AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 2131–2162.
DOI:10.3934/math.2025100
Received: 08 November 2024
Revised: 03 January 2025
Accepted: 15 January 2025
Published: 08 February 2025

Research article

Modern classes of fuzzy α-covering via rough sets over two distinct finite sets

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Abstract: Following the research of Yang and Atef proposing new classes of fuzzy β-covering via rough sets types over 2-featured universes, we present some modern classes of fuzzy α-covering via rough sets over two distinct finite sets using fuzzy α-neighborhoods for two distinct points over 2-distinct finite universes. Throughout this research, we present the ideas of the fuzzy α-neighborhood system and the fuzzy α-neighborhood for two distinct points over two distinct finite sets and investigate the relations of the fuzzy α-neighborhood system, fuzzy α-minimal and α-maximal descriptions over two distinct finite sets. Moreover, some kinds of fuzzy α-neighborhoods are proposed. In addition, some new types of fuzzy α-coverings over two finite sets are established. Finally, numerous topological characteristics of fuzzy α-covering via rough set types are investigated.

Keywords: rough sets; fuzzy α-neighborhoods; fuzzy α-coverings
Mathematics Subject Classification: 54A40, 54C55, 54D20, 60L20

1. Introduction

Pawlak [1, 2] was the first one who proposed the rough set theory (RST), which was written in terms of a pair of sets giving the lower and the upper approximations of the conventional set. The equivalence relations are used in the model of Pawlak's rough set. A partition of a finite set is formed by all equivalence classes. Some restrictions are imposed on different applications by an equivalence relation [3–5]. Ideas such as general relations have replaced equivalence relations over recent years [6–9]. In addition, some authors have used neighborhood systems [10–13] and coverings of finite sets [14–16] instead of equivalence relations.

The evolution and application of some rough set models were made by the following examples. Adaptive multi-granulation decision-theoretic rough sets represent an advanced and flexible approach to decision-making under uncertainty, where multiple levels of abstraction (granulation) are used, and the system adapts to the specific characteristics of the data or decision problem. By combining rough set theory with decision theory and adaptive granulation, this framework provides a powerful tool for handling uncertainty in complex decision-making environments. If one is working on a research project or application involving this concept, it would be useful to explore specific algorithms and methodologies that implement these ideas, as well as their real-world applications in areas like machine learning, artificial intelligence, and decision support systems [17]. Covering-based multi-granulation rough fuzzy sets offer a robust framework for managing data characterized by uncertainty, imprecision, and overlap. This approach integrates the versatility of covering relations, the adaptability of multi-granulation, and the capability of rough and fuzzy sets in modeling uncertainty, making it highly effective for applications such as feature selection, data mining, pattern recognition, and decision-making in uncertain environments [18]. The matrix-based fast granularity reduction algorithm for multi-granulation rough sets offers an innovative solution to minimize the computational demands of rough set models while preserving their effectiveness in managing uncertainty and imprecision. Leveraging matrix operations to streamline the model's granularity, the algorithm enhances the speed and efficiency of processing large datasets. This makes it especially valuable for applications such as feature selection, data mining, pattern recognition, and decision support systems [19]. The multi-scale information fusion-based multiple correlations method for unsupervised attribute selection offers an effective strategy for identifying relevant features in complex datasets. By integrating multi-scale analysis with multiple correlation techniques, it effectively captures both local and global data patterns, enhancing feature selection and optimizing performance in subsequent tasks. This approach is particularly advantageous in high-dimensional, noisy, or intricate data scenarios where conventional feature selection techniques may struggle [20]. The concept of variable precision multi-granulation covering rough intuitionistic fuzzy sets provides a robust framework for addressing uncertainty, vagueness, and imprecision in complex datasets. By integrating multi-granulation, covering relations, intuitionistic fuzzy sets, and variable precision, this approach enables a more flexible and detailed representation of data. It is well-suited for various applications in machine learning, data mining, and decision support systems. This framework effectively handles complex, uncertain, and noisy data, offering valuable insights in scenarios where traditional methods may be less effective [21].

The evolution of the covering-based rough set (CBRS) and the fuzzy covering-based rough set (FCBRS) is shown. CBRS is considered as an important subject to researchers since it can be applied to extract data, particularly in incomplete information systems. CBRS patterns and the relations between them were studied by Zhu [22], and Zhu and Wang [23–25]. Additional CBRS patterns were suggested by Tsang et al. [26] and Xu and Zhang [27]. Liu and Sai [28] compared the CBRS patterns from Zhu and the CBRS patterns from Xu and Zhang. The two ideas of a neighborhood and a complementary neighborhood were used to evolve some neighborhood CBRS patterns by Ma [29].

Fuzzy set theory (FST) [30] can solve the problem of rough sets [9] which handles qualitative (discrete) data by giving each element in the set a value between 0 and 1. The ideas of rough fuzzy sets and fuzzy rough sets are explained [31]. General fuzzy rough sets can be made by using various methods from many researchers. A new way to produce fuzzy rough sets using lattice theory was presented by Deng et al. [32]. Two fuzzy rough approximation factors were structured by Li et al. [33]. Other researchers [34–36] presented an introduction of the subject (FCBRS). The authors of [37] presented the definitions of fuzzy β-minimal description (Fβmd) and fuzzy β-maximal description (FβMD) over 2-finite sets and examined some of their properties. The motivation for writing this research was to investigate the idea of Yang and Atef [37] on the neighborhood in the research of Zhang et al. [38] and induce the results.

This paper discusses the fuzzy α-neighborhood of x and v and the fuzzy α-neighborhood system FαNS over two distinct finite sets, which provides us with the relation among FαNS, Fαmd, and FαMD. Also in this section, we presents some kinds of fuzzy α-neighborhoods, FαNs, and α-neighborhoods. The next section offers four types of fuzzy α-coverings over two finite sets and the relationships between them. Finally, we display some topological features of fuzzy α-covering depending on rough set patterns.

2. Preliminaries

In this section, we introduce some ideas on FST, fuzzy α-covering-based rough sets, and fuzzy α-neighborhoods that are used in our study.

Definition 2.1. *[22] Let W be a set and $\mathfrak{V}$ be a family of subsets of W. If $\mathfrak{V} \neq \phi$ and $W = \bigcup_{\epsilon\in\mathfrak{V}} \epsilon$, then $\mathfrak{V}$ is called a covering of W, and the covering approximation space will be denoted $(W, \mathfrak{V})$.*

Definition 2.2. *[39] Let W be a finite set. $\mathfrak{V} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\}$ is called a fuzzy α-covering of W where $\epsilon_i \in \mathscr{F}(W), (i = 1, 2, \cdots, m)$ and for each $\alpha \in (0, 1]$, if $(\bigcup_i^m \epsilon_i)(u) \geq \alpha$ for each $u \in W$. $(W, \mathfrak{V})$ is called a fuzzy α-covering-based approximation space (FαCAS).*

Definition 2.3. *In [40], FST has the Zadeh's extension principle as an important instrument. The family of all functions from W_1 to W_2 is denoted by $Fun(W_1, W_2)$, and the family of all onto functions from W_1 to W_2 is denoted by $Onto(W_1, W_2)$. Let W_1, W_2 be two finite sets and let $\mathscr{N}(W_1)$ and $\mathscr{N}(W_2)$ be the fuzzy power sets of W_1 and W_2, respectively; $\mathbb{P} \in \mathscr{N}(W_1)$, $\mathbb{Q} \in \mathscr{N}(W_2)$; and $n \in Fun(W_1, W_2)$. Then a fuzzy function n can be induced from $\mathscr{N}(W_1)$ to $\mathscr{N}(W_2)$, i.e.,*

$$n(\mathbb{P})(y) = \begin{cases} \bigvee_{u\in n^{-1}(y)} \mathbb{P}(u), & y \in n(W_1); \\ 0, & y \notin n(W_1), \end{cases}$$

and a fuzzy function n^{-1} can be induced from $\mathscr{N}(W_2)$ to $\mathscr{N}(W_1)$, i.e., $n^{-1}(\mathbb{Q})(u) = (\mathbb{Q})(n(u)), u \in W_1$. Also, we use $n(\mathfrak{V}) = \{n(\epsilon_1), n(\epsilon_2), \cdots, n(\epsilon_m)\}$, where $\mathfrak{V} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\} \subseteq \mathscr{N}(W_1)$.

Definition 2.4. *[37] Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS over two finite sets with $\mathfrak{V} = \{\epsilon_1, \epsilon_2, \ldots \cdots, \epsilon_m\}$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $u \in W_1$, Fαmd and FαMD are defined as:*
$md_{\mathfrak{V}}^{\alpha}(u) = \{f(\epsilon) \in f(\mathfrak{V}) : (\epsilon(u) \geq \alpha) \wedge (\forall f(K) \in f(\mathfrak{V}) \wedge K(u) \geq \alpha \wedge f(K) \subseteq f(\epsilon) \Rightarrow f(\epsilon) = f(K))\}$,
$MD_{\mathfrak{V}}^{\alpha}(u) = \{f(\epsilon) \in f(\mathfrak{V}) : (\epsilon(u) \geq \alpha) \wedge (\forall f(K) \in f(\mathfrak{V}) \wedge K(u) \geq \alpha \wedge f(K) \supseteq f(\epsilon) \Rightarrow f(\epsilon) = f(K))\}$.

Definition 2.5. *[38] For all $r_1, r_2 \in [0, 1]$, we define an R-implication operator I as $I(r_1, r_2) = min(1, 1 - r_1 + r_2)$.*

Definition 2.6. *[37] Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS over two finite sets with $\mathfrak{V} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\}$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $u \in W_1$, the fuzzy α-neighborhood system FαNS is defined as $\widetilde{N}_{\mathfrak{V}}^{\alpha}(u) = \{f(\epsilon) \in f(\mathfrak{V}) : \epsilon(u) \geq \alpha\}$.*

Definition 2.7. *[41] Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS on two finite sets with $\mathfrak{V} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\} \subseteq \mathscr{F}(W_1)$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $u \in W_1$, the fuzzy α-neighborhood of u is $\widetilde{N}_u^{\alpha} = \bigcap\{f(\epsilon) \in f(\mathfrak{V}) : \epsilon_i(u) \geq \alpha\}$.*

Definition 2.8. *[37] Let* $(W_1, W_2, \mathfrak{B})$ *be a FαCAS where* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $u \in W_1$*, the four kinds of fuzzy α-neighborhood of u are:*

(i) ${}_1N_u^\alpha = \bigcap\{f(\epsilon) \in f(\mathfrak{B}) : f(\epsilon) \in \widetilde{N}_{\mathfrak{B}}^\alpha(u)\}$.
(ii) ${}_2N_u^\alpha = \bigcup\{f(\epsilon) \in f(\mathfrak{B}) : f(\epsilon) \in md_{\mathfrak{B}}^\alpha(u)\}$.
(iii) ${}_3N_u^\alpha = \bigcap\{f(\epsilon) \in f(\mathfrak{B}) : f(\epsilon) \in MD_{\mathfrak{B}}^\alpha(u)\}$.
(iv) ${}_4N_u^\alpha = \bigcup\{f(\epsilon) \in f(\mathfrak{B}) : f(\epsilon) \in MD_{\mathfrak{B}}^\alpha(u)\}$.

Definition 2.9. *[41] Let* $(W_1, W_2, \mathfrak{B})$ *be a FαCAS on two finite sets where* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $B \in \mathscr{F}(W_2)$*,* $\mathbb{L}_0$ *(*$\mathbb{U}_0$*) is the symbol for the fuzzy covering lower (upper) approximations, which are*

$\mathbb{L}_0(B)(u) = \bigwedge_{z \in W_2} ([1 - \widetilde{N}_u^\alpha(z)] \bigvee B(z)), u \in W_1,$
$\mathbb{U}_0(B)(u) = \bigvee_{z \in W_2} (\widetilde{N}_u^\alpha(z) \bigwedge B(z)), u \in W_1.$
B is called a FCBRS if $\mathbb{L}_0(B) \neq \mathbb{U}_0(B)$; otherwise; it is definable.

3. Fuzzy α-neighborhood of x and v and the fuzzy α-neighborhood system FαNS over two distinct finite sets

We introduce in this section the definitions of FαNS and fuzzy α-neighborhood of x and v on two different finite sets W_1 and W_2, in FαCAS $(W_1, W_2, \mathfrak{B})$, and we point to some properties of them with Fαmd and FαMD. Furthermore, some kinds of fuzzy neighborhood operators are deduced from Fαmd, FαMD, and FαNS on two different finite sets in FαCAS $(W_1, W_2, \mathfrak{B})$. In the end, the properties of these fuzzy neighborhood operators are studied.

3.1. Relations among FαNS, Fαmd, and FαMD

To propose the fuzzy neighborhood operators in a given FαCAS $(W_1, W_2, \mathfrak{B})$, we define the FαNS as an extension of the neighborhood of $x, v \in W_1$ and $f \in$ Onto(W_1, W_2).

Definition 3.1. *Let* $(W_1, W_2, \mathfrak{B})$ *be FαCAS with* $\mathfrak{B} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\} \subseteq \mathscr{F}(W_1)$ *for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $x, v \in W_1$*, we define the fαNS as* $N_{\mathfrak{B}}^\alpha(x)(v) = \{I[f(\epsilon_i)(y_1), f(\epsilon_i)(y_2)] : f(\epsilon_i)(x)) \in md_{\mathfrak{B}}^\alpha(x), y_1 = f(x), y_2 = f(v), i \in \{1, 2, ..., m\}\}$ *.*

Definition 3.2. *Let* $(W_1, W_2, \mathfrak{B})$ *be a FαCAS with* $\mathfrak{B} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_m\} \subseteq \mathscr{F}(W_1)$ *for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $x, v \in W_1$*, we define Fαmd and FαMD as* $md_{\mathfrak{B}}^\alpha(x)(v) = \{I(f(\epsilon(x)), f(\epsilon(v))) \in N_{\mathfrak{B}}^\alpha(x)(v) : (f(\epsilon(x)) \geq \alpha) \wedge (\forall I(f(K(x)), f(K(v))) \in N_{\mathfrak{B}}^\alpha(x)(v) \wedge f(K(x)) \geq \alpha \wedge I(f(K(x)), f(K(v))) \leq I(f(\epsilon(x)), f(\epsilon(v))) \Rightarrow I(f(K(x)), f(K(v))) = I(f(\epsilon(x)), f(\epsilon(v)))\}$*,*
$MD_{\mathfrak{B}}^\alpha(x)(v) = \{I(f(\epsilon(x)), f(\epsilon(v))) \in N_{\mathfrak{B}}^\alpha(x)(v) : (f(\epsilon(x)) \geq \alpha) \wedge (\forall I(f(K(x)), f(K(v))) \in N_{\mathfrak{B}}^\alpha(x)(v) \wedge f(K(x)) \geq \alpha \wedge I(f(K(x)), f(K(v))) \geq I(f(\epsilon(x)), f(\epsilon(v))) \Rightarrow I(f(K(x)), f(K(v))) = I(f(\epsilon(x)), f(\epsilon(v)))\}$*.*

Definition 3.3. *Let* $(W_1, W_2, \mathfrak{B})$ *be a FαCAS with* $\mathfrak{B} = \{\epsilon_1, \epsilon_2, ..., \epsilon_m\} \subseteq \mathscr{F}(W_1)$ *for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $x, v \in W_1$*, we define the fuzzy α-neighborhood of x and v as follows:*

$$\widetilde{N}_{\mathfrak{B}}^\alpha(x)(v) = \bigwedge_{f(K_i(x)) \in md_{\mathfrak{B}}^\alpha(x)(v)} I(f(K_i(x)), f(K_i(v))), i \in N = \{1, 2, 3, ...\}.$$

Example 3.4. *Let* $X = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ *and* $Y = \{y_1, y_2, y_3, y_4\}$ *with* $f : X \longrightarrow Y$ *such that*

$$f(u) = \begin{cases} y_1, & u \in \{u_1, u_3\}; \\ y_2, & u \in \{u_2, u_5\}; \\ y_3, & u = u_4; \\ y_4, & u = u_6. \end{cases}$$

Let $\mathfrak{B} = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, where $\epsilon_1 = \frac{0.3}{u_1} + \frac{0.7}{u_2} + \frac{0.4}{u_3} + \frac{0.1}{u_4} + \frac{1}{u_5} + \frac{0.7}{u_6}$,
$\epsilon_2 = \frac{0.2}{u_1} + \frac{0.9}{u_2} + \frac{0.5}{u_3} + \frac{0.4}{u_4} + \frac{0.3}{u_5} + \frac{0.5}{u_6}$,
$\epsilon_3 = \frac{0.4}{u_1} + \frac{0.6}{u_2} + \frac{0.7}{u_3} + \frac{1}{u_4} + \frac{0.2}{u_5} + \frac{0.7}{u_6}$,
$\epsilon_4 = \frac{0.5}{u_1} + \frac{0.4}{u_2} + \frac{0.4}{u_3} + \frac{0.1}{u_4} + \frac{0.4}{u_5} + \frac{0.5}{u_6}$.

Then, by Zadeh's extension principle, we have
$f(\epsilon_1) = \frac{0.4}{y_1} + \frac{1}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}$,
$f(\epsilon_2) = \frac{0.5}{y_1} + \frac{0.9}{y_2} + \frac{0.4}{y_3} + \frac{0.5}{y_4}$,
$f(\epsilon_3) = \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}$,
$f(\epsilon_4) = \frac{0.5}{y_1} + \frac{0.4}{y_2} + \frac{0.1}{y_3} + \frac{0.5}{y_4}$.

Now, let $\alpha = 0.4$. We then have $(\bigcup_{i=1}^{4} \epsilon_i) = \frac{0.5}{u_1} + \frac{0.9}{u_2} + \frac{0.7}{u_3} + \frac{1}{u_4} + \frac{1}{u_5} + \frac{0.7}{u_6}$, i.e., each member of u_1, u_2, u_3, u_4, u_5, and u_6 is greater than 0.4. Then, $\mathfrak{B}$ is a fuzzy α-covering of X. Similarly, $\bigcup_{i=1}^{4} f(\epsilon_i) = \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}$, i.e., each member of y_1, y_2, y_3, and y_4 is greater than 0.4. So, $f(\mathfrak{B})$ is a fuzzy α-covering on Y.

On the other hand, using Tables 1 and 2, we have:

Table 1. $md_{\mathfrak{B}}^{\alpha}(x)$.

u_i	u_1	u_2	u_3	u_4	u_5	u_6
$md_{\mathfrak{B}}^{\alpha}(u_i)$	$f(\epsilon_4)$	$f(\epsilon_1)$,$f(\epsilon_4)$	$f(\epsilon_1)$,$f(\epsilon_4)$	$f(\epsilon_2)$,$f(\epsilon_3)$	$f(\epsilon_1)$,$f(\epsilon_4)$	$f(\epsilon_1)$,$f(\epsilon_4)$

Table 2. FαNS, Fαmd, and FαMD, i,j=1,2,3,4.5.6.

U	$N_{\mathfrak{B}}^{0.4}(u_i)(u_j)$	$md_{\mathfrak{B}}^{0.4}(u_i)(u_j)$	$MD_{\mathfrak{B}}^{0.4}(u_i)(u_j)$
u_1	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$
u_2	$I(f(\epsilon_1(u_2)), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_2)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_1(u_2)), f(\epsilon_1(u_j)))$	$I(f(\epsilon_4(u_2)), f(\epsilon_4(u_j)))$
u_3	$I(f(\epsilon_1(u_3)), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_3)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_4(u_3)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_1(u_3)), f(\epsilon_1(u_j)))$
u_4	$I(f(\epsilon_2(u_4)), f(\epsilon_2(u_j))), I(f(\epsilon_3(u_4)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_3(u_4)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_2(u_4)), f(\epsilon_2(u_j)))$
u_5	$I(f(\epsilon_1(u_5)), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_5)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_1(u_5)), f(\epsilon_1(u_j)))$	$I(f(\epsilon_4(u_5)), f(\epsilon_4(u_j)))$
u_6	$I(f(\epsilon_1(u_6))), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_6)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_1(u_6)), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_6)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_1(u_6)), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_6)), f(\epsilon_4(u_j)))$

$\widetilde{N}_{\mathfrak{B}}^{0.4}(u_1)(u_1) = I(f(\epsilon_4(u_1)), f(\epsilon_4(u_1))) = min(1, 1 - (f(\epsilon_4))(y_1) + (f(\epsilon_4))(y_1)) = min(1, 1 - 0.5 + 0.5) = 1 = \widetilde{N}_{\mathfrak{B}}^{0.4}(u_1)(u_3)$.
$\widetilde{N}_{\mathfrak{B}}^{0.4}(u_1)(u_2) = I(f(\epsilon_4(u_1)), f(\epsilon_4(u_2))) = min(1, 1 - (f(\epsilon_4))(y_1) + (f(\epsilon_4))(y_2)) = min(1, 1 - 0.5 + 0.4) = 0.9 = \widetilde{N}_{\mathfrak{B}}^{0.4}(u_1)(u_5)$.

$\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1)(u_4) = I(f(\epsilon_4(u_1)), f(\epsilon_4(u_4))) = min(1, 1-(f(\epsilon_4))(y_1)+(f(\epsilon_4))(y_3)) = min(1, 1-0.5+0.1) = 0.6.$
$\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1)(u_6) = I(f(\epsilon_4(u_1)), f(\epsilon_4(u_6))) = min(1, 1-(f(\epsilon_4))(y_1)+(f(\epsilon_4))(y_4)) = min(1, 1-0.5+0.5) = 1.$
So, $\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1) = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$

In the same manner, using Tables 3 and 4, we have:

Table 3. $md^{\alpha}_{\mathfrak{B}}(x)$.

u_i	u_1	u_2	u_3	u_4	u_5	u_6
$md^{\alpha}_{\mathfrak{B}}(u_i)$	$f(\epsilon_1), f(\epsilon_4)$	$f(\epsilon_4)$	$f(\epsilon_2)$	$f(\epsilon_1)$,$f(\epsilon_3)$	$f(\epsilon_1)$,$f(\epsilon_3)$,$f(\epsilon_4)$	$f(\epsilon_1)$,$f(\epsilon_3)$

Table 4. FαNS, Fαmd and FαMD.

U	$N^{0.5}_{\mathfrak{B}}(u_i)(u_j)$	$md^{0.5}_{\mathfrak{B}}(u_i)(u_j)$	$MD^{0.5}_{\mathfrak{B}}(u_i)(u_j)$
u_1	$I(f(\epsilon_1(u_1)), f(\epsilon_1(u_j))), I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_1(u_1)), f(\epsilon_1(u_j)))$
u_2	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j))))$	$I(f(\epsilon_4(u_1)), f(\epsilon_4(u_j)))$
u_3	$I(f(\epsilon_2(u_3)), f(\epsilon_2(u_j)))$	$I(f(\epsilon_2(u_3)), f(\epsilon_2(u_j)))$	$I(f(\epsilon_2(u_3))), f(\epsilon_2(u_j)))$
u_4	$I(f(\epsilon_1(u_4)), f(\epsilon_1(u_j))), I(f(\epsilon_3(u_4)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_3(u_4)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_1(u_4)), f(\epsilon_1(u_j)))$
u_5	$I(f(\epsilon_1(u_5)), f(\epsilon_1(u_j))), I(f(\epsilon_3(u_5)), f(\epsilon_3(u_j))), I(f(\epsilon_4(u_5)), f(\epsilon_4(u_j)))$	$I(f(\epsilon_3(u_5)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_4(u_5)), f(\epsilon_4(u_j)))$
u_6	$I(f(\epsilon_1(u_6)), f(\epsilon_1(u_j))), I(f(\epsilon_3(u_6)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_3(u_6)), f(\epsilon_3(u_j)))$	$I(f(\epsilon_1(u_6)), f(\epsilon_1(u_j)))$

$\widetilde{N}^{0.4}_{\mathfrak{B}}(u_2) = \frac{0.4}{y_1} + \frac{1}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}$, $\quad \widetilde{N}^{0.4}_{\mathfrak{B}}(u_3) = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$,
$\widetilde{N}^{0.4}_{\mathfrak{B}}(u_4) = \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}$, $\quad \widetilde{N}^{0.4}_{\mathfrak{B}}(u_5) = \frac{0.4}{y_1} + \frac{1}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}$,
$\widetilde{N}^{0.4}_{\mathfrak{B}}(u_6) = \frac{0.7}{y_1} + \frac{0.9}{y_2} + \frac{0.4}{y_3} + \frac{1}{y_4}$.

Definition 3.5. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \in \mathscr{F}(W_2)$, let $\mathbb{R}_*$ be the fuzzy covering upper approximation and $\mathbb{O}_*$ be the fuzzy covering lower approximation, which are defined as:*

$$\mathbb{R}_*(\mathscr{B})(y) = \bigvee_{z \in W_2} (\widetilde{N}^{\alpha}_{\mathfrak{B}}(y)(z) \wedge \mathscr{B}(z)), \qquad y \in W_1,$$
$$\mathbb{O}_*(\mathscr{B})(y) = \bigwedge_{z \in W_2} ([1 - \widetilde{N}^{\alpha}_{\mathfrak{B}}(y)(z)] \vee \mathscr{B}(z)), \quad y \in W_1.$$

If $\mathbb{O}_*(\mathscr{B}) \neq \mathbb{R}_*(\mathscr{B})$, then $\mathscr{B}$ is called a fuzzy covering via rough set theory; otherwise, it is definable.

Example 3.6. *From Example 3.4, we assume that $\mathscr{B} = \frac{0.2}{y_1} + \frac{0.6}{y_2} + \frac{0.4}{y_3} + \frac{0.9}{y_4}$; therefore,*

$\mathbb{R}_*(\mathscr{B})(u_1) = (\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1)(y_1) \wedge \mathscr{B})(y_1)) \bigvee (\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1)(y_2) \wedge \mathscr{B})(y_2)) \bigvee (\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1)(y_3) \wedge \mathscr{B})(y_3)) \bigvee (\widetilde{N}^{0.4}_{\mathfrak{B}}(u_1)(y_4) \wedge \mathscr{B})(y_4)) = (1 \wedge 0.2) \bigvee (0.9 \wedge 0.6) \bigvee (0.6 \wedge 0.4) \bigvee 1 \wedge 0.9) = 0.2 \vee 0.6 \vee 0.4 \vee 0.9 = 0.9.$

In the same manner for u_2 to u_6, we have $\mathbb{U}_*(\mathscr{B}) = \frac{0.9}{u_1} + \frac{0.7}{u_2} + \frac{0.9}{u_3} + \frac{0.7}{u_4} + \frac{0.7}{u_5} + \frac{0.9}{u_6}$.

$\mathbb{L}_*(\mathscr{B})(u_1) = ([1-1]\vee 0.2)\wedge([1-0.9]\vee 0.6)\wedge([1-0.6]\vee 0.4)\wedge([1-1]\vee 0.9) = 0.2\wedge 0.6\wedge 0.4\wedge 0.9 = 0.2$. In the same manner for u_2 to u_6, we have $\mathbb{L}_*(\mathscr{B}) = \frac{0.2}{u_1} + \frac{0.6}{u_2} + \frac{0.2}{u_3} + \frac{0.3}{u_4} + \frac{0.6}{u_5} + \frac{0.3}{u_6}$.

For $\mathscr{B} = \phi = \frac{0}{y_1}+\frac{0}{y_2}+\frac{0}{y_3}+\frac{0}{y_4}$, we have $\mathbb{U}_*(\phi) = \frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}+\frac{0}{u_4}+\frac{0}{u_5}+\frac{0}{u_6}$ and $\mathbb{L}_*(\phi) = \frac{0}{u_1}+\frac{0}{u_2}+\frac{0}{u_3}+\frac{0}{u_4}+\frac{0}{u_5}+\frac{0}{u_6}$. For $\mathscr{B} = V = \frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}$, we have $\mathbb{U}_*(V) = \frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}+\frac{1}{u_4}+\frac{1}{u_5}+\frac{1}{u_6}$ and $\mathbb{L}_*(V) = \frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}+\frac{1}{u_4}+\frac{1}{u_5}+\frac{1}{u_6}$.

The following results explain the relations among $N^\alpha_{\mathfrak{B}}(x)(v), md^\alpha_{\mathfrak{B}}(x)$, and $MD^\alpha_{\mathfrak{B}}(x)$ of $x \in W_1$.

Proposition 3.7. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$, $f \in Onto(W_1, W_2)$, $x, v \in W_1$, and $I(f(\epsilon(x)), f(\epsilon(v))) \in N^\alpha_{\mathfrak{B}}(x)(v)$. Then, some $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \in md^\alpha_{\mathfrak{B}}(x)(v)$ and $I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^\alpha_{\mathfrak{B}}(x)(v)$ exist such that $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \leq I(f(\epsilon(x)), f(\epsilon(v))) \leq I(f(\epsilon_2(x))$ and $f(\epsilon_2(v)))$.*

Proof. If $I(f(\epsilon_0(x)), f(\epsilon_0(v))) \nleq I(f(\epsilon(x)), f(\epsilon(v)))$ for any $I(f(\epsilon_0(x)), f(\epsilon_0(v))) \in md^\alpha_{\mathfrak{B}}(x)(v) - I(f(\epsilon(x)), f(\epsilon(v)))$, then it follows from $f(\epsilon(x)) \geq \alpha$ that $I(f(\epsilon(x)), f(\epsilon(v))) \in md^\alpha_{\mathfrak{B}}(x)(v)$. Then some $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \in md^\alpha_{\mathfrak{B}}(x)(v)$ exist such that $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \leq I(f(\epsilon(x)), f(\epsilon(v)))$. Similarly, if $I(f(\epsilon(x)), f(\epsilon(v))) \nleq I(f(\epsilon_*(x)), f(\epsilon_*(v)))$ for any $I(f(\epsilon_*(x)), f(\epsilon_*(v))) \in MD^\alpha_{\mathfrak{B}}(x)(v) - I(f(\epsilon(x)), f(\epsilon(v)))$, then it follows from $f(\epsilon(x)) \geq \alpha$ that $I(f(\epsilon(x)), f(\epsilon(v))) \in MD^\alpha_{\mathfrak{B}}(x)(v)$. Thus, some $I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^\alpha_{\mathfrak{B}}(x)(v)$ exists such that $I(f(\epsilon(x)), f(\epsilon(v))) \leq I(f(\epsilon_2(x)), f(\epsilon_2(v))$. Hence, $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \leq I(f(\epsilon(x)), f(\epsilon(v))) \leq I(f(\epsilon_2(x)), f(\epsilon_2(v)))$.

Proposition 3.8. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $x, v \in W_1$, the following statements hold.*

(i) $md^\alpha_{\mathfrak{B}}(x)(v) \subseteq N^\alpha_{\mathfrak{B}}(x)(v)$ and $MD^\alpha_{\mathfrak{B}}(x)(v) \subseteq N^\alpha_{\mathfrak{B}}(x)(v)$.
(ii) $\bigwedge N^\alpha_{\mathfrak{B}}(x)(v) = \bigwedge md^\alpha_{\mathfrak{B}}(x)(v)$ and $\bigvee N^\alpha_{\mathfrak{B}}(x)(v) = \bigvee MD^\alpha_{\mathfrak{B}}(x)(v)$.

Proof. (i) Definitions 3.1 and 3.2 can be easily used to prove this item.
(ii) Using Proposition 3.8, if $x, v \in W_1$ and $I(f(\epsilon(x)), f(\epsilon(v))) \in N^\alpha_{\mathfrak{B}}(x)(v)$, then $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \in md^\alpha_{\mathfrak{B}}(x)(v)$ and $I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^\alpha_{\mathfrak{B}}(x)(v)$ exist such that $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \leq I(f(\epsilon(x)), f(\epsilon(v))) \leq I(f(\epsilon_2(x)), f(\epsilon_2(v)))$. Then $\bigwedge N^\alpha_{\mathfrak{B}}(x)(v) \geq \bigwedge md^\alpha_{\mathfrak{B}}(x)(v)$ and $\bigvee N^\alpha_{\mathfrak{B}}(x)(v) \leq \bigvee MD^\alpha_{\mathfrak{B}}(x)(v)$. On the other hand, it follows from (1) that, $\bigwedge N^\alpha_{\mathfrak{B}}(x)(v) \leq \bigwedge md^\alpha_{\mathfrak{B}}(x)(v)$ and $\bigvee N^\alpha_{\mathfrak{B}}(x)(v) \geq \bigvee MD^\alpha_{\mathfrak{B}}(x)(v)$. Hence, $\bigwedge N^\alpha_{\mathfrak{B}}(x)(v) = \bigwedge md^\alpha_{\mathfrak{B}}(x)(v)$ and $\bigvee N^\alpha_{\mathfrak{B}}(x)(v) = \bigvee MD^\alpha_{\mathfrak{B}}(x)(v)$ for any $x, v \in W_1$.

Corollary 3.9. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $x, v \in W_1$, if $|N^\alpha_{\mathfrak{B}}(x)(v)| = 1$, then $N^\alpha_{\mathfrak{B}}(x)(v) = md^\alpha_{\mathfrak{B}}(x)(v) = MD^\alpha_{\mathfrak{B}}(x)(v)$.*

Proof. If $|N^\alpha_{\mathfrak{B}}(x)(v)| = 1$, then the number of implicators, say, $I(f(\epsilon(x)), f(\epsilon(v)))$ in $N^\alpha_{\mathfrak{B}}(x)(v)$ equals 1. But $md^\alpha_{\mathfrak{B}}(x)(v) \subseteq N^\alpha_{\mathfrak{B}}(x)(v)$ and $MD^\alpha_{\mathfrak{B}}(x)(v) \subseteq N^\alpha_{\mathfrak{B}}(x)(v)$ by Proposition 3.10. Hence, $N^\alpha_{\mathfrak{B}}(x)(v) = md^\alpha_{\mathfrak{B}}(x)(v) = MD^\alpha_{\mathfrak{B}}(x)(v)$.

Proposition 3.10 shows some features of the Fαmd and the FαMD in the FαCAS $(W_1, W_2, \mathfrak{B})$.

Proposition 3.10. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. If $0 < \alpha_1 \leq \alpha_2 \leq \alpha$, then*

(i) $N^{\alpha_2}_{\mathfrak{B}}(x)(v) \subseteq N^{\alpha_1}_{\mathfrak{B}}(x)(v)$ for any $x, v \in W_1$;
(ii) $md^{\alpha_2}_{\mathfrak{B}}(x)(v) \subseteq md^{\alpha_1}_{\mathfrak{B}}(x)(v)$ for any $x, v \in W_1$;
(iii) $MD^{\alpha_2}_{\mathfrak{B}}(x)(v) \subseteq MD^{\alpha_1}_{\mathfrak{B}}(x)(v)$ for any $x, v \in W_1$.

Proof. (i) By Definition 3.1, $N^{\alpha_2}_{\mathfrak{V}}(x)(v) = I(f(K_i(x)), f(K_i(v)))$, $f(K_i) \in md^{\alpha_2}_{\mathfrak{V}}(x)$ where $md^{\alpha_2}_{\mathfrak{V}}(x) = \{f(\epsilon) \in f(\mathfrak{V}) : (\epsilon(x) \geq \alpha \geq \alpha_2) \wedge (\forall f(K) \in f(\mathfrak{V}) \wedge K(x) \geq \alpha \geq \alpha_2 \wedge f(K) \subseteq f(\epsilon) \Rightarrow f(\epsilon) = f(K))\}$. On the other hand, $md^{\alpha_1}_{\mathfrak{V}}(x) = \{f(\epsilon) \in f(\mathfrak{V}) : (\epsilon(x) \geq \alpha \geq \alpha_2 \geq \alpha_1) \wedge (\forall f(K) \in f(\mathfrak{V}) \wedge K(x) \geq \alpha \geq \alpha_2 \geq \alpha_1 \wedge f(K) \subseteq f(\epsilon) \Rightarrow f(\epsilon) = f(K))\}$ gives $N^{\alpha_1}_{\mathfrak{V}}(x)(v) = I(f(K_i(x)), f(K_i(v)))$, $f(K_i) \in md^{\alpha_1}_{\mathfrak{V}}(x)$. It is clear that the number of $f(\epsilon)$ in $md^{\alpha_2}_{\mathfrak{V}}(x)$ is less than the number of $f(\epsilon)$ in $md^{\alpha_1}_{\mathfrak{V}}(x)$. Therefore, the number of implicators of $N^{\alpha_2}_{\mathfrak{V}}(x)(v)$ is less than the number of the implicators of $N^{\alpha_1}_{\mathfrak{V}}(x)(v)$. Hence, $N^{\alpha_2}_{\mathfrak{V}}(x)(v) \subseteq N^{\alpha_1}_{\mathfrak{V}}(x)(v)$ for any $x, v \in W_1$; (ii) and (iii) are proved in the same manner.

3.2. Some kinds of fuzzy α-neighborhoods FαNs and α-neighborhoods

In this subsection, some kinds of FαNs based on some concepts offered in the previous subsection are suggested and their features are discussed.

Definition 3.11. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $x, v \in W_1$, we define the kinds of FαNs for x as*

(i) ${}_1\widetilde{N}^{\alpha}_x = \bigwedge\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N^{\alpha}_{\mathfrak{V}}(x)(v)\}$.
(ii) ${}_2\widetilde{N}^{\alpha}_x = \bigvee\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in md^{\alpha}_{\mathfrak{V}}(x)(v)\}$.
(iii) ${}_3\widetilde{N}^{\alpha}_x = \bigwedge\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in MD^{\alpha}_{\mathfrak{V}}(x)(v)\}$.
(iv) ${}_4\widetilde{N}^{\alpha}_x = \bigvee\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in MD^{\alpha}_{\mathfrak{V}}(x)(v)\}$.

Example 3.12. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$ in Example 3.4. In this case,*

(i) For the first kind of FαN, we have ${}_1\widetilde{N}^{0.4}_{u_1} = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$.
(ii) For the second kind of FαN, we have ${}_2\widetilde{N}^{0.4}_{u_1} = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$.
(iii) For the third kind of FαN, we have ${}_3\widetilde{N}^{0.4}_{u_1} = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$.
(iv) For the fourth kind of FαN, we have ${}_4\widetilde{N}^{0.4}_{u_1} = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$.

The following proposition shows the properties of the FαNs ${}_i\widetilde{N}^{\alpha}_x$, $(i = 1, 2, 3, 4)$ of $x \in W_1$.

Proposition 3.13. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$, $\epsilon_j(x) \in \mathfrak{V}$ and $f \in Onto(W_1, W_2)$. Then, for all $i, j \in \{1, 2, 3, 4\}$, the following properties are hold. For any $x \in W_1$, ${}_i\widetilde{N}^{\alpha}_x(\bigwedge I(f(\epsilon_j(x)), f(\epsilon_j(x)))) \geq \alpha$.*

Proof. For $i = 1$, ${}_1\widetilde{N}^{\alpha}_x = \bigwedge\{I(f(\epsilon_j(x)), f(\epsilon_j(x))) : I(f(\epsilon_j(x)), f(\epsilon_j(x))) \in N^{\alpha}_{\mathfrak{V}}(x)(x)\}$ by Definition 3.11. Then, by Definition 3.1, ${}_1\widetilde{N}^{\alpha}_x = \bigwedge\{min(1, 1 - f(\epsilon_j(x)) + f(\epsilon_j(x))) = 1 \geq \alpha$ for $j \in \{1, 2, 3, 4\}$. For $i = \{2, 3, 4\}$, we have a similar proof.

Remark 3.14. *For any $x, y, z \in W_1$, if ${}_i\widetilde{N}^{\alpha}_x(\bigcap I(f(\epsilon_j(x)), f(\epsilon_j(y)))) \geq \alpha$, and ${}_i\widetilde{N}^{\alpha}_y(\bigcap I(f(\epsilon_j(y)), f(\epsilon_j(z)))) \geq \alpha$, then ${}_i\widetilde{N}^{\alpha}_x(\bigcap I(f(\epsilon_j(x)), f(\epsilon_j(z)))) \ngeq \alpha$.*

Example 3.15. *From Table 5, ${}_1\widetilde{N}^{0.4}_{x_2}(\bigcap I(f(\epsilon_j(x_2)), f(\epsilon_j(x_3)))) \geq 0.4$, and ${}_1\widetilde{N}^{0.4}_{x_3}(\bigcap I(f(\epsilon_j(x_3)), f(\epsilon_j(x_4)))) \geq 0.4$, but ${}_1\widetilde{N}^{0.4}_{x_2}(\bigcap I(f(\epsilon_j(x_2)), f(\epsilon_j(x_4)))) \ngeq 0.4$.*

Table 5. ${}_1\widetilde{N}^{0.4}_{x}(\bigcap I(f(\epsilon_j(x)), f(\epsilon_j(y))))$.

y	${}_1\widetilde{N}^{0.4}_{x_1}Ix_1y$	${}_1\widetilde{N}^{0.4}_{x_2}Ix_2y$	${}_1\widetilde{N}^{0.4}_{x_3}Ix_3y$	${}_1\widetilde{N}^{0.4}_{x_4}Ix_4y$	${}_1\widetilde{N}^{0.4}_{x_5}Ix_5y$	${}_1\widetilde{N}^{0.4}_{x_6}Ix_6y$
x_1	1	0.4	1	0.7	0.4	0.7
x_2	0.9	1	0.9	0.6	1	0.9
x_3	1	0.4	1	0.7	0.4	0.7
x_4	0.6	0.1	0.6	1	0.1	0.4
x_5	0.9	1	0.9	0.6	1	0.9
x_6	1	0.7	1	0.7	0.7	1

Depending on the four kinds of FαNs of $x \in W_1$ in FαCAS $(W_1, W_2, \mathfrak{B})$, four kinds of α-neighborhoods of $x \in W_1$ in FαCAS $(W_1, W_2, \mathfrak{B})$ will be introduced.

Definition 3.16. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $x, v \in W_1$, four kinds of α-neighborhoods will be defined as follows*

(i) ${}_1\overline{\widetilde{N}^{\alpha}_{x}} = \{y \in W_2 : {}_1\widetilde{N}^{\alpha}_{x}(y) \geq \alpha\}$.
(ii) ${}_2\overline{\widetilde{N}^{\alpha}_{x}} = \{y \in W_2 : {}_2\widetilde{N}^{\alpha}_{x}(y) \geq \alpha\}$.
(iii) ${}_3\overline{\widetilde{N}^{\alpha}_{x}} = \{y \in W_2 : {}_3\widetilde{N}^{\alpha}_{x}(y) \geq \alpha\}$.
(iv) ${}_4\overline{\widetilde{N}^{\alpha}_{x}} = \{y \in W_2 : {}_4\widetilde{N}^{\alpha}_{x}(y) \geq \alpha\}$.

Example 3.17. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$ in Example 3.12. We then have*

(i) ${}_1\overline{\widetilde{N}^{0.4}_{u_1}} = {}_1\overline{\widetilde{N}^{0.4}_{u_3}} = {}_1\overline{\widetilde{N}^{0.4}_{u_4}} = {}_1\overline{\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$ and ${}_1\overline{\widetilde{N}^{0.4}_{u_2}} = {}_1\overline{\widetilde{N}^{0.4}_{u_5}} = \{y_1, y_2, y_4\}$;
(ii) ${}_2\overline{\widetilde{N}^{0.4}_{u_1}} = {}_2\overline{\widetilde{N}^{0.4}_{u_3}} = {}_2\overline{\widetilde{N}^{0.4}_{u_4}} = {}_2\overline{\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$ and ${}_2\overline{\widetilde{N}^{0.4}_{u_2}} = {}_2\overline{\widetilde{N}^{0.4}_{u_5}} = \{y_1, y_2, y_4\}$;
(iii) ${}_3\overline{\widetilde{N}^{0.4}_{u_1}} = {}_3\overline{\widetilde{N}^{0.4}_{u_2}} = {}_3\overline{\widetilde{N}^{0.4}_{u_3}} = {}_3\overline{\widetilde{N}^{0.4}_{u_4}} = {}_3\overline{\widetilde{N}^{0.4}_{u_5}} = {}_3\overline{\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$;
(iv) ${}_4\overline{\widetilde{N}^{0.4}_{u_1}} = {}_4\overline{\widetilde{N}^{0.4}_{u_2}} = {}_4\overline{\widetilde{N}^{0.4}_{u_3}} = {}_4\overline{\widetilde{N}^{0.4}_{u_4}} = {}_4\overline{\widetilde{N}^{0.4}_{u_5}} = {}_4\overline{\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$.

Proposition 3.18. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. Then, for all $i \in \{1, 2, 3, 4\}$ and for any $x, v \in W_1$, $\bigcap I(f(\epsilon(x)), f(\epsilon(v))) \in {}_i\overline{\widetilde{N}^{\alpha}_{x}}$.*

Proof. We prove for $i = 1$ and the same proof for $i = \{2, 3, 4\}$. From Definition 3.11, we have ${}_1\widetilde{N}^{\alpha}_{x} = \bigcap\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N^{\alpha}_{\mathfrak{B}}(x)(v)\}$. But, by Definition 3.16, ${}_1\overline{\widetilde{N}^{\alpha}_{x}} = \{a = (\bigcap I(f(\epsilon_j(x)), f(\epsilon_j(v)))) \in W_2 : {}_1\widetilde{N}^{\alpha}_{x}(a) \geq \alpha\}$.

Remark 3.19. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$, $i = \{1, 2, 3, 4\}$. Therefore,*

(i) If $\bigcap I(f(\epsilon(x)), f(\epsilon(v))) \in {}_i\overline{\widetilde{N}^{\alpha}_{y}}$, then ${}_i\overline{\widetilde{N}^{\alpha}_{x}} \nsubseteq {}_i\overline{\widetilde{N}^{\alpha}_{y}}$ for any $x, v, y \in W_1$.
(ii) If $\bigcap I(f(\epsilon(x)), f(\epsilon(v))) \in {}_i\overline{\widetilde{N}^{\alpha}_{y}}$ and $\bigcap I(f(\epsilon(y)), f(\epsilon(v))) \in {}_i\overline{\widetilde{N}^{\alpha}_{z}}$, then $\bigcap I(f(\epsilon(x)), f(\epsilon(v))) \notin {}_i\overline{\widetilde{N}^{\alpha}_{z}}$ for any $x, v, y, z \in W_1$.
(iii) ${}_i\widetilde{N}^{\alpha}_{x} \subseteq {}_i\widetilde{N}^{\alpha}_{y} \not\Leftrightarrow {}_i\overline{\widetilde{N}^{\alpha}_{x}} \subseteq {}_i\overline{\widetilde{N}^{\alpha}_{y}}$ for any $x, v, y \in W_1$.

Example 3.20. *From Example 3.17 (i),*

(i) $\bigcap I(f(\epsilon_j(u_1)), f(\epsilon_j(v))) = y_1 \in {}_1\overline{\widetilde{N}^{0.4}_{u_2}}$ then ${}_1\overline{\widetilde{N}^{0.4}_{u_1}} \nsubseteq {}_1\overline{\widetilde{N}^{0.4}_{u_2}}$.

(ii) $\bigcap I(f(\epsilon(u_4)), f(\epsilon(v))) = y_3 \in {}_1\overline{\widetilde{N}^{0.4}_{u_6}}$ and $\bigcap I(f(\epsilon(u_6)), f(\epsilon(v))) = y_4 \in {}_1\overline{\widetilde{N}^{0.4}_{u_5}}$, then $\bigcap I(f(\epsilon(u_4)), f(\epsilon(v))) \notin {}_1\overline{\widetilde{N}^{0.4}_{u_5}}$.

(iii) ${}_1\overline{\widetilde{N}^{0.4}_{u_2}} = \{y_1, y_2, y_4\} \subseteq {}_1\overline{\widetilde{N}^{0.4}_{u_1}} = \{y_1, y_2, y_3, y_4\} \nRightarrow {}_1\widetilde{N}^{0.4}_{u_2} = \frac{0.4}{y_1} + \frac{1}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4} \subseteq {}_1\widetilde{N}^{0.4}_{u_1} = \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$.

Definition 3.21. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a, l, a_s \in W_1$, where $s \in \sigma$ and σ is a random index set, we define the following*

(i) $({}_i\widetilde{N}^\alpha_a \times {}_i\widetilde{N}^\alpha_l)(a', l') = ({}_i\widetilde{N}^\alpha_a(a')) \bigwedge ({}_i\widetilde{N}^\alpha_l(l'))$, where $(a', l') \in W_2 \times W_2$;

(ii) $\overline{({}_i\widetilde{N}^\alpha_a \times {}_i\widetilde{N}^\alpha_l)} = \{(a', l') \in W_2 \times W_2 : {}_i\widetilde{N}^\alpha_a(a') \geq \alpha, {}_i\widetilde{N}^\alpha_l(l') \geq \alpha\}$;

(iii) $\overline{(({}_i\widetilde{N}^\alpha_a) \bigcap ({}_i\widetilde{N}^\alpha_l))} = \{a' \in W_2 : ({}_i\widetilde{N}^\alpha_a(a')) \bigwedge ({}_i\widetilde{N}^\alpha_l(a')) \geq \alpha\}$;

(iv) $\overline{(({}_i\widetilde{N}^\alpha_a) \bigcup ({}_i\widetilde{N}^\alpha_l))} = \{a' \in W_2 : ({}_i\widetilde{N}^\alpha_a(a')) \bigvee ({}_i\widetilde{N}^\alpha_l(a')) \geq \alpha\}$;

(v) $\overline{(\bigcap ({}_i\widetilde{N}^\alpha_{a_s}))} = \{a' \in W_2 : \bigwedge_{s \in \sigma} ({}_i\widetilde{N}^\alpha_{a_s}(a')) \geq \alpha\}$;

(vi) $\overline{(\bigcup ({}_i\widetilde{N}^\alpha_{a_s}))} = \{a' \in W_2 : \bigvee_{s \in \sigma} ({}_i\widetilde{N}^\alpha_{a_s}(a')) \geq \alpha\}$, where $i \in \{1, 2, 3, 4\}$.

Example 3.22. *From Example 3.12, all the computations here are for when $i = 1$:*

1) $({}_1\widetilde{N}^{0.4}_{u_1} \times {}_1\widetilde{N}^{0.4}_{u_2})(u'_1, u'_2) = \frac{0.4}{(y_1,y_1)} + \frac{1}{(y_1,y_2)} + \frac{0.1}{(y_1,y_3)} + \frac{0.7}{(y_1,y_4)} + \frac{0.4}{(y_2,y_1)} + \frac{0.9}{(y_2,y_2)} + \frac{0.1}{(y_2,y_3)} + \frac{0.7}{(y_2,y_4)} + \frac{0.4}{(y_3,y_1)} + \frac{0.6}{(y_3,y_2)} + \frac{0.1}{(y_3,y_3)} + \frac{0.6}{(y_3,y_4)} + \frac{0.4}{(y_4,y_1)} + \frac{1}{(y_4,y_2)} + \frac{0.1}{(y_4,y_3)} + \frac{0.7}{(y_4,y_4)}$.

2) From (1), $\overline{({}_1\widetilde{N}^{0.4}_{u_1} \times {}_1\widetilde{N}^{0.4}_{u_2})} = \{(y_1, y_1), (y_1, y_2), (y_1, y_4), (y_2, y_1), (y_2, y_2), (y_2, y_4), (y_3, y_1), (y_3, y_2), (y_3, y_4), (y_4, y_1), (y_4, y_2), (y_4, y_4)\}$.

3) From (1), $\overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_2}))} = \{y_1, y_2, y_4\}$.

4) From Example 3.12, $({}_1\widetilde{N}^{0.4}_{u_1}(u'_1)) \bigvee ({}_1\widetilde{N}^{0.4}_{u_2}(u'_2)) = \frac{1}{(y_1,y_1)}, \frac{1}{(y_1,y_2)}, \frac{1}{(y_1,y_3)}, \frac{1}{(y_1,y_4)}, \frac{0.9}{(y_2,y_1)}, \frac{1}{(y_2,y_2)}, \frac{0.9}{(y_2,y_3)}, \frac{0.9}{(y_2,y_4)}, \frac{0.6}{(y_3,y_1)}, \frac{1}{(y_3,y_2)}, \frac{0.6}{(y_3,y_3)}, \frac{0.7}{(y_3,y_4)}, \frac{1}{(y_4,y_1)}, \frac{1}{(y_4,y_2)}, \frac{1}{(y_4,y_3)}, \frac{1}{(y_4,y_4)}$. We then have
$\overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_2}))} = \{y_1, y_2, y_3, y_4\}$.

5) From Example 3.12, $\overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_3}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_4}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_6}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_3}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_4}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_3}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_6}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_4}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_6}))} = \{y_1, y_2, y_3, y_4\}$;

$\overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_2}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_3}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_4}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_5}) \bigcap ({}_1\widetilde{N}^{0.4}_{u_6}))} =$
$\{y_1, y_2, y_4\}$.

6) From Example 3.12, $\overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_2}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_3}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_4}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_5}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_1}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_6}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_2}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_3}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_2}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_4}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_2}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_6}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_3}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_4}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_3}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_5}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_3}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_6}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_4}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_5}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_4}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_6}))} = \overline{(({}_1\widetilde{N}^{0.4}_{u_5}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_6}))} = \{y_1, y_2, y_3, y_4\}$.
$\overline{(({}_1\widetilde{N}^{0.4}_{u_2}) \bigcup ({}_1\widetilde{N}^{0.4}_{u_5}))} = \{y_1, y_2, y_4\}$.

$\{y_1, y_2, y_3, y_4\}$ is the answer for the other groups which contains three or four or five, or six of elements, respectively, with the union operation among them.

Proposition 3.23. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a, l \in W_1$ and for all $i \in \{1, 2, 3, 4\}$, the following properties hold:*

(i) $\overline{({}_i\widetilde{N}^{\alpha}_a \times {}_i\widetilde{N}^{\alpha}_l)} = \overline{{}_i\widetilde{N}^{\alpha}_a} \times \overline{{}_i\widetilde{N}^{\alpha}_l}$,

(ii) $\overline{({}_i\widetilde{N}^{\alpha}_a \bigcup {}_i\widetilde{N}^{\alpha}_l)} = \overline{{}_i\widetilde{N}^{\alpha}_a} \bigcup \overline{{}_i\widetilde{N}^{\alpha}_l}$,

(iii) $\overline{({}_i\widetilde{N}^{\alpha}_a \bigcap {}_i\widetilde{N}^{\alpha}_l)} = \overline{{}_i\widetilde{N}^{\alpha}_a} \bigcap \overline{{}_i\widetilde{N}^{\alpha}_l}$.

Proof. (i) For any $a, l \in W_1$, we have $(a', l') \in \overline{({}_i\widetilde{N}^{\alpha}_a \times {}_i\widetilde{N}^{\alpha}_l)} \iff {}_i\widetilde{N}^{\alpha}_a(a') \geq \alpha, {}_i\widetilde{N}^{\alpha}_l(l') \geq \alpha, \iff a' \in \overline{{}_i\widetilde{N}^{\alpha}_a}, l' \in \overline{{}_i\widetilde{N}^{\alpha}_l} \iff (a', l') \in \overline{{}_i\widetilde{N}^{\alpha}_a} \times \overline{{}_i\widetilde{N}^{\alpha}_l}$. Thus, $\overline{({}_i\widetilde{N}^{\alpha}_a \times {}_i\widetilde{N}^{\alpha}_l)} = \overline{{}_i\widetilde{N}^{\alpha}_a} \times \overline{{}_i\widetilde{N}^{\alpha}_l}$ holds for any $a, l \in W_1$ and $i \in \{1, 2, 3, 4\}$.

(ii) For any $a, l \in W_1$, we have $\overline{({}_i\widetilde{N}^{\alpha}_a \bigcup {}_i\widetilde{N}^{\alpha}_l)} = \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_a(a')) \bigvee ({}_i\widetilde{N}^{\alpha}_l(a')) \geq \alpha\} = \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_a(a')) \geq \alpha$ or $({}_i\widetilde{N}^{\alpha}_l(a')) \geq \alpha\} = \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_a(a')) \geq \alpha\} \bigcup \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_l(a')) \geq \alpha\} = \overline{{}_i\widetilde{N}^{\alpha}_a} \bigcup \overline{{}_i\widetilde{N}^{\alpha}_l}$. Thus, $\overline{({}_i\widetilde{N}^{\alpha}_a \bigcup {}_i\widetilde{N}^{\alpha}_l)} = \overline{{}_i\widetilde{N}^{\alpha}_a} \bigcup \overline{{}_i\widetilde{N}^{\alpha}_l}$ holds for any $a, l \in W_1$ and $i \in \{1, 2, 3, 4\}$.

(iii) For any $a, l \in W_1$, we have $\overline{({}_i\widetilde{N}^{\alpha}_a \bigcap {}_i\widetilde{N}^{\alpha}_l)} = \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_a(a')) \bigwedge ({}_i\widetilde{N}^{\alpha}_l(a')) \geq \alpha\} = \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_a(a')) \geq \alpha$ and $({}_i\widetilde{N}^{\alpha}_l(a')) \geq \alpha\} = \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_a(a')) \geq \alpha\} \bigcap \{a' \in W_2 : ({}_i\widetilde{N}^{\alpha}_l(a')) \geq \alpha\} = \overline{{}_i\widetilde{N}^{\alpha}_a} \bigcap \overline{{}_i\widetilde{N}^{\alpha}_l}$. Thus, $\overline{({}_i\widetilde{N}^{\alpha}_a \bigcap {}_i\widetilde{N}^{\alpha}_l)} = \overline{{}_i\widetilde{N}^{\alpha}_a} \bigcap \overline{{}_i\widetilde{N}^{\alpha}_l}$ holds for any $a, l \in W_1$ and $i \in \{1, 2, 3, 4\}$.

Proposition 3.24. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a_s \in W_1$, where $s \in \sigma$ and σ is a random index set and for all $i \in \{1, 2, 3, 4\}$, the following properties hold*

1) $\overline{(\bigcup_{r=1}^{n} {}_i\widetilde{N}^{\alpha}_{a_r})} = \bigcup_{r=1}^{n} \overline{{}_i\widetilde{N}^{\alpha}_{a_r}}$.

2) $\overline{(\bigcap_{r=1}^{n} {}_i\widetilde{N}^{\alpha}_{a_r})} = \bigcap_{r=1}^{n} \overline{{}_i\widetilde{N}^{\alpha}_{a_r}}$.

3) $\overline{(\bigcup_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})} \supseteq \bigcup_{s\in\sigma} \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$.

4) $\overline{(\bigcap_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})} = \bigcap_{s\in\sigma} \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$.

Proof. 1) By induction,

(a) By Proposition 3.23, the statement is true for two neighborhoods, that is $\overline{({}_i\widetilde{N}^{\alpha}_{a_1} \bigcup {}_i\widetilde{N}^{\alpha}_{a_2})} = \overline{{}_i\widetilde{N}^{\alpha}_{a_1}} \bigcup \overline{{}_i\widetilde{N}^{\alpha}_{a_2}}$.

(b) Suppose that the statement is true for $n = k$, i.e. $\overline{(\bigcup_{r=1}^{k} {}_i\widetilde{N}^{\alpha}_{a_r})} = \bigcup_{r=1}^{k} \overline{{}_i\widetilde{N}^{\alpha}_{a_r}}$.

(c) We prove that it is true for $n = k + 1$, $\overline{(\bigcup_{r=1}^{k+1} {}_i\widetilde{N}^{\alpha}_{a_r})} = \overline{((\bigcup_{r=1}^{k} {}_i\widetilde{N}^{\alpha}_{a_r}) \bigcup {}_i\widetilde{N}^{\alpha}_{a})} = \overline{(\bigcup_{r=1}^{k} {}_i\widetilde{N}^{\alpha}_{a_r})} \bigcup \overline{{}_i\widetilde{N}^{\alpha}_{a}} = \bigcup_{r=1}^{k} \overline{{}_i\widetilde{N}^{\alpha}_{a_r}} \bigcup \overline{{}_i\widetilde{N}^{\alpha}_{a}} = \bigcup_{r=1}^{k+1} \overline{{}_i\widetilde{N}^{\alpha}_{a_r}}$.

2) The same proof as 1) above.

3) Let $x \in \bigcup_{s\in\sigma} \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$. Then $s \in \sigma$ exists such that $x \in \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$. Therefore, ${}_i\widetilde{N}^{\alpha}_{a_s} \geq \alpha$. Hence, $(\bigcup_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})(x) = \bigvee_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s}(x) \geq \alpha$. Therefore, $x \in \overline{(\bigcup_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})}$, i.e., $\overline{(\bigcup_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})} \supseteq \bigcup_{s\in\sigma} \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$ is satisfied for $\forall\, i \in \{1, 2, 3, 4\}$.

4) For any $x \in W_2$, $x \in \bigcap_{s\in\sigma} \overline{{}_i\widetilde{N}^{\alpha}_{a_s}} \iff x \in \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$ for every $s \in \sigma \iff {}_i\widetilde{N}^{\alpha}_{a_s} \geq \alpha$ for every $s \in \sigma \iff \bigwedge_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s}(x) \geq \alpha \iff x \in \overline{(\bigcap_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})}$. Therefore, $\overline{(\bigcap_{s\in\sigma} {}_i\widetilde{N}^{\alpha}_{a_s})} = \bigcap_{s\in\sigma} \overline{{}_i\widetilde{N}^{\alpha}_{a_s}}$ is satisfied for $\forall\, i \in \{1, 2, 3, 4\}$.

4. Four types of fuzzy α-coverings over 2-finite sets

Here, we discuss four kinds of fuzzy α-coverings depending on rough set theory for 2-finite sets, and their properties will be studied, using Definition 3.11.

4.1. The first pattern

Definition 4.1. *Let* $(W_1, W_2, \mathfrak{B})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $\mathscr{B} \in \mathscr{F}(W_2)$*, the first fuzzy α-lower* $\mathbb{L}_I$ *and fuzzy α-upper* $\mathbb{U}_I$ *approximations will be defined as:*

$\mathbb{L}_I(\mathscr{B})(y) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{B}(z)), \quad y \in W_1,$
$\mathbb{U}_I(\mathscr{B})(y) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{B}(z)), \quad y \in W_1.$
$\mathscr{B}$ is definable, except if $\mathbb{L}_I(\mathscr{B}) \neq \mathbb{U}_I(\mathscr{B})$, when $\mathscr{B}$ will be called the first kind of fuzzy α-covering depending rough set.

Example 4.2. *Using Example 3.12, let* $\mathscr{B} = \frac{0.3}{y_1} + \frac{0.7}{y_2} + \frac{0.5}{y_3} + \frac{0.2}{y_4}$*. We then have,*

$\mathbb{L}_I(\mathscr{B})(u_1) = ([1 - {}_1\widetilde{N}^{0.4}_{u_1}(y_1)] \bigvee \mathscr{B}(y_1)) \bigwedge([1 - {}_1\widetilde{N}^{0.4}_{u_1}(y_2)] \bigvee \mathscr{B}(y_2)) \bigwedge([1 - {}_1\widetilde{N}^{0.4}_{u_1}(y_3)] \bigvee \mathscr{B}(y_3))$
$\bigwedge([1 - {}_1\widetilde{N}^{0.4}_{u_1}(y_4)] \bigvee \mathscr{B}(y_4)) = 0.3 \wedge 0.7 \wedge 0.5 \wedge 0.2 = 0.2.$
Continuing in the same manner for u_2 to u_6, we get $\mathbb{L}_I(\mathscr{B}) = \frac{0.2}{u_1} + \frac{0.3}{u_2} + \frac{0.2}{u_3} + \frac{0.3}{u_4} + \frac{0.3}{u_5} + \frac{0.2}{u_6}$.
$\mathbb{U}_I(\mathscr{B})(u_1) = ({}_1\widetilde{N}^{0.4}_{u_1}(y_1) \bigwedge \mathscr{B}(y_1)) \bigvee({}_1\widetilde{N}^{0.4}_{u_1}(y_2) \bigwedge \mathscr{B}(y_2)) \bigvee({}_1\widetilde{N}^{0.4}_{u_1}(y_3) \bigwedge \mathscr{B}(y_3)) \bigvee$
$({}_1\widetilde{N}^{0.4}_{u_1}(y_4) \bigwedge \mathscr{B}(y_4)) = 0.3 \vee 0.7 \vee 0.5 \vee 0.2 = 0.7.$
Continuing in the same manner for u_2 to u_6, we get $\mathbb{U}_I(\mathscr{B}) = \frac{0.7}{u_1} + \frac{0.7}{u_2} + \frac{0.7}{u_3} + \frac{0.6}{u_4} + \frac{0.7}{u_5} + \frac{0.7}{u_6}$.

Proposition 4.3 provides the characteristics of the first fuzzy α-covering depending rough set pattern.

Proposition 4.3. *Let* $(W_1, W_2, \mathfrak{B})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $\mathscr{A}, \mathscr{H} \in \mathscr{F}(W_2)$*, we have*

1) $\mathbb{L}_I(\mathscr{A}^c) = (\mathbb{U}_I(\mathscr{A}))^c$.
2) $\mathbb{L}_I(W_2) = W_1$.
3) $\mathbb{L}_I(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_I(\mathscr{A}) \cap \mathbb{L}_I(\mathscr{H})$.
4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{L}_I(\mathscr{A}) \subseteq \mathbb{L}_I(\mathscr{H})$.
5) $\mathbb{L}_I(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_I(\mathscr{A}) \cup \mathbb{L}_I(\mathscr{H})$.
6) If $1 - {}_1\widetilde{N}^{\alpha}_y(z) \leq \mathscr{A}(z) \leq {}_1\widetilde{N}^{\alpha}_y(z)$ for any $z \in W_2$, then $\mathbb{L}_I(\mathscr{A}) \subseteq \mathbb{U}_I(\mathscr{A})$.
7) $\mathbb{U}_I(\mathscr{A}^c) = (\mathbb{L}_I(\mathscr{A}))^c$.
8) $\mathbb{U}_I(\phi) = \phi$.
9) $\mathbb{U}_I(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_I(\mathscr{A}) \cup \mathbb{U}_I(\mathscr{H})$.
10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{U}_I(\mathscr{A}) \subseteq \mathbb{U}_I(\mathscr{H})$.
11) $\mathbb{U}_I(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_I(\mathscr{A}) \cap \mathbb{U}_I(\mathscr{H})$.

Proof. 1) For any $y \in W_1$, $\mathbb{L}_I(\mathscr{A}^c)(y) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{A}^c(z)) = \bigwedge_{z\in W_2}(({}_1\widetilde{N}^{\alpha}_y(z))^c \bigvee \mathscr{A}^c(z)) = \bigwedge_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{A}(z))^c = (\bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{A}(z)))^c = 1 - \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{A}(z)) = 1 - \mathbb{U}_1(\mathscr{A})(y) = (\mathbb{U}_1(\mathscr{A}))^c(y)$.

2) For any $y \in W_1$, $\mathbb{L}_I(W_2)(y) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee W_2(z))$. $W_2(z) = 1$ for any $z \in W_2$; therefore, $\mathbb{L}_I(W_2)(y) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee 1) = 1 = W_1(y)$, i.e. $\mathbb{L}(W_2) = W_1$.

3) For any $y \in W_1$, $\mathbb{L}_I(\mathscr{A} \cap \mathscr{H})(y) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee [\mathscr{A} \cap \mathscr{H}](z)) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee [\mathscr{A}(z) \bigwedge \mathscr{H}(z)]) = (\bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{A}(z))) \bigwedge (\bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{H}(z))) = (\mathbb{L}_I(\mathscr{A}) \cap \mathbb{L}_I(\mathscr{H}))(y)$. Then, $\mathbb{L}_I(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_I(\mathscr{A}) \cap \mathbb{L}_I(\mathscr{H})$.
4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, we have $(\bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{A}(z))) \leq (\bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{H}(z)))$. Thus, $\mathbb{L}_I(\mathscr{A})(y) \leq \mathbb{L}_I(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{L}_I(\mathscr{A}) \subseteq \mathbb{L}_I(\mathscr{H})$.
5) Since $\mathscr{A} \subseteq \mathscr{A} \cup \mathscr{H}$ and $\mathscr{H} \subseteq \mathscr{A} \cup \mathscr{H}$, by (4), we have $\mathbb{L}_I(\mathscr{A}) \subseteq \mathbb{L}_I(\mathscr{A} \cup \mathscr{H})$ and $\mathbb{L}_I(\mathscr{H}) \subseteq \mathbb{L}_I(\mathscr{A} \cup \mathscr{H})$. Hence, $\mathbb{L}_I(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_I(\mathscr{A}) \cup \mathbb{L}_I(\mathscr{H})$.
6) For any $y \in W_1$, there is $1 - {}_1\widetilde{N}^{\alpha}_y(z) \leq \mathscr{A}(z) \leq {}_1\widetilde{N}^{\alpha}_y(z)$, for any $z \in W_2$; then $\mathscr{A}(z) = {}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}(z) \leq \bigvee_{x\in W_2}({}_1\widetilde{N}^{\alpha}_y(x) \wedge \mathscr{A}(z)) = \mathbb{U}_I(\mathscr{A})(y)$ and $\mathscr{A}(z) = ([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{A}(z)) \geq \bigwedge_{x\in W_2}([1 -_1 \widetilde{N}^{\alpha}_y(x)] \vee \mathscr{A}(x)) = \mathbb{L}_I(\mathscr{A})(y)$. Thus, $\mathbb{L}_I(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathbb{U}_I(\mathscr{A})$.
7) For any $y \in W_1$, $\mathbb{U}_I(\mathscr{A}^c) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}^c(z)) = \bigvee_{z\in W_2}(({}_1\widetilde{N}^{\alpha}_y(z))^c \vee \mathscr{A}(z))^c = (\bigwedge_{z\in W_2}[1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{A}(z))^c = (\mathbb{L}_I(\mathscr{A}))^c$.
8) $\mathbb{U}_I(\phi) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \phi), \quad y \in W_1$
$= \bigvee_{z\in W_2} \phi, \quad y \in W_1$
$= \phi$.
9) For any $y \in W_1$, we have $\mathbb{U}_I(\mathscr{A} \cup \mathscr{H})(y) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge (\mathscr{A} \cup \mathscr{H})(z)) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge [\mathscr{A}(z) \vee \mathscr{H}(z)]) = (\bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}(z))) \vee (\bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{H}(z))) = (\mathbb{U}_I(\mathscr{A}) \cup \mathbb{U}_I(\mathscr{H}))(y)$. Then, $\mathbb{U}_I(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_I(\mathscr{A}) \cup \mathbb{U}_I(\mathscr{H})$.
10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, we have $\bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}(z)) \leq \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{H}(z))$. Thus, $\mathbb{U}_I(\mathscr{A})(y) \leq \mathbb{U}_I(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{U}_I(\mathscr{A}) \subseteq \mathbb{U}_I(\mathscr{H})$.
11) Since $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{A}$ and $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{H}$, by (10), we have that $\mathbb{U}_I(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_I(\mathscr{A})$ and $\mathbb{U}_I(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_I(\mathscr{H})$. Hence, $\mathbb{U}_I(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_I(\mathscr{A}) \wedge \mathbb{U}_I(\mathscr{H})$.

Proposition 4.4. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $\mathscr{K} \in \mathscr{F}(W_2)$ and $k \in [0, 1]$, the following hold:*

1) $\mathbb{L}_I(\mathscr{K} \cup kW_2) = \mathbb{L}_I(\mathscr{K}) \cup kW_1$.
2) $\mathbb{L}_I(\mathscr{K} \cap kW_2) = \mathbb{L}_I(\mathscr{K}) \cap kW_1$.

Proof. 1) For any $y \in W_1$, we have $\mathbb{L}_I(\mathscr{K} \cup kW_2)(y) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee [\mathscr{K} \cup kW_2](z)) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z) \vee kW_2(z)) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z) \vee k) = (\bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z))) \vee (\bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}^{\alpha}_y(z)] \vee k)) = \mathbb{L}_I(\mathscr{K})(y) \vee k = \mathbb{L}_I(\mathscr{K}) \cup kW_1$.
2) For any $y \in W_1$, we have $\mathbb{U}_I(\mathscr{K} \cap kW_2)(y) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge [\mathscr{K} \cap kW_2](z)) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z) \wedge kW_2(z)) = \bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z) \wedge k) = (\bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z))) \wedge (\bigvee_{z\in W_2}({}_1\widetilde{N}^{\alpha}_y(z) \wedge k)) = \mathbb{U}_I(\mathscr{K})(y) \wedge k = \mathbb{U}_I(\mathscr{K}) \cap kW_1$.

Proposition 4.5. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a \in W_1, c \in W_2$, and $X \subseteq W_2$,*

1) $\mathbb{U}_I(1_c)(a) = {}_1\widetilde{N}^{\alpha}_a(c)$.
2) $\mathbb{L}_I(1_{W_2-\{c\}})(a) = 1 - {}_1\widetilde{N}^{\alpha}_a(c)$.

3) $\mathbb{U}_I(1_X)(a) = \bigvee_{c\in X} {}_1\widetilde{N}_a^\alpha(c)$.
4) $\mathbb{L}_I(1_X)(a) = \bigwedge_{c\notin X}(1 - {}_1\widetilde{N}_a^\alpha(c))$.

Proof. 1) For any $a \in W_1$, $a', c \in W_2$, and from the definition of 1_c, we have $1_c(a') = 0$ for $a' \neq c$. Hence, $\mathbb{U}_I(1_c)(a) = \bigvee_{z\in W_2}[{}_1\widetilde{N}_a^\alpha(z) \wedge 1_c(z)] = {}_1\widetilde{N}_a^\alpha(c)$.

2) $\mathbb{L}_I(1_{W_2-\{c\}})(a) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}_a^\alpha(z)] \bigvee 1_{W_2-\{c\}}(z)) = (\bigwedge_{z\in W_2-\{c\}}([1 - {}_1\widetilde{N}_a^\alpha(z)] \bigvee 1_{W_2-\{c\}}(z))) \bigwedge(\bigwedge_{z=c}([1 - {}_1\widetilde{N}_a^\alpha(z)] \bigvee 1_{W_2-\{c\}}(z))) = 1 \bigwedge([1 - {}_1\widetilde{N}_a^\alpha(c)] \bigvee 0) = 1 - {}_1\widetilde{N}_a^\alpha(c)$.

3) For any $c \in W_2$, $X \subseteq W_2$ and from the definition of 1_X, we have $1_X(c) = 0$ if and only if $c \notin X$. Hence, for any $x \in W_1$, we have $\mathbb{U}_I(1_X)(a) = \bigvee_{c\in W_2}[{}_1\widetilde{N}_a^\alpha(c) \wedge 1_X(c)] = (\bigvee_{c\in X}[{}_1\widetilde{N}_a^\alpha(c) \wedge 1_X(c)]) \bigvee(\bigvee_{c\notin X}[{}_1\widetilde{N}_a^\alpha(c) \wedge 1_X(c)]) = \bigvee_{c\in X} {}_1\widetilde{N}_a^\alpha(c)$.

4) $\mathbb{L}_I(1_X)(a) = \bigwedge_{c\in W_2}([1 - {}_1\widetilde{N}_a^\alpha(c)] \bigvee 1_X(c)) = (\bigwedge_{c\in X}([1 - {}_1\widetilde{N}_a^\alpha(c)] \bigvee 1_X(c))) \bigwedge(\bigwedge_{c\notin X}([1 - {}_1\widetilde{N}_a^\alpha(c)] \bigvee 1_X(c))) = 1 \bigwedge(\bigwedge_{c\notin X}([1 - {}_1\widetilde{N}_a^\alpha(c)] \bigvee 0)) = \bigwedge_{c\notin X}(1 - {}_1\widetilde{N}_a^\alpha(c))$.

Depending on Pawlak's rough set pattern, the fuzzy covering depending on the rough set pattern over 2-finite sets is provided by using the idea of an α-neighborhood.

Definition 4.6. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \subseteq W_2$, the first upper $\overline{\mathbb{U}_I}$ and lower $\overline{\mathbb{L}_I}$ approximations are defined, respectively, as*

$\overline{\mathbb{U}_I}(\mathscr{B}) = \{x \in W_1 : \overline{{}_1\widetilde{N}_x^\alpha}(y) \cap \mathscr{B} \neq \phi\}$.
$\overline{\mathbb{L}_I}(\mathscr{B}) = \{x \in W_1 : \overline{{}_1\widetilde{N}_x^\alpha}(y) \subseteq \mathscr{B}\}$.
$\mathscr{B}$ is definable, except if $\overline{\mathbb{U}_I}(\mathscr{B}) \neq \overline{\mathbb{L}_I}(\mathscr{B})$. $\mathscr{B}$ is called the first kind of fuzzy covering depending on the rough set.

Example 4.7. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. From Example 3.17, we have $\overline{{}_1\widetilde{N}_{u_1}^{0.4}} = \overline{{}_1\widetilde{N}_{u_3}^{0.4}} = \overline{{}_1\widetilde{N}_{u_4}^{0.4}} = \overline{{}_1\widetilde{N}_{u_6}^{0.4}} = \{y_1, y_2, y_3, y_4\}$ and $\overline{{}_1\widetilde{N}_{u_2}^{0.4}} = \overline{{}_1\widetilde{N}_{u_5}^{0.4}} = \{y_1, y_2, y_4\}$.*

1) Let $\mathscr{B} = \{y_1, y_4\}$. Then $\overline{\mathbb{U}_I}(\mathscr{B}) = \{u_1, u_2, u_3, u_4, u_5, u_6\} = W_1$ and $\overline{\mathbb{L}_I}(\mathscr{B}) = \phi$.
2) Let $\mathscr{B} = \{y_3\}$. Then $\overline{\mathbb{U}_I}(\mathscr{B}) = \{u_1, u_3, u_4, u_6\}$ and $\overline{\mathbb{L}_I}(\mathscr{B}) = \phi$.

4.2. The second pattern

Definition 4.8. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \in \mathscr{F}(W_2)$, the second fuzzy α-lower $\mathbb{L}_{II}$ and fuzzy α-upper $\mathbb{U}_{II}$ approximations, respectively, are defined as*

$\mathbb{L}_{II}(\mathscr{B})(y) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}_y^\alpha(z)] \bigvee \mathscr{B}(z)), \quad y \in W_1$,
$\mathbb{U}_{II}(\mathscr{B})(y) = \bigvee_{z\in W_2}({}_2\widetilde{N}_y^\alpha(z) \bigwedge \mathscr{B}(z)), \quad y \in W_1$.
$\mathscr{B}$ is definable, except if $\mathbb{L}_{II}(\mathscr{B}) \neq \mathbb{U}_{II}(\mathscr{B})$. $\mathscr{B}$ is called the second kind of fuzzy α-covering depending on the rough set.

Example 4.9. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. Using the information in Examples 3.12 and 4.2, we have*

$\mathbb{L}_{II}(\mathscr{B})(u_1) = ([1 - {}_2\widetilde{N}_{u_1}^{0.4}(y_1)] \bigvee \mathscr{B}(y_1)) \bigwedge([1 - {}_2\widetilde{N}_{u_1}^{0.4}(y_2)] \bigvee \mathscr{B}(y_2)) \bigwedge([1 - {}_2\widetilde{N}_{u_1}^{0.4}(y_3)] \bigvee \mathscr{B}(y_3))$
$\bigwedge([1 - {}_2\widetilde{N}_{u_1}^{0.4}(y_4)] \bigvee \mathscr{B}(y_4)) = 0.3 \wedge 0.7 \wedge 0.5 \wedge 0.2 = 0.2$.

In the same manner for u_2 to u_6, we get $\mathbb{L}_{II}(\mathscr{B}) = \frac{0.2}{y_1} + \frac{0.3}{y_2} + \frac{0.2}{y_3} + \frac{0.3}{y_4} + \frac{0.3}{y_5} + \frac{0.2}{y_6}$.
$\mathbb{U}_{II}(\mathscr{B})(u_1) = ({}_2\widetilde{N}^{0.4}_{u_1}(y_1) \bigwedge \mathscr{B}(y_1)) \bigvee ({}_2\widetilde{N}^{0.4}_{u_1}(y_2) \bigwedge \mathscr{B}(y_2)) \bigvee ({}_2\widetilde{N}^{0.4}_{u_1}(y_3) \bigwedge \mathscr{B}(y_3)) \bigvee ({}_2\widetilde{N}^{0.4}_{u_1}(y_4) \bigwedge \mathscr{B}(y_4)) = 0.3 \vee 0.7 \vee 0.5 \vee 0.2 = 0.7$.
In the same manner for u_2 to u_6, we get $\mathbb{U}_{II}(\mathscr{B}) = \frac{0.7}{y_1} + \frac{0.7}{y_2} + \frac{0.7}{y_3} + \frac{0.6}{y_4} + \frac{0.7}{y_5} + \frac{0.7}{y_6}$.

Proposition 4.10 provides the characteristics of the second fuzzy α-covering depending on the rough set pattern.

Proposition 4.10. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{A}, \mathscr{H} \in \mathscr{F}(W_2)$, we have*

1) $\mathbb{L}_{II}(\mathscr{A}^c) = (\mathbb{U}_{II}(\mathscr{A}))^c$.
2) $\mathbb{L}_{II}(W_2) = W_1$.
3) $\mathbb{L}_{II}(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_{II}(\mathscr{A}) \cap \mathbb{L}_{II}(\mathscr{H})$.
4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{L}_{II}(\mathscr{A}) \subseteq \mathbb{L}_{II}(\mathscr{H})$.
5) $\mathbb{L}_{II}(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_{II}(\mathscr{A}) \cup \mathbb{L}_{II}(\mathscr{H})$.
6) If $1 - {}_2\widetilde{N}^{\alpha}_y(z) \leq \mathscr{A}(z) \leq {}_2\widetilde{N}^{\alpha}_y(z)$ for any $z \in W_2$, then $\mathbb{L}_{II}(\mathscr{A}) \subseteq \mathbb{U}_{II}(\mathscr{A})$.
7) $\mathbb{U}_{II}(\mathscr{A}^c) = (\mathbb{L}_{II}(\mathscr{A}))^c$.
8) $\mathbb{U}_{II}(\phi) = \phi$.
9) $\mathbb{U}_{II}(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_{II}(\mathscr{A}) \cup \mathbb{U}_{II}(\mathscr{H})$.
10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{U}_{II}(\mathscr{A}) \subseteq \mathbb{U}_{II}(\mathscr{H})$.
11) $\mathbb{U}_{II}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{II}(\mathscr{A}) \cap \mathbb{U}_{II}(\mathscr{H})$.

Proof. 1) For any $y \in W_1$, we have $\mathbb{L}_{II}(\mathscr{A}^c)(y) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{A}^c(z)) = \bigwedge_{z\in W_2}(({}_2\widetilde{N}^{\alpha}_y(z))^c \bigvee \mathscr{A}^c(z)) = \bigwedge_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{A}(z))^c = (\bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{A}(z)))^c = 1 - \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \bigwedge \mathscr{A}(z)) = 1 - \mathbb{U}_{II}(\mathscr{A})(y) = (\mathbb{U}_{II}(\mathscr{A}))^c(y)$.

2) For any $y \in W_1$, we have $\mathbb{L}_{II}(W_2)(y) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee W_2(z))$. Since $W_2(z) = 1$ for any $z \in W_2$, $\mathbb{L}_{II}(W_2)(y = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee 1) = 1 = W_1(y)$, i.e., $\mathbb{L}(W_2) = W_1$.

3) For any $y \in W_1$, we have $\mathbb{L}_{II}(\mathscr{A} \cap \mathscr{H})(y) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee [\mathscr{A} \cap \mathscr{H}](z)) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee [\mathscr{A}(z) \bigwedge \mathscr{H}(z)]) = (\bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{A}(z))) \bigwedge (\bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{H}(z))) = (\mathbb{L}_{II}(\mathscr{A}) \cap \mathbb{L}_{II}(\mathscr{H}))(y)$. Then $\mathbb{L}_{II}(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_{II}(\mathscr{A}) \cap \mathbb{L}_{II}(\mathscr{H})$.

4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, we have $(\bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{A}(z))) \leq (\bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \bigvee \mathscr{H}(z)))$. Thus, $\mathbb{L}_{II}(\mathscr{A})(y) \leq \mathbb{L}_{II}(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{L}_{II}(\mathscr{A}) \subseteq \mathbb{L}_{II}(\mathscr{H})$.

5) Since $\mathscr{A} \subseteq \mathscr{A} \cup \mathscr{H}$ and $\mathscr{H} \subseteq \mathscr{A} \cup \mathscr{H}$, by (4), we have that $\mathbb{L}_{II}(\mathscr{A}) \subseteq \mathbb{L}_{II}(\mathscr{A} \cup \mathscr{H})$ and $\mathbb{L}_{II}(\mathscr{H}) \subseteq \mathbb{L}_{II}(\mathscr{A} \cup \mathscr{H})$. Hence, $\mathbb{L}_{II}(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_{II}(\mathscr{A}) \cup \mathbb{L}_{II}(\mathscr{H})$.

6) For any $y \in W_1$, there is $1 - {}_2\widetilde{N}^{\alpha}_y(z) \leq \mathscr{A}(z) \leq {}_2\widetilde{N}^{\alpha}_x(z)$ for any $z \in W_2$; therefore, $\mathscr{A}(z) = {}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}(z) \leq \bigvee_{x\in W_2}({}_2\widetilde{N}^{\alpha}_y(x) \wedge \mathscr{A}(z)) = \mathbb{U}_{II}(\mathscr{A})(y)$ and $\mathscr{A}(z) = ([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{A}(z)) \geq \bigwedge_{x\in W_2}([1 -_2 \widetilde{N}^{\alpha}_y(x)] \vee \mathscr{A}(x)) = \mathbb{L}_{II}(\mathscr{A})(y)$. Thus, $\mathbb{L}_{II}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathbb{U}_{II}(\mathscr{A})$.

7) For any $y \in W_1$, $\mathbb{U}_{II}(\mathscr{A}^c) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}^c(z)) = \bigvee_{z\in W_2}(({}_2\widetilde{N}^{\alpha}_y(z))^c \vee \mathscr{A}(z))^c = (\bigwedge_{z\in W_2}[1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{A}(z))^c = (\mathbb{L}_{II}(\mathscr{A}))^c$.

8) $\mathbb{U}_{II}(\phi) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \phi), \quad y \in W_1$
$= \bigvee_{z\in W_2} \phi, \quad y \in W_1$
$= \phi$.

9) For any $y \in W_1$, we have $\mathbb{U}_{II}(\mathscr{A} \cup \mathscr{H})(y) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge (\mathscr{A} \cup \mathscr{H})(z)) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge [\mathscr{A}(z) \vee \mathscr{H}(z)]) = (\bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}(z))) \vee (\bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{H}(z))) = (\mathbb{U}_{II}(\mathscr{A}) \cup \mathbb{U}_{II}(\mathscr{H}))(y)$. Then $\mathbb{U}_{II}(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_{II}(\mathscr{A}) \cup \mathbb{U}_{II}(\mathscr{H})$.
10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \le \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, we have $\bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{A}(z)) \le \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{H}(z))$. Thus, $\mathbb{U}_{II}(\mathscr{A})(y) \le \mathbb{U}_{II}(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{U}_{II}(\mathscr{A}) \subseteq \mathbb{U}_{II}(\mathscr{H})$.
11) Since $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{A}$ and $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{H}$, by (10), we have $\mathbb{U}_{II}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{II}(\mathscr{A})$ and $\mathbb{U}_{II}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{II}(\mathscr{H})$. Hence, $\mathbb{U}_{II}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{II}(\mathscr{A}) \wedge \mathbb{U}_{II}(\mathscr{H})$.

Proposition 4.11. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $\mathscr{K} \in \mathscr{F}(W_2)$ and $k \in [0, 1]$,*

1) $\mathbb{L}_{II}(\mathscr{K} \cup kW_2) = \mathbb{L}_{II}(\mathscr{K}) \cup kW_1$.
2) $\mathbb{L}_{II}(\mathscr{K} \cap kW_2) = \mathbb{L}_{II}(\mathscr{K}) \cap kW_1$.

Proof. 1) For any $y \in W_1$, $\mathbb{L}_{II}(\mathscr{K} \cup kW_2)(y) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee [\mathscr{K} \cup kW_2](z)) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z) \vee kW_2(z)) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z) \vee k) = (\bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z))) \vee (\bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_y(z)] \vee k)) = \mathbb{L}_{II}(\mathscr{K})(y) \vee k = \mathbb{L}_{II}(\mathscr{K}) \cup kW_1$.

2) For any $y \in W_1$, $\mathbb{U}_{II}(\mathscr{K} \cap kW_2)(y) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge [\mathscr{K} \cap kW_2](z)) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z) \wedge kW_2(z)) = \bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z) \wedge k) = (\bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z))) \wedge (\bigvee_{z\in W_2}({}_2\widetilde{N}^{\alpha}_y(z) \wedge k)) = \mathbb{U}_{II}(\mathscr{K})(y) \wedge k = \mathbb{U}_{II}(\mathscr{K}) \cap kW_1$.

Proposition 4.12. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a \in W_1, c \in W_2$, and $X \subseteq W_2$,*

1) $\mathbb{U}_{II}(1_c)(a) = {}_2\widetilde{N}^{\alpha}_a(c)$.
2) $\mathbb{L}_{II}(1_{W_2-\{c\}})(a) = 1 - {}_2\widetilde{N}^{\alpha}_a(c)$.
3) $\mathbb{U}_{II}(1_X)(a) = \bigvee_{c\in X} {}_2\widetilde{N}^{\alpha}_a(c)$.
4) $\mathbb{L}_{II}(1_X)(a) = \bigwedge_{c\notin X}(1 - {}_2\widetilde{N}^{\alpha}_a(c))$.

Proof. 1) For any $a \in W_1$, $a', c \in W_2$, and from the definition of 1_c, we have $1_c(a') = 0$ for $a' \neq c$. Hence, $\mathbb{U}_{II}(1_c)(a) = \bigvee_{z\in W_2}[{}_2\widetilde{N}^{\alpha}_a(z) \wedge 1_c(z)] = {}_2\widetilde{N}^{\alpha}_a(c)$.

2) $\mathbb{L}_{II}(1_{W_2-\{c\}})(a) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z)) = (\bigwedge_{z\in W_2-\{c\}}([1 - {}_2\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z))) \bigwedge (\bigwedge_{z=c}([1 - {}_2\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z))) = 1 \bigwedge ([1 - {}_2\widetilde{N}^{\alpha}_a(c)] \bigvee 0) = 1 - {}_2\widetilde{N}^{\alpha}_a(c)$.

3) For any $c \in W_2$ and $X \subseteq W_2$, from the definition of 1_X, we have $1_X(c) = 0$ if and only if $c \notin X$. Hence, for any $x \in W_1$, we have $\mathbb{U}_{II}(1_X)(a) = \bigvee_{c\in W_2}[{}_2\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)] = (\bigvee_{c\in X}[{}_2\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)]) \bigvee (\bigvee_{c\notin X}[{}_2\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)]) = \bigvee_{c\in X} {}_2\widetilde{N}^{\alpha}_a(c)$.

4) $\mathbb{L}_{II}(1_X)(a) = \bigwedge_{c\in W_2}([1 - {}_2\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c)) = (\bigwedge_{c\in X}([1 - {}_2\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c))) \bigwedge (\bigwedge_{c\notin X}([1 - {}_2\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c))) = 1 \bigwedge (\bigwedge_{c\notin X}([1 - {}_2\widetilde{N}^{\alpha}_a(c)] \bigvee 0)) = \bigwedge_{c\notin X}(1 - {}_2\widetilde{N}^{\alpha}_a(c))$.

Depending on Pawlak's rough set pattern, the fuzzy covering depending on the rough set pattern over 2-finite sets is provided by using the idea of an α-neighborhood.

Definition 4.13. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $\mathscr{B} \subseteq W_2$*, the second upper* $\overline{\mathbb{U}_{II}}$ *and lower* $\overline{\mathbb{L}_{II}}$ *approximations, respectively, are defined as:*

$\overline{\mathbb{U}_{II}}(\mathscr{B}) = \{x \in W_1 : \overline{{}_2\widetilde{N}^{\alpha}_x}(y) \cap \mathscr{B} \neq \phi\}$.
$\overline{\mathbb{L}_{II}}(\mathscr{B}) = \{x \in W_1 : \overline{{}_2\widetilde{N}^{\alpha}_x}(y) \subseteq \mathscr{B}\}$.
$\mathscr{B}$ is definable, except if $\overline{\mathbb{U}_{II}}(\mathscr{B}) \neq \overline{\mathbb{L}_{II}}(\mathscr{B})$. $\mathscr{B}$ is called the second kind of fuzzy covering depending on the rough set.

Example 4.14. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. From Example 3.17, we have* $\overline{{}_2\widetilde{N}^{0.4}_{u_1}} = \overline{{}_2\widetilde{N}^{0.4}_{u_3}} = \overline{{}_2\widetilde{N}^{0.4}_{u_4}} = \overline{{}_2\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$ *and* $\overline{{}_2\widetilde{N}^{0.4}_{u_2}} = \overline{{}_2\widetilde{N}^{0.4}_{u_5}} = \{y_1, y_2, y_4\}$*.*

1) Let $\mathscr{B} = \{y_1, y_4\}$. Then, $\overline{\mathbb{U}_{II}}(\mathscr{B}) = \{u_1, u_2, u_3, u_4, u_5, u_6\} = W_1$, and $\overline{\mathbb{L}_{II}}(\mathscr{B}) = \phi$.
2) Let $\mathscr{B} = \{y_3\}$. Then, $\overline{\mathbb{U}_{II}}(\mathscr{B}) = \{u_1, u_3, u_4, u_6\}$, and $\overline{\mathbb{L}_{II}}(\mathscr{B}) = \phi$.

4.3. The third pattern

Definition 4.15. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $\mathscr{B} \in \mathscr{F}(W_2)$*, the third fuzzy α-lower* $\mathbb{L}_{III}$ *and fuzzy α-upper* $\mathbb{U}_{III}$ *approximations, respectively, are defined as*

$\mathbb{L}_{III}(\mathscr{B})(x) = \bigwedge_{y \in W_2}([1 - {}_3\widetilde{N}^{\alpha}_x(y)] \bigvee \mathscr{B}(y)), \quad x \in W_1,$
$\mathbb{U}_{III}(\mathscr{B})(x) = \bigvee_{y \in W_2}({}_3\widetilde{N}^{\alpha}_x(y) \bigwedge \mathscr{B}(y)), \quad x \in W_1.$
$\mathscr{B}$ is definable, except if $\mathbb{L}_{III}(\mathscr{B}) \neq \mathbb{U}_{III}(\mathscr{B})$. $\mathscr{B}$ is called the third kind of fuzzy α-covering depending on the rough set.

Example 4.16. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. Using the information in Examples 3.12 and 4.2, we have*

$\mathbb{L}_{III}(\mathscr{B})(u_1) = ([1 - {}_3\widetilde{N}^{0.4}_{u_1}(y_1)] \bigvee \mathscr{B}(y_1)) \bigwedge([1 - {}_3\widetilde{N}^{0.4}_{u_1}(y_2)] \bigvee \mathscr{B}(y_2)) \bigwedge([1 - {}_3\widetilde{N}^{0.4}_{u_1}(y_3)] \bigvee \mathscr{B}(y_3))$
$\bigwedge([1 - {}_3\widetilde{N}^{0.4}_{u_1}(y_4)] \bigvee \mathscr{B}(y_4)) = 0.3 \wedge 0.7 \wedge 0.5 \wedge 0.2 = 0.2.$
In the same manner for u_2 to u_6, we get $\mathbb{L}_{III}(\mathscr{B}) = \frac{0.2}{y_1} + \frac{0.2}{y_2} + \frac{0.2}{y_3} + \frac{0.2}{y_4} + \frac{0.2}{y_5} + \frac{0.2}{y_6}$.
$\mathbb{U}_{III}(\mathscr{B})(u_1) = ({}_3\widetilde{N}^{0.4}_{u_1}(y_1) \bigwedge \mathscr{B}(y_1)) \bigvee({}_3\widetilde{N}^{0.4}_{u_1}(y_2) \bigwedge \mathscr{B}(y_2)) \bigvee({}_3\widetilde{N}^{0.4}_{u_1}(y_3) \bigwedge \mathscr{B}(y_3)) \bigvee$
$({}_3\widetilde{N}^{0.4}_{u_1}(y_4) \bigwedge \mathscr{B}(y_4)) = 0.3 \vee 0.7 \vee 0.5 \vee 0.2 = 0.7.$
In the same manner for u_2 to u_6, we get $\mathbb{U}_{III}(\mathscr{B}) = \frac{0.7}{y_1} + \frac{0.7}{y_2} + \frac{0.7}{y_3} + \frac{0.7}{y_4} + \frac{0.7}{y_5} + \frac{0.7}{y_6}$.

Proposition 4.17 provides the characteristics of the third fuzzy α-covering depending on the rough set pattern.

Proposition 4.17. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$*. For each* $\mathscr{A}, \mathscr{H} \in \mathscr{F}(W_2)$*, we have*

1) $\mathbb{L}_{III}(\mathscr{A}^c) = (\mathbb{U}_{III}(\mathscr{A}))^c$.
2) $\mathbb{L}_{III}(W_2) = W_1$.
3) $\mathbb{L}_{III}(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_{III}(\mathscr{A}) \cap \mathbb{L}_{III}(\mathscr{H})$.
4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{L}_{III}(\mathscr{A}) \subseteq \mathbb{L}_{III}(\mathscr{H})$.
5) $\mathbb{L}_{III}(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_{III}(\mathscr{A}) \cup \mathbb{L}_{III}(\mathscr{H})$.
6) If $1 - {}_3\widetilde{N}^{\alpha}_y(z) \leq \mathscr{A}(z) \leq {}_3\widetilde{N}^{\alpha}_y(z)$ for any $z \in W_2$, then $\mathbb{L}_{III}(\mathscr{A}) \subseteq \mathbb{U}_{III}(\mathscr{A})$.

7) $\mathbb{U}_{III}(\mathscr{A}^c) = (\mathbb{L}_{III}(\mathscr{A}))^c$.
8) $\mathbb{U}_{III}(\phi) = \phi$.
9) $\mathbb{U}_{III}(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_{III}(\mathscr{A}) \cup \mathbb{U}_{III}(\mathscr{H})$.
10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{U}_{III}(\mathscr{A}) \subseteq \mathbb{U}_{III}(\mathscr{H})$.
11) $\mathbb{U}_{III}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{III}(\mathscr{A}) \cap \mathbb{U}_{III}(\mathscr{H})$.

Proof. 1) For any $y \in W_1$, $\mathbb{L}_{III}(\mathscr{A}^c)(y) = \bigwedge_{z\in W_2}([1-{}_3\widetilde{N}^\alpha_y(z)] \bigvee \mathscr{A}^c(z)) = \bigwedge_{z\in W_2}(({}_3\widetilde{N}^\alpha_y(z))^c \bigvee \mathscr{A}^c(z)) = \bigwedge_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \bigwedge \mathscr{A}(z))^c = (\bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \bigwedge \mathscr{A}(z)))^c = 1 - \bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \bigwedge \mathscr{A}(z)) = 1 - \mathbb{U}_{III}(\mathscr{A})(y) = (\mathbb{U}_{III}(\mathscr{A}))^c(y)$.

2) For any $y \in W_1$, $\mathbb{L}_{III}(W_2)(y) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee W_2(z))$. Since $W_2(z) = 1$ for any $z \in W_2$, $\mathbb{L}_{III}(W_2)(y) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee 1) = 1 = W_1(y)$, i.e. $\mathbb{L}(W_2) = W_1$.

3) For any $y \in W_1$, $\mathbb{L}_{III}(\mathscr{A} \cap \mathscr{H})(x) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee[\mathscr{A} \cap \mathscr{H}](z)) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee[\mathscr{A}(z) \bigwedge \mathscr{H}(z)]) = (\bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee \mathscr{A}(z))) \bigwedge(\bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee \mathscr{H}(z))) = (\mathbb{L}_{III}(\mathscr{A}) \cap \mathbb{L}_{III}(\mathscr{H}))(y)$. Then $\mathbb{L}_{III}(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_{III}(\mathscr{A}) \cap \mathbb{L}_{III}(\mathscr{H})$.

4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, $(\bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee \mathscr{A}(z))) \leq (\bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^\alpha_y(z)] \bigvee \mathscr{H}(z)))$. Thus, $\mathbb{L}_{III}(\mathscr{A})(y) \leq \mathbb{L}_{III}(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{L}_{III}(\mathscr{A}) \subseteq \mathbb{L}_{III}(\mathscr{H})$.

5) Since $\mathscr{A} \subseteq \mathscr{A} \cup \mathscr{H}$ and $\mathscr{H} \subseteq \mathscr{A} \cup \mathscr{H}$, by (4), $\mathbb{L}_{III}(\mathscr{A}) \subseteq \mathbb{L}_{III}(\mathscr{A} \cup \mathscr{H})$ and $\mathbb{L}_{III}(\mathscr{H}) \subseteq \mathbb{L}_{III}(\mathscr{A} \cup \mathscr{H})$. Hence, $\mathbb{L}_{III}(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_{III}(\mathscr{A}) \cup \mathbb{L}_{III}(\mathscr{H})$.

6) For any $y \in W_1$, there is $1 - {}_3\widetilde{N}^\alpha_y(z) \leq \mathscr{A}(z) \leq {}_3\widetilde{N}^\alpha_y(z)$, for any $z \in W_2$ and therefore, $\mathscr{A}(z) = {}_3\widetilde{N}^\alpha_y(z) \wedge \mathscr{A}(z) \leq \bigvee_{x\in W_2}({}_3\widetilde{N}^\alpha_y(x) \wedge \mathscr{A}(z)) = \mathbb{U}_{III}(\mathscr{A})(y)$ and $\mathscr{A}(z) = ([1 - {}_3\widetilde{N}^\alpha_y(z)] \vee \mathscr{A}(z)) \geq \bigwedge_{x\in W_2}([1 -{}_3 \widetilde{N}^\alpha_y(x)] \vee \mathscr{A}(x)) = \mathbb{L}_{III}(\mathscr{A})(y)$. Thus, $\mathbb{L}_{III}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathbb{U}_{III}(\mathscr{A})$.

7) For any $y \in W_1$, $\mathbb{U}_{III}(\mathscr{A}^c) = \bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge \mathscr{A}^c(z)) = \bigvee_{z\in W_2}(({}_3\widetilde{N}^\alpha_y(z))^c \vee \mathscr{A}(z))^c = (\bigwedge_{z\in W_2}[1 - {}_3\widetilde{N}^\alpha_y(z)] \vee \mathscr{A}(z))^c = (\mathbb{L}_{III}(\mathscr{A}))^c$.

8) $\mathbb{U}_{III}(\phi) = \bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge \phi), \quad y \in W_1$
$= \bigvee_{z\in W_2} \phi, \quad y \in W_1$
$= \phi$.

9) For any $y \in W_1$, we have $\mathbb{U}_{III}(\mathscr{A} \cup \mathscr{H})(y) = \bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge (\mathscr{A} \cup \mathscr{H})(z)) = \bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge [\mathscr{A}(z) \vee \mathscr{H}(z)]) = (\bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge \mathscr{A}(z))) \vee (\bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge \mathscr{H}(z))) = (\mathbb{U}_{III}(\mathscr{A}) \cup \mathbb{U}_{III}(\mathscr{H}))(y)$. Then, $\mathbb{U}_{III}(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_{III}(\mathscr{A}) \cup \mathbb{U}_{III}(\mathscr{H})$.

10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, we have $\bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge \mathscr{A}(z)) \leq \bigvee_{z\in W_2}({}_3\widetilde{N}^\alpha_y(z) \wedge \mathscr{H}(z))$. Thus, $\mathbb{U}_{III}(\mathscr{A})(y) \leq \mathbb{U}_{III}(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{U}_{III}(\mathscr{A}) \subseteq \mathbb{U}_{III}(\mathscr{H})$.

11) Since $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{A}$ and $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{H}$, by (10) we have that $\mathbb{U}_{III}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{III}(\mathscr{A})$ and $\mathbb{U}_{III}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{III}(\mathscr{H})$. Hence, $\mathbb{U}_{III}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{III}(\mathscr{A}) \wedge \mathbb{U}_{III}(\mathscr{H})$.

Proposition 4.18. *Let $(W_1, W_2, \mathfrak{V})$ be a $F\alpha CAS$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $\mathscr{K} \in \mathscr{F}(W_2)$ and $k \in [0, 1]$,*

1) $\mathbb{L}_{III}(\mathscr{K} \cup kW_2) = \mathbb{L}_{III}(\mathscr{K}) \cup kW_1$.
2) $\mathbb{L}_{III}(\mathscr{K} \cap kW_2) = \mathbb{L}_{III}(\mathscr{K}) \cap kW_1$.

Proof. 1) For any $y \in W_1$, $\mathbb{L}_{III}(\mathscr{K} \cup kW_2)(y) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_y(z)] \vee [\mathscr{K} \cup kW_2](z)) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z) \vee kW_2(z)) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z) \vee k) = (\bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_y(z)] \vee \mathscr{K}(z))) \vee (\bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_y(z)] \vee k)) = \mathbb{L}_{III}(\mathscr{K})(y) \vee k = \mathbb{L}_{III}(\mathscr{K}) \cup kW_1$.

2) For any $y \in W_1$, $\mathbb{U}_{III}(\mathscr{K} \cap kW_2)(y) = \bigvee_{z\in W_2}({}_3\widetilde{N}^{\alpha}_y(z) \wedge [\mathscr{K} \cap kW_2](z)) = \bigvee_{z\in W_2}({}_3\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z) \wedge kW_2(z)) = \bigvee_{z\in W_2}({}_3\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z) \wedge k) = (\bigvee_{z\in W_2}({}_3\widetilde{N}^{\alpha}_y(z) \wedge \mathscr{K}(z))) \wedge (\bigvee_{z\in W_2}({}_3\widetilde{N}^{\alpha}_y(z) \wedge k)) = \mathbb{U}_{III}(\mathscr{K})(y) \wedge k = \mathbb{U}_{III}(\mathscr{K}) \cap kW_1$.

Proposition 4.19. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a \in W_1, c \in W_2$, and $X \subseteq W_2$,*

1) $\mathbb{U}_{III}(1_c)(a) = {}_3\widetilde{N}^{\alpha}_a(c)$.
2) $\mathbb{L}_{III}(1_{W_2-\{c\}})(a) = 1 - {}_3\widetilde{N}^{\alpha}_a(c)$.
3) $\mathbb{U}_{III}(1_X)(a) = \bigvee_{c\in X} {}_3\widetilde{N}^{\alpha}_a(c)$.
4) $\mathbb{L}_{III}(1_X)(a) = \bigwedge_{c\notin X}(1 - {}_3\widetilde{N}^{\alpha}_a(c))$.

Proof. 1) For any $a \in W_1$, $a', andc \in W_2$, from the definition of 1_c, we have $1_c(a') = 0$ for $a' \neq c$. Hence, $\mathbb{U}_{III}(1_c)(a) = \bigvee_{z\in W_2}[{}_3\widetilde{N}^{\alpha}_a(z) \wedge 1_c(z)] = {}_3\widetilde{N}^{\alpha}_a(c)$.

2) $\mathbb{L}_{III}(1_{W_2-\{c\}})(a) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z)) = (\bigwedge_{z\in W_2-\{c\}}([1 - {}_3\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z))) \bigwedge(\bigwedge_{z=c}([1 - {}_3\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z))) = 1 \bigwedge([1 - {}_3\widetilde{N}^{\alpha}_a(c)] \bigvee 0) = 1 - {}_3\widetilde{N}^{\alpha}_a(c)$.

3) For any $c \in W_2$ and $X \subseteq W_2$, from the definition of 1_X, we have $1_X(c) = 0$ if and only if $c \notin X$. Hence, for any $x \in W_1$, we have $\mathbb{U}_{III}(1_X)(a) = \bigvee_{c\in W_2}[{}_3\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)] = (\bigvee_{c\in X}[{}_3\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)]) \bigvee(\bigvee_{c\notin X}[{}_3\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)]) = \bigvee_{c\in X} {}_3\widetilde{N}^{\alpha}_a(c)$.

4) $\mathbb{L}_{III}(1_X)(a) = \bigwedge_{c\in W_2}([1 - {}_3\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c)) = (\bigwedge_{c\in X}([1 - {}_3\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c))) \bigwedge(\bigwedge_{c\notin X}([1 - {}_3\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c))) = 1 \bigwedge(\bigwedge_{c\notin X}([1 - {}_3\widetilde{N}^{\alpha}_a(c)] \bigvee 0)) = \bigwedge_{c\notin X}(1 - {}_3\widetilde{N}^{\alpha}_a(c))$.

Depending on Pawlak's rough set pattern, the fuzzy covering depending on the rough set pattern over 2-finite sets is provided by using the idea of an α-neighborhood.

Definition 4.20. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \subseteq W_2$, the third upper $\overline{\mathbb{U}_{III}}$ and lower $\overline{\mathbb{L}_{III}}$ approximations, respectively, are defined as*

$\overline{\mathbb{U}_{III}}(\mathscr{B}) = \{x \in W_1 : \overline{{}_3\widetilde{N}^{\alpha}_x}(y) \cap \mathscr{B} \neq \phi\}$.
$\overline{\mathbb{L}_{III}}(\mathscr{B}) = \{x \in W_1 : \underline{{}_3\widetilde{N}^{\alpha}_x(y)} \subseteq \mathscr{B}\}$.
$\mathscr{B}$ is definable, except if $\overline{\mathbb{U}_{III}}(\mathscr{B}) \neq \overline{\mathbb{L}_{III}}(\mathscr{B})$. $\mathscr{B}$ is called the third kind of fuzzy covering depending on the rough set.

Example 4.21. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. From Example 3.17, we have $\overline{{}_3\widetilde{N}^{0.4}_{u_1}} = \overline{{}_3\widetilde{N}^{0.4}_{u_2}} = \overline{{}_3\widetilde{N}^{0.4}_{u_3}} = \overline{{}_3\widetilde{N}^{0.4}_{u_4}} = \overline{{}_3\widetilde{N}^{0.4}_{u_5}} = \overline{{}_3\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$.*

1) Let $\mathscr{B} = \{y_1, y_4\}$. Then $\overline{\mathbb{U}_{III}}(\mathscr{B}) = \{u_1, u_2, u_3, u_4, u_5, u_6\} = W_1$ and $\overline{\mathbb{L}_{III}}(\mathscr{B}) = \phi$.
2) Let $\mathscr{B} = \{y_3\}$. Then $\overline{\mathbb{U}_{III}}(\mathscr{B}) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $\overline{\mathbb{L}_{III}}(\mathscr{B}) = \phi$.

4.4. The fourth pattern

Definition 4.22. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \in \mathscr{F}(W_2)$, the fourth fuzzy α-lower $\mathbb{L}_{IV}$ and fuzzy α-upper $\mathbb{U}_{IV}$ approximations, respectively, are defined as*

$\mathbb{L}_{IV}(\mathscr{B})(y) = \bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee \mathscr{B}(z)), \quad y \in W_1,$
$\mathbb{U}_{IV}(\mathscr{B})(y) = \bigvee_{z \in W_2}({}_4\widetilde{N}_y^{\alpha}(z) \bigwedge \mathscr{B}(z)), \quad y \in W_1.$
$\mathscr{B}$ is definable, except if $\mathbb{L}_{IV}(\mathscr{B}) \neq \mathbb{U}_{IV}(\mathscr{B})$. $\mathscr{B}$ is called the fourth kind of fuzzy α-covering depending on the rough set.

Example 4.23. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. Using the information in Examples 3.12 and 4.2, we have*

$\mathbb{L}_{IV}(\mathscr{B})(u_1) = ([1 - {}_4\widetilde{N}_{u_1}^{0.4}(y_1)] \bigvee \mathscr{B}(y_1)) \bigwedge([1 - {}_4\widetilde{N}_{u_1}^{0.4}(y_2)] \bigvee \mathscr{B}(y_2)) \bigwedge([1 - {}_4\widetilde{N}_{u_1}^{0.4}(y_3)] \bigvee \mathscr{B}(y_3))$
$\bigwedge([1 - {}_4\widetilde{N}_{u_1}^{0.4}(y_4)] \bigvee \mathscr{B}(y_4)) = 0.3 \wedge 0.7 \wedge 0.5 \wedge 0.2 = 0.2.$
In the same manner for u_2 to u_6, we get $\mathbb{L}_{IV}(\mathscr{B}) = \frac{0.2}{y_1} + \frac{0.2}{y_2} + \frac{0.2}{y_3} + \frac{0.2}{y_4} + \frac{0.2}{y_5} + \frac{0.2}{y_6}$.
$\mathbb{U}_{IV}(\mathscr{B})(u_1) = ({}_4\widetilde{N}_{u_1}^{0.4}(y_1) \bigwedge \mathscr{B}(y_1)) \bigvee({}_4\widetilde{N}_{u_1}^{0.4}(y_2) \bigwedge \mathscr{B}(y_2)) \bigvee({}_4\widetilde{N}_{u_1}^{0.4}(y_3) \bigwedge \mathscr{B}(y_3)) \bigvee$
$({}_4\widetilde{N}_{u_1}^{0.4}(y_4) \bigwedge \mathscr{B}(y_4)) = 0.3 \vee 0.7 \vee 0.5 \vee 0.2 = 0.7.$
In the same manner for u_2 to u_6, we get $\mathbb{U}_{IV}(\mathscr{B}) = \frac{0.7}{y_1} + \frac{0.7}{y_2} + \frac{0.7}{y_3} + \frac{0.7}{y_4} + \frac{0.7}{y_5} + \frac{0.7}{y_6}$.

Proposition 4.24 provides the characteristics of the fourth fuzzy α-covering depending on the rough set pattern.

Proposition 4.24. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{A}, \mathscr{H} \in \mathscr{F}(W_2)$, we have*

1) $\mathbb{L}_{IV}(\mathscr{A}^c) = (\mathbb{U}_{IV}(\mathscr{A}))^c$.
2) $\mathbb{L}_{IV}(W_2) = W_1$.
3) $\mathbb{L}_{IV}(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_{IV}(\mathscr{A}) \cap \mathbb{L}_{IV}(\mathscr{H})$.
4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{L}_{IV}(\mathscr{A}) \subseteq \mathbb{L}_{IV}(\mathscr{H})$.
5) $\mathbb{L}_{IV}(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_{IV}(\mathscr{A}) \cup \mathbb{L}_{IV}(\mathscr{H})$.
6) If $1 - {}_4\widetilde{N}_y^{\alpha}(z) \leq \mathscr{A}(z) \leq {}_4\widetilde{N}_y^{\alpha}(z)$ for any $z \in W_2$, then $\mathbb{L}_{IV}(\mathscr{A}) \subseteq \mathbb{U}_{IV}(\mathscr{A})$.
7) $\mathbb{U}_{IV}(\mathscr{A}^c) = (\mathbb{L}_{IV}(\mathscr{A}))^c$.
8) $\mathbb{U}_{IV}(\phi) = \phi$.
9) $\mathbb{U}_{IV}(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_{IV}(\mathscr{A}) \cup \mathbb{U}_{IV}(\mathscr{H})$.
10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathbb{U}_{IV}(\mathscr{A}) \subseteq \mathbb{U}_{IV}(\mathscr{H})$.
11) $\mathbb{U}_{IV}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{IV}(\mathscr{A}) \cap \mathbb{U}_{IV}(\mathscr{H})$.

Proof.
1) For any $y \in W_1$, $\mathbb{L}_{IV}(\mathscr{A}^c)(y) = \bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee \mathscr{A}^c(z)) = \bigwedge_{z \in W_2}(({}_4\widetilde{N}_y^{\alpha}(z))^c \bigvee \mathscr{A}^c(z)) = \bigwedge_{z \in W_2}({}_4\widetilde{N}_y^{\alpha}(z) \bigwedge \mathscr{A}(z))^c = (\bigvee_{z \in W_2}({}_4\widetilde{N}_y^{\alpha}(z) \bigwedge \mathscr{A}(z)))^c = 1 - \bigvee_{z \in W_2}({}_4\widetilde{N}_y^{\alpha}(z) \bigwedge \mathscr{A}(z)) = 1 - \mathbb{U}_{IV}(\mathscr{A})(y) = (\mathbb{U}_{IV}(\mathscr{A}))^c(y)$.
2) For any $y \in W_1$, $\mathbb{L}_{IV}(W_2)(y) = \bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee W_2(z))$. Since $W_2(z) = 1$ for any $z \in W_2$, $\mathbb{L}_{IV}(W_2)(y) = \bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee 1) = 1 = W_1(y)$, i.e., $\mathbb{L}(W_2) = W_1$.
3) For any $y \in W_1$, $\mathbb{L}_{IV}(\mathscr{A} \cap \mathscr{H})(y) = \bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee [\mathscr{A} \cap \mathscr{H}](z)) = \bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee [\mathscr{A}(z) \bigwedge \mathscr{H}(z)]) = (\bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee \mathscr{A}(z))) \bigwedge (\bigwedge_{z \in W_2}([1 - {}_4\widetilde{N}_y^{\alpha}(z)] \bigvee \mathscr{H}(z))) = (\mathbb{L}_{IV}(\mathscr{A}) \cap \mathbb{L}_{IV}(\mathscr{H}))(y)$. Then $\mathbb{L}_{IV}(\mathscr{A} \cap \mathscr{H}) = \mathbb{L}_{IV}(\mathscr{A}) \cap \mathbb{L}_{IV}(\mathscr{H})$.

4) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, $(\bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \bigvee \mathscr{A}(z))) \leq (\bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \bigvee \mathscr{H}(z)))$. Thus, $\mathbb{L}_{IV}(\mathscr{A})(y) \leq \mathbb{L}_{IV}(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{L}_{IV}(\mathscr{A}) \subseteq \mathbb{L}_{IV}(\mathscr{H})$.

5) Since $\mathscr{A} \subseteq \mathscr{A} \cup \mathscr{H}$ and $\mathscr{H} \subseteq \mathscr{A} \cup \mathscr{H}$, by (4), $\mathbb{L}_{IV}(\mathscr{A}) \subseteq \mathbb{L}_{IV}(\mathscr{A} \cup \mathscr{H})$ and $\mathbb{L}_{IV}(\mathscr{H}) \subseteq \mathbb{L}_{IV}(\mathscr{A} \cup \mathscr{H})$. Hence, $\mathbb{L}_{IV}(\mathscr{A} \cup \mathscr{H}) \supseteq \mathbb{L}_{IV}(\mathscr{A}) \cup \mathbb{L}_{IV}(\mathscr{H})$.

6) For any $y \in W_1$, there is $1 - {}_4\widetilde{N}_y^\alpha(z) \leq \mathscr{A}(z) \leq {}_4\widetilde{N}_y^\alpha(z)$ for any $z \in W_2$, $\mathscr{A}(z) = {}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{A}(z) \leq \bigvee_{x\in W_2}({}_4\widetilde{N}_y^\alpha(x) \wedge \mathscr{A}(z)) = \mathbb{U}_{IV}(\mathscr{A})(y)$ and $\mathscr{A}(z) = ([1 - {}_4\widetilde{N}_y^\alpha(z)] \vee \mathscr{A}(z)) \geq \bigwedge_{x\in W_2}([1 -_4 \widetilde{N}_y^\alpha(x)] \vee \mathscr{A}(x)) = \mathbb{L}_{IV}(\mathscr{A})(y)$. Thus, $\mathbb{L}_{IV}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathbb{U}_{IV}(\mathscr{A})$.

7) For any $y \in W_1$, $\mathbb{U}_{IV}(\mathscr{A}^c) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{A}^c(z)) = \bigvee_{z\in W_2}(({}_4\widetilde{N}_y^\alpha(z))^c \vee \mathscr{A}(z))^c = (\bigwedge_{z\in W_2}[1 - {}_4\widetilde{N}_y^\alpha(z)] \vee \mathscr{A}(z))^c = (\mathbb{L}_{IV}(\mathscr{A}))^c$.

8) $\mathbb{U}_{IV}(\phi) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \phi), \quad y \in W_1$
$= \bigvee_{z\in W_2} \phi, \quad y \in W_1$
$= \phi$.

9) For any $y \in W_1$, we have $\mathbb{U}_{IV}(\mathscr{A} \cup \mathscr{H})(y) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge (\mathscr{A} \cup \mathscr{H})(z)) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge [\mathscr{A}(z) \vee \mathscr{H}(z)]) = (\bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{A}(z))) \vee (\bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{H}(z))) = (\mathbb{U}_{IV}(\mathscr{A}) \cup \mathbb{U}_{IV}(\mathscr{H}))(y)$. Then $\mathbb{U}_{IV}(\mathscr{A} \cup \mathscr{H}) = \mathbb{U}_{IV}(\mathscr{A}) \cup \mathbb{U}_{IV}(\mathscr{H})$.

10) If $\mathscr{A} \subseteq \mathscr{H}$, then $\mathscr{A}(z) \leq \mathscr{H}(z)$ for any $z \in W_2$. For any $y \in W_1$, $\bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{A}(z)) \leq \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{H}(z))$. Thus, $\mathbb{U}_{IV}(\mathscr{A})(y) \leq \mathbb{U}_{IV}(\mathscr{H})(y)$ holds for any $y \in W_1$, i.e., $\mathbb{U}_{IV}(\mathscr{A}) \subseteq \mathbb{U}_{IV}(\mathscr{H})$.

11) Since $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{A}$ and $\mathscr{A} \cap \mathscr{H} \subseteq \mathscr{H}$, by (10), $\mathbb{U}_{IV}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{IV}(\mathscr{A})$ and $\mathbb{U}_{IV}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{IV}(\mathscr{H})$. Hence, $\mathbb{U}_{IV}(\mathscr{A} \cap \mathscr{H}) \subseteq \mathbb{U}_{IV}(\mathscr{A}) \wedge \mathbb{U}_{IV}(\mathscr{H})$.

Proposition 4.25. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $\mathscr{K} \in \mathscr{F}(W_2)$ and $k \in [0, 1]$, the following hold:*

1) $\mathbb{L}_{IV}(\mathscr{K} \cup kW_2) = \mathbb{L}_{IV}(\mathscr{K}) \cup kW_1$.
2) $\mathbb{L}_{IV}(\mathscr{K} \cap kW_2) = \mathbb{L}_{IV}() \cap kW_1$.

Proof. 1) For any $y \in W_1$, $\mathbb{L}_{IV}(\mathscr{K} \cup kW_2)(y) = \bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \vee [\mathscr{K} \cup kW_2](z)) = \bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \vee \mathscr{K}(z) \vee kW_2(z)) = \bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \vee \mathscr{K}(z) \vee k) = (\bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \vee \mathscr{K}(z))) \vee (\bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^\alpha(z)] \vee k)) = \mathbb{L}_{IV}(\mathscr{K})(y) \vee k = \mathbb{L}_{IV}(\mathscr{K}) \cup kW_1$.

2) For any $y \in W_1$, $\mathbb{U}_{IV}(\mathscr{K} \cap kW_2)(y) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge [\mathscr{K} \cap kW_2](z)) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{K}(z) \wedge kW_2(z)) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge \mathscr{K}(z) \wedge k) = (\bigvee_{z\in W_2}({}_4\widetilde{N}_x^\alpha(z) \wedge \mathscr{K}(z))) \wedge (\bigvee_{z\in W_2}({}_4\widetilde{N}_y^\alpha(z) \wedge k)) = \mathbb{U}_{IV}(\mathscr{K})(y) \wedge k = \mathbb{U}_{IV}(\mathscr{K}) \cap kW_1$.

Proposition 4.26. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $a \in W_1, c \in W_2$, and $X \subseteq W_2$,*

1) $\mathbb{U}_{IV}(1_c)(a) = {}_4\widetilde{N}_a^\alpha(c)$.
2) $\mathbb{L}_{IV}(1_{W_2-\{c\}})(a) = 1 - {}_4\widetilde{N}_a^\alpha(c)$.
3) $\mathbb{U}_{IV}(1_X)(a) = \bigvee_{c\in X} {}_4\widetilde{N}_a^\alpha(c)$.
4) $\mathbb{L}_{IV}(1_X)(a) = \bigwedge_{c\notin X}(1 - {}_4\widetilde{N}_a^\alpha(c))$.

Proof. 1) For any $a \in W_1$, $a', c \in W_2$, from the definition of 1_c, we have $1_c(a') = 0$ for $a' \neq c$. Hence, $\mathbb{U}_{IV}(1_c)(a) = \bigvee_{z\in W_2}[{}_4\widetilde{N}^{\alpha}_a(z) \wedge 1_c(z)] = {}_4\widetilde{N}^{\alpha}_a(c)$.

2) $\mathbb{L}_{IV}(1_{W_2-\{c\}})(a) = \bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z)) = (\bigwedge_{z\in W_2-\{c\}}([1 - {}_4\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z))) \bigwedge(\bigwedge_{z=c}([1 - {}_4\widetilde{N}^{\alpha}_a(z)] \bigvee 1_{W_2-\{c\}}(z))) = 1 \bigwedge([1 - {}_4\widetilde{N}^{\alpha}_a(c)] \bigvee 0) = 1 - {}_4\widetilde{N}^{\alpha}_a(c)$.

3) For any $c \in W_2$, $X \subseteq W_2$, from the definition of 1_X, we have $1_X(c) = 0$ if and only if $c \notin X$. Hence, for any $x \in W_1$, we have $\mathbb{U}_{IV}(1_X)(a) = \bigvee_{c\in W_2}[{}_4\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)] = (\bigvee_{c\in X}[{}_4\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)]) \bigvee(\bigvee_{c\notin X}[{}_4\widetilde{N}^{\alpha}_a(c) \wedge 1_X(c)]) = \bigvee_{c\in X} {}_4\widetilde{N}^{\alpha}_a(c)$.

4) $\mathbb{L}_{IV}(1_X)(a) = \bigwedge_{c\in W_2}([1 - {}_4\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c)) = (\bigwedge_{c\in X}([1 - {}_4\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c))) \bigwedge(\bigwedge_{c\notin X}([1 - {}_4\widetilde{N}^{\alpha}_a(c)] \bigvee 1_X(c))) = 1 \bigwedge(\bigwedge_{c\notin X}([1 - {}_4\widetilde{N}^{\alpha}_a(c)] \bigvee 0)) = \bigwedge_{c\notin X}(1 - {}_4\widetilde{N}^{\alpha}_a(c))$.

Depending on Pawlak's rough set pattern, the fuzzy covering depending on the rough set pattern over 2-finite sets is provided by using the idea of an α-neighborhood.

Definition 4.27. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \subseteq W_2$, the fourth upper $\overline{\mathbb{U}_{IV}}$ and lower $\overline{\mathbb{L}_{IV}}$ approximations, respectively, are defined as:*

$\overline{\mathbb{U}_{IV}}(\mathscr{B}) = \{x \in W_1 : \overline{{}_4\widetilde{N}^{\alpha}_x}(y) \cap \mathscr{B} \neq \phi\}$.
$\overline{\mathbb{L}_{IV}}(\mathscr{B}) = \{x \in W_1 : \overline{{}_4\widetilde{N}^{\alpha}_x}(y) \subseteq \mathscr{B}\}$.
$\mathscr{B}$ is definable, except if $\overline{\mathbb{U}_{IV}}(\mathscr{B}) \neq \overline{\mathbb{L}_{IV}}(\mathscr{B})$. $\mathscr{B}$ is called the fourth kind of fuzzy covering depending on the rough set.

Example 4.28. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. From Example 3.17, we have $\overline{{}_4\widetilde{N}^{0.4}_{u_1}} = \overline{{}_4\widetilde{N}^{0.4}_{u_2}} = \overline{{}_4\widetilde{N}^{0.4}_{u_3}} = \overline{{}_4\widetilde{N}^{0.4}_{u_4}} = \overline{{}_4\widetilde{N}^{0.4}_{u_5}} = \overline{{}_4\widetilde{N}^{0.4}_{u_6}} = \{y_1, y_2, y_3, y_4\}$.*

1) Let $\mathscr{B} = \{y_1, y_4\}$. Then $\overline{\mathbb{U}_{IV}}(\mathscr{B}) = \{u_1, u_2, u_3, u_4, u_5, u_6\} = W_1$. and $\overline{\mathbb{L}_{IV}}(\mathscr{B}) = \phi$.
2) Let $\mathscr{B} = \{y_3\}$. Then $\overline{\mathbb{U}_{IV}}(\mathscr{B}) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. and $\overline{\mathbb{L}_{IV}}(\mathscr{B}) = \phi$.

We provide a definition of the accuracy degree to examine the four kinds of fuzzy α-covering depending on the rough set patterns introduced in this research.

Definition 4.29. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \in \mathscr{F}(W_2)$, the accuracy degree of $\mathscr{B}$ is defined as*

1) $\varsigma^{\alpha}_{I}(\mathscr{B}) = \frac{\Sigma_{x\in W_1}\mathbb{L}_{I}(\mathscr{B})(x)}{\Sigma_{x\in W_1}\mathbb{U}_{I}(\mathscr{B})(x)}$,
2) $\varsigma^{\alpha}_{II}(\mathscr{B}) = \frac{\Sigma_{x\in W_1}\mathbb{L}_{II}(\mathscr{B})(x)}{\Sigma_{x\in W_1}\mathbb{U}_{II}(\mathscr{B})(x)}$,
3) $\varsigma^{\alpha}_{III}(\mathscr{B}) = \frac{\Sigma_{x\in W_1}\mathbb{L}_{III}(\mathscr{B})(x)}{\Sigma_{x\in W_1}\mathbb{U}_{III}(\mathscr{B})(x)}$,
4) $\varsigma^{\alpha}_{IV}(\mathscr{B}) = \frac{\Sigma_{x\in W_1}\mathbb{L}_{IV}(\mathscr{B})(x)}{\Sigma_{x\in W_1}\mathbb{U}_{IV}(\mathscr{B})(x)}$.

Example 4.30 illustrates the idea of the definition above.

Example 4.30. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. Using the information in Examples 4.2, 4.9, 4.16, and 4.23, then the accuracy degrees for $\mathscr{B} = \frac{0.3}{y_1} + \frac{0.7}{y_2} + \frac{0.5}{y_3} + \frac{0.2}{y_4}$ are:*

1) $\varsigma^{0.4}_{I}(\mathscr{B}) = \frac{0.2+0.3+0.2+0.3+0.3+0.2}{0.7+0.7+0.7+0.6+0.7+0.7} = 0.37$,

2) $\varsigma_{II}^{0.4}(\mathscr{B}) = \frac{0.2+0.3+0.2+0.3+0.3+0.2}{0.7+0.7+0.7+0.6+0.7+0.7} = 0.37,$
3) $\varsigma_{III}^{0.4}(\mathscr{B}) = \frac{0.2+0.2+0.2+0.2+0.2+0.2}{0.7+0.7+0.7+0.7+0.7+0.7} = 0.29,$
4) $\varsigma_{IV}^{0.4}(\mathscr{B}) = \frac{0.2+0.2+0.2+0.2+0.2+0.2}{0.7+0.7+0.7+0.7+0.7+0.7} = 0.29.$

We provide a definition of the accuracy measure (am) for the four kinds of fuzzy α-covering depending on the rough set patterns introduced in Definitions 4.6, 4.13,4.20, and 4.27.

Definition 4.31. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$. *For each* $\mathscr{B} \in W_2$, *the am of* $\mathscr{B}$ *is defined as*

1) $\varrho_I^{\alpha}(\mathscr{B}) = \frac{|\overline{\mathbb{L}_I(\mathscr{B})}|}{|\overline{\mathbb{U}_I(\mathscr{B})}|},$
2) $\varrho_{II}^{\alpha}(\mathscr{B}) = \frac{|\overline{\mathbb{L}_{II}(\mathscr{B})}|}{|\overline{\mathbb{U}_{II}(\mathscr{B})}|},$
3) $\varrho_{III}^{\alpha}(\mathscr{B}) = \frac{|\overline{\mathbb{L}_{III}(\mathscr{B})}|}{|\overline{\mathbb{U}_{III}(\mathscr{B})}|},$
4) $\varrho_{IV}^{\alpha}(\mathscr{B}) = \frac{|\overline{\mathbb{L}_{IV}(\mathscr{B})}|}{|\overline{\mathbb{U}_{IV}(\mathscr{B})}|}$, where $|\mathscr{B}|$ represents the cardinality of $\mathscr{B}$.

Example 4.32 illustrates the definition above.

Example 4.32. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$. *Using the information in Examples 4.7,4.14,4.21, and 4.28 we have*

1) For $\mathscr{B} = \{y_1, y_4\}$,
   (a) $\varrho_I^{\alpha}(\mathscr{B}) = \frac{0}{6} = 0,$
   (b) $\varrho_{II}^{\alpha}(\mathscr{B}) = \frac{0}{6} = 0,$
   (c) $\varrho_{III}^{\alpha}(\mathscr{B}) = \frac{0}{6} = 0,$
   (d) $\varrho_{III}^{\alpha}(\mathscr{B}) = \frac{0}{6} = 0.$
2) For $\mathscr{B} = \{y_3\}$,
   (a) $\varrho_I^{\alpha}(\mathscr{B}) = \frac{0}{4} = 0,$
   (b) $\varrho_{II}^{\alpha}(\mathscr{B}) = \frac{0}{4} = 0,$
   (c) $\varrho_{III}^{\alpha}(\mathscr{B}) = \frac{0}{6} = 0,$
   (d) $\varrho_{III}^{\alpha}(\mathscr{B}) = \frac{0}{6} = 0.$

Definition 4.33. *Let* $(W_1, W_2, \mathfrak{V})$ *be a FαCAS for some* $\alpha_1 \in (0, 1]$ *and* $f \in Onto(W_1, W_2)$. *For each* $\mathscr{B} \in \mathscr{F}(W_2)$ *and* $0 < \alpha_1 < \alpha$, *the fuzzy* α_1*-covering depending on the rough set is defined as*

1) ${}^*\mathbb{L}_I(\mathscr{B}) = \bigwedge_{z\in W_2}([1 - {}_1\widetilde{N}_y^{\alpha_1}(z)] \bigvee \mathscr{B}(z)),$ ${}^*\mathbb{U}_I(\mathscr{B})(y) = \bigvee_{z\in W_2}({}_1\widetilde{N}_y^{\alpha_1}(z) \bigwedge \mathscr{B}(z)).$
2) ${}^*\mathbb{L}_{II}(\mathscr{B}) = \bigwedge_{z\in W_2}([1 - {}_2\widetilde{N}_y^{\alpha_1}(z)] \bigvee \mathscr{B}(z)),$ ${}^*\mathbb{U}_{II}(\mathscr{B})(y) = \bigvee_{z\in W_2}({}_2\widetilde{N}_y^{\alpha_1}(z) \bigwedge \mathscr{B}(z)).$
3) ${}^*\mathbb{L}_{III}(\mathscr{B}) = \bigwedge_{z\in W_2}([1 - {}_3\widetilde{N}_y^{\alpha_1}(z)] \bigvee \mathscr{B}(z)),$ ${}^*\mathbb{U}_{III}(\mathscr{B})(y) = \bigvee_{z\in W_2}({}_3\widetilde{N}_y^{\alpha_1}(z) \bigwedge \mathscr{B}(z)).$
4) ${}^*\mathbb{L}_{IV}(\mathscr{B}) = \bigwedge_{z\in W_2}([1 - {}_4\widetilde{N}_y^{\alpha_1}(z)] \bigvee \mathscr{B}(z)),$ ${}^*\mathbb{U}_{IV}(\mathscr{B})(y) = \bigvee_{z\in W_2}({}_4\widetilde{N}_y^{\alpha_1}(z) \bigwedge \mathscr{B}(z)).$

4.5. The relationships among the four patterns

In this subsection, we introduce the connections between the patterns which are provided above. Moreover, it gives a comparison of the patterns defined by Yao [42] and Yang and Hu [43].

The following propositions explain the relationships among different types of fuzzy α-neighborhoods.

Proposition 4.34. *Let $(W_1, W_2, \mathfrak{V})$ be a $F\alpha CAS$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $u_i, u_j \in W_1$, the following statements hold:*

1) $\bigcap md^{\alpha}_{\mathfrak{V}}(u_i)(u_j) = {}_1\widetilde{N}^{\alpha}_{u_i}$.
2) $\bigcup MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j) = {}_4\widetilde{N}^{\alpha}_{u_i}$.
3) If $md^{\alpha}_{\mathfrak{V}}(u_i)(u_j) \neq MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$, then ${}_2\widetilde{N}^{\alpha}_{\mathfrak{V}}(u_i) \subseteq {}_3\widetilde{N}^{\alpha}_{\mathfrak{V}}(u_i)$.
4) $md^{\alpha}_{\mathfrak{V}}(u_i)(u_j) \subseteq MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$.
5) ${}_1\widetilde{N}^{\alpha}_{u_i} \subseteq {}_2\widetilde{N}^{\alpha}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha}_{u_i}$.
6) ${}_1\widetilde{N}^{\alpha}_{u_i} \subseteq {}_3\widetilde{N}^{\alpha}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha}_{u_i}$.

Proof. 1) From Definition 3.11, ${}_1\widetilde{N}^{\alpha}_{u_i} = \bigcap\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} = \bigcap N^{\alpha}_{\mathfrak{V}}(u_i)(u_j) = \bigcap md^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$ by Proposition 3.10.

2) From Definition 3.11, ${}_4\widetilde{N}^{\alpha}_{u_i} = \bigcup\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} = \bigcup MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$.

3) Let $I(f(\epsilon(x)), f(\epsilon(v))) \in {}_2\widetilde{N}^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$. Then $I(f(\epsilon(x)), f(\epsilon(v))) \in md^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$. However, by the assumption $md^{\alpha}_{\mathfrak{V}}(u_i)(u_j) \neq MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$, $I(f(\epsilon(x)), f(\epsilon(v))) \subseteq I(f(\epsilon_1(x)), f(\epsilon_1(v)))$, where $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$ by Proposition 3.8. Therefore, $I(f(\epsilon(x)), f(\epsilon(v))) \in \bigcap MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$. That is, $I(f(\epsilon(x)), f(\epsilon(v))) \in {}_3\widetilde{N}^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$. Hence, ${}_2\widetilde{N}^{\alpha}_{\mathfrak{V}}(u_i)(u_j) \subseteq {}_3\widetilde{N}^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$.

4) This is clear from Proposition 3.8.

5) Let ${}_1\widetilde{N}^{\alpha}_{u_i} = \bigcap\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} = \bigcap N^{\alpha}_{\mathfrak{V}}(u_i)(u_j) = \bigcap md^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$ by Proposition 3.10. However, $\bigcap md^{\alpha}_{\mathfrak{V}}(u_i)(u_j) \subseteq \bigcup md^{\alpha}_{\mathfrak{V}}(u_i)(u_j) = \bigcup\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v)) \in md^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} = {}_2\widetilde{N}^{\alpha}_{u_i}$. Therefore, ${}_1\widetilde{N}^{\alpha}_{u_i} \subseteq {}_2\widetilde{N}^{\alpha}_{u_i}$. Moreover, by Proposition 3.8, since $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \subseteq I(f(\epsilon_2(x)), f(\epsilon_2(v)))$ where $I(f(\epsilon_1(x)), f(\epsilon_1(v))) \in md^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$ and $I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$, $\bigcup\{I(f(\epsilon_1(x)), f(\epsilon_1(v))) : I(f(\epsilon_1(x)), f(\epsilon_1(v))) \in md^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} \subseteq \bigcup\{I(f(\epsilon_2(x)), f(\epsilon_2(v))) : I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\}$. That is, ${}_2\widetilde{N}^{\alpha}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha}_{u_i}$. Hence, ${}_1\widetilde{N}^{\alpha}_{u_i} \subseteq {}_2\widetilde{N}^{\alpha}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha}_{u_i}$.

6) By Proposition 3.8, since $I(f(\epsilon(x)), f(\epsilon(v))) \subseteq I(f(\epsilon_2(x)), f(\epsilon_2(v)))$ where $I(f(\epsilon(x)), f(\epsilon(v))) \in N^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$ and $I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)$. Therefore, $\bigcap\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} \subseteq \bigcap\{I(f(\epsilon_2(x)), f(\epsilon_2(v))) : I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\}$. That is, ${}_1\widetilde{N}^{\alpha}_{u_i} \subseteq {}_3\widetilde{N}^{\alpha}_{u_i}$. Moreover, $\bigcap\{I(f(\epsilon_2(x)), f(\epsilon_2(v))) : I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\} \subseteq \bigcup\{I(f(\epsilon_2(x)), f(\epsilon_2(v))) : I(f(\epsilon_2(x)), f(\epsilon_2(v))) \in MD^{\alpha}_{\mathfrak{V}}(u_i)(u_j)\}$. That is, ${}_3\widetilde{N}^{\alpha}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha}_{u_i}$. Hence, ${}_1\widetilde{N}^{\alpha}_{u_i} \subseteq {}_3\widetilde{N}^{\alpha}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha}_{u_i}$.

Proposition 4.35. *Let $(W_1, W_2, \mathfrak{V})$ be a $F\alpha CAS$ and $f \in Onto(W_1, W_2)$. If $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha$. For any $u_i, u_j \in W_1, i, and j = \{1, 2, ..., n\}$, we have*

1) ${}_1\widetilde{N}^{\alpha_2}_{u_i} \supseteq {}_1\widetilde{N}^{\alpha_1}_{u_i}$.
2) ${}_2\widetilde{N}^{\alpha_2}_{u_i} \subseteq {}_2\widetilde{N}^{\alpha_1}_{u_i}$.
3) ${}_4\widetilde{N}^{\alpha_2}_{u_i} \subseteq {}_4\widetilde{N}^{\alpha_1}_{u_i}$.

Proof. We will prove only 1), and the proof for the others is similar.

1) If $u_i \in W_1$ and $\alpha_1 \leq \alpha_2$, then ${}_1\widetilde{N}_{u_i}^{\alpha_1} = \bigcap\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N_{\mathfrak{B}}^{\alpha_1}(u_i)(u_j)\} \subseteq \bigcap\{I(f(\epsilon(x)), f(\epsilon(v))) : I(f(\epsilon(x)), f(\epsilon(v))) \in N_{\mathfrak{B}}^{\alpha_2}(u_i)(u_j)\} = {}_1\widetilde{N}_{u_i}^{\alpha_2}$. Therefore, ${}_1\widetilde{N}_{u_i}^{\alpha_1} \subseteq {}_1\widetilde{N}_{u_i}^{\alpha_2}$ for any $u_i, u_j \in W_1$.

The next proposition explains the relationships among four kinds of FαCAS, which are given in Definitions 4.1, 4.8, 4.15, and 4.22 with the four types of the fuzzy α_1-covering based on the rough set in Definition 4.33.

Proposition 4.36. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha_1 \in (0, 1]$ and $f \in Onto(W_1, W_2)$. If $\alpha_1 \leq \alpha$, then for any $\mathscr{B} \in \mathscr{F}(W_2)$, we have*

1) $\mathbb{L}_I(\mathscr{B}) \subseteq {}^*\mathbb{L}_I(\mathscr{B})$, ${}^*\mathbb{U}_I(\mathscr{B}) \subseteq \mathbb{U}_I(\mathscr{B})$.
2) $\mathbb{L}_{II}(\mathscr{B}) \subseteq {}^*\mathbb{L}_{II}(\mathscr{B})$, ${}^*\mathbb{U}_{II}(\mathscr{B}) \subseteq \mathbb{U}_{II}(\mathscr{B})$.
3) $\mathbb{L}_{III}(\mathscr{B}) \subseteq {}^*\mathbb{L}_{III}(\mathscr{B})$, ${}^*\mathbb{U}_{III}(\mathscr{B}) \subseteq \mathbb{U}_{III}(\mathscr{B})$.
4) $\mathbb{L}_{IV}(\mathscr{B}) \subseteq {}^*\mathbb{L}_{IV}(\mathscr{B})$, ${}^*\mathbb{U}_{IV}(\mathscr{B}) \subseteq \mathbb{U}_{IV}(\mathscr{B})$.

Proof. We will prove only 1), and the proof for the others is similar.

1) If $\alpha_1 \leq \alpha$, then, by Proposition 4.35, we have that ${}_1\widetilde{N}_{u_i}^{\alpha_1} \subseteq {}_1\widetilde{N}_{u_i}^{\alpha}$. Thus, we have $\mathbb{L}_I(\mathscr{B}) \subseteq {}^*\mathbb{L}_I(\mathscr{B}), {}^*\mathbb{U}_I(\mathscr{B}) \subseteq \mathbb{U}_I(\mathscr{B})$ by their definitions.

5. Some topological features of fuzzy α-covering depending on the rough set patterns

Here, we investigate some topological characteristics of the fuzzy α-covering depending on the rough set patterns.

Definition 5.1. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS with $\mathfrak{B} = \{\epsilon_1, \epsilon_1, ..., \epsilon_m\}$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $u \in W_1$, the kinds of sub-base for the fuzzy topology are*

1) $S_1 = \{f(\epsilon) \in f(\mathfrak{B}) : {}_1\widetilde{N}_u^{\alpha} \subseteq f(\epsilon)\}$,
2) $S_2 = \{f(\epsilon) \in f(\mathfrak{B}) : {}_2\widetilde{N}_u^{\alpha} \subseteq f(\epsilon)\}$,
3) $S_3 = \{f(\epsilon) \in f(\mathfrak{B}) : {}_3\widetilde{N}_u^{\alpha} \subseteq f(\epsilon)\}$,
4) $S_4 = \{f(\epsilon) \in f(\mathfrak{B}) : {}_4\widetilde{N}_u^{\alpha} \subseteq f(\epsilon)\}$.

Example 5.2. *Let $(W_1, W_2, \mathfrak{B})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$ in Example 3.12. Then we have the following sub-bases*

$S_1 = \{{}_1\widetilde{N}_{u_1}^{0.4}, {}_1\widetilde{N}_{u_2}^{0.4}, {}_1\widetilde{N}_{u_4}^{0.4}, {}_1\widetilde{N}_{u_6}^{0.4}\}$,
$S_2 = \{{}_2\widetilde{N}_{u_1}^{0.4}, {}_2\widetilde{N}_{u_2}^{0.4}, {}_2\widetilde{N}_{u_4}^{0.4}, {}_2\widetilde{N}_{u_6}^{0.4}\}$,
$S_3 = \{{}_3\widetilde{N}_{u_1}^{0.4}, {}_3\widetilde{N}_{u_2}^{0.4}, {}_3\widetilde{N}_{u_4}^{0.4}, {}_3\widetilde{N}_{u_6}^{0.4}\}$,
$S_4 = \{{}_4\widetilde{N}_{u_1}^{0.4}, {}_4\widetilde{N}_{u_2}^{0.4}, {}_4\widetilde{N}_{u_4}^{0.4}, {}_4\widetilde{N}_{u_6}^{0.4}\}$.
Therefore, the bases will be:
$B_1 = \{1, {}_1\widetilde{N}_{u_1}^{0.4}, {}_1\widetilde{N}_{u_2}^{0.4}, {}_1\widetilde{N}_{u_4}^{0.4}, {}_1\widetilde{N}_{u_6}^{0.4}, \frac{0.4}{y_1} + \frac{0.9}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}, \frac{0.4}{y_1} + \frac{0.6}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{0.4}{y_3} + \frac{0.7}{y_4}\}$,
$B_2 = \{1, {}_2\widetilde{N}_{u_1}^{0.4}, {}_2\widetilde{N}_{u_2}^{0.4}, {}_2\widetilde{N}_{u_4}^{0.4}, {}_2\widetilde{N}_{u_6}^{0.4}, \frac{0.4}{y_1} + \frac{0.9}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}, \frac{0.4}{y_1} + \frac{0.6}{y_2} + \frac{0.1}{y_3} + \frac{0.7}{y_4}\}$,

$B_3 = \{1, {}_3\widetilde{N}^{0.4}_{u_1}, {}_3\widetilde{N}^{0.4}_{u_2}, {}_3\widetilde{N}^{0.4}_{u_6}\}$,
$B_4 = \{1, {}_4\widetilde{N}^{0.4}_{u_1}, {}_4\widetilde{N}^{0.4}_{u_2}, {}_4\widetilde{N}^{0.4}_{u_4}, {}_4\widetilde{N}^{0.4}_{u_6}\}$.
Hence, the topologies will be
$\tau_1 = \{0, 1, {}_1\widetilde{N}^{0.4}_{u_1}, {}_1\widetilde{N}^{0.4}_{u_2}, {}_1\widetilde{N}^{0.4}_{u_4}, {}_1\widetilde{N}^{0.4}_{u_6}, \frac{1}{y_1} + \frac{1}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}, \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}, \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{0.4}{y_3} + \frac{1}{y_4}, \frac{0.7}{y_1} + \frac{0.9}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}, \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{0.4}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{0.9}{y_2} + \frac{0.4}{y_3} + \frac{1}{y_4}, \frac{0.7}{y_1} + \frac{0.9}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}\}$,
$\tau_2 = \{0, 1, {}_2\widetilde{N}^{0.4}_{u_1}, {}_2\widetilde{N}^{0.4}_{u_2}, {}_2\widetilde{N}^{0.4}_{u_4}, {}_2\widetilde{N}^{0.4}_{u_6}, \frac{1}{y_1} + \frac{0.9}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}, \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{0.6}{y_3} + \frac{0.7}{y_4}, \frac{0.7}{y_1} + \frac{0.9}{y_2} + \frac{1}{y_3} + \frac{0.7}{y_4}\}$,
$\tau_3 = \{0, 1, {}_3\widetilde{N}^{0.4}_{u_1}, {}_3\widetilde{N}^{0.4}_{u_2}, {}_3\widetilde{N}^{0.4}_{u_6}\}$,
$\tau_4 = \{0, 1, {}_4\widetilde{N}^{0.4}_{u_1}, {}_4\widetilde{N}^{0.4}_{u_2}, {}_4\widetilde{N}^{0.4}_{u_6}\}$.
where $0 = \frac{0}{y_1} + \frac{0}{y_2} + \frac{0}{y_3} + \frac{0}{y_4}, 1 = \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4} \in \mathscr{F}(W_2)$ are two fuzzy sets for any $y_i \in W_2, i \in \{1, 2, 3, 4\}$.

Definition 5.3. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS with $\mathfrak{V} = \{\epsilon_1, \epsilon_1, ..., \epsilon_m\}$ for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. The complement of the open set which is an element in τ is called a closed set and is denoted by $\mathbb{C}_r = \{f(\epsilon) \in f(\mathfrak{V}) : (f(\epsilon))^c \in \tau_r\}$, where $r \in \{1, 2, 3, 4\}$.*

Example 5.4. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$ in Example 5.2. The closed sets will be*

$\mathbb{C}_1 = \{1, 0, \frac{0}{y_1} + \frac{0.1}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}, \frac{0.6}{y_1} + \frac{0}{y_2} + \frac{0.9}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0.4}{y_2} + \frac{0}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0.6}{y_3} + \frac{0}{y_4}, \frac{0}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}, \frac{0}{y_1} + \frac{0.1}{y_2} + \frac{0}{y_3} + \frac{0}{y_4}, \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.6}{y_3} + \frac{0}{y_4}, \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0}{y_3} + \frac{0}{y_4}, \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.6}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}\}$,
$\mathbb{C}_2 = \{1, 0, \frac{0}{y_1} + \frac{0.1}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}, \frac{0.6}{y_1} + \frac{0}{y_2} + \frac{0.9}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0.4}{y_2} + \frac{0}{y_3} + \frac{0.3}{y_4}, \frac{0}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}, \frac{0}{y_1} + \frac{0.1}{y_2} + \frac{0}{y_3} + \frac{0}{y_4}, \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0.3}{y_4}, \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0}{y_3} + \frac{0.3}{y_4}\}$,
$\mathbb{C}_3 = \{1, 0, \frac{0}{y_1} + \frac{0.1}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}, \frac{0}{y_1} + \frac{0}{y_2} + \frac{0.3}{y_3} + \frac{0}{y_4}, \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0.6}{y_3} + \frac{0}{y_4}\}$.
$\mathbb{C}_4 = \{1, 0, \frac{0}{y_1} + \frac{0.1}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}, \frac{0}{y_1} + \frac{0}{y_2} + \frac{0.3}{y_3} + \frac{0}{y_4}, \frac{0}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}\}$.

Definition 5.5. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $\mathscr{B} \in \mathscr{F}(W_2)$ and $s \in \{1, 2, 3, 4\}$, the fuzzy interior and fuzzy closure, respectively, are defined as*

$I_s(\mathscr{B}) = \bigcup\{P \in \tau_s : P \subseteq \mathscr{B}\}$,
$C_s(\mathscr{B}) = \bigcap\{G \in \mathbb{C}_s : \mathscr{B} \subseteq G\}$.

Example 5.6. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$ in Example 5.2. If $\mathscr{B} = \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$, then for $s = \{1\}$, we have the following:*

1) The fuzzy interior of $\mathscr{B}$ is $I_1(\mathscr{B}) = \frac{0.7}{y_1} + \frac{1}{y_2} + \frac{0.6}{y_3} + \frac{1}{y_4}$.
2) The fuzzy closure of $\mathscr{B}$ is $C_1(\mathscr{B}) = \phi$.

Theorem 5.7. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For any $\mathscr{Z}, \mathscr{Z}_1, \mathscr{Z}_2 \in \mathscr{F}(W_2)$ and $s \in \{1, 2, 3, 4\}$, we have the following properties:*

1) $I_s(\phi) = \phi, I_s(W_2) = W_2$.
2) $I_s(\mathscr{Z}) \subseteq \mathscr{Z}$.
3) $\mathscr{Z}$ is an open set $\Longleftrightarrow I_s(\mathscr{Z}) = \mathscr{Z}$.
4) $I_s(I_s(\mathscr{Z})) = I_s(\mathscr{Z})$.
5) If $\mathscr{Z}_1 \subseteq \mathscr{Z}_2$, then $I_s(\mathscr{Z}_1) \subseteq I_s(\mathscr{Z}_2)$.
6) $I_s(\mathscr{Z}_1 \cap \mathscr{Z}_2) = I_s(\mathscr{Z}_1) \cap I_s(\mathscr{Z}_2)$.
7) $I_s(\mathscr{Z}_1) \cup I_s(\mathscr{Z}_2) \subseteq I_s(\mathscr{Z}_1 \cup \mathscr{Z}_2)$.

8) $C_s(\phi) = \phi, C_s(W_1) = W_1$.
9) $\mathscr{Z} \subseteq C_s(\mathscr{Z})$.
10) $\mathscr{Z}$ is a closed set $\Longleftrightarrow C_s(\mathscr{Z}) = \mathscr{Z}$.
11) $C_s(C_s(\mathscr{Z})) = C_s(\mathscr{Z})$.
12) If $\mathscr{Z}_1 \subseteq \mathscr{Z}_2$, then $C_s(\mathscr{Z}_1) \subseteq C_s(\mathscr{Z}_2)$.
13) $C_s(\mathscr{Z}_1 \cap \mathscr{Z}_2) \subseteq C_s(\mathscr{Z}_1) \cap C_s(\mathscr{Z}_2)$.
14) $C_s(\mathscr{Z}_1) \cup C_s(\mathscr{Z}_2) = C_s(\mathscr{Z}_1 \cup \mathscr{Z}_2)$.

Proof. The following proof is for $s = 1$, and the same proof will be applicable to the others.

1) From Definition 5.5, we have
$I_1(\phi) = \bigcup\{P \in \tau_1 : P \subseteq \phi\} = \phi$,
$I_1(W2) = \bigcup\{P \in \tau_1 : P \subseteq W_2\} = W_2$.
2) Definition 5.5 illustrates the proof.
3) If $\mathscr{Z}$ is an open set, then the biggest open set contained in $\mathscr{Z}$ is $\mathscr{Z}$ itself. Therefore, $I_1(\mathscr{Z}) = \mathscr{Z}$. Moreover, let $I_1(\mathscr{Z}) = \mathscr{Z}$. As $I_1(\mathscr{Z})$ is an open set, $\mathscr{Z}$ is an open set.
4) From Definition 5.5, for any $\mathscr{Z} \in \mathscr{F}(W_2)$, we have
$I_1(I_1(\mathscr{Z})) = \bigcup\{P \in \tau_1 : P \subseteq I_1(\mathscr{Z})\} = \bigcup\{P \in \tau_1 : P \subseteq \bigcup[P \in \tau_1 : P \subseteq \mathscr{Z}]\} = \bigcup\{P \in \tau_1 : P \subseteq \mathscr{Z}\} = I_1(\mathscr{Z})$.
5) From Definition 5.5, for any $\mathscr{Z}_1, \mathscr{Z}_2 \in \mathscr{F}(W_2)$ we have
$\mathscr{Z}_1 \subseteq \mathscr{Z}_2 \Rightarrow I_1(\mathscr{Z}_1) = \bigcup\{P \in \tau_1 : P \subseteq \mathscr{Z}_1\} \subseteq \bigcup\{P \in \tau_1 : P \subseteq \mathscr{Z}_2\} = I_1(\mathscr{Z}_2)$.
6) From Definition 5.5, for any $\mathscr{Z}_1, \mathscr{Z}_2 \in \mathscr{F}(W_2)$ we have
$I_1(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \bigcup\{P \in \tau_1 : P \subseteq (\mathscr{Z}_1 \cap \mathscr{Z}_2)\} = \bigcup\{P \in \tau_1 : P \subseteq \mathscr{Z}_1$ and $P \subseteq \mathscr{Z}_2\} = (\bigcup\{P \in \tau_1 : P \subseteq \mathscr{Z}_1\}) \bigcap (\bigcup\{P \in \tau_1 : P \subseteq \mathscr{Z}_2\}) = I_1(\mathscr{Z}_1) \bigcap I_1(\mathscr{Z}_2)$.
7) Since $\mathscr{Z}_1 \subseteq \mathscr{Z}_1 \bigcup \mathscr{Z}_2$, by 5), we have $I_1(\mathscr{Z}_1) \subseteq I_1(\mathscr{Z}_1 \bigcup \mathscr{Z}_2)$. Moreover, as $\mathscr{Z}_2 \subseteq \mathscr{Z}_1 \bigcup \mathscr{Z}_2$, by 5), we have $I_1(\mathscr{Z}_2) \subseteq I_1(\mathscr{Z}_1 \bigcup \mathscr{Z}_2)$. Thus, $I_1(\mathscr{Z}_2) \bigcup I_1(\mathscr{Z}_2) \subseteq I_1(\mathscr{Z}_1 \bigcup \mathscr{Z}_2)$.
8) From Definition 5.5, we have $C_1(\phi) = \bigcap\{G \in \mathbb{C}_1 : \phi \subseteq G\} = \phi$, $C_1(W_1) = \bigcap\{G \in \mathbb{C}_1 : W_1 \subseteq G\} = W_1$.
9) Definition 5.5 illustrates the proof.
10) If $\mathscr{Z}$ is a closed set, then the smallest closed set containing $\mathscr{Z}$ is $\mathscr{Z}$ itself. Therefore, $C_1(\mathscr{Z}) = \mathscr{Z}$. Moreover, let $C_1(\mathscr{Z}) = \mathscr{Z}$. As $C_1(\mathscr{Z})$ is a closed set, $\mathscr{Z}$ is a closed set.
11) By Definition 5.5, for any $\mathscr{Z} \in \mathscr{F}(W_1)$, we have $C_1(C_1(\mathscr{Z})) = \bigcap\{G \in \mathbb{C}_1 : \mathbb{C}_1(\mathscr{Z}) \subseteq G\} = \bigcap\{G \in \mathbb{C}_1 : \bigcap[G \in \mathbb{C} : W_1 \subseteq G] \subseteq G\} = \bigcap\{G \in \mathbb{C}_1 : W_1 \subseteq G\} = C_1(\mathscr{Z})$.
12) From Definition 5.5, for any $\mathscr{Z}_1, \mathscr{Z}_2 \in \mathscr{F}(W_1)$, we have $\mathscr{Z}_1 \subseteq \mathscr{Z}_2 \Rightarrow C_1(\mathscr{Z}_1) = \bigcap\{G \in \mathbb{C}_1 : \mathscr{Z}_1 \subseteq G\} \subseteq \bigcap\{G \in \mathbb{C}_1 : \mathscr{Z}_2 \subseteq G\} = C_1(\mathscr{Z}_2)$.
13) Since $\mathscr{Z}_1 \bigcap \mathscr{Z}_2 \subseteq \mathscr{Z}_1$, by (12), we have $C_1(\mathscr{Z}_1 \bigcap \mathscr{Z}_2) \subseteq C_1(\mathscr{Z}_1)$. Moreover, as $\mathscr{Z}_1 \bigcap \mathscr{Z}_2 \subseteq \mathscr{Z}_2$, by (12), we have $C_1(\mathscr{Z}_1 \bigcap \mathscr{Z}_2) \subseteq C_1(\mathscr{Z}_2)$. Hence, $C_1(\mathscr{Z}_1 \bigcap \mathscr{Z}_2) \subseteq C_1(\mathscr{Z}_1) \bigcap C_1(\mathscr{Z}_2)$.
14) By Definition 5.5, for any $\mathscr{Z}_1, \mathscr{Z}_2 \in \mathscr{F}(W_1)$, we have $C_1(\mathscr{Z}_1) \bigcup C_1(\mathscr{Z}_2) = (\bigcap\{G \in \mathbb{C}_1 : \mathscr{Z}_1 \subseteq G\}) \bigcup (\bigcap\{G \in \mathbb{C}_1 : \mathscr{Z}_2 \subseteq G\}) = \bigcap\{G \in \mathbb{C}_1 : \mathscr{Z}_1 \subseteq G$ or $\mathscr{Z}_2 \subseteq G\} = \bigcap\{G \in \mathbb{C}_1 : (\mathscr{Z}_1 \bigcup \mathscr{Z}_2) \subseteq G = C_1(\mathscr{Z}_1 \bigcup \mathscr{Z}_2)$.

Definition 5.8. *Let $(W_1, W_2, \mathfrak{V})$ be a FαCAS for some $\alpha \in (0, 1]$ and $f \in Onto(W_1, W_2)$. For each $\mathscr{B} \in \mathscr{F}(W_2)$, the fuzzy boundary of $\mathscr{B}$ will be defined as $B_s(\mathscr{B}) = C_s(\mathscr{B}) \bigcap C_s(\mathscr{B}^c)$, where $s \in \{1, 2, 3, 4\}$.*

Example 5.9. *From the information in Examples 5.4 and 5.6, we have* $C_1(\mathscr{B}) = \phi$*. Moreover,* $\mathscr{B}^c = \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}$*. Then* $C_1(\mathscr{B}^c) = 1 \cap \frac{0.6}{y_1} + \frac{0}{y_2} + \frac{0.9}{y_3} + \frac{0.3}{y_4} \cap \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0.6}{y_3} + \frac{0}{y_4} \cap \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.6}{y_3} + \frac{0}{y_4} \cap \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0.3}{y_4} \cap \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.6}{y_3} + \frac{0.3}{y_4} \cap \frac{0.3}{y_1} + \frac{0.1}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4} = \frac{0.3}{y_1} + \frac{0}{y_2} + \frac{0.4}{y_3} + \frac{0}{y_4}$*. Therefore,* $B_s(\mathscr{B}) = C_1(\mathscr{B}) \cap C_1(\mathscr{B}^c) = \phi \cap (\frac{0.3}{y_1} + \frac{0}{y_1} + \frac{0.4}{y_1} + \frac{0}{y_1}) = \phi$*.*

Theorem 5.10. *Let* $(W_1, W_2, \mathfrak{B})$ *be a F*α*CAS for some* $\alpha \in (0, 1]$*, and* $f \in Onto(W_1, W_2)$*. For each* $\mathscr{Z} \in \mathscr{F}(W_2)$ *and* $s \in \{1, 2, 3, 4\}$*, we have*

1) $C_s(\mathscr{Z}) = \mathscr{Z} \bigcup B_s(\mathscr{Z})$.
2) $I_s(\mathscr{Z}) = \mathscr{Z} \bigcap (B_s(\mathscr{Z}))^c$.

Proof. We will prove the case for $s = 1$, and the proof will be the same for the others.

1) For any $\mathscr{Z} \in \mathscr{F}(W_2)$, we have $\mathscr{Z} \subseteq C_1(\mathscr{Z})$, and thus $\mathscr{Z} \cup B_1(\mathscr{Z}) = \mathscr{Z} \cup (C_1(\mathscr{Z}) \cap C_1(\mathscr{Z}^c)) = (\mathscr{Z} \cup C_1(\mathscr{Z})) \cap (\mathscr{Z} \cup C_1(\mathscr{Z}^c)) = C_1(\mathscr{Z}) \cap (\mathscr{Z} \cup C_1(\mathscr{Z}^c)) = C_1(\mathscr{Z})$.
2) For any $\mathscr{Z} \in \mathscr{F}(W_2)$, we have $(C_1(\mathscr{Z}))^c = I_1(\mathscr{Z}^c)$. Then $\mathscr{Z} \cap (B_1(\mathscr{Z}))^c = \mathscr{Z} \cap (C_1(\mathscr{Z}) \cap C_1(\mathscr{Z}^c))^c = \mathscr{Z} \cap [(C_1(\mathscr{Z}))^c \cup (C_1(\mathscr{Z}^c))^c] = (\mathscr{Z} \cap (C_1(\mathscr{Z}))^c) \cup (\mathscr{Z} \cap (C_1(\mathscr{Z}^c))^c) = (\mathscr{Z} \cap I_1(\mathscr{Z}^c)) \cup (\mathscr{Z} \cap I_1(\mathscr{Z})) = \phi \cup I_1(\mathscr{Z}) = I_1(\mathscr{Z})$.

Remark 5.11. *Figure 1 shows that the reversal of stocks is not achieved, and this has been made clear through Examples 3.15 and 3.20.*

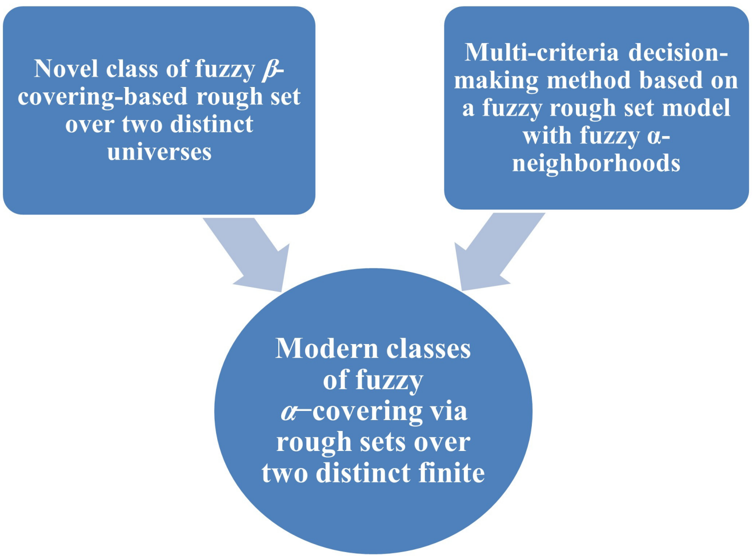

Figure 1. Comparison between results.

6. Conclusions and future work

In this research, the idea of the fuzzy α neighborhood $N^\alpha(x)$ of x, which was defined by Zhang et al. [38] for FαCAS was investigated over two finite sets (W_1, W_2). First, we introduced the definitions of FαNS, the fuzzy α-neighborhood of x, and v, Fαmd and FαMD. Then, the relations among FαNS, Fαmd, and FαMD were discussed. After that, we suggested some kinds of FαNs. On the other hand, four patterns of fuzzy α-covering over two finite sets were provided and the relationships among them were studied. Finally, we proposed some topological features of fuzzy α-covering depending on the rough set patterns. Future work could explore several key areas:

1) Expanding the framework of fuzzy α-neighborhoods (FαNS):

Investigate the properties of FαNS in more intricate or dynamic systems.

Explore applications in multi-dimensional or real-time data environments.

Develop efficient algorithms for computing FαNS in large-scale datasets.

2) Enhancing the understanding of relationships:

Broaden the analysis of the relationships among FαNS, Fαmd, and FαMD, including scenarios with non-linear or stochastic characteristics.

Assess the implications of these relationships in practical contexts like decision-making systems and pattern recognition.

3) Introducing variants of fuzzy α-neighborhoods:

Propose and evaluate new variants or modifications of FαNs to tackle specific challenges or requirements.

Test the effectiveness of these variants in applications such as clustering and classification.

4) Generalizing fuzzy α-coverings:

Extend the four patterns of fuzzy α-coverings to larger, more complex finite or infinite sets.

Study fuzzy α-coverings in relation to other mathematical structures, including lattices and graphs.

5) Advancing topological features in fuzzy rough sets:

Develop more detailed topological features of fuzzy α-coverings based on rough set theory.

Examine their practical applications in areas such as image processing and knowledge representation.

6) Exploring real-world applications:

Apply the proposed concepts in fields like artificial intelligence, machine learning, and data analysis.

Investigate the integration of fuzzy α-coverings into real-time systems, including dynamic decision-making frameworks.

7) Building interdisciplinary connections:

Establish links between fuzzy α-coverings and fields such as fuzzy logic, neural networks, and optimization.

Collaborate with experts to customize fuzzy α-concepts for industries like healthcare and finance.

Author contributions

Amal T. Abushaaban: Methodology, Writing the original draft preparation, Formal analysis; O. A. Embaby: Supervision, Formal analysis; Abdelfattah A. El-Atik: Methodology, Supervision, Formal analysis. All authors provided critical feedback and helped shape the research, analysis and manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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