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Research article

Two improved extrapolated gradient algorithms for pseudo-monotone variational inequalities

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Abstract: In this paper, the approximation problem for pseudo-monotonic variational inequalities is studied. With the help of the reciprocal form of parameters and the golden section ratio, two kinds of algorithms are introduced, and their strong convergence is proved under certain appropriate conditions.

Keywords: pseudo-monotonic variational inequality; subgradient projection; golden section ratio; strong convergence
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1. Introduction

Let C be a nonempty closed convex subset in the real Hilbert space H, and let $A : H \to H$ be a mapping. The variational inequality problem is to find $x^* \in C$ that is satisfied with the following inequality:

$$\langle A(x^*), y - x^* \rangle \geq 0, \forall y \in C.$$

In recent years, due to the wide application of variational inequalities in mathematics [1–2], physics [3–6], economics [7–10], and other fields, the theory of variational inequalities has become an important direction during the process of investigating the nonlinear analysis. Many academics have been studying it and put forward a multitude of findings [11–14]. Among the projection algorithms of variational inequalities, the Korpelevich's extragradient [15] algorithm is one of the most effective algorithms, that is, the following iterative formula for producing a sequence $\{x_n\}$:

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \end{cases}$$

in which L is the Lipschitz constant of the mapping A and $\lambda \in (0, \frac{1}{L})$. In each iteration, one needs to compute the projection onto the feasible set twice. It should be pointed out that if the structure of C is complex, the projection will be difficult to calculate, which in turn impacts the algorithm's efficiency. In 2000, Tseng introduced an extragradient method with a single projection; that is, one needs to calculate only a single projection onto the feasible set during each iteration [16]. The specific iteration formula was given as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = y_n - \lambda(A(y_n) - A(x_n)), \end{cases}$$

where L is the Lipschitz constant of the mapping A and $\lambda \in (0, \frac{1}{L})$. It is worth noting that the steps of the above algorithms should all depend on the Lipschitz constant of mapping A. However, the Lipschitz constant is often difficult to calculate or approximate. In order to avoid estimating the Lipschitz constant, it is common to utilize the Armijo linear search to get the step size [17,18]. However, determining the appropriate set size can demand significant computational effort at each iteration, making the linear search with Armijo's rule rather costly. Recently, Yang and Liu proposed an adaptive external gradient algorithm [19], in which the step size was updated step by step by simple calculation instead of estimating the Lipschitz constant.

Based on literature [20,21], this paper introduces two new projection algorithms. Under the assumption that the mapping A is pseudo-monotone and Lipschitz continuous, it is proved that the sequence generated by the algorithm converges strongly to the solution of the variational inequality. Compared with literature [15–19], the step size of the algorithm proposed in this paper is updated step by step through simple calculation without the need to estimate the Lipschitz constant.

2. Preliminaries

Let H be a real Hilbert space, and $C \subset H$ is a nonempty closed convex subset. If $\{x_n\} \subset C$ and $\{x_n\}$ weakly converges to x, which is denoted as $x_n \rightharpoonup x(n \to \infty)$; $\{x_n\}$ strongly converges to x, which is denoted as $x_n \to x(n \to \infty)$. For all $x, y \in H$ and $\alpha \in \mathbb{R}$, we have

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle \tag{2.1}$$

and

$$||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2. \tag{2.2}$$

Definition 2.1. *Let H be a real Hilbert space, and $A : H \to H$ is a nonlinear operator, and then we call A as*
(1) $L-$Lipschitz continuous $(L > 0)$ if

$$||Ax - Ay|| \leq L||x - y||, \forall x, y \in H. \tag{2.3}$$

(2) Monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \forall x, y \in H. \tag{2.4}$$

(3) Pseudo-monotone if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \forall x, y \in H. \tag{2.5}$$

(4) Weakly sequentially continuous if for all sequences $\{x_n\} \in H$, when $n \to \infty$, we have

$$x_n \rightharpoonup x \Rightarrow Ax_n \rightharpoonup Ax. \tag{2.6}$$

Definition 2.2. *[22] Let H be a real Hilbert space, and $C \subset H$ is a nonempty closed convex subset. For all $x \in H$, there exists a unique approximation point in C, which is subjects to $\|x - P_C(x)\| \leq \|x - y\|$, $\forall y \in C$. We denote it by $P_C(x)$, and we call it a metric projection.*

Lemma 2.1. *[22] Let H be a real Hilbert space, and $C \subset H$ is a nonempty closed convex subset. For all $x \in H$, $z \in C$, we have*

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \forall y \in C.$$

Lemma 2.2. *[22] Let H be a real Hilbert space, and $C \subset H$ is a nonempty closed convex subset, and $x \in H$, we have*

(1) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \forall y \in C$,

(2) $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2, \forall y \in C$,

(3) $\langle (I - P_C)(x) - (I - P_C)(y), x - y\rangle \geq \|(I - P_C)(x) - (I - P_C)(y)\|^2, \forall y \in C$.

Lemma 2.3. *[12] For all $x \in H, \alpha \geq \beta > 0$, the following inequality holds:*

(1) $\|x - P_C[x - \beta A(x)]\| \leq \|x - P_C[x - \alpha A(x)]\|$,

(2) $\dfrac{\|x - P_C[x - \alpha A(x)]\|}{\alpha} \leq \dfrac{\|x - P_C[x - \beta A(x)]\|}{\beta}$.

Lemma 2.4. *[23] Let $\{x_n\} \subset (0, 1)$ be a real number sequence, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and satisfy:*

$$x_{n+1} \leq (1 - \alpha_n)x_n + \alpha_n y_n, \forall n \geq 1.$$

If $\limsup\limits_{k\to\infty} y_{n_k} \leq 0$, and for every subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$, we have $\liminf\limits_{k\to\infty}(x_{n_k+1} - x_{n_k}) \geq 0$, then $\lim\limits_{n\to\infty} x_n = 0$.

Lemma 2.5. *[24] Let $A : H \to H$ be $L-$Lipschitz continuous in H's bounded subset, and M is a bounded subset in H; then $A(M)$ is bounded.*

Lemma 2.6. *[25] Let $A : C \to H$ be a pseudo-monotone continuous mapping, and $x^* \in C$, then*

$$x^* \in VI(C, A) \Leftrightarrow \langle A(x), x - x^*\rangle \geq 0, \forall x \in C.$$

Lemma 2.7. *[26] Let $\{x_{n_j}\}$ be a subsequence of the nonnegative real number sequence $\{x_n\}$, and satisfy for all $j \in \mathbb{N}$. We have $x_{n_j} < x_{n_j+1}$, then there exists a monotonically increasing subsequence of integers $\{m_k\}$ that subjects to $\lim\limits_{k\to\infty} m_k = \infty$. When $k \in \mathbb{N}$ is large enough, we have*

$$x_{m_k} \leq x_{m_k+1}, x_k \leq x_{m_k+1}.$$

In fact, m_k is the largest number in the set $\{1, 2, \cdots, k\}$ and satisfies $x_n < x_{n+1}$.

Lemma 2.8. *[27] Let $\{x_n\}$ be a sequence of nonnegative real number that satisfies*

$$x_{n+1} \leq (1 - \gamma_n)x_n + \gamma_n y_n, \forall n \geq 0,$$

where $\{\gamma_n\}$, $\{y_n\}$ meet

$$\{\gamma_n\} \subset (0, 1), \{y_n\} \subset \mathbb{R}, \sum_{n=1}^{\infty} \gamma_n = \infty, \limsup_{n\to\infty} y_n = 0,$$

then

$$\lim_{n\to\infty} x_n = 0.$$

Lemma 2.9. *[21] Let $\{p^k\} \subset C$ be a bounded sequence. We suppose p is a cluster point of $\{p^k\}$; when $\lim_{k\to\infty} \|p^k - p\|$ exists, $\{p^k\}$ is convergent.*

3. Improved adaptive subgradient outer gradient algorithm and convergence proof

Assumpton 3.1.
(C1) Feasible set C is a nonempty closed convex subset in Hilbert space H. Solution set $VI(C, A)$ is nonempty.
(C2) Operator $A : H \to H$ is pseudo-monotone in C 's bounded set, $L-$ Lipschitz continuous, and weakly sequentially continuous.
(C3) $f : H \to H$ is a contraction mapping ($\rho \in [0, 1)$), the sequence $\{\beta_n\} \subset (0, 1]$ is satisfied with:

$$\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty.$$

Next, the improved adaptive subgradient outer gradient algorithm is given.
Algorithm 3.1. Improved adaptive subgradient outer gradient algorithm.
Step 0: Given $\lambda_0 > 0$, $\mu > 2$, $\xi \in (0, 1]$. $\forall x_0, x_1 \in H$, $\{\tau_n\}$ is a positive real number sequence. $\lim_{n\to\infty} \frac{\tau_n}{\lambda_n} = \eta$, η is a constant, and $\eta \in (\frac{2}{\mu}, 1]$.
Step 1: Compute

$$\omega_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Step 2: Compute

$$y_n = P_C[\omega_n - \frac{\xi}{\tau_n} A(\omega_n)],$$

$$z_n = P_{T_n}[\omega_n - \frac{\xi}{\lambda_n} A(y_n)],$$

$$T_n \triangleq \{x \in H : \langle \omega_n - \frac{\xi}{\tau_n} A(\omega_n) - y_n, x - y_n \rangle \leq 0\}.$$

If $\omega_n = y_n$ or $A(y_n) = 0$, STOP. Otherwise, go to Step 3.
Step 3: Compute

$$x_{n+1} = \beta_n f(z_n) + (1 - \beta_n) z_n.$$

Step 4: Compute

$$\lambda_{n+1} = \begin{cases} \max\{\mu \frac{\langle A(\omega_n) - A(y_n), z_n - y_n \rangle}{\|\omega_n - y_n\|^2 + \|z_n - y_n\|^2}, \lambda_n\}, & if \langle A(\omega_n) - A(y_n), z_n - y_n \rangle > 0, \\ \lambda_n, & else. \end{cases} \tag{3.1}$$

Set $n := n + 1$, and go to Step 1.

Lemma 3.1. *Suppose that (C1)–(C3) hold; then $\{\lambda_n\}$ is a nondecreasing sequence generated by (3.1) and satisfies*

$$0 < \lim_{n\to\infty} \lambda_n = \lambda \leq \max\{\lambda, \frac{\mu L}{2}\}.$$

Proof. According to (3.1), we gain that $\{\lambda_n\}$ is a nondecreasing sequence. Since A is $L - Lipschitz$ continuous, we can also gain $\|A(x_n) - A(y_n)\| \leq L\|x_n - y_n\|$. When $\langle A(\omega_n) - A(y_n), z_n - y_n\rangle > 0$, we have

$$\mu\frac{\langle A(\omega_n) - A(y_n), z_n - y_n\rangle}{\|\omega_n - y_n\|^2 + \|z_n - y_n\|^2} \leq \mu\frac{\|A(\omega_n) - A(y_n)\| \, \|z_n - y_n\|}{2\|\omega_n - y_n\| \, \|z_n - y_n\|} \leq \frac{\mu L}{2},$$

so $\{\lambda_n\}$ is nondecreasing and has an upper bound, then there exists $\lim_{n\to\infty} \lambda_n = \lambda$, and $\lambda_{n+1} \leq \max\{\lambda, \frac{\mu L}{2}\}$. Based on the mathematical induction, we obtain

$$\lambda_{n+1} \leq \max\{\lambda, \frac{\mu L}{2}\}.$$

When $\langle A(\omega_n) - A(y_n), z_n - y_n\rangle \leq 0$, we get the conclusion.

Lemma 3.2. *Suppose (C1)–(C3) are set up; $\{z_n\}$ is the sequence generated by the Algorithm 3.1. Then*

$$\begin{aligned}\|z_n - p\|^2 \leq \|\omega_n - p\|^2 &- (1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 \\ &- (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2.\end{aligned} \tag{3.2}$$

Proof. We know that it can be obtained that $p \in VI(C, A) \subset C \subset T_n$. According to Lemma 2.2, we have

$$\begin{aligned}\|z_n - p\|^2 &= \|P_{T_n}[\omega_n - \frac{\xi}{\lambda_n}A(y_n)] - p\|^2 \\ &\leq \|\omega_n - \frac{\xi}{\lambda_n}A(y_n) - p\|^2 - \|\omega_n - \frac{\xi}{\lambda_n}A(y_n) - z_n\|^2 \\ &= \|\omega_n - p\|^2 + \|\frac{\xi}{\lambda_n}A(y_n)\|^2 - 2\frac{\xi}{\lambda_n}\langle A(y_n), \omega_n - p\rangle - \|\omega_n - z_n\|^2 \\ &- \|\frac{\xi}{\lambda_n}A(y_n)\|^2 + 2\frac{\xi}{\lambda_n}\langle A(y_n), \omega_n - z_n\rangle \\ &= \|\omega_n - p\|^2 - \|\omega_n - z_n\|^2 - 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - p\rangle \\ &= \|\omega_n - p\|^2 - \|\omega_n - z_n\|^2 - 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n + y_n - p\rangle \\ &= \|\omega_n - p\|^2 - \|\omega_n - z_n\|^2 - 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle - 2\frac{\xi}{\lambda_n}\langle A(y_n), y_n - p\rangle.\end{aligned}$$

For $y_n \in C$, we gain $\langle A(p), y_n - p\rangle \geq 0$. Because A is a pseudo-monotone operator, then $\langle A(y_n), y_n - p\rangle \geq 0$. Hence

$$\|z_n - p\|^2 \leq \|\omega_n - p\|^2 - \|\omega_n - z_n\|^2 - 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle, \tag{3.3}$$

in which

$$
\begin{aligned}
&\|\omega_n - z_n\|^2 + 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \\
&= \|\omega_n - y_n + y_n - z_n\|^2 + 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \\
&= \|\omega_n - y_n\|^2 + \|y_n - z_n\|^2 + 2\langle \omega_n - y_n, y_n - z_n\rangle + 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \\
&= \|\omega_n - y_n\|^2 + \|y_n - z_n\|^2 + 2\langle y_n - \omega_n + \frac{\xi}{\lambda_n}A(y_n), z_n - y_n\rangle.
\end{aligned}
\tag{3.4}
$$

Furthermore, combined with $y_n = P_C[\omega_n - \frac{\xi}{\tau_n}A(\omega_n)]$ and $z_n \in T_n$, we have

$$
\begin{aligned}
&2\langle y_n - \omega_n + \frac{\xi}{\lambda_n}A(y_n), z_n - y_n\rangle \\
&= 2\langle y_n - \omega_n + \frac{\xi}{\tau_n}A(\omega_n) - \frac{\xi}{\tau_n}A(\omega_n) + \frac{\xi}{\tau_n}A(y_n) - \frac{\xi}{\tau_n}A(y_n) + \frac{\xi}{\lambda_n}A(y_n), z_n - y_n\rangle \\
&= 2\langle y_n - \omega_n + \frac{\xi}{\tau_n}A(\omega_n), z_n - y_n\rangle + 2\frac{\xi}{\tau_n}\langle A(y_n) - A(\omega_n), z_n - y_n\rangle + 2(\frac{\xi}{\lambda_n} - \frac{\xi}{\tau_n})\langle A(y_n), z_n - y_n\rangle \\
&\geq 2\frac{\xi}{\tau_n}\langle A(y_n) - A(\omega_n), z_n - y_n\rangle + \frac{\tau_n - \lambda_n}{\tau_n} \cdot \frac{2\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle.
\end{aligned}
\tag{3.5}
$$

Substituting (3.5) into (3.4), one has

$$
\begin{gathered}
\|\omega_n - z_n\|^2 + 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \\
\geq \|\omega_n - y_n\|^2 + \|y_n - z_n\|^2 + 2\frac{\xi}{\tau_n}\langle A(y_n) - A(\omega_n), z_n - y_n\rangle + (1 - \frac{\lambda_n}{\tau_n}) \cdot \frac{2\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle,
\end{gathered}
$$

that is to say

$$
\begin{aligned}
&[1 - (1 - \frac{\lambda_n}{\tau_n})] \cdot 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \\
&\geq \|\omega_n - y_n\|^2 + \|y_n - z_n\|^2 - \|\omega_n - z_n\|^2 + 2\frac{\xi}{\tau_n}\langle A(y_n) - A(\omega_n), z_n - y_n\rangle.
\end{aligned}
\tag{3.6}
$$

Multiply both sides of (3.6) with $\frac{\tau_n}{\lambda_n}$, we have

$$
\begin{aligned}
2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \geq{}& \frac{\tau_n}{\lambda_n}\|\omega_n - y_n\|^2 + \frac{\tau_n}{\lambda_n}\|y_n - z_n\|^2 \\
&- \frac{\tau_n}{\lambda_n}\|\omega_n - z_n\|^2 + 2\frac{\xi}{\lambda_n}\langle A(y_n) - A(\omega_n), z_n - y_n\rangle.
\end{aligned}
\tag{3.7}
$$

It follows from the definition of $\{\lambda_n\}$ that

$$
\langle A(\omega_n) - A(y_n), z_n - y_n\rangle \leq \frac{\lambda_{n+1}}{\mu}\|\omega_n - y_n\|^2 + \frac{\lambda_{n+1}}{\mu}\|z_n - y_n\|^2. \tag{3.8}
$$

In fact, if $\langle A(\omega_n) - A(y_n), z_n - y_n\rangle \le 0$, then the inequality (3.8) holds. Otherwise, from (3.1), it infers

$$\lambda_{n+1} = \max\{\mu\frac{\langle A(\omega_n) - A(y_n), z_n - y_n\rangle}{\|\omega_n - y_n\|^2 + \|z_n - y_n\|^2}, \lambda_n\} \ge \mu\frac{\langle A(\omega_n) - A(y_n), z_n - y_n\rangle}{\|\omega_n - y_n\|^2 + \|z_n - y_n\|^2},$$

which implies that (3.8) holds. In addition, combining (3.7) and (3.8), we have

$$\begin{aligned} 2\frac{\xi}{\lambda_n}\langle A(y_n), z_n - y_n\rangle \ge & \frac{\tau_n}{\lambda_n}\|\omega_n - y_n\|^2 + \frac{\tau_n}{\lambda_n}\|y_n - z_n\|^2 - \frac{\tau_n}{\lambda_n}\|\omega_n - z_n\|^2 \\ & - 2\frac{\xi}{\lambda_n}\cdot\frac{\lambda_{n+1}}{\mu}\|\omega_n - y_n\|^2 - 2\frac{\xi}{\lambda_n}\cdot\frac{\lambda_{n+1}}{\mu}\|z_n - y_n\|^2 \\ & = (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 + (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2 \\ & - \frac{\tau_n}{\lambda_n}\|\omega_n - z_n\|^2. \end{aligned} \tag{3.9}$$

Putting (3.9) into (3.3), one obtain

$$\begin{aligned} \|z_n - p\|^2 \le \|\omega_n - p\|^2 - \|\omega_n - z_n\|^2 - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 \\ -(\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2 + \frac{\tau_n}{\lambda_n}\|\omega_n - z_n\|^2 \\ = \|\omega_n - p\|^2 - (1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 \\ -(\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2. \end{aligned}$$

The proof is completed.

Lemma 3.3. *Suppose (C1)–(C3) are set up; $\{\omega_n\}$ is the sequence generated by Algorithm 3.1. If there exists a subsequence $\{\omega_{n_k}\} \subset \{\omega_n\}$ weakly converging to $z \in H$ such that*

$$\lim_{k\to\infty} \|\omega_{n_k} - y_{n_k}\| = 0,$$

then we have $z \in VI(C, A)$.

Proof. According to the definition of T_n, we have

$$\langle \omega_{n_k} - \frac{\xi}{\tau_{n_k}} A(\omega_{n_k}) - y_{n_k}, x - y_{n_k}\rangle \le 0, \forall x \in C,$$

which equals

$$\frac{\tau_{n_k}}{\xi}\langle \omega_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le \langle A(\omega_{n_k}), x - y_{n_k}\rangle, \forall x \in C.$$

After expanding the right-hand side of the above equation, we can obtain

$$\frac{\tau_{n_k}}{\xi}\langle \omega_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le \langle A(\omega_{n_k}), x - \omega_{n_k}\rangle + \langle A(\omega_{n_k}), \omega_{n_k} - y_{n_k}\rangle, \forall x \in C.$$

Following transposition, we have

$$\frac{\tau_{n_k}}{\xi}\langle \omega_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle A(\omega_{n_k}), y_{n_k} - \omega_{n_k}\rangle \leq \langle A(\omega_{n_k}), x - \omega_{n_k}\rangle, \forall x \in C. \tag{3.10}$$

Because $\{\omega_n\}$ weakly converges to z, we know $\{\omega_{n_k}\}$ is bounded. Once more, due to the $A's$ Lipschitz continuity, we have $\{A(\omega_{n_k})\}$ is bounded. As $\|\omega_{n_k} - y_{n_k}\| \to 0$, $\{y_{n_k}\}$ is also bounded, and $0 < \tau_n \leq \lambda_n \leq \max\{\lambda, \frac{\mu L}{2}\}$.

Set $k \to \infty$ in (3.10), we gain

$$\liminf_{k\to\infty}\langle A(\omega_{n_k}), x - \omega_{n_k}\rangle \geq 0, \forall x \in C. \tag{3.11}$$

In addition, we have

$$\begin{aligned}\langle A(y_{n_k}), x - y_{n_k}\rangle &= \langle A(y_{n_k}) - A(\omega_{n_k}), x - \omega_{n_k}\rangle \\ &\quad + \langle A(\omega_{n_k}), x - \omega_{n_k}\rangle + \langle A(y_{n_k}), \omega_{n_k} - y_{n_k}\rangle.\end{aligned} \tag{3.12}$$

It follows from $\lim_{k\to\infty}\|\omega_{n_k} - y_{n_k}\| = 0$ and the $L-$ Lipschitz continuity of A in H, one obtain

$$\lim_{k\to\infty}\|A(\omega_{n_k}) - A(y_{n_k})\| = 0.$$

Combining the above equation with (3.11) and (3.12), we have

$$\liminf_{k\to\infty}\langle A(y_{n_k}), x - y_{n_k}\rangle \geq 0.$$

Next, we prove that $z \in VI(C, A)$. Choose decreasing sequences of positive terms $\{\epsilon_k\}$ and $\liminf_{k\to\infty}\epsilon_k = 0$. For any k, N_k denotes the smallest positive integer such that

$$\langle A(y_{n_j}), x - y_{n_j}\rangle + \epsilon_k \geq 0, \forall j \geq N_k. \tag{3.13}$$

Due to $\{\epsilon_k\}$ being decreasing, then N_k is also obviously decreasing. Moreover, for any k, since $\{y_{N_k}\} \subset C$, we can suppose $A(y_{N_k}) \neq 0$(Otherwise, y_{N_k} is the solution) and set

$$v_{N_k} = \frac{A(y_{N_k})}{\|A(y_{N_k})\|^2},$$

which infers that $\langle A(y_{N_k}), v_{N_k}\rangle = 1$. Now, it follows from (3.13) that for any k,

$$\langle A(y_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle \geq 0.$$

Since A is pseudo-monotone, the above equation can be reduced to

$$\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle \geq 0,$$

which implies that

$$\langle A(x), x - y_{N_k}\rangle \geq \langle A(x) - A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle - \epsilon_k\langle A(x), v_{N_k}\rangle. \tag{3.14}$$

Next, we will prove that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Since $\omega_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|\omega_{n_k} - y_{n_k}\| = 0$, we can get $y_{N_k} \rightharpoonup z (k \to \infty)$. While $\{y_n\} \subset C$, we have $z \in C$. According to A, it is weakly sequence continuous in H, and we have $\{A(y_{N_k})\}$ weakly converges to $A(z)$. Assume that $A(z) \neq 0$(Otherwise, z is the solution). Since the norm map is weakly sequential lower semicontinuous, one has

$$0 < \|A(z)\| \leq \liminf_{k\to\infty} \|A(y_{N_k})\|.$$

Combined with $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0 (k \to \infty)$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty} (\frac{\epsilon_k}{\|A(y_{n_k})\|}) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|A(y_{n_k})\|} = 0,$$

which infers $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$.

Let $k \to \infty$ in (3.13); since A is continuous, and the right-hand side tends to 0, $\{\omega_{N_k}\}$ and $\{v_{N_k}\}$ are bounded, and $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Therefore, we have

$$\liminf_{k\to\infty} \langle A(x), x - y_{N_k}\rangle \geq 0.$$

Since $\forall x \in C$, we have

$$\langle A(x), x - z\rangle = \lim_{k\to\infty} \langle A(x), x - y_{N_k}\rangle = \liminf_{k\to\infty} \langle A(x), x - y_{N_k}\rangle \geq 0.$$

From Lemma 2.6, one obtain

$$z \in VI(C, A).$$

The proof is completed.

Theorem 3.1. *Suppose that (C1)–(C3) hold, the sequence $\{\alpha_n\}$ is satisfied with:*

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| = 0.$$

Then the sequence $\{x_n\}$ generated by Algorithm 3.1 strongly converges to $p \in VI(C, A)$ and $p = P_{VI(C,A)} \circ f(p)$.

Proof. For the convenience of the proof, we divide it into the following four steps.

Step 1: Prove that $\{x_n\}$ is bounded.

In fact, it follows from Lemma 3.2 and the definition of $\{\tau_n\}$ that there exists $N_1 \in \mathbb{N}$ subject to

$$\begin{aligned} \|z_n - p\|^2 &\leq \|\omega_n - p\|^2 - (1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu} \cdot \frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 \\ &\quad - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu} \cdot \frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2 \leq \|\omega_n - p\|^2. \end{aligned} \tag{3.15}$$

Hence

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n f(z_n) + (1 - \beta_n) z_n - p\| = \|\beta_n [f(z_n) - p] + (1 - \beta_n)(z_n - p)\| \\ &\leq \beta_n \|f(z_n) - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leq \beta_n \|f(z_n) - f(p)\| + \beta_n \|f(p) - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leq \beta_n \rho \|z_n - p\| + \beta_n \|f(p) - p\| + (1 - \beta_n)\|z_n - p\| \\ &= [1 - \beta_n(1 - \rho)] \cdot \|z_n - p\| + \beta_n \|f(p) - p\| \\ &\leq [1 - \beta_n(1 - \rho)] \cdot \|\omega_n - p\| + \beta_n \|f(p) - p\|. \end{aligned} \tag{3.16}$$

Otherwise

$$
\begin{aligned}
\|\omega_n - p\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \\
&= \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\| \\
&\le \|x_n - p\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.17}
$$

Since $\lim\limits_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, there exist $N_2 \in \mathbb{N}$ and constant $M_1 > 0$ such that

$$
\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1, \forall n \ge N_2. \tag{3.18}
$$

Simultaneous (3.17) and (3.18), we have

$$
\|\omega_n - p\| \le \|x_n - p\| + \beta_n M_1, \forall n \ge N_2. \tag{3.19}
$$

Set $N = \max\{N_1, N_2\}$. Combining (3.16) and (3.19), we then gain

$$
\begin{aligned}
\forall x{>}N, \|x_{n+1} - p\| &\le [1 - \beta_n(1-\rho)]\|x_n - p\| + \beta_n M_1 + \beta_n\|f(p) - p\| \\
&= [1 - \beta_n(1-\rho)]\|x_n - p\| + \beta_n(1-\rho)\frac{\|f(p) - p\| + M_1}{1-\rho} \\
&\le \max\{\|x_n - p\|, \frac{\|f(p) - p\| + M_1}{1-\rho}\} \\
&\le \cdots \\
&\le \max\{\|x_N - p\|, \frac{\|f(p) - p\| + M_1}{1-\rho}\}.
\end{aligned}
$$

According to the above discussion, it can be proved that $\{x_n\}$ is bounded.

Step 2: Prove

$$
\begin{aligned}
(1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 + (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu} \cdot \frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 + (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu} \cdot \frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2 \\
\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n M_4,
\end{aligned} \tag{3.20}
$$

in which $M_4 > 0$ is a constant. In fact,

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n f(z_n) + (1-\beta_n)z_n - p\|^2 = \|\beta_n[f(z_n) - p] + (1-\beta_n)(z_n - p)\|^2 \\
&\le \beta_n\|f(z_n) - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n(\|f(z_n) - f(p)\| + \|f(p) - p\|)^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n(\rho\|z_n - p\| + \|f(p) - p\|)^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n(\|z_n - p\| + \|f(p) - p\|)^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n\|z_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + \beta_n M_2 \\
&= \|z_n - p\|^2 + \beta_n M_2,
\end{aligned} \tag{3.21}
$$

and $M_2 \triangleq \sup\limits_{n\in\mathbb{N}}\{2\|z_n - p\| \cdot \|f(p) - p\| + \|f(p) - p\|^2 > 0\}$. By substituting (3.2) into (3.21), it can be obtained that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \le \|\omega_n - p\|^2 - (1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu} \cdot \frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 \\
- (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu} \cdot \frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2 + \beta_n M_2.
\end{aligned} \tag{3.22}
$$

From (3.19), one has

$$\begin{aligned}\|\omega_n - p\|^2 &\le (\|x_n - p\| + \beta_n M_1)^2 = \|x_n - p\|^2 + \beta_n(2M_1\|x_n - p\| + \beta_n M_1^2) \\ &\le \|x_n - p\|^2 + \beta_n M_3,\end{aligned} \tag{3.23}$$

in which $M_3 \triangleq \sup_{n\in\mathbb{N}}\{2M_1\|x_n - p\| + \beta_n M_1^2\} > 0$. Substitute (3.23) into (3.22), we gain

$$\begin{aligned}\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 &+ \beta_n M_3 + \beta_n M_2 - (1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 \\ &- (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 - (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2,\end{aligned}$$

which infers that

$$\begin{aligned}(1 - \frac{\tau_n}{\lambda_n})\|\omega_n - z_n\|^2 + (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|\omega_n - y_n\|^2 + (\frac{\tau_n}{\lambda_n} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n+1}}{\lambda_n})\|z_n - y_n\|^2 \\ \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n M_4,\end{aligned}$$

and $M_4 \triangleq M_2 + M_3$.

Step 3: Prove that there exists some constant $M > 0$ such that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \le [1 - \beta_n(1-\rho)]\|x_n - p\|^2 \\ &+ \beta_n(1-\rho)[\frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle + \frac{2M}{1-\rho}\cdot\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|].\end{aligned}$$

In fact, applying (2.1) and $\{\omega_n\}$'s definition, we have

$$\begin{aligned}\|\omega_n - p\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\|^2 \le \|x_n - p\|^2 + 2\alpha_n\langle x_n - x_{n-1}, \omega_n - p\rangle \\ &\le \|x_n - p\|^2 + 2\alpha_n\|x_n - x_{n-1}\|\cdot\|\omega_n - p\|.\end{aligned} \tag{3.24}$$

Combined with (2.1), (2.2), and (3.15), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n f(z_n) + (1-\beta_n)z_n - p\|^2 \\ &= \|\beta_n[f(z_n) - f(p)] + (1-\beta_n)(z_n - p) + \beta_n[f(p) - p]\|^2 \\ &\le \|\beta_n[f(z_n) - f(p)] + (1-\beta_n)(z_n - p)\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le \beta_n\|f(z_n) - f(p)\|^2 + (1-\beta_n)\|z_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le \beta_n\rho^2\|z_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le [1 - \beta_n(1-\rho)]\|z_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\le [1 - \beta_n(1-\rho)]\|\omega_n - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{3.25}$$

Since $\|\omega_n - p\|^2 \le \|x_n - p\|^2 + 2\alpha_n\|x_n - x_{n-1}\|\cdot\|\omega_n - p\|$ holds, and we put (3.24) into (3.25), one has

$$\begin{aligned}&\|x_{n+1} - p\|^2 \le [1 - \beta_n(1-\rho)]\|x_n - p\|^2 + 2\alpha_n\|x_n - x_{n-1}\|\cdot\|\omega_n - p\| + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\ &= [1 - \beta_n(1-\rho)]\|x_n - p\|^2 + \beta_n(1-\rho)\cdot\frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle + 2\alpha_n\|x_n - x_{n-1}\|\cdot\|\omega_n - p\| \\ &\le [1 - \beta_n(1-\rho)]\|x_n - p\|^2 + \beta_n(1-\rho)\cdot\frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle + 2M\alpha_n\|x_n - x_{n-1}\| \\ &= [1 - \beta_n(1-\rho)]\|x_n - p\|^2 + \beta_n(1-\rho)[\frac{2}{1-\rho}\langle f(p) - p, x_{n+1} - p\rangle + \frac{2M}{1-\rho}\cdot\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|],\end{aligned}$$

and $M \triangleq \sup\limits_{n\in\mathbb{N}}\{\|x_n - p\|\} > 0$.

Step 4: Prove $\{\|x_n - p\|^2\}$ converges to 0.

In fact, according to Lemma 2.4, we only need to prove the sequence $\{\|x_n - p\|\}$'s subsequence, which satisfies $\liminf\limits_{k\to\infty}(\|x_{n_{k+1}} - p\| - \|x_{n_k} - p\|) \geq 0$, $\{\|x_{n_k} - p\|\}$ meet

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_{k+1}} - p\rangle \leq 0.$$

Thus, assume that $\{\|x_{n_k} - p\|\}$ is the subsequence of $\{\|x_n - p\|\}$ and satisfies $\liminf\limits_{k\to\infty}(\|x_{n_{k+1}} - p\| - \|x_{n_k} - p\|) \geq 0$, then

$$\begin{aligned}
&\liminf_{k\to\infty}(\|x_{n_{k+1}} - p\|^2 - \|x_{n_k} - p\|^2)\\
=&\liminf_{k\to\infty}[(\|x_{n_{k+1}} - p\| - \|x_{n_k} - p\|)(\|x_{n_{k+1}} - p\| + \|x_{n_k} - p\|)] \geq 0.
\end{aligned}$$

According to Step 2, we have

$$\begin{aligned}
&\limsup_{k\to\infty}[(1 - \frac{\tau_{n_k}}{\lambda_{n_k}})\|\omega_{n_k} - z_{n_k}\|^2 + (\frac{\tau_{n_k}}{\lambda_{n_k}} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}})\|\omega_{n_k} - y_{n_k}\|^2\\
&+ (\frac{\tau_{n_k}}{\lambda_{n_k}} - \frac{2\xi}{\mu}\cdot\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}})\|z_{n_k} - y_{n_k}\|^2]\\
&\leq \limsup_{k\to\infty}[\|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2 + \beta_{n_k}M_4]\\
&\leq \limsup_{k\to\infty}[\|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2] + \limsup_{k\to\infty}\beta_{n_k}M_4\\
&= -\liminf_{k\to\infty}[\|x_{n_{k+1}} - p\|^2 - \|x_{n_k} - p\|^2] \leq 0.
\end{aligned}$$

As the coefficients of $\|\omega_{n_k} - z_{n_k}\|^2, \|\omega_{n_k} - y_{n_k}\|^2, \|z_{n_k} - y_{n_k}\|^2$ are all positive, hence

$$\lim_{k\to\infty}\|\omega_{n_k} - z_{n_k}\| = 0, \lim_{k\to\infty}\|\omega_{n_k} - y_{n_k}\| = 0, \lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0. \tag{3.26}$$

Next, we will prove that

$$\|x_{n_{k+1}} - x_{n_k}\| \to 0 (k\to\infty).$$

As $k\to\infty$, we have

$$\|x_{n_{k+1}} - z_{n_k}\| = \beta_{n_k}\|z_{n_k} - f(z_{n_k})\| \to 0, \tag{3.27}$$

and

$$\|x_{n_k} - \omega_{n_k}\| = \alpha_{n_k}\|x_{n_k} - x_{n_{k-1}}\| = \beta_{n_k}\cdot\frac{\alpha_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_{k-1}}\| \to 0. \tag{3.28}$$

From (3.26)–(3.28), one gets

$$\|x_{n_{k+1}} - x_{n_k}\| \leq \|x_{n_{k+1}} - z_{n_k}\| + \|z_{n_k} - \omega_{n_k}\| + \|\omega_{n_k} - x_{n_k}\| \to 0 (k\to\infty).$$

Since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ weakly converging to $z \in H$, thereby

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, z - p\rangle. \tag{3.29}$$

Apply (3.28), then $\omega_{n_k} \rightharpoonup z(k \to \infty)$. Combining (3.26) and Lemma 3.3, we have $z \in VI(C, A)$. Available from (3.29) and $p = P_{VI(C,A)\circ f(p)}$, we gain

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, z - p\rangle \leq 0. \tag{3.30}$$

Then

$$\begin{aligned}&\limsup_{k\to\infty}\langle f(p) - p, x_{n_{k+1}} - p\rangle \\ \leq &\limsup_{k\to\infty}\langle f(p) - p, x_{n_{k+1}} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle \leq 0.\end{aligned} \tag{3.31}$$

Hence it follows from Lemma 2.4, (3.24), and (3.31) that $\lim\limits_{n\to\infty} \|x_n - p\| = 0$. The proof is completed.

Remark 3.1. The algorithm combines the subgradient outer gradient method, the inertia method, and the viscosity method. Under appropriate conditions, different parameters are introduced to improve the algorithm, and the selection of $\{\lambda_n\}$ and $\{\tau_n\}$ in the existing algorithm is improved to the reciprocal form to avoid excessive iteration and control convergence. The second class of algorithms is given next, and the convergence is proved.

4. Golden section algorithm and convergence proof

Algorithm 4.1. Golden section algorithm for pseudo-monotone variational inequalities.

Step 0: For given $p^0, p^1 \in C, \lambda_0 > 0, \overline{\lambda} > 0, \phi \in (1, \varphi], \varphi$ is, golden section ratio constant $\dfrac{\sqrt{5}+1}{2}$.

Let $\overline{p}^0 = p^1, \theta_0 = 1, \rho = \dfrac{1}{\phi} + \dfrac{1}{\phi^2}, k = 1.$

Step 1: Compute $x^k = A(p^k), x^{k-1} = A(p^{k-1})$,

$$\lambda_k = \min\{\rho\lambda_{k-1}, \frac{\phi\theta_{k-1}}{4\lambda_{k-1}}\frac{\|p^k - p^{k-1}\|^2}{\|x^k - x^{k-1}\|^2}, \overline{\lambda}\}. \tag{4.1}$$

Step 2: Compute

$$\overline{p}^k = \frac{(\phi - 1)p^k + \overline{p}^{k-1}}{\phi},$$
$$p^{k+1} = P_C(\overline{p}^k - \lambda_k x^k).$$

Step 3: Compute

$$\theta_k = \frac{\lambda_k}{\lambda_{k-1}}\phi.$$

Set $k := k + 1$, and go to Step 1.

Remark 4.1. *Two key inequalities can be obtained from (4.1). First, we have* $\lambda_k \leq (\frac{1}{\phi} + \frac{1}{\phi^2})\lambda_{k-1}$, *it means* $\theta_k - 1 - \frac{1}{\phi} \leq 0$. *Second, combining* $\frac{\phi\theta_{k-1}}{\lambda_{k-1}} = \frac{\theta_k\theta_{k-1}}{\lambda_k}$ *and (4.1), we can gain*

$$\lambda_k \leq \frac{\phi\theta_{k-1}}{4\lambda_{k-1}}\frac{\|p^k - p^{k-1}\|^2}{\|x^k - x^{k-1}\|^2}.$$

Then we have

$$\lambda_k^2\|x^k - x^{k-1}\|^2 \leq \frac{\theta_k\theta_{k-1}}{4}\|p^k - p^{k-1}\|^2. \tag{4.2}$$

Lemma 4.1. *Suppose that the mapping $A : H \to H$ is $L-$ Lipschitz continuous. If the sequence $\{p^k\}$ generated by Algorithm 4.1 is bounded, then both $\{\lambda_k\}$ and $\{\theta_k\}$ are also bounded and separate from 0.*

Proof. Obviously, $\{\lambda_k\}$ is decreasing and positive, so $\{\lambda_k\}$ is bounded. Next, the mathematical induction will be utilized to prove that $\{\lambda_k\}$ separates from 0.

Since $\{p^k\}$ is bounded, then there exists $L > 0$ such that

$$\|x^k - x^{k-1}\| = \|A(p^k) - A(p^{k-1})\| \leq L\|p^k - p^{k-1}\|.$$

Set L large enough, subject to $\lambda_i \geq \frac{\phi^2}{4L^2\overline{\lambda}}(i = 0, 1)$. Suppose that for all $i = 0, 1, \ldots, k-1$, $\lambda_i \geq \frac{\phi^2}{4L^2\overline{\lambda}}$ holds, then one has

$$\lambda_k = \rho\lambda_{k-1} \geq \lambda_{k-1} \geq \frac{\phi^2}{4L^2\overline{\lambda}},$$

or

$$\lambda_k = \frac{\phi^2}{4\lambda_{k-2}}\frac{\|p^k - p^{k-1}\|^2}{\|x^k - x^{k-1}\|^2} \geq \frac{\phi^2}{4\lambda_{k-2}L^2} \geq \frac{\phi^2}{4L^2\overline{\lambda}}.$$

So regardless of what the expression for $\{\lambda_k\}$is we have $\lambda_k \geq \frac{\phi^2}{4L^2\overline{\lambda}}$ holds, hence $\{\lambda_k\}$ separates from 0. While $\theta_k = \frac{\lambda_k}{\lambda_{k-1}}\phi$, by the similar method, itus easy to figure out that $\{\theta_k\}$ is also bounded and separates from 0.

To simplify the proof that follows, we define $\Psi(u, v) = \langle u^*, v - u\rangle$, and $u^* = A(u)$.

Theorem 4.1. *Suppose that $A : H \to H$ is pseudo-monotone and satisfies $L-$ Lipschitz continuity. For every $p^1, \overline{p}^0 \in C, \lambda \in (0, \frac{\varphi}{2L}]$, the sequences $\{p^k\}$ and $\{\overline{p}^k\}$ generated by Algorithm 4.1 converge to $VI(C, A)$.*

Proof. Combining $p^{k+1} = P_C(\overline{p}^k - \lambda_k x^k)$ and Lemma 2.1, we have

$$\langle p^{k+1} - \overline{p}^k + \lambda_k x^k, z - p^{k+1}\rangle \geq 0, \forall z \in C. \tag{4.3}$$

Set $k = k - 1$ in (4.3), and take $z = p^{k+1}$, one obtains

$$\langle p^k - \overline{p}^{k-1} + \lambda_{k-1}x^{k-1}, p^{k+1} - p^k\rangle \geq 0. \tag{4.4}$$

Multiply both sides of (4.4) with $\frac{\lambda_k}{\lambda_{k-1}}$, and utilize the equation $\frac{\lambda_k}{\lambda_{k-1}}(p^k - \overline{p}^{k-1}) = \theta_k(p^k - \overline{p}^k)$; it infers

$$\langle \theta_k(p^k - \overline{p}^k) + \lambda_k x^{k-1}, p^{k+1} - p^k\rangle \geq 0. \tag{4.5}$$

Combining (4.3) and (4.5), we obtain

$$\begin{aligned} &\langle p^{k+1} - \overline{p}^k, z - p^{k+1}\rangle + \theta_k\langle p^k - \overline{p}^k, p^{k+1} - p^k\rangle + \lambda_k\langle x^k - x^{k-1}, p^k - p^{k+1}\rangle \\ &\geq \lambda_k\langle x^k, p^k - z\rangle \geq \lambda_k\langle x, p^k - z\rangle = \lambda_k\Psi(z, p^k). \end{aligned} \tag{4.6}$$

Writing the first two terms of (4.6) in the form of a norm, we can obtain

$$\|p^{k+1}-z\|^2 \le \|\overline{p}^k - z\|^2 - \|p^{k+1}-\overline{p}^k\|^2 + 2\lambda_k\langle x^k - x^{k-1}, p^k - p^{k+1}\rangle$$
$$+\theta_k(\|p^{k+1}-\overline{p}^k\|^2 - \|p^{k+1}-p^k\|^2 - \|p^k-\overline{p}^k\|^2) - 2\lambda_k\Psi(z,p^k). \tag{4.7}$$

According to $\overline{p}^{k+1} = \frac{(\phi-1)p^{k+1}+\overline{p}^k}{\phi}$, we have

$$\|p^{k+1}-z\|^2 = \frac{\phi}{\phi-1}\|\overline{p}^{k+1}-z\|^2 - \frac{1}{\phi-1}\|\overline{p}^k - z\|^2 + \frac{1}{\phi}\|p^{k+1}-\overline{p}^k\|^2.$$

The above equation combined with (4.7), one has

$$\frac{\phi}{\phi-1}\|\overline{p}^{k+1}-z\|^2 \le \frac{\phi}{\phi-1}\|\overline{p}^k - z\|^2 + (\theta_k - 1 - \frac{1}{\phi})\|p^{k+1}-\overline{p}^k\|^2$$
$$-2\lambda_k\Psi(z,p^k) - \theta_k(\|p^{k+1}-p^k\|^2 + \|p^k-\overline{p}^k\|^2) + 2\lambda_k\langle x^k - x^{k-1}, p^k - p^{k+1}\rangle. \tag{4.8}$$

While $\theta_k \le 1 + \frac{1}{\phi}$, it follows from (4.2) that the rightmost term in (4.8) can be scaled to

$$\begin{aligned}2\lambda_k\langle x^k - x^{k-1}, p^k - p^{k+1}\rangle &\le 2\lambda_k\|x^k - x^{k-1}\|\,\|p^k - p^{k+1}\|\\ &\le \sqrt{\theta_k\theta_{k-1}}\|p^k - p^{k-1}\|\,\|p^k - p^{k+1}\|\\ &\le \frac{\theta_k}{2}\|p^{k+1}-p^k\|^2 + \frac{\theta_{k-1}}{2}\|p^k - p^{k-1}\|^2.\end{aligned}$$

However, after scaling (4.8) with this formula, it infers

$$\frac{\phi}{\phi-1}\|\overline{p}^{k+1}-z\|^2 + \frac{\theta_k}{2}\|p^{k+1}-p^k\|^2 + 2\lambda_k\Psi(z,p^k)$$
$$\le \frac{\phi}{\phi-1}\|\overline{p}^k - z\|^2 + \frac{\theta_{k-1}}{2}\|p^k - p^{k-1}\|^2 - \theta_k\|p^k - \overline{p}^k\|^2. \tag{4.9}$$

After iteratively accumulating the above equation, we have

$$\frac{\phi}{\phi-1}\|\overline{p}^{k+1}-z\|^2 + \frac{\theta_k}{2}\|p^{k+1}-p^k\|^2 + \sum_{i=2}^{k}\theta_i\|p^i - \overline{p}^i\|^2 + 2\sum_{i=1}^{k}\lambda_i\Psi(z,p^i)$$
$$\le \frac{\phi}{\phi-1}\|\overline{p}^2 - z\|^2 + \frac{\theta_1}{2}\|p^2 - p^1\|^2 - \theta_2\|p^2 - \overline{p}^2\|^2 + 2\lambda_1\Psi(z,p^1). \tag{4.10}$$

Set $z = p \in VI(C,A)$, then the last term on the left in (4.10) is nonnegative. It can gain $\{\overline{p}^k\}$ and $\{p^k\}$ are bounded, and $\theta_k\|p^k - \overline{p}^k\| \to 0$. According to the proof of Lemma 4.1, we have $\lambda_k \ge \frac{\phi^2}{4L^2\overline{\lambda}}$ and $\{\theta_k\}$ separates from 0. Hence, $\lim\limits_{k\to+\infty}\|p^k - \overline{p}^k\| = 0$ holds. It means that $p^k - \overline{p}^{k-1} \to 0$ and $p^{k+1} - p^k \to 0$.

Next, we will prove that every cluster point of $\{p^k\}$ and $\{\overline{p}^k\}$ belongs to $VI(C,A)$.

Set subsequence $\{k_i\}$ subjects to $p^{k_i} \to \widetilde{p}$ and $\lambda_{k_i} \to \lambda > 0 (i \to +\infty)$ holds. Obviously, both $p^{k_i+1} \to \widetilde{p}$ and $\overline{p}^{k_i} \to \widetilde{p}$ hold. Substituting $k = k_i$ into (4.3), as i increases to infinity, we have

$$\lambda\langle\widetilde{x}, z - \widetilde{p}\rangle \ge 0, \widetilde{x} = A(\widetilde{p}), \forall z \in C.$$

Hence, $\widetilde{p} \in VI(C,A)$. According to (4.9), we can get $\frac{\phi}{\phi-1}\|\overline{p}^k - z\|^2 + \frac{\theta_{k-1}}{2}\|p^k - p^{k-1}\|^2$ is non-monotonically increasing, which implies $\lim\limits_{k\to+\infty}\|\overline{p}^k - z\|$ exists. Since $z \in VI(C,A)$ is arbitrary, it follows from Lemma 2.9 states that there exist sequences $\{p^k\}$ and $\{\overline{p}^k\}$ converging to the certain elements of $VI(C,A)$. The proof is completed.

5. Numerical experiments

In order to evaluate the performance of the proposed algorithms, we present numerical experiments relative to the variational inequality. In this section, we provide an example to compare Algorithms 3.1 and 4.1 with the modified projection and contraction algorithm [28], see Figure 1.

Example 5.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, be defined by $f(x) = Ax + b$, where $A = Z^T Z, Z = (z_{ij})_{n\times n}$ and $b = (b_i) \in \mathbb{R}^n$ where $z_{ij} \in [1, 100]$ and $b_i \in [-100, 0]$ are generated randomly.*

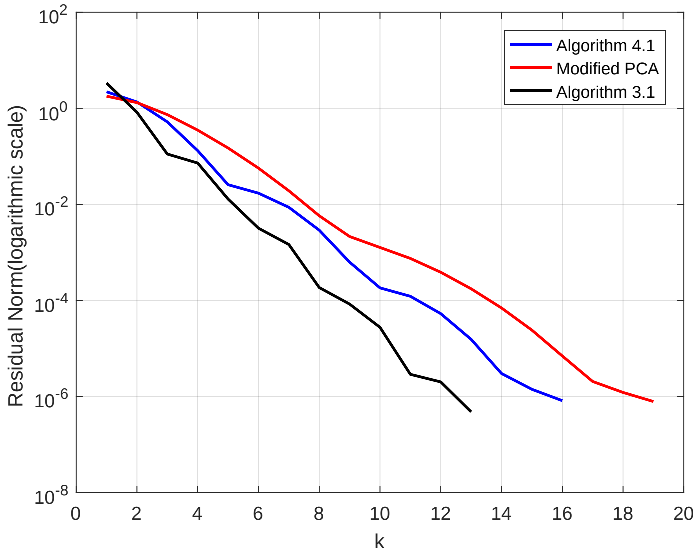

Figure 1. Comparison of Algorithm 3.1, Algorithm 4.1, and modified projection and contraction algorithm for Example 5.1.

Under the given parameters, we can obtain the convergence rate ratio of Algorithms 3.1 and 4.1 which is faster than Modified PCA.

Remark 5.1. *First, we begin with a comparison of the stochastic projection and contraction algorithm in [29] and the fast alternated inertial projection algorithms in [30]. But the variational inequality conditions of these two literatures are too different from those of this literature, so I choose the literature [28] with more similar conditions and conclusions to compare.*

6. Conclusions

In this paper, the approximation problem has been investigated for the pseudo-monotonic variational inequalities. By taking advantage of the reciprocal form of parameters and the golden section ratio, two improved extrapolated gradient algorithms have been proposed, in which the strong convergence of the two defined algorithms for solving pseudo-monotone variational inequalities problems is achieved by extending the existing results of Thong [17] and Malitsky [21]. Finally, one numerical example is provided to illustrate the effectiveness of the proposed results.

Author contribution

Haoran Tang: Formal analysis, Writing editing, Visualization, Data curation; Weiqiang Gong: Funding acquisition, Visualization, Conceptualization, Supervision. All the authors have agreed and given their consent for the publication of this research paper.

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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