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Research article

A hyperbolic polyharmonic system in an exterior domain

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Abstract: A nonlinear hyperbolic polyharmonic system in an exterior domain of $\mathbb{R}^N$ is considered under inhomogeneous Navier-type boundary conditions. Using nonlinear capacity estimates specifically adapted to the polyharmonic operator $(-\Delta)^m$, the geometry of the domain, and the boundary conditions, a sharp criterium for the nonexistence of weak solutions is obtained. Next, an optimal nonexistence result for the corresponding stationary problem is deduced.

Keywords: hyperbolic systems; polyharmonic operator; exterior domain; inhomogeneous Navier-type boundary conditions; weak solutions; nonexistence
Mathematics Subject Classification: 35A01, 35B33, 35L55, 31B30

1. Introduction

This article studies the questions of existence and nonexistence of weak solutions to the system of polyharmonic wave inequalities

$$\begin{cases} u_{tt} + (-\Delta)^m u \geq |x|^a |v|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1}, \\ v_{tt} + (-\Delta)^m v \geq |x|^b |u|^q, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1}. \end{cases} \tag{1.1}$$

Here, $(u,v) = (u(t,x), v(t,x))$, $N \geq 2$, B_1 is the open unit ball of $\mathbb{R}^N$, $m \geq 1$ is an integer, $a, b \geq -2m$, $(a,b) \neq (-2m,-2m)$, and $p, q > 1$. We will investigate (1.1) under the Navier-type boundary conditions

$$\begin{cases} (-\Delta)^i u \geq f_i(x), \ \ i = 0, \cdots, m-1, \ \ (t,x) \in (0,\infty) \times \partial B_1, \\ (-\Delta)^i v \geq g_i(x), \ \ i = 0, \cdots, m-1, \ \ (t,x) \in (0,\infty) \times \partial B_1, \end{cases} \tag{1.2}$$

where $f_i, g_i \in L^1(\partial B_1)$ and $(-\Delta)^0$ is the identity operator. Notice that no restriction on the signs of f_i or g_i is imposed.

The study of semilinear wave inequalities in $\mathbb{R}^N$ was firstly considered by Kato [1] and Pohozaev & Véron [2]. It was shown that the problem

$$u_{tt} - \Delta u \geq |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \tag{1.3}$$

possesses a critical exponent $p_K = \frac{N+1}{N-1}$ in the following sense:
(i) If $N \geq 2$ and $1 < p \leq p_K$, then (1.3) possesses no global weak solution, provided

$$\int_{\mathbb{R}^N} u_t(0,x)\,dx > 0. \tag{1.4}$$

(ii) If $p > p_K$, there are global positive solutions satisfying (1.4).

Caristi [3] studied the higher-order evolution polyharmonic inequality

$$\frac{\partial^j u}{\partial t^j} - |x|^\alpha \Delta^m u \geq |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N, \tag{1.5}$$

where $\alpha \leq 2m$. Caristi discussed separately the cases $\alpha = 2m$ and $\alpha < 2m$. For instance, when $j = 2$ and $\alpha = 0$, it was shown that, if $N \geq m+1$ and $1 < p \leq \frac{N+m}{N-m}$, then (1.5) possesses no global weak solution, provided (1.4) holds. Other existence and nonexistence results for evolution inequalities involving the polyharmonic operator in the whole space can be found in [4–6].

The study of the blow-up for semilinear wave equations in exterior domains was firstly considered by Zhang [7]. Namely, among many other problems, Zhang investigated the equation

$$u_{tt} - \Delta u = |x|^a |u|^p, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash D, \tag{1.6}$$

where $N \geq 3$, $a > -2$, and D is a smooth bounded subset of $\mathbb{R}^N$. It was shown that (1.6) under the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = f(x) \geq 0, \quad (t,x) \in (0,\infty) \times \partial D,$$

admits a critical exponent $\frac{N+a}{N-2}$ in the following sense:
(i) If $1 < p < \frac{N+a}{N-2}$, then (1.6) admits no global solution, provided $f \not\equiv 0$.
(ii) If $p > \frac{N+a}{N-2}$, then (1.6) admits global solutions for some $f > 0$.

In [8, 9], it was shown that the critical value $p = \frac{N+a}{N-2}$ belongs to case (i). Furthermore, the same result holds true, if (1.6) is considered under the Dirichlet boundary condition

$$u = f(x) \geq 0, \quad (t,x) \in (0,\infty) \times \partial D,$$

where $D = \overline{B_1}$.

In [10], the authors considered the system of wave inequalities (1.1) in the case $m = 1$. The system was studied under different types of inhomogeneous boundary conditions. In particular, under the boundary conditions (1.2) with $m = 1$ (Dirichlet-type boundary conditions), the authors obtained the following result: Assume that $a, b \geq -2$, $(a,b) \neq (-2,-2)$, $I_{f_0} := \int_{\partial B_1} f_0\,dS_x \geq 0$, $I_{g_0} := \int_{\partial B_1} g_0\,dS_x \geq 0$, $(I_{f_0}, I_{g_0}) \neq (0,0)$, and $p, q > 1$. If $N = 2$; or $N \geq 3$ and

$$N < \max\left\{\operatorname{sgn}(I_{f_0})\frac{2p(q+1)+pb+a}{pq-1}, \operatorname{sgn}(I_{g_0})\frac{2q(p+1)+qa+b}{pq-1}\right\},$$

then (1.1)-(1.2) (with $m = 1$) admits no weak solution. Moreover, the authors pointed out the sharpness of the above condition.

In the case $m = 2$, the system (1.1) was recently studied in [11] under different types of boundary conditions. In particular, under the boundary conditions (1.2) with $f_0 \equiv 0$ and $g_0 \equiv 0$, i.e.,

$$\begin{cases} u \geq 0,\ -\Delta u \geq f_1(x), & (t,x) \in (0,\infty) \times \partial B_1, \\ v \geq 0,\ -\Delta v \geq g_1(x), & (t,x) \in (0,\infty) \times \partial B_1. \end{cases} \tag{1.7}$$

Namely, the following result was obtained: Let $N \geq 2$, $a, b \geq -4$, $(a,b) \neq (-4,-4)$, $\int_{\partial B_1} f_1\, dS_x > 0$, $\int_{\partial B_1} g_1\, dS_x > 0$, and $p, q > 1$. If $N \in \{2,3,4\}$; or

$$N \geq 5,\ N < \max\left\{\frac{4p(q+1)+pb+a}{pq-1}, \frac{4q(p+1)+qa+b}{pq-1}\right\},$$

then (1.1) (with $m = 2$) under the boundary conditions (1.7) admits no weak solution. Moreover, it was shown that the above condition is sharp.

Further results related to the existence and nonexistence of solutions for evolution problems in exterior domains can be found in [12–17].

The present work aims to extend the obtained results in [10,11] from $m \in \{1,2\}$ to an arbitrary $m \geq 1$. Before presenting our main results, we need to define weak solutions to the considered problem.

Let

$$Q = (0,\infty) \times \mathbb{R}^N \backslash B_1, \quad \Sigma_Q = (0,\infty) \times \partial B_1.$$

Notice that $\Sigma_Q \subset Q$.

Definition 1.1. We say that φ is an admissible test function, if

(i) $\varphi \in C^{2,2m}_{t,x}(Q)$;
(ii) $\mathrm{supp}(\varphi) \subset\subset Q$ (φ is compactly supported in Q);
(iii) $\varphi \geq 0$;
(iv) For all $j = 0, 1, \cdots, m-1$,

$$\Delta^j \varphi|_{\Sigma_Q} = 0,\ (-1)^j \frac{\partial(\Delta^j \varphi)}{\partial \nu}|_{\Sigma_Q} \leq 0,$$

where ν denotes the outward unit normal vector on ∂B_1, relative to $\mathbb{R}^N \backslash B_1$.

The set of all admissible test functions is denoted by Φ.

Definition 1.2. We say that the pair (u,v) is a weak solution to (1.1)-(1.2), if

$$(u,v) \in L^q_{\mathrm{loc}}(Q) \times L^p_{\mathrm{loc}}(Q),$$

$$\int_Q |x|^a |v|^p \varphi\, dx\, dt - \sum_{i=0}^{m-1} \int_{\Sigma_Q} f_i(x) \frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial \nu}\, d\sigma\, dt \leq \int_Q u(-\Delta)^m \varphi\, dx\, dt + \int_Q u\varphi_{tt}\, dx\, dt \tag{1.8}$$

and

$$\int_Q |x|^b |u|^q \varphi\, dx\, dt - \sum_{i=0}^{m-1} \int_{\Sigma_Q} g_i \frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial \nu}\, d\sigma\, dt \leq \int_Q v(-\Delta)^m \varphi\, dx\, dt + \int_Q v\varphi_{tt}\, dx\, dt \tag{1.9}$$

for every $\varphi \in \Phi$.

Notice that, if (u, v) is a regular solution to (1.1)-(1.2), then (u, v) is a weak solution in the sense of Definition 1.2.

For every function $f \in L^1(\partial B_1)$, we set

$$I_f = \int_{\partial B_1} f(x)\, d\sigma.$$

Our first main result is stated in the following theorem.

Theorem 1.1. Let $p, q > 1$, $N \geq 2$, and $a, b \geq -2m$ with $(a, b) \neq (-2m, -2m)$. Let $f_i, g_i \in L^1(\partial B_1)$ for every $i = 0, \cdots, m-1$. Assume that $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$ and $(I_{f_{m-1}}, I_{g_{m-1}}) \neq (0, 0)$. If $N \leq 2m$; or $N \geq 2m + 1$ and

$$N < \max \left\{ \operatorname{sgn}(I_{f_{m-1}}) \times \frac{2mp(q+1) + pb + a}{pq - 1}, \operatorname{sgn}(I_{g_{m-1}}) \times \frac{2mq(p+1) + qa + b}{pq - 1} \right\}, \tag{1.10}$$

then (1.1)-(1.2) possesses no weak solution.

Remark 1.1. Notice that (1.10) is equivalent to

$$N - 2m < \alpha, \; I_{f_{m-1}} > 0; \text{ or } \quad N - 2m < \beta, \; I_{g_{m-1}} > 0, \tag{1.11}$$

where

$$\alpha = \frac{a + 2m + p(b + 2m)}{pq - 1} \tag{1.12}$$

and

$$\beta = \frac{b + 2m + q(a + 2m)}{pq - 1}. \tag{1.13}$$

On the other hand, due to the condition $a, b \geq -2m$ and $(a, b) \neq (-2m, -2m)$, we have $\alpha, \beta > 0$, which shows that, if $N \leq 2m$, then (1.10) is always satisfied.

The proof of Theorem 1.1 is based on the construction of a suitable admissible test function and integral estimates. The construction of the admissible test function is specifically adapted to the polyharmonic operator $(-\Delta)^m$, the geometry of the domain, and the Navier-type boundary conditions (1.2).

Remark 1.2. By Theorem 1.1, we recover the nonexistence result obtained in [10] in the case $m = 1$. We also recover the nonexistence result obtained in [11] in the case $m = 2$.

Next, we are concerned with the existence of solutions to (1.1)-(1.2). Our second main result shows the sharpness of condition (1.10).

Theorem 1.2. Let $p, q > 1$ and $a, b \geq -2m$ with $(a, b) \neq (-2m, -2m)$. If

$$N - 2m > \max \{\alpha, \beta\}, \tag{1.14}$$

where α and β are given by (1.12) and (1.13), then (1.1)-(1.2) admits stationary solutions for some $f_i, g_i \in L^1(\partial B_1)$ $(i = 0, \cdots, m-1)$ with $I_{f_{m-1}}, I_{g_{m-1}} > 0$.

Theorem 1.2 will be proved by the construction of explicit stationary solutions to (1.1)-(1.2).

Remark 1.3. At this moment, we don't know whether there is existence or nonexistence in the critical case $N \geq 2m + 1$,

$$N = \max\left\{\operatorname{sgn}(I_{f_{m-1}}) \times \frac{2mp(q+1) + pb + a}{pq - 1}, \operatorname{sgn}(I_{g_{m-1}}) \times \frac{2mq(p+1) + qa + b}{pq - 1}\right\}.$$

This question is left open.

From Theorem 1.1, we deduce the following nonexistence result for the corresponding stationary polyharmonic system

$$\begin{cases} (-\Delta)^m u \geq |x|^a |v|^p, & x \in \mathbb{R}^N \backslash \overline{B_1}, \\ (-\Delta)^m v \geq |x|^b |u|^q, & x \in \mathbb{R}^N \backslash \overline{B_1}, \end{cases} \tag{1.15}$$

under the Navier-type boundary conditions

$$\begin{cases} (-\Delta)^i u \geq f_i(x), \ i = 0, \cdots, m-1, \ x \in \partial B_1, \\ (-\Delta)^i v \geq g_i(x), \ i = 0, \cdots, m-1, \ x \in \partial B_1. \end{cases} \tag{1.16}$$

Corollary 1.1. Let $p, q > 1$, $N \geq 2$, and $a, b \geq -2m$ with $(a, b) \neq (-2m, -2m)$. Let $f_i, g_i \in L^1(\partial B_1)$ for every $i = 0, \cdots, m-1$. Assume that $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$ and $(I_{f_{m-1}}, I_{g_{m-1}}) \neq (0, 0)$. If $N \leq 2m$; or $N \geq 2m + 1$ and (1.10) holds, then (1.15)-(1.16) possesses no weak solution.

The rest of this manuscript is organized as follows: Section 2 is devoted to some auxiliary results. Namely, we first construct an admissible test function in the sense of Definition 1.1. Next, we establish some useful integral estimates involving the constructed test function. The proofs of Theorems 1.1 and 1.2 are provided in Section 3.

Throughout this paper, the letter C denotes a positive constant that is independent of the scaling parameters T, τ, and the solution (u, v). The value of C is not necessarily the same from one line to another.

2. Auxiliary results

In this section, we establish some auxiliary results that will be used later in the proof of our main result.

2.1. Admissible test function

Let us introduce the radial function H defined in $\mathbb{R}^N \backslash B_1$ by

$$H(x) = \begin{cases} \ln |x| & \text{if} \quad N = 2, \\ 1 - |x|^{2-N} & \text{if} \quad N \geq 3. \end{cases} \tag{2.1}$$

We collect below some useful properties of the function H.

Lemma 2.1. The function H satisfies the following properties:

(i) $H \geq 0$;

(ii) $H \in C^{2m}(\mathbb{R}^N \backslash B_1)$;
(iii) $H|_{\partial B_1} = 0$;
(iv) $\Delta H = 0$ in $\mathbb{R}^N \backslash B_1$;
(v) For all $j \geq 1$,

$$\Delta^j H|_{\partial B_1} = \frac{\partial(\Delta^j H)}{\partial \nu}|_{\partial B_1} = 0;$$

(vi) $\dfrac{\partial H}{\partial \nu}|_{\partial B_1} = -C$.

Proof. (i)–(v) follow immediately from (2.1). On the other hand, we have

$$\frac{\partial H}{\partial \nu}|_{\partial B_1} = \begin{cases} -1 & \text{if} \quad N = 2, \\ -(N-2) & \text{if} \quad N \geq 3, \end{cases}$$

which proves (vi). □

We next consider a cut-off function $\xi \in C^\infty(\mathbb{R})$ satisfying the following properties:

$$0 \leq \xi \leq 1, \ \xi(s) = 1 \text{ if } |s| \leq 1, \ \xi(s) = 0 \text{ if } |s| \geq 2. \tag{2.2}$$

For all $\tau \gg 1$, let

$$\xi_\tau(x) = \xi\left(\frac{|x|}{\tau}\right), \quad x \in \mathbb{R}^N \backslash B_1,$$

that is (from (2.2)),

$$\xi_\tau(x) = \begin{cases} 1 & \text{if} \quad 1 \leq |x| \leq \tau, \\ \xi\left(\frac{|x|}{\tau}\right) & \text{if} \quad \tau \leq |x| \leq 2\tau, \\ 0 & \text{if} \quad |x| \geq 2\tau. \end{cases} \tag{2.3}$$

For $k \gg 1$, we introduce the function

$$\zeta_\tau(x) = H(x)\xi_\tau^k(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{2.4}$$

We now introduce a second cut-off function $G \in C^\infty(\mathbb{R})$ satisfying the following properties:

$$G \geq 0, \quad \text{supp}(G) \subset\subset (0, 1). \tag{2.5}$$

For $T > 0$ and $k \gg 1$, let

$$G_T(t) = G^k\left(\frac{t}{T}\right), \quad t \geq 0. \tag{2.6}$$

Let φ be the function defined by

$$\varphi(t, x) = G_T(t)\zeta_\tau(x), \quad (t, x) \in Q. \tag{2.7}$$

By Lemma 2.1, (2.3)–(2.7), we obtain the following result.

Lemma 2.2. The function φ belongs to Φ.

2.2. *A priori estimates*

For all $\lambda > 1$, $\mu \geq -2m$, and $\varphi \in \Phi$, we consider the integral terms

$$J(\lambda,\mu,\varphi) = \int_Q |x|^{\frac{-\mu}{\lambda-1}} \varphi^{\frac{-1}{\lambda-1}} |(-\Delta)^m \varphi|^{\frac{\lambda}{\lambda-1}}\, dx\, dt \tag{2.8}$$

and

$$K(\lambda,\mu,\varphi) = \int_Q |x|^{\frac{-\mu}{\lambda-1}} \varphi^{\frac{-1}{\lambda-1}} |\varphi_{tt}|^{\frac{\lambda}{\lambda-1}}\, dx\, dt. \tag{2.9}$$

Lemma 2.3. Let φ be the admissible test function defined by (2.7). Assume that

(i) $J(p,a,\varphi), J(q,b,\varphi), K(p,a,\varphi), K(q,b,\varphi) < \infty$;
(ii) $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$.

If (u, v) is a weak solution to (1.1)-(1.2), then

$$I_{f_{m-1}} \leq CT^{-1} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}} \right)^{\frac{p}{pq-1}} \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}} \right)^{\frac{pq}{pq-1}} \tag{2.10}$$

and

$$I_{g_{m-1}} \leq CT^{-1} \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}} \right)^{\frac{q}{pq-1}} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}} \right)^{\frac{pq}{pq-1}}. \tag{2.11}$$

Proof. Let (u, v) be a weak solution to (1.1)-(1.2) and φ be the admissible test function defined by (2.7). By (1.8), we have

$$\int_Q |x|^a |v|^p \varphi\, dx\, dt - \sum_{i=0}^{m-1} \int_{\Sigma_Q} f_i(x) \frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial \nu}\, d\sigma\, dt \leq \int_Q u(-\Delta)^m \varphi\, dx\, dt + \int_Q u\varphi_{tt}\, dx\, dt.$$

On the other hand, by Lemma 2.1: (v), (vi), (2.5)–(2.7), we have

$$\begin{aligned}
\sum_{i=0}^{m-1} \int_{\Sigma_Q} f_i(x) \frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial \nu}\, d\sigma\, dt &= \int_{\Sigma_Q} f_{m-1}(x) \frac{\partial \varphi}{\partial \nu}\, d\sigma\, dt \\
&= -C \int_{\Sigma_Q} f_{m-1}(x) G_T(t)\, d\sigma\, dt \\
&= -C \left(\int_0^\infty G_T(t)\, dt \right) \int_{\partial B_1} f_{m-1}(x)\, d\sigma \\
&= -C \left(\int_0^\infty G^k\left(\frac{t}{T}\right) dt \right) I_{f_{m-1}} \\
&= -CT \left(\int_0^1 G^k(s)\, ds \right) I_{f_{m-1}} \\
&= -CT I_{f_{m-1}}.
\end{aligned}$$

Consequently, we obtain

$$\int_Q |x|^a |v|^p \varphi\, dx\, dt + CT I_{f_{m-1}} \leq \int_Q u(-\Delta)^m \varphi\, dx\, dt + \int_Q u\varphi_{tt}\, dx\, dt. \tag{2.12}$$

Similarly, by (1.9), we obtain

$$\int_Q |x|^b |u|^q \varphi\,dx\,dt + CTI_{g_{m-1}} \le \int_Q v(-\Delta)^m \varphi\,dx\,dt + \int_Q v\varphi_{tt}\,dx\,dt. \tag{2.13}$$

Furthermore, by Hölder's inequality, we have

$$\begin{aligned}\int_Q u(-\Delta)^m \varphi\,dx\,dt &\le \int_Q |u|\,|(-\Delta)^m\varphi|\,dx\,dt \\ &= \int_Q \left(|x|^{\frac{b}{q}}|u|\varphi^{\frac{1}{q}}\right)\left(|x|^{\frac{-b}{q}}\,|(-\Delta)^m\varphi|\,\varphi^{\frac{-1}{q}}\right) dx\,dt \\ &\le \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}} \left(\int_Q |x|^{\frac{-b}{q-1}}\,|(-\Delta)^m\varphi|^{\frac{q}{q-1}}\,\varphi^{\frac{-1}{q-1}}\,dx\,dt\right)^{\frac{q-1}{q}},\end{aligned}$$

that is,

$$\int_Q u(-\Delta)^m\varphi\,dx\,dt \le \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}} [J(q,b,\varphi)]^{\frac{q-1}{q}}. \tag{2.14}$$

Similarly, we obtain

$$\int_Q u\varphi_{tt}\,dx\,dt \le \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}} [K(q,b,\varphi)]^{\frac{q-1}{q}}. \tag{2.15}$$

Thus, it follows from (2.12), (2.14), and (2.15) that

$$\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}} \le \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}} \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}}\right). \tag{2.16}$$

Using (2.13) and proceeding as above, we obtain

$$\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}} \le \left(\int_Q |x|^a|v|^p\varphi\,dx\,dt\right)^{\frac{1}{p}} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}}\right). \tag{2.17}$$

Using (2.16)-(2.17), and taking into consideration that $I_{g_{m-1}} \ge 0$, we obtain

$$\begin{aligned}&\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}} \\ &\le \left(\int_Q |x|^a|v|^p\varphi\,dx\,dt\right)^{\frac{1}{pq}} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{1}{q}} \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}}\right).\end{aligned}$$

Then, by Young's inequality, it holds that

$$\begin{aligned}&\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}} \\ &\le \frac{1}{pq}\int_Q |x|^a|v|^p\varphi\,dx\,dt \\ &\quad + \frac{pq-1}{pq}\left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{q(pq-1)}} \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}}.\end{aligned}$$

Consequently, we have

$$\left(1-\frac{1}{pq}\right)\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}}$$
$$\leq \frac{pq-1}{pq}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{p}{pq-1}}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}},$$

which yields (2.10). Similarly, using (2.16)-(2.17), and taking into consideration that $I_{f_{m-1}} \geq 0$, we obtain

$$\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}}$$
$$\leq \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{pq}}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right),$$

which implies by Young's inequality that

$$\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}}$$
$$\leq \frac{1}{pq}\int_Q |x|^b|u|^q\varphi\,dx\,dt$$
$$+\frac{pq-1}{pq}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{p(pq-1)}}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}}.$$

Thus, it holds that

$$\left(1-\frac{1}{pq}\right)\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}}$$
$$\leq \frac{pq-1}{pq}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}},$$

which yields (2.11). □

2.3. *Estimates of $J(\lambda,\mu,\varphi)$ and $K(\lambda,\mu,\varphi)$*

The aim of this subsection is to estimate the integral terms $J(\lambda,\mu,\varphi)$ and $K(\lambda,\mu,\varphi)$, where $\lambda > 1$, $\mu \geq -2m$, and φ is the admissible test function defined by (2.7) with $\tau, k \gg 1$.

The following result follows immediately from (2.5) and (2.6).

Lemma 2.4. We have

$$\int_0^\infty G_T(t)\,dt = CT.$$

Lemma 2.5. We have

$$\int_0^\infty G_T^{\frac{-1}{\lambda-1}}\left|\frac{d^2G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}}\,dt \leq CT^{1-\frac{2\lambda}{\lambda-1}}. \tag{2.18}$$

Proof. By (2.5) and (2.6), we have

$$\int_0^\infty G_T^{\frac{-1}{\lambda-1}} \left|\frac{d^2 G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}} dt = \int_0^T G_T^{\frac{-1}{\lambda-1}} \left|\frac{d^2 G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}} dt \tag{2.19}$$

and

$$\frac{d^2 G_T}{dt^2}(t) = kT^{-2} G^{k-2}\left(\frac{t}{T}\right)\left((k-1)G'^2\left(\frac{t}{T}\right) + G\left(\frac{t}{T}\right)G''\left(\frac{t}{T}\right)\right)$$

for all $t \in (0,T)$. The above inequality yields

$$\left|\frac{d^2 G_T}{dt^2}(t)\right| \le CT^{-2} G^{k-2}\left(\frac{t}{T}\right), \quad t \in (0,T),$$

which implies that

$$G_T^{\frac{-1}{\lambda-1}} \left|\frac{d^2 G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}} \le CT^{\frac{-2\lambda}{\lambda-1}} G^{k-\frac{2\lambda}{\lambda-1}}\left(\frac{t}{T}\right), \quad t \in (0,T).$$

Then, by (2.19), it holds that

$$\begin{aligned}\int_0^\infty G_T^{\frac{-1}{\lambda-1}} \left|\frac{d^2 G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}} dt &\le CT^{\frac{-2\lambda}{\lambda-1}} \int_0^T G^{k-\frac{2\lambda}{\lambda-1}}\left(\frac{t}{T}\right) dt \\ &= CT^{1-\frac{2\lambda}{\lambda-1}} \int_0^1 G^{k-\frac{2\lambda}{\lambda-1}}(s)\, ds \\ &= CT^{1-\frac{2\lambda}{\lambda-1}},\end{aligned}$$

which proves (2.18). □

To estimate $J(\lambda,\mu,\varphi)$ and $K(\lambda,\mu,\varphi)$, we consider separately the cases $N \ge 3$ and $N = 2$.

2.3.1. The case $N \ge 3$

Lemma 2.6. We have

$$\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}}\, dx \le C\tau^{N-\frac{\mu+2m\lambda}{\lambda-1}}. \tag{2.20}$$

Proof. Since H and ξ_τ are radial functions (see (2.1) and (2.3)), to simplify writing, we set

$$H(x) = H(r),\ \ \xi_\tau(x) = \xi_\tau(r),$$

where $r = |x|$. By (2.4) and making use of Lemma 2.1 (iv), one can show that for all $x \in \mathbb{R}^N\setminus B_1$, we have

$$\begin{aligned}\Delta^m \zeta_\tau(x) &= \Delta^m\left(H(x)\xi_\tau^k(x)\right) \\ &= \sum_{i=0}^{2m-1} \frac{d^i H}{dr^i}(r) \sum_{j=1}^{2m-i} C_{i,j} \frac{d^j \xi_\tau^k}{dr^j}(r) r^{i+j-2m},\end{aligned}$$

where $C_{i,j}$ are some constants, which implies by (2.3) that

$$\operatorname{supp}(\Delta^m \zeta_\tau) \subset \left\{x \in \mathbb{R}^N : \tau \le |x| \le 2\tau\right\} \tag{2.21}$$

and

$$|\Delta^m \zeta_\tau(x)| \le C \sum_{i=0}^{2m-1} \left|\frac{d^i H}{dr^i}(r)\right| \sum_{j=1}^{2m-i} \left|\frac{d^j \xi_\tau^k}{dr^j}(r)\right| r^{i+j-2m}, \quad x \in \operatorname{supp}(\Delta^m \zeta_\tau). \tag{2.22}$$

On the other hand, for all $x \in \operatorname{supp}(\Delta^m \zeta_\tau)$, we have by (2.1) and (2.3) that

$$\left|\frac{d^i H}{dr^i}(r)\right| = \begin{cases} H(r) & \text{if} \quad i = 0, \\ C r^{2-N-i} & \text{if} \quad i = 1, \cdots, 2m-1 \end{cases} \tag{2.23}$$

and (we recall that $0 \le \xi_\tau \le 1$)

$$\begin{aligned} \left|\frac{d^j \xi_\tau^k}{dr^j}(r)\right| &\le C\tau^{-j} \xi_\tau^{k-j}(r) \\ &\le C\tau^{-j} \xi_\tau^{k-2m}(r), \;\; j = 1, \cdots, 2m-i. \end{aligned} \tag{2.24}$$

Then, in view of (2.1), (2.21)–(2.24), we have

$$\begin{aligned} |\Delta^m \zeta_\tau(x)| &\le C \xi_\tau^{k-2m}(r) \left(H(r) \sum_{j=1}^{2m} \tau^{-j} r^{j-2m} + r^{2-N} \sum_{i=1}^{2m-1} \sum_{j=1}^{2m-i} \tau^{-j} r^{j-2m} \right) \\ &\le C \xi_\tau^{k-2m}(r) \left(\tau^{-2m} + \tau^{2-N-2m} \right) \\ &\le C \tau^{-2m} \xi_\tau^{k-2m}(x) \end{aligned}$$

for all $x \in \operatorname{supp}(\Delta^m \zeta_\tau)$. Taking into consideration that $H \ge C$ for all $x \in \operatorname{supp}(\Delta^m \zeta_\tau)$, the above estimate yields

$$|x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \le C \tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x), \quad x \in \operatorname{supp}(\Delta^m \zeta_\tau). \tag{2.25}$$

Finally, by (2.21) and (2.25), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \, dx &= \int_{\tau<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \, dx \\ &\le C \tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \int_{\tau<|x|<2\tau} \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x) \, dx \\ &\le C \tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \int_{r=\tau}^{2\tau} r^{N-1} \, dr \\ &= C \tau^{N-\frac{\mu+2m\lambda}{\lambda-1}}, \end{aligned}$$

which proves (2.20). □

Lemma 2.7. We have

$$J(\lambda, \mu, \varphi) \le C T \tau^{N-\frac{\mu+2m\lambda}{\lambda-1}}.$$

Proof. By (2.7) and (2.8), we have

$$J(\lambda, \mu, \varphi) = \left(\int_0^\infty G_T(t) \, dt \right) \left(\int_{\mathbb{R}^N \setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \, dx \right).$$

Then, using Lemmas 2.4 and 2.6, we obtain the desired estimate. □

Lemma 2.8. We have

$$\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau(x)\,dx \le C\left(\tau^{N-\frac{\mu}{\lambda-1}}+\ln\tau\right). \tag{2.26}$$

Proof. By (2.1)–(2.4), we have

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau(x)\,dx &= \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}}\left(1-|x|^{2-N}\right)\xi^\kappa\left(\frac{|x|}{\tau}\right)dx\\
&\le \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}}\,dx\\
&= C\int_{r=1}^{2\tau} r^{N-1-\frac{\mu}{\lambda-1}}\,dr\\
&\le \begin{cases} C\tau^{N-\frac{\mu}{\lambda-1}} & \text{if } \; N-\frac{\mu}{\lambda-1}>0,\\ C\ln\tau & \text{if } \; N-\frac{\mu}{\lambda-1}=0,\\ C & \text{if } \; N-\frac{\mu}{\lambda-1}<0\end{cases}\\
&\le C\left(\tau^{N-\frac{\mu}{\lambda-1}}+\ln\tau\right),
\end{aligned}$$

which proves (2.26). □

Lemma 2.9. We have

$$K(\lambda,\mu,\varphi)\le CT^{1-\frac{2\lambda}{\lambda-1}}\left(\tau^{N-\frac{\mu}{\lambda-1}}+\ln\tau\right).$$

Proof. By (2.7) and (2.9), we have

$$K(\lambda,\mu,\varphi)=\left(\int_0^\infty G_T^{\frac{-1}{\lambda-1}}\left|\frac{d^2G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}}dt\right)\left(\int_{\mathbb{R}^N\setminus B_1}|x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau(x)\,dx\right).$$

Then, using Lemmas 2.5 and 2.7, we obtain the desired estimate. □

2.3.2. The case $N=2$

Lemma 2.10. We have

$$\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau^{\frac{-1}{\lambda-1}}|(-\Delta)^m\zeta_\tau|^{\frac{\lambda}{\lambda-1}}\,dx\le C\tau^{2-\frac{2m\lambda+\mu}{\lambda-1}}\ln\tau. \tag{2.27}$$

Proof. Proceeding as in the proof of Lemma 2.6, we obtain

$$\operatorname{supp}(\Delta^m\zeta_\tau)\subset\left\{x\in\mathbb{R}^2:\tau\le|x|\le 2\tau\right\}$$

and

$$|\Delta^m\zeta_\tau(x)|\le C\tau^{-2m}\ln\tau\,\xi_\tau^{k-2m}(x),\quad x\in\operatorname{supp}(\Delta^m\zeta_\tau).$$

The above estimate yields

$$|x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau^{\frac{-1}{\lambda-1}}|(-\Delta)^m\zeta_\tau|^{\frac{\lambda}{\lambda-1}}\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}}\ln\tau\,\xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x),\quad x\in\operatorname{supp}(\Delta^m\zeta_\tau).$$

Then, it holds that

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}}\, dx &\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \ln\tau \int_{\tau<|x|<2\tau} \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x)\, dx \\ &\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \ln\tau \int_{r=\tau}^{2\tau} r\, dr \\ &\le C\tau^{2-\frac{2m\lambda+\mu}{\lambda-1}} \ln\tau,\end{aligned}$$

which proves (2.27). □

Using (2.7)-(2.8), Lemma 2.4, and Lemma 2.10, we obtain the following estimate of $J(\lambda,\mu,\varphi)$.

Lemma 2.11. We have

$$J(\lambda,\mu,\varphi) \le CT\tau^{2-\frac{2m\lambda+\mu}{\lambda-1}} \ln\tau.$$

Lemma 2.12. We have

$$\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau(x)\, dx \le C\ln\tau \left(\tau^{2-\frac{\mu}{\lambda-1}} + \ln\tau\right). \tag{2.28}$$

Proof. By (2.1)–(2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau(x)\, dx &= \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}} \ln|x|\, \xi^\kappa\left(\frac{|x|}{\tau}\right) dx \\ &\le \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}} \ln|x|\, dx \\ &= C\int_{r=1}^{2\tau} r^{1-\frac{\mu}{\lambda-1}} \ln r\, dr \\ &\le \begin{cases} C\tau^{2-\frac{\mu}{\lambda-1}} \ln\tau & \text{if} \quad 2-\frac{\mu}{\lambda-1}>0, \\ C(\ln\tau)^2 & \text{if} \quad 2-\frac{\mu}{\lambda-1}=0, \\ C\ln\tau & \text{if} \quad 2-\frac{\mu}{\lambda-1}<0 \end{cases} \\ &\le C\ln\tau \left(\tau^{2-\frac{\mu}{\lambda-1}} + \ln\tau\right),\end{aligned}$$

which proves (2.28). □

Using (2.7), (2.9), Lemma 2.5, and Lemma 2.12, we obtain the following estimate of $K(\lambda,\mu,\varphi)$.

Lemma 2.13. We have

$$K(\lambda,\mu,\varphi) \le CT^{1-\frac{2\lambda}{\lambda-1}} \ln\tau \left(\tau^{2-\frac{\mu}{\lambda-1}} + \ln\tau\right).$$

3. Proofs of the main results

This section is devoted to the proofs of Theorems 1.1 and 1.2.

3.1. Proof of Theorem 1.1

By Remark 1.1, (1.10) is equivalent to (1.11). Without restriction of the generality, we assume that

$$N - 2m < \alpha, \quad I_{f_{m-1}} > 0. \tag{3.1}$$

Indeed, exchanging the roles of $(I_{f_{m-1}}, a, p)$ and $(I_{g_{m-1}}, b, q)$, the case

$$N - 2m < \beta, \quad I_{g_{m-1}} > 0$$

reduces to (3.1).

We use the contradiction argument. Namely, let us suppose that (u, v) is a weak solution to (1.1)-(1.2) (in the sense of Definition 1.2). For $k, T, \tau \gg 1$, let φ be the admissible test function defined by (2.7). Then, by Lemma 2.3, we have

$$I_{f_{m-1}}^{\frac{pq-1}{p}} \le CT^{-\frac{pq-1}{p}} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}}\right)\left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{q}. \tag{3.2}$$

Making use of Lemmas 2.7 and 2.12, we obtain that for all $N \ge 2$,

$$J(\lambda, \mu, \varphi) \le CT\tau^{N-\frac{\mu+2m\lambda}{\lambda-1}} \ln \tau, \quad \lambda > 1,\ \mu \ge -2m. \tag{3.3}$$

Similarly, by Lemmas 2.9 and 2.13, we obtain that for all $N \ge 2$,

$$K(\lambda, \mu, \varphi) \le CT^{1-\frac{2\lambda}{\lambda-1}} \left(\tau^{N-\frac{\mu}{\lambda-1}} + \ln \tau\right) \ln \tau, \quad \lambda > 1,\ \mu \ge -2m. \tag{3.4}$$

In particular, for $(\lambda, \mu) = (p, a)$, we obtain by (3.3) and (3.4) that

$$\begin{aligned}
&[J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}} \\
&\le C\left[T^{\frac{p-1}{p}} \tau^{\left(N-\frac{a+2mp}{p-1}\right)\frac{p-1}{p}} (\ln\tau)^{\frac{p-1}{p}} + T^{\left(1-\frac{2p}{p-1}\right)\frac{p-1}{p}} \left(\tau^{N-\frac{a}{p-1}} + \ln\tau\right)^{\frac{p-1}{p}} (\ln\tau)^{\frac{p-1}{p}}\right] \\
&= CT^{\frac{p-1}{p}} \tau^{\left(N-\frac{a+2mp}{p-1}\right)\frac{p-1}{p}} (\ln\tau)^{\frac{p-1}{p}} \left[1 + T^{-2}\left(\tau^{\frac{2mp}{p-1}} + \tau^{-\left(N-\frac{a+2mp}{p-1}\right)} \ln\tau\right)^{\frac{p-1}{p}}\right].
\end{aligned} \tag{3.5}$$

Furthermore, taking $T = \tau^{\theta}$, where

$$\theta > \max\left\{m, \left(\frac{a+2mp}{p-1} - N\right)\frac{p-1}{p}\right\}, \tag{3.6}$$

we obtain

$$1 + T^{-2}\left(\tau^{\frac{2mp}{p-1}} + \tau^{-\left(N-\frac{a+2mp}{p-1}\right)} \ln\tau\right)^{\frac{p-1}{p}} \le C.$$

Then, from (3.5), we deduce that

$$[J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}} \le C\left[\tau^{\theta+N-\frac{a+2mp}{p-1}} \ln\tau\right]^{\frac{p-1}{p}}. \tag{3.7}$$

Similarly, for

$$\theta > \max\left\{m, \left(\frac{b+2mq}{q-1} - N\right)\frac{q-1}{q}\right\}, \tag{3.8}$$

we obtain

$$\left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^q \le C\left[\tau^{\theta+N-\frac{b+2mq}{q-1}} \ln\tau\right]^{q-1}. \tag{3.9}$$

Thus, for $T = \tau^\theta$, where θ satisfies (3.6) and (3.8), we obtain by (3.2), (3.7), and (3.9) that

$$I_{f_{m-1}}^{\frac{pq-1}{p}} \le C\tau^{-\frac{\theta(pq-1)}{p}} \left[\tau^{\theta+N-\frac{a+2mp}{p-1}} \ln\tau\right]^{\frac{p-1}{p}} \left[\tau^{\theta+N-\frac{b+2mq}{q-1}} \ln\tau\right]^{q-1},$$

that is,

$$I_{f_{m-1}}^{\frac{pq-1}{p}} \le C\tau^{\delta}(\ln\tau)^{\frac{pq-1}{p}}, \tag{3.10}$$

where

$$\begin{aligned}\delta &= \frac{pq-1}{p}\left[N - \frac{(b+2mq)p + a + 2mp}{pq-1}\right]\\ &= \frac{pq-1}{p}(N-2m-\alpha).\end{aligned}$$

Since $N - 2m < \alpha$, we have $\delta < 0$. Then, passing to the limit as $\tau \to \infty$ in (3.10), we reach a contradiction with $I_{f_{m-1}} > 0$. This completes the proof of Theorem 1.1. □

3.2. *Proof of Theorem 1.2*

Let us introduce the family of polynomial functions $\{P_i\}_{0\le i\le m}$, where

$$P_i(z) = \begin{cases} 1 & \text{if } \ i = 0,\\ \prod_{j=0}^{i-1}(z+2j)\prod_{j=1}^{i}(N-2j-z) & \text{if } \ i = 1,\cdots,m. \end{cases}$$

From (1.14), we deduce that

$$N - 2j > \max\{\alpha,\beta\}, \quad j = 1,\cdots,m.$$

Furthermore, because $a, b \ge -2m$ and $(a,b) \ne (-2m,-2m)$, we have $\alpha,\beta > 0$. Then,

$$P_i(z) > 0, \quad i = 0,1,\cdots,m, \quad z \in \{\alpha,\beta\}. \tag{3.11}$$

For all

$$0 < \varepsilon \le \min\left\{[P_m(\alpha)]^{\frac{1}{p-1}}, [P_m(\beta)]^{\frac{1}{q-1}}\right\}, \tag{3.12}$$

we consider functions of the forms

$$u_\varepsilon(x) = \varepsilon|x|^{-\alpha}, \quad x \in \mathbb{R}^N\backslash B_1 \tag{3.13}$$

and

$$v_\varepsilon(x) = \varepsilon|x|^{-\beta}, \quad x \in \mathbb{R}^N\backslash B_1. \tag{3.14}$$

Since u_ε and v_ε are radial functions, elementary calculations show that

$$(-\Delta)^i u_\varepsilon(x) = \varepsilon P_i(\alpha)|x|^{-\alpha-2i}, \quad i = 0,1,\cdots,m, \quad x \in \mathbb{R}^N\backslash B_1 \tag{3.15}$$

and

$$(-\Delta)^i v_\varepsilon(x) = \varepsilon P_i(\beta)|x|^{-\beta-2i}, \quad i = 0, 1, \cdots, m, \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.16}$$

Taking $i = m$ in (3.15), using (3.11)–(3.14), we obtain

$$\begin{aligned}(-\Delta)^m u_\varepsilon(x) &= \varepsilon P_m(\alpha)|x|^{-\alpha-2m}\\ &= |x|^a \varepsilon^p |x|^{-\beta p}\left(\varepsilon^{1-p} P_m(\alpha)|x|^{-\alpha-2m-a+\beta p}\right)\\ &\geq |x|^a v_\varepsilon^p(x)|x|^{-\alpha-2m-a+\beta p}.\end{aligned}$$

On the other hand, by (1.12) and (1.13), one can show that

$$-\alpha - 2m - a + \beta p = 0.$$

Then, we obtain

$$(-\Delta)^m u_\varepsilon(x) \geq |x|^a v_\varepsilon^p(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.17}$$

Similarly, taking $m = i$ in (3.16), using (3.11)–(3.14), we obtain

$$\begin{aligned}(-\Delta)^m v_\varepsilon(x) &= \varepsilon P_m(\beta)|x|^{-\beta-2m}\\ &= |x|^b \varepsilon^q |x|^{-\alpha q}\left(\varepsilon^{1-q} P_m(\beta)|x|^{-\beta-2m-b+\alpha q}\right)\\ &\geq |x|^b u_\varepsilon^q(x)|x|^{-\beta-2m-b+\alpha q}.\end{aligned}$$

Using that

$$-\beta - 2m - b + \alpha q = 0,$$

we obtain

$$(-\Delta)^m v_\varepsilon(x) \geq |x|^b u_\varepsilon^q(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.18}$$

Furthermore, by (3.11) and (3.15), for all $i = 0, \cdots, m-1$, we have

$$(-\Delta)^i u_\varepsilon(x) = \varepsilon P_i(\alpha) > 0, \quad x \in \partial B_1. \tag{3.19}$$

Similarly, by (3.11) and (3.16), for all $i = 0, \cdots, m-1$, we have

$$(-\Delta)^i v_\varepsilon(x) = \varepsilon P_i(\beta) > 0, \quad x \in \partial B_1. \tag{3.20}$$

Finally, (3.17)–(3.20) show that for all ε satisfying (3.12), the pair of functions $(u_\varepsilon, v_\varepsilon)$ given by (3.13) and (3.14) is a stationary solution to (1.1)-(1.2) with $f_i \equiv \varepsilon P_i(\alpha)$ and $g_i \equiv \varepsilon P_i(\beta)$ for all $i = 0, \cdots, m-1$. The proof of Theorem 1.2 is then completed.

□

4. Conclusions

The system of polyharmonic wave inequalities (1.1) under the inhomogeneous Navier-type boundary conditions (1.2) was investigated. First, we established a nonexistence criterium for the nonexistence of weak solutions (see Theorem 1.1). Namely, under condition (1.10), we proved that (1.1)-(1.2) possesses no weak solution, provided $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$ and $(I_{f_{m-1}}, I_{g_{m-1}}) \neq (0, 0)$. Next, we proved the sharpness of the obtained criterium (1.10) by showing that under condition (1.14), (1.1)-(1.2) possesses weak solutions (stationary solutions) for some $f_i, g_i \in L^1(\partial B_1)$ $(i = 0, \cdots, m-1)$ with $I_{f_{m-1}}, I_{g_{m-1}} > 0$ (see Theorem 1.2). From Theorem 1.1, we deduced an optimal criterium for the nonexistence of weak solutions to the corresponding stationary polyharmonic system (1.15) under the Navier-type boundary conditions (1.16) (see Corollary 1.1).

In this study, the critical case $N \geq 2m+1$,

$$N = \max\left\{\operatorname{sgn}(I_{f_{m-1}}) \times \frac{2mp(q+1)+pb+a}{pq-1}, \operatorname{sgn}(I_{g_{m-1}}) \times \frac{2mq(p+1)+qa+b}{pq-1}\right\}$$

is not investigated. It would be interesting to know whether there is existence or nonexistence of weak solutions in this case.

Author contributions

Manal Alfulaij: validation, investigation, writing review and editing; Mohamed Jleli: Conceptualization, methodology, investigation and formal analysis; Bessem Samet: Conceptualization, methodology, validation and investigation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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