p-1} (m-p-1)!}{p!(-p+m-s)!} \times \\ & (-2bp(p+s) + ap(-m+p+s) + bm(2p+s))\ FL_{m-s-2p}^{a,b}(x). \end{aligned} \tag{4.6}$$

Remark 4.1. *If we consider the two important special cases in* (3.3)*, then some specific derivative formulas can be deduced. In the following two corollaries, we will demonstrate these results.*

Corollary 4.2. *Consider the two positive integers, s and m with $m \geq s$. The following derivative expressions hold:*

$$D^s F_{m+1}(x) = \frac{1}{c\, s!}\sum_{p=0}^{m-s} \frac{(-1)^p\,(m-p)!\,(p+s)!}{p!\,(m-p-s)!}\ FL_{m-s-2p}^{c,d}(x), \tag{4.7}$$

$$D^s L_m(x) = \frac{m}{c\,(s-1)!}\sum_{p=0}^{m-s} \frac{(-1)^p\,(m-p-1)!(p+s-1)!}{p!\,(m-p-s)!}\ FL_{m-s-2p}^{c,d}(x), \tag{4.8}$$

$$\begin{aligned} D^s FL_m^{a,b}(x) =& \sum_{p=0}^{m-s} \frac{(-1)^p\,(s+1)_{p-1} (m-p-1)!}{p!(-p+m-s)!} \times \\ & (-2bp(p+s) + ap(-m+p+s) + bm(2p+s))\ F_{m-s-2p+1}(x), \end{aligned} \tag{4.9}$$

$$\begin{aligned} D^s FL_m^{a,b}(x) =& \frac{1}{2}\sum_{p=0}^{m-s} \frac{(-1)^p\,(s+1)_{p-1} (m-p-1)!}{p!(-p+m-s)!} \times \\ & (-2bp(p+s) + ap(-m+p+s) + bm(2p+s))\ L_{m-s-2p}(x). \end{aligned} \tag{4.10}$$

Proof. Substitution by $a = b = 1$ and $a = 2, b = 1$, respectively, in (4.6) leads to (4.7) and (4.8), while substitution by $c = d = 1$ and $c = 2, d = 1$, in (4.6) leads, respectively, to formulas (4.9) and (4.10). □

Other specific derivative formulas can be deduced taking into account the two special cases considered in (3.3).

Corollary 4.3. *Consider the two positive integers s and m with $m \geq s$. The following expressions for $D^s FL_m^{a,b}(x)$ hold:*

$$D^s F_{m+1}(x) = \frac{1}{s!}\sum_{p=0}^{m-s} \frac{(-1)^p\,(m-p)!\,(p+s)!}{p!\,(m-p-s)!}\ F_{m-s-2p+1}(x), \tag{4.11}$$

$$D^s L_m(x) = \frac{m}{2\,(s-1)!}\sum_{p=0}^{m-s} \frac{(-1)^p\,(m-p-1)!(p+s-1)!}{p!\,(m-p-s)!}\ L_{m-s-2p}(x), \tag{4.12}$$

$$D^s L_m(x) = \frac{m}{(s-1)!}\sum_{p=0}^{m-s} \frac{(-1)^p(m-p-1)!(p+s-1)!}{p!(m-p-s)!}\ F_{m-s-2p+1}(x), \tag{4.13}$$

$$D^s F_{m+1}(x) = \frac{1}{2\,s!} \sum_{p=0}^{m-s} \frac{(-1)^p (m-p)!(p+s)!}{p!(m-p-s)!} L_{m-s-2p}(x). \tag{4.14}$$

Proof. Substitution by $c = d = 1$ and $c = 2, d = 1$ respectively in (4.7) and (4.8) leads to (4.11) and (4.12), while substitution by $a = 2, b = 1$ and $a = b = 1$, respectively, in (4.9) and (4.10) leads to formulas (4.13) and (4.14). □

5. Derivatives of the MFLPs in terms of different polynomials

This section is confined to the establishment of some new derivative expressions of the MFLPs in terms of other polynomials. In this concern, we state and prove two theorems. The first theorem expresses the derivative of the MFLPs in terms of any symmetric polynomials, while the second theorem gives the analog expression for the non-symmetric polynomials. Furthermore, some new specific formulas are developed based on the two theorems.

5.1. Expressions for $D^s FL_m^{a,b}(x)$ in terms of any symmetric polynomial

In this section, we give a general formula for the derivatives of the MFLPs in terms of any symmetric polynomials that can be expressed as in (2.9).

Theorem 5.1. *Consider the two positive integers s and m with $m \geq s$. Let $\chi_i(x)$ be the symmetric polynomials that can be expressed by* (2.9)*. In terms of $\chi_i(x)$, $D^s FL_m^{a,b}(x)$ has the following expression:*

$$D^s FL_m^{a,b}(x) = \sum_{p=0}^{\lfloor \frac{m-s}{2} \rfloor} \left(\sum_{r=0}^{p} \frac{(b(m-2r)+ar)(m-r-1)!}{r!(m-2r-s)!} \right) \bar{G}_{p-r,m-2r-s}\, \chi_{m-s-2p}(x), \tag{5.1}$$

where $\bar{G}_{p,s}$ are the inversion coefficients of $\chi_j(x)$ that appear in (2.11).

Proof. As in (4.3), $D^s FL_m^{a,b}(x)$ may be expressed as

$$D^s FL_m^{a,b}(x) = \sum_{r=0}^{\lfloor \frac{m-s}{2} \rfloor} \frac{(b(m-2r)+ar)(m-r-1)!}{r!(m-2r-s)!}\, x^{m-2r-s}. \tag{5.2}$$

In terms of the symmetric polynomials $\chi_j(x)$, we can write

$$x^\ell = \sum_{j=0}^{\lfloor \frac{\ell}{2} \rfloor} \bar{G}_{j,\ell}\, \chi_{\ell-2j}(x), \tag{5.3}$$

and thus, we have

$$\begin{aligned} D^s FL_m^{a,b}(x) = \sum_{r=0}^{\lfloor \frac{m-s}{2} \rfloor} & \frac{(b(m-2r)+ar)(m-r-1)!}{r!(m-2r-s)!} \times \\ & \sum_{t=0}^{\lfloor \frac{m-s}{2} \rfloor - r} G_{t,m-2r-s}\, \chi_{m-2r-s-2t}(x). \end{aligned} \tag{5.4}$$

An alternative formula to formula (5.4) after rearranging the terms is

$$D^s FL_m^{a,b}(x) = \sum_{p=0}^{\lfloor \frac{m-s}{2} \rfloor} R_{p,s,m} \chi_{m-s-2p}(x), \tag{5.5}$$

and $R_{p,s,m}$ is given by

$$R_{p,s,m} = \sum_{r=0}^{p} \frac{(b(m-2r)+ar)(m-r-1)!}{r!(m-2r-s)!} G_{p-r,m-2r-s}.$$

This proves formula (5.1). ◻

Remark 5.1. *From Theorem 5.1, and considering some celebrated symmetric polynomials, some derivative expressions can be deduced. More precisely, the following corollaries display these results.*

Corollary 5.1. *Consider the two positive integers s and m with $m \geq s$. In terms of $U_j^{(\lambda)}(x)$, $D^s FL_m^{a,b}(x)$ has the following expression:*

$$\begin{aligned} D^s FL_m^{a,b}(x) = & \frac{2^{1-m+s-2\lambda}\, b\, \sqrt{\pi}\, m!\, \Gamma(m-s+2\lambda)}{(m-s)!\, \Gamma\left(\frac{1}{2}+\lambda\right) \Gamma(m-s+\lambda)} U_{m-s}^{(\lambda)}(x) \\ & + \frac{2^{1-m+s-2\lambda}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\lambda\right)} \sum_{p=1}^{\lfloor \frac{m-s}{2} \rfloor} \frac{\Gamma(-s-2p+m+2\lambda)}{(p-1)!p!(m-s-2p)!\Gamma(-s-p+m+\lambda)\Gamma(1-s-p+m+\lambda)} R_{p,m,s}\, U_{m-s-2p}^{(\lambda)}(x), \end{aligned} \tag{5.6}$$

and $R_{p,s,m}$ is given by

$$\begin{aligned} R_{p,s,m} = & 4(a-2b)(m-2p-s+\lambda)(m-2)!p!\Gamma(1+m-p-s+\lambda) \\ & \times {}_2F_1\left(\begin{matrix} 1-p, 1-m+p+s-\lambda \\ 2-m \end{matrix} \middle| -4 \right) \\ & + b\, m!\, ((m-s+\lambda)(p-1)! - 2p!)\Gamma(m-p-s+\lambda) \\ & \times {}_2F_1\left(\begin{matrix} -p, -m+p+s-\lambda \\ 1-m \end{matrix} \middle| -4 \right). \end{aligned}$$

Proof. It is an immediate outcome of Theorem 5.1 considering the inversion coefficients of $U_\ell^{(\lambda)}(x)$ in the third row of Table 1. ◻

Corollary 5.2. *Consider the two positive integers s and m with $m \geq s$. In terms of $F_j^{A,B}(x)$, $D^s FL_m^{a,b}(x)$ has the following expression:*

$$\begin{aligned} D^s FL_m^{a,b}(x) = & (m-2)! \sum_{p=0}^{\lfloor \frac{m-s}{2} \rfloor} \frac{(-1)^{p+1} A^{-m+s} B^{p-1}(-1-m+2p+s)}{p!(m-p-s+1)!} \times \\ & \left(A^2(a-2b)\, p\, (-1-m+p+s)\, {}_2F_1\left(\begin{matrix} 1-p, -m+p+s \\ 2-m \end{matrix} \middle| \frac{A^2}{B} \right) \right. \\ & \left. + b\, B\, (m-1) m\, {}_2F_1\left(\begin{matrix} -p, -1-m+p+s \\ 1-m \end{matrix} \middle| \frac{A^2}{B} \right) \right) F_{m-s-2p}^{A,B}(x). \end{aligned} \tag{5.7}$$

Proof. It is an immediate outcome of Theorem 5.1, considering the inversion coefficients of $F_{\ell}^{a,b}(x)$ in the fifth row of Table 1. □

Corollary 5.3. *Consider the two positive integers s and m with $m \geq s$. In terms of $L_j^{\bar{A},\bar{B}}(x)$, $D^s FL_m^{a,b}(x)$ has the following expression:*

$$\begin{aligned} D^s FL_m^{a,b}(x) =&(m-2)! \sum_{p=0}^{\lfloor \frac{m-s}{2} \rfloor} \frac{(-1)^p \bar{A}^{s-m} \bar{B}^{p-1} c_{m-2p-s}}{p!(m-p-s)!} \times \\ &\Bigg(\bar{A}^2 (a-2b) p(-m+p+s)\, {}_2F_1\left(\begin{matrix} 1-p, 1-m+p+s \\ 2-m \end{matrix} \,\middle|\, \frac{\bar{A}^2}{\bar{B}} \right) \\ &+ b\,\bar{B}(m-1)m\, {}_2F_1\left(\begin{matrix} -p, -m+p+s \\ 1-m \end{matrix} \,\middle|\, \frac{\bar{A}^2}{\bar{B}} \right) \Bigg) L_{m-s-2p}^{\bar{A},\bar{B}}(x). \end{aligned} \tag{5.8}$$

Proof. It is an immediate outcome of Theorem 5.1 considering the inversion coefficients of $L_{\ell}^{a,b}(x)$ in the sixth row of Table 1. □

Corollary 5.4. *Consider the two positive integers s and m with $m \geq s$. In terms of $H_j(x)$, $D^s FL_m^{a,b}(x)$ has the following expression:*

$$\begin{aligned} D^s FL_m^{a,b}(x) =&(m-2)! \sum_{p=0}^{\lfloor \frac{m-s}{2} \rfloor} \frac{1}{p!(m-2p-s)!} 2^{-m+s} \times \\ &(4(a-2b)p\, {}_1F_1(1-p; 2-m; 4) + b(-1+m)m\, {}_1F_1(-p; 1-m; 4))\ H_{m-2s}(x). \end{aligned} \tag{5.9}$$

Proof. It is an immediate outcome of Theorem 5.1 considering the inversion coefficients of $H_{\ell}(x)$ in the fourth row of Table 1. □

5.2. Expressions for $D^s FL_m^{a,b}(x)$ in terms of any non-symmetric polynomial

In this section, we give a general formula for the derivatives of the MFLPs in terms of the non-symmetric polynomials that can be expressed by (2.10).

Theorem 5.2. *Let $\theta_i(x)$ be the non-symmetric polynomials that are represented by* (2.10). *We have the following expression:*

$$\begin{aligned} D^s FL_m^{a,b}(x) =& \sum_{p=0}^{\lfloor \frac{m-s}{2} \rfloor} \sum_{r=0}^{p} \frac{(b(m-2r)+2br)(1+m-2r)_{r-1}(1+m-2r-s)_s}{r!} H_{2p-2r,m,r,s}\, \theta_{m-s-2p}(x) \\ &+ \sum_{p=0}^{\lfloor \frac{1}{2}(m-s-1) \rfloor} \sum_{r=0}^{p} \frac{(b(m-2r)+2br)(1+m-2r)_{r-1}(1+m-2r-s)_s}{r!} H_{2p-2r+1,m,r,s}\, \theta_{m-s-2p-1}(x), \end{aligned} \tag{5.10}$$

where $H_{t,m,\ell,s} = \bar{Y}_{t,m-\ell-2s}$ and $\bar{Y}_{\ell,m}$ are the coefficients in (2.12).

Proof. Starting from (5.2), and taking into consideration the inversion formula of the non-symmetric polynomials $\theta_i(x)$ given by (2.12), the following formula is obtained:

$$D^s FL_m^{a,b}(x) = \sum_{\ell=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{(b(m-2\ell)+2b\ell)(1+m-2\ell)_{\ell-1}(1+m-2\ell-s)_s}{\ell!} \sum_{t=0}^{m-2\ell-s} H_{t,m,\ell,s}\, \theta_{m-2\ell-s-t}(x). \tag{5.11}$$

Some tedious computations lead to formula (5.10). □

In the following, and depending on Theorem 5.2, we can get derivative formulas for MFLPs in terms of Bernoulli polynomials and shifted Jacobi polynomials.

Corollary 5.5. *In terms of Bernoulli polynomials, the following formula is valid for $a \neq 2\,b$:*

$$\begin{aligned} D^s FL_m^{a,b}(x) &= b\,m! \sum_{p=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{1}{(2p+1)!(m-2p-s)!} R_{p,m} B_{m-s-2p}(x) \\ &+ \sum_{p=0}^{\lfloor\frac{1}{2}(m-s-1)\rfloor} \frac{-(b(-2+m-2p)+a(p+1))(m-p-2)!(2p+2)!+bm!(p+1)!\,\bar{R}_{p,m}}{(p+1)!(2p+2)!(m-2p-s-1)!} B_{m-s-2p-1}(x), \end{aligned} \tag{5.12}$$

where

$$R_{p,m} = {}_3F_2\left(\left. \begin{array}{c} -p, -\frac{1}{2}-p, 1+\frac{bm}{a-2b} \\ 1-m, \frac{bm}{a-2b} \end{array} \right| -4 \right), \tag{5.13}$$

$$\bar{R}_{p,m} = {}_3F_2\left(\left. \begin{array}{c} -p-1, -\frac{1}{2}-p, 1+\frac{bm}{a-2b} \\ 1-m, \frac{bm}{a-2b} \end{array} \right| -4 \right), \tag{5.14}$$

while the formula corresponding to the case $a = 2\,b$ is given by

$$\begin{aligned} D^s FL_m^{a,b}(x) &= bm! \sum_{p=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{1}{(2p+1)!(m-2p-s)!}\, {}_2F_1\left(\left. \begin{array}{c} -p, -\frac{1}{2}-p \\ 1-m \end{array} \right| -4 \right) B_{m-s-2p}(x) \\ &+ b\,m \sum_{p=0}^{\lfloor\frac{1}{2}(m-s-1)\rfloor} \frac{-(m-p-2)!(2p+2)!+(m-1)!(p+1)!\; {}_2F_1\left(\left. \begin{array}{c} -p-1, -\frac{1}{2}-p \\ 1-m \end{array} \right| -4 \right)}{(p+1)!(2p+2)!(m-2p-s-1)!} B_{m-s-2p-1}(x). \end{aligned} \tag{5.15}$$

Proof. It is an immediate consequence of Theorem 5.2 considering the inversion coefficients of $B_\ell(x)$ in the last row of Table 1. □

Corollary 5.6. *In terms of shifted Jacobi polynomials, the following formula is valid for $a \neq 2\,b$:*

$$D^s FL_m^{a,b}(x) = \frac{bm!\Gamma(1+m-s+\beta)}{\Gamma(1+\alpha)}\times$$
$$\left(\sum_{p=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{(1+2m-4p-2s+\alpha+\beta)\Gamma(1+m-2p-s+\alpha)\Gamma(1+m-2p-s+\alpha+\beta)}{(2p)!(m-2p-s)!\Gamma(1+m-2p-s+\beta)\Gamma(2+2m-2p-2s+\alpha+\beta)} Z_{p,m,s}\, V_{m-s-2p}^{(\alpha,\beta)}(x)\right.$$
$$\left. + \sum_{p=0}^{\lfloor\frac{1}{2}(m-s-1)\rfloor} \frac{(-1+2m-4p-2s+\alpha+\beta)\Gamma(m-2p-s+\alpha)\Gamma(m-2p-s+\alpha+\beta)}{(1+2p)!(-1+m-2p-s)!\Gamma(m-2p-s+\beta)\Gamma(1+2m-2p-2s+\alpha+\beta)} \bar{Z}_{p,m,s}\, V_{m-s-2p-1}^{(\alpha,\beta)}(x)\right), \tag{5.16}$$

where

$$Z_{p,m,s} = {}_5F_4\left(\left.\begin{array}{c} -p, \frac{1}{2}-p, 1+\frac{bm}{a-2b}, -\frac{1}{2}-m+p+s-\frac{\alpha}{2}-\frac{\beta}{2}, -m+p+s-\frac{\alpha}{2}-\frac{\beta}{2} \\ 1-m, \frac{bm}{a-2b}, -\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2}, \frac{1}{2}-\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2} \end{array}\right| -4\right), \tag{5.17}$$

$$\bar{Z}_{p,m,s} = {}_5F_4\left(\left.\begin{array}{c} -p, -\frac{1}{2}-p, 1+\frac{bm}{a-2b}, -m+p+s-\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}-m+p+s-\frac{\alpha}{2}-\frac{\beta}{2} \\ 1-m, \frac{bm}{a-2b}, -\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2}, \frac{1}{2}-\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2} \end{array}\right| -4\right), \tag{5.18}$$

while the formula corresponding to the case $a = 2\,b$ is given by

$$D^s FL_m^{2b,b}(x) = \frac{bm!\Gamma(1+m-s+\beta)}{\Gamma(1+\alpha)}\times$$
$$\left(\sum_{p=0}^{\lfloor\frac{m-s}{2}\rfloor} \frac{(1+2m-4p-2s+\alpha+\beta)\Gamma(1+m-2p-s+\alpha)\Gamma(1+m-2p-s+\alpha+\beta)}{(2p)!(m-2p-s)!\Gamma(1+m-2p-s+\beta)\Gamma(2+2m-2p-2s+\alpha+\beta)} M_{p,m,s}\, V_{m-s-2p}^{(\alpha,\beta)}(x)\right.$$
$$\left. + \sum_{p=0}^{\lfloor\frac{1}{2}(m-s-1)\rfloor} \frac{(-1+2m-4p-2s+\alpha+\beta)\Gamma(m-2p-s+\alpha)\Gamma(m-2p-s+\alpha+\beta)}{(1+2p)!(-1+m-2p-s)!\Gamma(m-2p-s+\beta)\Gamma(1+2m-2p-2s+\alpha+\beta)} \bar{M}_{p,m,s}\, V_{m-s-2p-1}^{(\alpha,\beta)}(x)\right), \tag{5.19}$$

where

$$M_{p,s,m} = {}_4F_3\left(\left.\begin{array}{c} -p, \frac{1}{2}-p, -\frac{1}{2}-m+p+s-\frac{\alpha}{2}-\frac{\beta}{2}, -m+p+s-\frac{\alpha}{2}-\frac{\beta}{2} \\ 1-m, -\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2}, \frac{1}{2}-\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2} \end{array}\right| -4\right), \tag{5.20}$$

$$\bar{M}_{p,s,m} = {}_4F_3\left(\left.\begin{array}{c} -p, -\frac{1}{2}-p, -m+p+s-\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}-m+p+s-\frac{\alpha}{2}-\frac{\beta}{2} \\ 1-m, -\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2}, \frac{1}{2}-\frac{m}{2}+\frac{s}{2}-\frac{\beta}{2} \end{array}\right| -4\right). \tag{5.21}$$

Proof. It is an immediate result of Theorem 5.2 considering the inversion coefficients of $V_\ell^{(\alpha,\beta)}(x)$ in the first row of Table 1. □

Remark 5.2. *From the derivatives formulas developed in Section 5, many connection formulas can be deduced only by setting $q = 0$. In the following corollary, we give the connection formula of the MFLPs with Bernoulli polynomials. Other connection formulas can also be deduced.*

Corollary 5.7. *The connection formula between the MFLPs and Bernoulli polynomials are given as follows:*

$$FL_m^{a,b}(x)\,dx = \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} V_{p,m}\, B_{m-2p}(x) + \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} \bar{V}_{p,m} B_{m-2p-1}(x), \quad m \geq 2, \tag{5.22}$$

where $V_{p,m}$ and $\bar{V}_{p,m}$ have the following explicit forms:

$$V_{p,m} = \begin{cases} \dfrac{bm!}{(2p+1)!(m-2p)!}\, {}_2F_1\left(\begin{matrix} -p, -\frac{1}{2}-p \\ 1-m \end{matrix} \middle| -4 \right), & a = 2b, \\ \dfrac{bm!}{(m-2p)!(2p+1)!}\, {}_3F_2\left(\begin{matrix} -p, -\frac{1}{2}-p, 1+\frac{bm}{a-2b} \\ 1-m, \frac{bm}{a-2b} \end{matrix} \middle| -4 \right), & a \neq 2b, \end{cases} \tag{5.23}$$

$$\bar{V}_{p,m} = \begin{cases} \dfrac{bm}{(m-2p-1)!}\left(-\dfrac{(m-p-2)!}{(p+1)!} + \dfrac{(m-1)!\, {}_2F_1\left(\begin{matrix} -1-p, -\frac{1}{2}-p \\ 1-m \end{matrix} \middle| -4 \right)}{(2p+2)!} \right), & a = 2b, \\ \dfrac{1}{(m-2p-1)!}\left(-\dfrac{(b(-2+m-2p)+a(p+1))(m-p-2)!}{(p+1)!} \right. \\ \qquad \left. + \dfrac{b\, m!\, {}_3F_2\left(\begin{matrix} -1-p, -\frac{1}{2}-p, 1+\frac{bm}{a-2b} \\ 1-m, \frac{bm}{a-2b} \end{matrix} \middle| -4 \right)}{(2p+2)!} \right), & a \neq 2b. \end{cases} \tag{5.24}$$

Proof. Simply by setting $q = 0$, respectively, in (5.12) and (5.15). □

6. Derivatives of different polynomials in terms of the MFLPs

In this section, we will state and prove two theorems in which the derivatives of the symmetric and non-symmetric polynomials are given as combinations of the MFLPs.

6.1. Expressions for the derivatives of symmetric polynomials

We will give an expression for the derivatives of the symmetric polynomials $\chi_m(x)$ that are expressed by (2.9).

Theorem 6.1. *Let $\chi_m(x)$ be the symmetric polynomials that are expressed by* (2.9). *The derivatives of $\chi_m(x)$ can be expressed in terms of $FL_m^{a,b}(x)$ as*

$$D^s\chi_m(x) = \frac{1}{a}\sum_{p=0}^{m-s}\left(\sum_{r=0}^{p}(-1)^{p-r}\binom{m-s-2r}{p-r}(1+m-s-2r)_s G_{r,m}\right) FL_{m-s-2p}^{a,b}(x), \tag{6.1}$$

where $G_{r,m}$ are the power form coefficients that appear in (2.9).

Proof. The analytic form of $\chi_m(x)$ leads to the following formula:

$$D^s\chi_m(x) = \sum_{r=0}^{\lfloor\frac{m-s}{2}\rfloor} G_{r,m}\,(m-2r-s+1)_s\, x^{m-2r-s}. \tag{6.2}$$

The inversion formula of (3.11) converts (6.2) into

$$D^s\chi_m(x) = \frac{1}{a}\sum_{r=0}^{\lfloor\frac{m-s}{2}\rfloor} G_{r,m}\,(m-2r-s+1)_s \sum_{t=0}^{m-2r-s}(-1)^t\binom{m-s-2r}{t} FL^{a,b}_{m-2r-s-2t}(x), \tag{6.3}$$

which can be turned into

$$D^s\chi_m(x) = \frac{1}{a}\sum_{p=0}^{m-s}\left(\sum_{r=0}^{p}(-1)^{p-r}\binom{m-s-2r}{p-r}(1+m-s-2r)_s G_{r,m}\right) FL^{a,b}_{m-s-2p}(x). \tag{6.4}$$

Theorem 6.1 is now proved. □

Corollary 6.1. *Consider the two positive integers m and s with $m \geq s$. The following derivative formulas hold:*

$$\begin{aligned} D^sU^{(\lambda)}_m(x) =& \frac{2^{-1+m+2\lambda}\, m!\,\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma(m+\lambda)}{a\,\sqrt{\pi}\,\Gamma(m+2\lambda)}\times \\ & \sum_{p=0}^{m-s}\frac{(-1)^p}{p!(m-p-s)!}\,{}_2F_1\left(\begin{matrix}-p, -m+p+s\\ 1-m-\lambda\end{matrix}\,\middle|\, -\frac{1}{4}\right) FL^{a,b}_{m-s-2p}(x), \end{aligned} \tag{6.5}$$

$$D^sH_m(x) = \frac{m!}{a}\sum_{p=0}^{m-s}\frac{(-1)^p 2^{m-2p}}{p!}\,{}_1\tilde{F}_1(-p;1+m-2p-s;-4)\, FL^{a,b}_{m-s-2p}(x), \tag{6.6}$$

$$D^sF^{A,B}_m(x) = \frac{A^m m!}{a}\sum_{p=0}^{m-s}\frac{(-1)^p}{p!(m-p-s)!}\,{}_2F_1\left(\begin{matrix}-p, -m+p+s\\ -m\end{matrix}\,\middle|\, \frac{B}{A^2}\right) FL^{a,b}_{m-s-2p}(x), \tag{6.7}$$

$$D^sL^{\bar{A},\bar{B}}_m(x) = \frac{\bar{A}^m\, m!}{a}\sum_{p=0}^{m-s}\frac{(-1)^p}{p!(m-p-s)!}\,{}_2F_1\left(\begin{matrix}-p, -m+p+s\\ 1-m\end{matrix}\,\middle|\, \frac{\bar{B}}{\bar{A}^2}\right) FL^{a,b}_{m-s-2p}(x). \tag{6.8}$$

Proof. Formulas (6.5)–(6.8) are direct consequences of Theorem 6.1. □

6.2. Expressions for the derivatives of non-symmetric polynomials

Theorem 6.2. *Let $\theta_m(x)$ be the non-symmetric polynomials that are expressed by* (2.10). *The derivatives of $\theta_m(x)$ can be expressed in terms of $FL^{a,b}_m(x)$ as*

$$\begin{aligned} D^s\theta_m(x) =& \frac{1}{a}\sum_{p=0}^{m-s}\sum_{\ell=0}^{p}(-1)^{p-\ell}Y_{2\ell,m}\binom{m-2\ell-s}{p-\ell}(1+m-2\ell-s)_s\, FL^{a,b}_{m-s-2p}(x) \\ &+\frac{1}{a}\sum_{p=0}^{m-s}\sum_{\ell=0}^{p}(-1)^{p-\ell}Y_{2\ell+1,m}\binom{-1+m-2\ell-s}{p-\ell}(m-2\ell-s)_s\, FL^{a,b}_{m-s-2p-1}(x), \end{aligned} \tag{6.9}$$

where $Y_{\ell,m}$ are the power form coefficients that appear in (2.10).

Proof. The analytic form of $\theta_m(x)$ leads to the following formula:

$$D^s\chi_m(x) = \sum_{r=0}^{m-s} Y_{r,m}\,(m-r-s+1)_s\, x^{m-r-s}. \tag{6.10}$$

The application of the inversion formula (3.11) leads to

$$D^s\theta_m(x) = \frac{1}{a}\sum_{r=0}^{m-s} Y_{r,m}\,(m-r-s+1)_s \sum_{t=0}^{m-r-s}(-1)^t\binom{m-s-r}{t} FL^{a,b}_{m-r-s-2t}(x), \tag{6.11}$$

which can be turned into

$$\begin{aligned} D^s\theta_m(x) =& \frac{1}{a}\sum_{p=0}^{m-s}\sum_{\ell=0}^{p}(-1)^{p-\ell}Y_{2\ell,m}\binom{m-2\ell-s}{p-\ell}(1+m-2\ell-s)_s\, FL^{a,b}_{m-s-2p}(x)\\ &+\frac{1}{a}\sum_{p=0}^{m-s}\sum_{\ell=0}^{p}(-1)^{p-\ell}Y_{2\ell+1,m}\binom{-1+m-2\ell-s}{p-\ell}(m-2\ell-s)_s\, FL^{a,b}_{m-s-2p-1}(x). \end{aligned} \tag{6.12}$$

This proves Theorem 6.2. □

As a consequence of Theorem 6.2, the following corollary gives two expressions for the derivative formulas of the normalized shifted Jacobi polynomials and the Schröder polynomials in terms of the MFLPs.

Corollary 6.2. *Consider two positive integers m and s with m ≥ s. The following derivative formulas hold:*

$$\begin{aligned} D^s V_m^{(\alpha,\beta)}(x) =& \frac{m!\,\Gamma(1+\alpha)\,\Gamma(2m+\alpha+\beta)}{a\,\Gamma(1+m+\alpha)\,\Gamma(1+m+\alpha+\beta)}\times\\ &\left(\sum_{p=0}^{m-s}\frac{(-1)^p(2m+\alpha+\beta)}{p!(m-p-s)!}\,{}_4F_3\left(\left.\begin{matrix}-p,-m+p+s,-\frac{m}{2}-\frac{\beta}{2},\frac{1}{2}-\frac{m}{2}-\frac{\beta}{2}\\ \frac{1}{2},-m-\frac{\alpha}{2}-\frac{\beta}{2},\frac{1}{2}-m-\frac{\alpha}{2}-\frac{\beta}{2}\end{matrix}\right|-\frac{1}{4}\right)FL^{a,b}_{m-s-2p}(x)\right.\\ &\left.+\sum_{p=0}^{m-s}\frac{(-1)^{p+1}(m+\beta)}{p!(m-p-s-1)!}\,{}_4F_3\left(\left.\begin{matrix}-p,1-m+p+s,\frac{1}{2}-\frac{m}{2}-\frac{\beta}{2},1-\frac{m}{2}-\frac{\beta}{2}\\ \frac{3}{2},\frac{1}{2}-m-\frac{\alpha}{2}-\frac{\beta}{2},1-m-\frac{\alpha}{2}-\frac{\beta}{2}\end{matrix}\right|-\frac{1}{4}\right)FL^{a,b}_{m-s-2p-1}(x)\right), \end{aligned} \tag{6.13}$$

$$\begin{aligned} D^s S_m(x) =& \frac{(2m)!}{a(m+1)!}\sum_{p=0}^{m-s}\frac{(-1)^p}{p!(m-p-s)!}\,{}_4F_3\left(\left.\begin{matrix}-\frac{1}{2}-\frac{m}{2},-\frac{m}{2},-p,-m+p+s\\ \frac{1}{2},\frac{1}{2}-m,-m\end{matrix}\right|-\frac{1}{4}\right)FL^{a,b}_{m-s-2p}(x)\\ &+\frac{(2m-1)!}{am!}\sum_{p=0}^{m-s}\frac{(-1)^p}{p!(m-p-s-1)!}\,{}_4F_3\left(\left.\begin{matrix}\frac{1}{2}-\frac{m}{2},-\frac{m}{2},-p,1-m+p+s\\ \frac{3}{2},\frac{1}{2}-m,1-m\end{matrix}\right|-\frac{1}{4}\right)FL^{a,b}_{m-s-2p-1}(x). \end{aligned} \tag{6.14}$$

Proof. It is an immediate outcome of Theorem 6.2 considering the power form coefficients of the $V_\ell^{(\alpha,\beta)}(x)$ in the first row of Table 1. □

7. Linearization formulas involving the MFLPs

This section is confined to presenting a new linearization formula (LF) of the MFLPs. In addition, other linearization formulas (LFs) involving MFLPs will be given. More precisely, the following formulas will be established:

- The LF of two classes of the MFLPs.
- The LFs of the MFLPs with any symmetric polynomial whose expression is as in (2.9).
- The product formula of the MFLPs with any non-symmetric polynomials whose expression is as in (2.10).

Theorem 7.1. *Let i and j be two positive integers. The following product formula is valid:*

$$FL_i^{a,b}(x)\,FL_j^{c,d}(x) = b\left(FL_{i+j}^{c,d}(x) + (-1)^i\,FL_{j-i}^{c,d}(x)\right) + (2b-a)\sum_{m=0}^{i-2}(-1)^{m+1}FL_{j+i-2m-2}^{c,d}(x). \tag{7.1}$$

Proof. It is possible to write the product $FL_i^{a,b}(x)\,FL_j^{c,d}(x)$ using the analytic form (3.9) along with the moment formula (3.10) as

$$\begin{aligned} FL_i^{a,b}(x)\,FL_j^{c,d}(x) = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \frac{(i-2m+1)_{m-1}((i-2m)b+ma)}{m!}\times \\ \sum_{r=0}^{i-2m}(-1)^r\binom{i-2m}{r}FL_{i+j-2m-2r}^{c,d}(x). \end{aligned} \tag{7.2}$$

Some tedious manipulations lead to the following formula:

$$\begin{aligned} &FL_i^{a,b}(x)\,FL_j^{c,d}(x) = b\,FL_{i+j}^{c,d}(x) + (-1)^i\,FL_{j-i}^{c,d}(x) \\ &+ \sum_{p=0}^{i-2}\sum_{\ell=0}^{p+1}\frac{(-1)^{1-\ell+p}(b(i-2\ell)+a\ell)\binom{i-2\ell}{1-\ell+p}(1+i-2\ell)_{\ell-1}}{\ell!}\,FL_{i+j-2p-2}^{c,d}(x). \end{aligned} \tag{7.3}$$

Now, if we let

$$H_{p,i,j} = \sum_{\ell=0}^{p+1}\frac{(-1)^{1-\ell+p}(b(i-2\ell)+a\ell)\binom{i-2\ell}{1-\ell+p}(1+i-2\ell)_{\ell-1}}{\ell!},$$

then, it is not difficult to show that $H_{p,i,j}$ satisfies the following recursive formula:

$$H_{p+1,i,j} + H_{p,i,j} = 0, \quad H_{0,i,j} = a - 2\,b.$$

Thus, it is easy to note that $H_{p,i,j}$ has the following expression:

$$H_{p,i,j} = (-1)^p(a-2\,b),$$

and accordingly, the following product formula is obtained:

$$FL_i^{a,b}(x)\,FL_j^{c,d}(x) = b\left(FL_{i+j}^{c,d}(x) + (-1)^i\,FL_{j-i}^{c,d}(x)\right) + (2b-a)\sum_{m=0}^{i-2}(-1)^{m+1}FL_{j+i-2m-2}^{c,d}(x).$$

This finalizes the proof of Theorem 7.1. □

Some new particular product formulas can be deduced as consequences of Theorem 7.1. In the following section, we present some of these results.

Corollary 7.1. *Consider two positive integers i and j. The following linearization formula holds:*

$$FL_i^{a,b}(x)\,FL_j^{a,b}(x) = b\left(FL_{i+j}^{a,b}(x) + (-1)^i\,FL_{j-i}^{a,b}(x)\right) + (2b-a)\sum_{m=0}^{i-2}(-1)^{m+1}FL_{j+i-2m-2}^{a,b}(x). \tag{7.4}$$

Proof. The proof is a special case of formula (7.1) corresponding to the choice: $c = a$ and $d = b$. □

Corollary 7.2. *Consider two positive integers i and j. The following linearization formulas hold:*

$$F_{i+1}(x)L_j(x) = L_{i+j}(x) + (-1)^i L_{j-i}(x) + \sum_{m=0}^{i-2}(-1)^{m+1}L_{j+i-2m-2}(x), \tag{7.5}$$

$$L_i(x)F_{j+1}(x) = F_{i+j+1}(x) + (-1)^i F_{j-i+1}(x), \tag{7.6}$$

$$F_{i+1}(x)F_{j+1}(x) = F_{i+j+1}(x) + (-1)^i F_{j-i+1}(x) + \sum_{m=0}^{i-2}(-1)^{m+1}F_{j+i-2m-1}(x), \tag{7.7}$$

$$L_i(x)L_j(x) = L_{i+j}(x) + (-1)^i L_{j-i}(x). \tag{7.8}$$

Proof. Relations (7.5)–(7.8) are special cases of (7.1) taking into consideration (3.3). □

Theorem 7.2. *Let i and j be two non-negative integers, and consider the symmetric polynomials $\chi_i(x)$ that can be represented by* (2.9)*. The following product formula applies:*

$$FL_i^{a,b}(x)\chi_j(x) = \sum_{p=0}^{j}\sum_{\ell=0}^{p} G_{\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell}FL_{i+j-2p}^{a,b}(x), \tag{7.9}$$

where $G_{\ell,j}$ are the power form coefficients in (2.9).

Proof. The analytic form of $\chi_i(x)$ in (2.9) enables one to write

$$FL_i^{a,b}(x)\chi_j(x) = \sum_{m=0}^{\lfloor \frac{j}{2}\rfloor} G_{m,j}\,x^{j-2m}FL_i^{a,b}(x). \tag{7.10}$$

Thanks to the moment formula (3.10), the last formula turns into

$$FL_i^{a,b}(x)\chi_j(x) = \sum_{m=0}^{\lfloor \frac{j}{2}\rfloor} G_{m,j}\sum_{r=0}^{j-2m}(-1)^r\binom{j-2m}{r}FL_{i+j-2m-2r}^{a,b}(x), \tag{7.11}$$

which can be converted into

$$FL_i^{a,b}(x)\chi_j(x) = \sum_{p=0}^{j}\sum_{\ell=0}^{p} G_{\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell}FL_{i+j-2p}^{a,b}(x). \tag{7.12}$$

This proves Theorem 7.2. □

In the following corollary, we display some product formulas of the MFLPs and some symmetric polynomials.

Corollary 7.3. *Consider positive integers i and j. The following LFs hold:*

$$FL_i^{a,b}(x)\,C_j^{(\lambda)}(x) = \frac{2^{-1+j+2\lambda}\Gamma(j+\lambda)\Gamma\left(\frac{1}{2}+\lambda\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)}\times \sum_{p=0}^{j}(-1)^p\binom{j}{p}\,{}_2F_1\left(\begin{matrix}-p,p-j\\1-j-\lambda\end{matrix}\,\middle|\,-\frac{1}{4}\right)FL_{i+j-2p}^{a,b}(x), \tag{7.13}$$

$$FL_i^{a,b}(x)\,F_j^{A,B}(x) = A^j\sum_{p=0}^{j}(-1)^p\binom{j}{p}\,{}_2F_1\left(\begin{matrix}-p,p-j\\-j\end{matrix}\,\middle|\,\frac{B}{A^2}\right)FL_{i+j-2p}^{a,b}(x), \tag{7.14}$$

$$FL_i^{a,b}(x)\,L_j^{\bar{A},\bar{B}}(x) = \bar{A}^j\sum_{p=0}^{j}(-1)^p\binom{j}{p}\,{}_2F_1\left(\begin{matrix}-p,p-j\\1-j\end{matrix}\,\middle|\,\frac{\bar{B}}{\bar{A}^2}\right)FL_{i+j-2p}^{a,b}(x), \tag{7.15}$$

$$FL_i^{a,b}(x)\,H_j(x) = \sum_{p=0}^{j}(-2)^{-j+2p}\binom{j}{p}U(-j+p,1-j+2p,-4)FL_{i+j-2p}^{a,b}(x), \tag{7.16}$$

where $U(a,b,z)$ is the confluent hypergeometric function [48].

Proof. The results of Corollary 7.3 are consequences of Theorem 7.2. □

Theorem 7.3. *Let i and j be two non-negative integers. For the non-symmetric polynomials $\theta_i(x)$ that are represented by* (2.10)*, the following product formula applies:*

$$FL_i^{a,b}(x)\,\theta_j(x) = \sum_{p=0}^{j}\sum_{\ell=0}^{p}Y_{2\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell}FL_{i+j-2p}^{a,b}(x) + \sum_{p=0}^{j-1}\sum_{\ell=0}^{p}Y_{2\ell+1,j}\,(-1)^{p-\ell}\binom{-1+j-2\ell}{p-\ell}FL_{i+j-2p-1}^{a,b}(x), \tag{7.17}$$

where $Y_{\ell,j}$ are the power form coefficients in (2.10).

Proof. Applying the analytic form of the non-symmetric polynomials $\theta_i(x)$ in (2.9) enables one to write

$$FL_i^{a,b}(x)\,\theta_j(x) = \sum_{m=0}^{j}Y_{m,j}\,x^{j-m}\,FL_i^{a,b}(x). \tag{7.18}$$

Thanks to the moment formula (3.10), the last formula turns into

$$FL_i^{a,b}(x)\chi_j(x) = \sum_{m=0}^{j}Y_{m,j}\sum_{r=0}^{j-m}(-1)^r\binom{j-m}{r}F_{i+j-m-2r}^{a,b}(x). \tag{7.19}$$

Some lengthy computations convert the last formula into

$$\begin{aligned} FL_i^{a,b}(x)\chi_j(x) = &\sum_{p=0}^{j}\sum_{\ell=0}^{p} Y_{2\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell} FL^{a.b}_{i+j-2p}(x) \\ &+\sum_{p=0}^{j-1}\sum_{\ell=0}^{p} Y_{2\ell+1,j}\,(-1)^{p-\ell}\binom{-1+j-2\ell}{p-\ell} FL^{a.b}_{i+j-2p-1}(x). \end{aligned} \tag{7.20}$$

The proof is now complete. □

Remark 7.1. *We comment here that from Theorem 7.3, we can find linearization formulas of $FL_i^{a,b}(x)$ with some celebrated non-symmetric polynomials. The following corollary exhibits two linearization formulas as two applications of this theorem.*

Corollary 7.4. *Consider the two positive integers i and j. The following two LFs apply:*

$$\begin{aligned} &FL_i^{a,b}(x)\,P_j^{(\alpha,\beta)}(x) = \frac{\Gamma(1+\alpha)\Gamma(1+2j+\alpha+\beta)}{\Gamma(1+j+\alpha)\Gamma(1+j+\alpha+\beta)}\times \\ &\sum_{p=0}^{j}(-1)^p\binom{j}{p}\,{}_4F_3\left(\left.\begin{matrix} -p,-j+p,-\frac{j}{2}-\frac{\beta}{2},\frac{1}{2}-\frac{j}{2}-\frac{\beta}{2} \\ \frac{1}{2},-j-\frac{\alpha}{2}-\frac{\beta}{2},\frac{1}{2}-j-\frac{\alpha}{2}-\frac{\beta}{2} \end{matrix}\right| -\frac{1}{4}\right) FL^{a,b}_{i+j-2p}(x) \\ &+\frac{(j+\beta)j!\Gamma(1+\alpha)\Gamma(2j+\alpha+\beta)}{\Gamma(1+j+\alpha)\Gamma(1+j+\alpha+\beta)}\times \\ &\sum_{p=0}^{j-1}\frac{(-1)^{p+1}}{(-1+j-p)!p!}\,{}_4F_3\left(\left.\begin{matrix} -p,1-j+p,\frac{1}{2}-\frac{j}{2}-\frac{\beta}{2},1-\frac{j}{2}-\frac{\beta}{2} \\ \frac{3}{2},\frac{1}{2}-j-\frac{\alpha}{2}-\frac{\beta}{2},1-j-\frac{\alpha}{2}-\frac{\beta}{2} \end{matrix}\right| -\frac{1}{4}\right) FL^{a,b}_{i+j-2p-1}(x), \end{aligned} \tag{7.21}$$

$$\begin{aligned} FL_i^{a,b}(x)\,S_j(x) = &\sum_{p=0}^{j}(-1)^p\binom{j}{p}\,{}_6F_5\left(\left.\begin{matrix} -p,-j+p,\frac{1}{4}-\frac{j}{4},\frac{1}{2}-\frac{j}{4},\frac{3}{4}-\frac{j}{4},-\frac{j}{4} \\ \frac{1}{2},1,\frac{3}{2},\frac{1}{2}-\frac{j}{2},-\frac{j}{2} \end{matrix}\right| -4\right) FL^{a,b}_{i+j-2p}(x) \\ &+\frac{1}{2}(j-1)j\sum_{p=0}^{j-1}(-1)^p\binom{j-1}{p}\,{}_6F_5\left(\left.\begin{matrix} -p,1-j+p,\frac{1}{2}-\frac{j}{4},\frac{3}{4}-\frac{j}{4},1-\frac{j}{4},\frac{5}{4}-\frac{j}{4} \\ \frac{3}{2},\frac{3}{2},2,\frac{1}{2}-\frac{j}{2},1-\frac{j}{2} \end{matrix}\right| -4\right) FL^{a,b}_{i+j-2p-1}(x). \end{aligned} \tag{7.22}$$

Proof. The results of Corollary 7.4 are consequences of Theorem 7.3. □

8. An application to the derived expressions

This section presents an application of some of the derived expressions. More precisely, some new definite integral formulas can be deduced based on derivatives, connection, and linearization coefficients.

Corollary 8.1. *Let $m \geq 2$. The following integral formula is valid:*

$$\int_0^1 FL_m^{a,b}(x)\,dx = W_m^{a,b} =$$
$$\begin{cases} a, & m=0,\\ \frac{b}{2}, & m=1,\\ \frac{b}{m+1}\,{}_3F_2\left(\begin{matrix} -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, 1+\frac{bm}{a-2b} \\ 1-m, \frac{bm}{a-2b}\end{matrix}\middle| -4\right), & m\ \text{even}, a\neq 2b,\\ \frac{-2(a-2b+am)}{m^2-1} + \frac{b}{m+1}\,{}_3F_2\left(\begin{matrix} -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2}, 1+\frac{bm}{a-2b} \\ 1-m, \frac{bm}{a-2b}\end{matrix}\middle| -4\right), & m\ \text{odd}, a\neq 2b,\\ \frac{b}{m+1}\,{}_2F_1\left(\begin{matrix} -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2} \\ 1-m\end{matrix}\middle| -4\right), & m\ \text{even}, a=2b,\\ bm\left(\frac{-4}{m^2-1} + \frac{1}{m(m+1)}\,{}_2F_1\left(\begin{matrix} -\frac{1}{2}-\frac{m}{2}, -\frac{m}{2} \\ 1-m\end{matrix}\middle| -4\right)\right), & m\ \text{odd}, a=2b.\end{cases} \tag{8.1}$$

Proof. Form the connection formula (5.22), we have

$$FL_m^{a,b}(x)\,dx = \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor} V_{p,m}\,B_{m-2p}(x) + \sum_{p=0}^{\lfloor \frac{m-1}{2}\rfloor} \bar{V}_{p,m} B_{m-2p-1}(x), \quad m\geq 2, \tag{8.2}$$

where the coefficients $V_{p,m}$, and $\bar{V}_{p,m}$ are given by (5.23) and (5.24). Now, if we integrate both sides of (5.22) from 0 to 1, then we obtain

$$\int_0^1 FL_m^{a,b}(x)\,dx = \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor} V_{p,m}\int_0^1 B_{m-2p}(x)\,dx + \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor} \bar{V}_{p,m}\int_0^1 B_{m-2p-1}(x)\,dx. \tag{8.3}$$

The well-known integral [52]

$$\int_0^1 B_\ell(x)\,dx = \begin{cases} 1, & \ell=0,\\ 0, & \ell>0,\end{cases}$$

leads to converting (8.3) into

$$\int_0^1 FL_m^{a,b}(x)\,dx = \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor} V_{p,m}\,\delta_{m-2p,0} + \sum_{p=0}^{\lfloor \frac{m}{2}\rfloor} \bar{V}_{p,m}\,\delta_{m-2p-1,0}, \tag{8.4}$$

which implies the following integral formula:

$$\int_0^1 FL_m^{a,b}(x)\,dx = \begin{cases} V_{\frac{m}{2},m}, & m\ \text{even},\\ \bar{V}_{\frac{m-1}{2},m}, & m\ \text{odd}.\end{cases} \tag{8.5}$$

The two cases corresponding to $m = 0, 1$ are easy. So now, formula (8.1) can be obtained. □

Corollary 8.2. *Let m and n be two non-negative integers $n \geq m$. We have the following integral formula:*

$$\int_0^1 FL_m^{a,b}(x)\, FL_n^{a,b}(x)\, dx = b\,(W_{m+n} + (-1)^m W_{n-m}) + (2b-a)\sum_{\ell=1}^{m-1}(-1)^\ell\, W_{m+n-2\ell}, \tag{8.6}$$

and in particular, for $a = 2b$, this formula reduces to

$$\int_0^1 FL_m^{2b,b}(x)\, FL_n^{2b,b}(x)\, dx = b\,(W_{m+n} + (-1)^m\, W_{n-m})\,. \tag{8.7}$$

Proof. Formula (8.6) is a consequence of the linearization formula (7.4) together with formula (8.1). Formula (8.7) is a specific formula of (8.6). □

Corollary 8.3. *Let $\chi_j(x)$ be the symmetric polynomial that is represented by* (2.9)*. The following formula is valid for two non-negative integers i and j:*

$$\int_0^1 FL_i^{a,b}(x)\chi_j(x)\, dx = \sum_{p=0}^{j}\sum_{\ell=0}^{p} G_{\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell} W_{i+j-2p}, \tag{8.8}$$

where W_i are given in (8.1)*.*

Proof. Using formula (7.12), we can write

$$\int_0^1 FL_i^{a,b}(x)\chi_j(x)\, dx = \sum_{p=0}^{j}\sum_{\ell=0}^{p} G_{\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell}\int_0^1 FL_{i+j-2p}(x)\, dx, \tag{8.9}$$

which is equivalent to the form in (8.8). □

Corollary 8.4. *Let i, j be two positive numbers. The following integral formulas hold:*

$$\int_0^1 FL_i^{a,b}(x)\, C_j^{(\lambda)}(x)\, dx = \frac{2^{-1+j+2\lambda}\Gamma(j+\lambda)\Gamma\left(\frac{1}{2}+\lambda\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)}\sum_{p=0}^{j}(-1)^p\binom{j}{p}\, {}_2F_1\left(\begin{matrix}-p, p-j\\ 1-j-\lambda\end{matrix}\,\middle|\, -\frac{1}{4}\right) W_{i+j-2p},$$

$$\int_0^1 FL_i^{a,b}(x)\, F_j^{A,B}(x)\, dx = A^j\sum_{p=0}^{j}(-1)^p\binom{j}{p}{}_2F_1\left(\begin{matrix}-p, p-j\\ -j\end{matrix}\,\middle|\,\frac{B}{A^2}\right) W_{i+j-2p},$$

$$\int_0^1 FL_i^{a,b}(x)\, L_j^{\bar{A},\bar{B}}(x) = \bar{A}^j\sum_{p=0}^{j}(-1)^p\binom{j}{p}{}_2F_1\left(\begin{matrix}-p, p-j\\ 1-j\end{matrix}\,\middle|\,\frac{\bar{B}}{\bar{A}^2}\right) W_{i+j-2p},$$

$$\int_0^1 FL_i^{a,b}(x)\, H_j(x)\, dx = \sum_{p=0}^{j}(-2)^{-j+2p}\binom{j}{p} U(-j+p, 1-j+2p, -4)\, W_{i+j-2p}.$$

Proof. Direct application to the result in (8.8) together with the product formulas (7.13)–(7.16). □

Corollary 8.5. *Let $\theta_j(x)$ be the non-symmetric polynomials that are represented by* (2.10). *The following formula is valid for two non-negative integers i and j:*

$$\begin{aligned}\int_0^1 FL_i^{a,b}(x)\,\theta_j(x)\,dx &= \sum_{p=0}^{j}\sum_{\ell=0}^{p} Y_{2\ell,j}\,(-1)^{p-\ell}\binom{j-2\ell}{p-\ell} W_{i+j-2p}\\ &+ \sum_{p=0}^{j-1}\sum_{\ell=0}^{p} Y_{2\ell+1,j}\,(-1)^{p-\ell}\binom{-1+j-2\ell}{p-\ell} W_{i+j-2p-1}.\end{aligned} \tag{8.10}$$

Proof. Making use of formula (7.17) together with the integral formula (8.1), formula (8.10) can be obtained. □

As consequences of Corollary 8.5, we give the integral formulas of the MFLPs with shifted Jacobi and Schröder polynomials.

Corollary 8.6. *Let i, j be two positive numbers. The following integral formula holds:*

$$\int_0^1 FL_i^{a,b}(x)\,\tilde{P}_j^{(\alpha,\beta)}(x)\,dx = \sum_{p=0}^{j} R_{p,i,j}\,W_{i+j-2p} + \sum_{p=0}^{j-1} \bar{R}_{p,i,j}\,W_{i+j-2p-1}, \tag{8.11}$$

where

$$\begin{aligned}R_{p,i,j} &= \frac{\Gamma(1+\alpha)\Gamma(1+2j+\alpha+\beta)(-1)^p\binom{j}{p}}{\Gamma(1+j+\alpha)\Gamma(1+j+\alpha+\beta)}\ {}_4F_3\left(\begin{matrix}-p,-j+p,-\frac{j}{2}-\frac{\beta}{2},\frac{1}{2}-\frac{j}{2}-\frac{\beta}{2}\\ \frac{1}{2},-j-\frac{\alpha}{2}-\frac{\beta}{2},\frac{1}{2}-j-\frac{\alpha}{2}-\frac{\beta}{2}\end{matrix}\,\middle|\,-\frac{1}{4}\right),\\ \bar{R}_{p,i,j} &= \frac{(-1)^{p+1}(j+\beta)j!\Gamma(1+\alpha)\Gamma(2j+\alpha+\beta)}{(-1+j-p)!p!\Gamma(1+j+\alpha)\Gamma(1+j+\alpha+\beta)}\ {}_4F_3\left(\begin{matrix}-p,1-j+p,\frac{1}{2}-\frac{j}{2}-\frac{\beta}{2},1-\frac{j}{2}-\frac{\beta}{2}\\ \frac{3}{2},\frac{1}{2}-j-\frac{\alpha}{2}-\frac{\beta}{2},1-j-\frac{\alpha}{2}-\frac{\beta}{2}\end{matrix}\,\middle|\,-\frac{1}{4}\right).\end{aligned}$$

Proof. Formula (8.11) comes from the direct application of (7.21) together with (8.1). □

Corollary 8.7. *Let i, j be two positive numbers. The following integral formula holds:*

$$\int_0^1 FL_i^{a,b}(x)\,S_j(x)\,dx = \sum_{p=0}^{j} G_{p,i,j}\,W_{i+j-2p} + \sum_{p=0}^{j-1} \bar{G}_{p,i,j}\,W_{i+j-2p-1}, \tag{8.12}$$

where

$$\begin{aligned}G_{p,i,j} &= (-1)^p\binom{j}{p}\,{}_6F_5\left(\begin{matrix}-p,p-j,\frac{1}{4}-\frac{j}{4},\frac{1}{2}-\frac{j}{4},\frac{3}{4}-\frac{j}{4},-\frac{j}{4}\\ \frac{1}{2},1,\frac{3}{2},\frac{1}{2}-\frac{j}{2},-\frac{j}{2}\end{matrix}\,\middle|\,-4\right),\\ \bar{G}_{p,i,j} &= \frac{1}{2}(j-1)j(-1)^p\binom{j-1}{p}\,{}_6F_5\left(\begin{matrix}-p,1-j+p,\frac{1}{2}-\frac{j}{4},\frac{3}{4}-\frac{j}{4},1-\frac{j}{4},\frac{5}{4}-\frac{j}{4}\\ \frac{3}{2},\frac{3}{2},2,\frac{1}{2}-\frac{j}{2},1-\frac{j}{2}\end{matrix}\,\middle|\,-4\right).\end{aligned}$$

Proof. Formula (8.12) comes from the direct application of (7.22) together with (8.1). □

9. Conclusions

In this study, we have established a novel polynomial sequence that combines Fibonacci and Lucas sequences. Thus, these polynomials are significant since they are extensions of the two sequences of polynomials. Developing certain basic formulas for these polynomials led to the establishment of additional important related formulas. We established several new formulas for linearization, connection, and derivatives of the MFLPs. The application of some of the established formulas led to the deduction of additional formulas for the computation of certain integrals. We wish to investigate further sequences of polynomials shortly. We anticipate that these polynomials may be used as basis functions for approximating different differential equations in numerical analysis. This is a worthy point to investigate.

Author contributions

W. M. Abd-Elhameed: Conceptualization, methodology, validation, formal analysis, funding acquisition, investigation, project administration, supervision, writing – original draft, writing - review editing; O. M. Alqubori: Methodology, funding acquisition, validation and investigation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

The authors would like to thank the referees for their constructive comments, which helped us to enhance the paper in its present form.

Conflict of interest

The authors declare that they have no competing interests.

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