Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(2): 2021–2041.
DOI: 10.3934/math.2025095
Received: 30 October 2024
Revised: 20 January 2025
Accepted: 22 January 2025
Published: 08 February 2025

Research article

On the composition operator with variable integrability

Carlos F. Álvarez[1,*], Javier Henríquez-Amador[2], John Millán G.[3] and Eiver Rodríguez[4]

[1] Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile

[2] Departamento de Matemáticas, Universidad del Valle, Ciudadela Universitaria Meléndez Edificio E20, Cali, Colombia

[3] Área de Básicas Exactas, Universidad del Sinú, Seccional Cartagena, Av. El Bosque, Transv. 54 Nº 30-729, Cartagena, Colombia

[4] Departamento de Matematicas, Universidad de Puerto Rico, 17 AVE Universidad STE 1701, San Juan, PR, USA

* **Correspondence:** Email: carlos.alvarez.e@pucv.cl.

Abstract: In this article, we considered a class of composition operators on Lebesgue spaces with variable exponents over metric measure spaces. Taking advantage of the compatibility between the metric-measurable structure and the regularity properties of the variable exponent, we provided necessary and sufficient conditions for this class of operators to be bounded and compact, respectively. In addition, we showed the usefulness of the variable change to study weak compactness properties in the framework of non-standard spaces.

Keywords: composition operators; variable exponent spaces; metric measure spaces; continuity; compactness
Mathematics Subject Classification: 47B33, 46BE30

1. Introduction

Let (X, d, μ) be a metric measure space equipped with a metric d and the Borel regular measure μ. If $\varphi : X \to X$ is a non-singular map, i.e.,

$$\mu(\varphi^{-1}(E)) = 0, \quad \text{for all Borel } \mu\text{-measurable sets } E \subset X \text{ with } \mu(E) = 0, \tag{1.1}$$

then we can define the composition operator

$$C_\varphi : f \mapsto f \circ \varphi, \quad \text{for every measurable function } f \text{ on } X.$$

According to (1.1), the measure $(\mu \circ \varphi^{-1})(\cdot) := \mu(\varphi^{-1}(\cdot))$ is absolutely continuous with respect to the measure μ. From the Radon-Nikodym theorem, there exists a measurable function $u_\varphi : X \to [0, \infty+]$ such that

$$\mu(\varphi^{-1}(E)) = \int_E u_\varphi(x)\, d\mu, \quad \text{for all Borel } \mu\text{-measurable sets } E \subset X \text{ with } \mu(E) = 0. \tag{1.2}$$

The composition operator C_φ appears naturally in the context of variable change and has recently been applied to dynamical systems, partial differential equations, and data science. For more details, see [3, 8, 16, 32]. For seminal applications, refer to [23, 24].

In L^p spaces, characterizing of the boundedness of composition operators is a fundamental problem (e.g., see [5, 27–29]). For other function spaces, see [1, 6, 7, 13, 20, 25, 30, 31]. It is well-known that C_φ continuously maps $L^p(X)$ into itself if and only if the function u_φ is essentially bounded on X. In such a case, φ is said to induce a composition operator on $L^p(X)$.

The first question that we address in this article is:

Q.1 What kind of control should be imposed on the variable exponent $p : X \to \mathbb{R}$ such that the map φ induces a composition operator C_φ on $L^{p(\cdot)}(X)$?

A fruitful analysis of variable integrability spaces $L^{p(\cdot)}(X)$ is obtained when X is a Euclidean domain, e.g., [9, 11, 12]. Unfortunately, the same cannot be said for general metric measure spaces. However, the authors in [15] achieved boundedness of the maximal operator with a notion of "dimension" through a local uniformity condition:

$$\mu(B(x, r)) \approx r^{p(x)}, \quad x \in X, \tag{1.3}$$

where, $B(x, r) := \{z \in X : d(z, x) < r\}$, $r \geq 0$, see also [22] by endowing X.

An essential difficulty in proving the boundedness of C_φ over all $L^{p(\cdot)}(X)$ is that Cavalier's principle (see [4, Lemma 1.10]) does not hold. This motivates our question about appropriate control conditions (Q.1). A control on the variable exponent $p(\cdot)$ compatible with a condition like (1.3), which we propose in this article, is:

$$\inf_{x\in B} p(x) \leq \inf_{x\in\varphi^{-1}(B)} p(x) \leq \sup_{x\in\varphi^{-1}(B)} p(x) \leq \sup_{x\in B} p(x), \tag{1.4}$$

for every ball $B \subset X$. Note that when $p(\cdot)$ is constant, the condition (1.4) is trivial. It seems natural to impose a "control" on the local supremum and infimum of the variable exponent as a new ingredient in our analysis.

Another important point is that our boundedness result for C_φ over $L^{p(\cdot)}(X)$ can be extended in several directions including:

- n-dimensional Euclidean domains on $\mathbb{R}^n$ with Lebesgue measure and Euclidean distance.
- Complete Riemannian manifolds of positive Riemannian measure and distance.
- Locally compact and separable group equipped with a left-invariant metric and left-invariant Haar measure.

Currently, the boundedness of C_φ on variable integrability spaces such as $L^{p(\cdot)}(X)$ is not fully understood, even for n-dimensional Euclidean domains. However, boundedness results for these operators are known in more regular function spaces, such as variable exponent Bergman spaces (see [26]) and, recently, in holomorphic function spaces [21]. Additionally, the inequality

$$\int |C_\varphi f(x)|^{p(x)}\, d\mu \leq C \int |f(x)|^{p(x)}\, d\mu, \tag{1.5}$$

does not hold in $L^{p(\cdot)}(X)$ unless $p(\varphi(\cdot)) = p(\cdot)$ almost everywhere in X.

Regarding compactness, it is well-known that $L^p(X)$ with $p \geq 1$ does not support compact composition operators if X has no atoms. In this paper, we show that metric measure spaces satisfying a local uniform property as in (1.3) have no atoms. This allows us to provide a different proof from the constant exponent case, using recent developments on precompact sets on $L^{p(\cdot)}(X)$ obtained in [14].

We also study a related class of operators:

$$T_\varphi : f \mapsto f \circ \varphi, \quad D(T_\varphi) := \left\{ f \in L^{p(\cdot)}(X) : f \circ \varphi \in L^{(p\circ\varphi)(\cdot)}(X) \right\}.$$

This class T_φ represents the natural extension to the variable exponent case. Recent studies on boundedness, compactness, and closed-range properties for such operators have been conducted in [2, 10] for bounded exponents defined on complete σ-finite spaces.

In this paper, we study the boundedness of T_φ by providing a simple proof on $L^{p(\cdot)}(X)$, where X is a metric measure space with a doubling measure μ and unbounded exponents $p(\cdot)$. On the other hand, the non-compactness of T_φ is obtained when X is a connected space. Particularly, in some cases, T_φ maps $L^{p(\cdot)}(X)$ to $L^{p(\cdot)}(X)$. For example, when

$$T_\varphi : L^{p(\cdot)}(X) \to L^{(p\circ\varphi)(\cdot)}(X) \quad \text{and} \quad L^{(p\circ\varphi)(\cdot)}(X) \hookrightarrow L^{p(\cdot)}(X),$$

the right-hand embedding implies, in particular, that $p(\varphi(x)) \geq p(x)$ a.e. in $x \in X$. Moreover, this implies the right inequality in (1.4). In this sense, the results concerning the operator C_φ are obtained with a weaker hypothesis by replacing the embedding with regularity of log-continuous type, which we adapt to the environment of metric measure spaces.

Finally, another property that has not been explored in the framework of spaces with variable integrability is the weak compactness of the operator T_φ, even in the Euclidean case. In this direction, we show that the operator T_φ behaves well on weakly compact sets in $L^{p(\cdot)}([0,1])$, and some results are obtained in the non-reflexive setting $p^- = 1$. In fact, our partial results allow us to state the following conjecture: Let $p^- = 1$,

$$T_\varphi \quad \text{is weakly compact on} \quad L^{p(\cdot)}([0,1]) \quad \text{if and only if} \quad \inf_{x\in[0,1]} u_\varphi(x) = 0.$$

Let us now describe the organization of the article. In Section 2, we fix the notations and recall the definitions and a few results that will be important in our work. In Section 3, we study continuity and compactness for the operator C_φ. Finally, we study some properties of the operator T_φ in Section 4.

2. Preliminaries and notations

We assume throughout the paper that $\mathcal{B}_X$ is the σ-algebra of Borel generated by μ-measurable open sets in X, the measure μ of every open nonempty set is positive, and the measure of every bounded set is finite on X.

2.1. Doubling measure and Q-Ahlfors property

We define the property of doubling measure, which endows a metric measure space with good properties; for more details, see [4, 17].

Definition 1. *A measure μ is said to satisfy the doubling condition if there exists a positive constant C such that*

$$\mu(B(x,2r)) \leq C\,\mu(B(x,r)), \quad \textit{for every ball } B(x,r).$$

Another fruitful property for a metric measure space is the regular Q-Ahlfors property, which in some cases is stronger than the doubling property (see [15]).

Definition 2. *We say that the measure μ is lower Ahlfors Q-regular if there exists a positive constant C such that $\mu(B) \leq C\ diam(B)^Q$ for every ball $B \subset X$ with $diam B \leq 2\ diam X$. We say that μ is upper Ahlfors Q-regular if there exists a positive constant C such that $\mu(B) \geq C\ diam(B)^Q$ for every ball $B \subset X$ with $diam B \leq 2\ diam X$. The measure μ is Ahlfors Q-regular if it is upper and lower Ahlfors Q-regular, i.e., if*

$$\mu(B) \approx diam(B)^Q \text{ for every ball } B \subset X \text{ with } diam\ B \leq 2\ diam\ X.$$

2.2. The variable exponents class

The class of variable exponents, denoted by $\mathcal{P}(X)$, is defined by

$$\mathcal{P}(X) := \{p : X \to [1,\infty) : p(\cdot) \text{ is Borel measurable}\}.$$

Given $A \subset X$ and $p(\cdot) \in \mathcal{P}(X)$, we put

$$p_A^+ := \operatorname*{ess\,sup}_{x\in A} p(x) \text{ and } p_A^- := \operatorname*{ess\,inf}_{x\in A} p(x).$$

When the domain is clear, we simply write $p^+ = p^+(X)$ and $p^- = p^-(X)$. Some properties of regularity at infinity, relative to variable exponent, are useful for studying composition operators within the framework of non-standard functional spaces. For Euclidean domains and metric measures spaces, these properties are provided in [9, 11] and [14, 15], respectively.

Definition 3. *Let $p(\cdot) \in \mathcal{P}(X)$. We say that*

1. *$p(\cdot)$ is locally log-Hölder continuous, denoted by $p(\cdot) \in LH_0(X)$, if there exists a constant K_0 such that for all $x, y \in X$, $d(x,y) < \frac{1}{2}$,*

$$|p(x) - p(y)| \leq \frac{K_0}{-\log(d(x,y))}.$$

2. *$p(\cdot)$ is log-Hölder continuous at infinity with point base $x_0 \in X$, denoted by $p(\cdot) \in LH_\infty(X)$, if there are constants K_∞ and p_∞ such that for all $x \in X$,*

$$|p(x) - p_\infty| \leq \frac{K_\infty}{\log(e + d(x,x_0))}.$$

When $p(\cdot)$ is log-Hölder continuous both locally and at infinity, we denote this by $p(\cdot) \in LH(X)$.

In this work, we introduce a class of exponents associated with the φ-map. Denote by $\mathcal{B}_0$ the set of all open balls in X.

Definition 4. *Let $\varphi : X \to X$ be a Borel measurable map. We define the following kinds of exponents:*

$$\mathcal{P}^{log}_{\varphi^+}(X) := \{p(\cdot) \in LH(X) : [\varphi]_{p^+} \geq 1\},$$

$$\mathcal{P}^{log}_{\varphi^-}(X) := \{p(\cdot) \in LH_0(X) : [\varphi]_{p^-} \leq 1\},$$

$$\mathcal{P}^{log}_{\varphi}(X) := \mathcal{P}^{log}_{\varphi^+}(X) \cap \mathcal{P}^{log}_{\varphi^-}(X),$$

where

$$[\varphi]_{p^+} := \inf\left\{\frac{p_B^+}{p^+_{\varphi^{-1}(B)}} : B \in \mathcal{B}_0\right\}, \quad [\varphi]_{p^-} := \sup\left\{\frac{p_B^-}{p^-_{\varphi^{-1}(B)}} : B \in \mathcal{B}_0\right\}.$$

In the following example, we show that $[\varphi]_{p^+} \geq 1$ for countable sub-covers, in fact, this is sufficient for the continuity result that we will obtain below (see Theorem 2).

Example 1. *Let $X := \mathbb{R}^+$ and define $p : \mathbb{R}^+ \to [1, +\infty)$ as variable exponent given by*

$$p(x) := \begin{cases} 1 + x^2, & x \in (0, 1) \\ 1 + \frac{1}{2x-1}, & x \geq 1 \end{cases}.$$

It is not difficult to show that

$$0 \leq p(x) - 1 \leq \frac{\ln(1 + e)}{\ln(|x| + e)}, \ \ \textit{for all} \ \ x \in \mathbb{R}^+.$$

That is, $p(\cdot) \in LH_\infty(\mathbb{R}^+)$. In addition, it is easy to see that $p(\cdot) \in LH_0(\mathbb{R}^+)$; for $x \in (0, 1)$ and $y \geq 1$ such that it follows that

$$|p(x) - p(y)| = \left|\frac{2x^2y - x^2 - 1}{2y - 1}\right| \leq 2x^2y - 2x^2 \leq 2\,(y - 1) \leq 2\,(y - x) = 2\,|x - y|.$$

Similarly, for $x \geq 1$ and $y \in (0, 1)$. This is sufficient to show that $p(\cdot)$ is Lipschitz continuous on $\mathbb{R}^+$ and, therefore, $p(\cdot) \in LH_0(\mathbb{R}^+)$. Now, consider $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ to be a non-singular map such that $0 \leq \varphi(x) \leq x$ for all $x \in \mathbb{R}^+$. On the other hand, let $\epsilon \in (0, +\infty)$ and we consider

$$\mathcal{A} := \{(x - \epsilon, x + \epsilon) : x \in \mathbb{R}^+\}.$$

Let $\{q_j\}_j \subset \mathbb{Q}^+$ be such that

$$q_j \to +\infty \ \ \textit{as} \ \ j \to +\infty, \ \ \textit{and} \ \ \mathbb{R}^+ = \bigcup_j I_j, \ \ I_j := (q_j - 5\epsilon, q_j + 5\epsilon).$$

Then, by using LH_∞-regularity, we have

$$\sup_{x\in\varphi^{-1}(I_j)} p(x) \ \ \leq \ \ \sup_{x\in\varphi^{-1}(I_j)} |p(x) - p_\varphi(x)| + \sup_{x\in\varphi^{-1}(I_j)} p_\varphi(x)$$

$$\leq \sup_{x\in\varphi^{-1}(I_j)} \frac{C_\infty}{\ln(e+\varphi(x))} + p^+_{I_j}$$
$$\leq \frac{C_\infty}{\ln(e+q_j-\epsilon)} + p^+_{I_j}.$$

Hence, since $q_j \to +\infty$ as $j \to +\infty$

$$\limsup_j p^+_\varphi(I_j) \leq \limsup_j p^+(I_j) < +\infty, \quad as \quad j \to +\infty.$$

Therefore,

$$0 \leq p^+(I_{q_{n_j}}) - p^+_\varphi(I_{q_{n_j}}), \quad for\ many\ infinite \quad n_j \in \mathbb{N}.$$

Finally, in particular, note that $L^{p(\cdot)}(\mathbb{R}^+)$ is not embedded in $L^{p_\varphi(\cdot)}(\mathbb{R}^+)$ and $p(\cdot) \in \mathcal{P}^{\log}_{\varphi^+}(\mathbb{R}^+)$ (in the sense of accounting coverages).

2.3. Variable integrability spaces

The space $L^{p(\cdot)}(X)$ is the classical variable Lebesgue space on X. Some of its basic properties have been studied, for instance, in [15].

Definition 5. *Let $p(\cdot) \in \mathcal{P}(X)$ and $L_0(X)$ be the set of measurable functions on X. The Lebesgue space with variable exponent $L^{p(\cdot)}(X)$ is defined by*

$$L^{p(\cdot)}(X) := \left\{ f \in L_0(X) : \rho\left(\frac{f}{\eta}\right) < \infty \text{ for some } \eta > 0 \right\},$$

where

$$\rho(f) := \int_X |f(x)|^{p(x)} d\mu,$$

is called the associated modular with $p(\cdot)$. It is known that $L^{p(\cdot)}(X)$ is a Banach space equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \eta > 0 : \rho\left(\frac{f}{\eta}\right) \leq 1 \right\}.$$

Additionally, when $p(\cdot)$ is bounded, we obtain the inequality that provides a relation between the norm and the modular:

$$\min\left\{ \|f\|^{p^+}_{p(\cdot)}, \|f\|^{p^-}_{p(\cdot)} \right\} \leq \rho(f) \leq \max\left\{ \|f\|^{p^+}_{p(\cdot)}, \|f\|^{p^-}_{p(\cdot)} \right\}, \tag{2.1}$$

for $f \in L^{p(\cdot)}(X)$. Also, in the case $p^+ < +\infty$, the space $L^{p(\cdot)}(X)$ can be defined as all measurable functions $f : X \to \mathbb{R}$ such that $\rho(f) < +\infty$, and the dual space is identified with $L^{p'(\cdot)}(X)$, where $p'(\cdot)$ is the conjugate exponent relative to $p(\cdot)$, that is,

$$\frac{1}{p'(x)} + \frac{1}{p(x)} = 1, \quad x \in X \setminus \{x : p(x) = 1\}.$$

It is well-know that the space of essentially bounded functions with compact support $L^{p(\cdot)}_c(X)$ is dense in $L^{p(\cdot)}(X)$.

3. Some properties for C_φ

In this section, we start analyzing the behavior and properties of the operator C_φ on $L^{p(\cdot)}(X)$ based on the following results in the space $L^p(X)$:

P.1 $C_\varphi : L^p(X) \to L^p(X)$ is bounded if and only if $u_\varphi \in L^\infty(X)$. In particular,

$$\int_X |(C_\varphi f)(x)|^p \, d\mu \;\leq \|u_\varphi\|_\infty \int_X |f(x)|^p \, d\mu, \text{ for all } f \in L^p(X).$$

In the second part, we completely characterize the continuity and compactness of operator C_φ on $L^{p(\cdot)}(X)$.

3.1. *On the property P.1 in $L^{p(\cdot)}(X)$*

Our first result shows that the integral inequality does not hold in general for any variable exponent, unless the exponent is both bounded and invariant under dilations or contractions induced by φ. The variable exponent induced by φ is denoted by $p_\varphi(\cdot) := p(\varphi(\cdot))$. Additionally, we associate φ with the number

$$\mathcal{U}(\varphi) := \sup_{B\in\mathcal{B}_0} \frac{\mu(\varphi^{-1}(B))}{\mu(B)}.$$

Theorem 1. *Assume $p(\cdot) \in \mathcal{P}(X)$ with $p^+ < +\infty$ and let $\varphi : X \to X$ be a non-singular measurable transformation. Then, the following statements are equivalent:*

M.1 There exists a constant $C > 0$ such that

$$\int_X |(C_\varphi f)(x)|^{p(x)} \, d\mu \;\leq C \int_X |f(x)|^{p(x)} \, d\mu, \textit{ for all } f \in L^{p(\cdot)}(X). \tag{3.1}$$

M.2 The function $u_\varphi : X \to \mathbb{R}$ is essentially bounded on X, and

$$p(x) = p_\varphi(x), \quad \textit{a.e. in } x \in X.$$

Proof. It is clear that $(M2) \Rightarrow (M1)$ with $C := \|u_\varphi\|_\infty$. To prove $(M1) \Rightarrow (M2)$, first suppose that $p^+ < \infty$; since φ is non-singular, we have $p \circ \varphi \in L^\infty(X)$ so

$$\frac{p_\varphi(\cdot)}{p(\cdot)} \in L^\infty(X).$$

Assume that $(M.1)$ holds, so for $B \in \mathcal{B}_0$, with $\mu(B) < +\infty$. Considering the function $f = \mathcal{X}_B \in L^{p(\cdot)}(X)$, from inequality (3.1), we deduce that

$$\frac{C\,\mu(B)}{\mu[\varphi^{-1}(B)]} \geq 1. \tag{3.2}$$

Now, let $\Omega_\varphi := \{x \in X : p(x) \neq p_\varphi(x)\}$ and suppose that $\mu(\Omega_\varphi) > 0$. Thus, if $E_\varphi := \{x \in X : p(x) > p_\varphi(x)\} \in \mathcal{B}$, then by σ-additivity, we have

$$\mu(E_\varphi) > 0 \;\text{ or }\; \mu(\Omega_\varphi \setminus E_\varphi) > 0.$$

So, we see only the case $\mu(E_\varphi) > 0$ because the case $\mu(\Omega_\varphi \setminus E_\varphi) > 0$ is analogue with minor settings. In fact, since $\mu(E_\varphi) > 0$, then by [17, Lemma 3.3.31], we can take $y \in E_\varphi$ such that it satisfies

$$\mu(E_\varphi \cap B_y) > 0, \quad \text{for every ball } \; B_y \in \mathcal{B}_0.$$

Hence, since $\varphi^{-1}(X) = X$ from (3.2), we can choose $B_y \in \mathcal{B}_0$ (sufficiently large rate) such that

$$0 < \mu(E_\varphi \cap \varphi^{-1}(B_y)) < +\infty. \tag{3.3}$$

Hence, using (3.1) with the functions $f_n(\cdot) = n^{\frac{1}{p(\cdot)}} \mathcal{X}_{B_y}(\cdot) \in L^{p(\cdot)}(X)$, we obtain that

$$\begin{aligned} \int_{E_\varphi \cap \varphi^{-1}(B_y)} |n|^{\frac{p(x)}{p_\varphi(x)}} \, d\mu &= \int_X |(f_n \circ \varphi)(x)|^{p(x)} \, d\mu \\ &\leq C \int_X |f_n(x)|^{p(x)} \, d\mu \\ &= C \, n \, \mu(B_y). \end{aligned}$$

Thus,

$$\int_{E_\varphi \cap \varphi^{-1}(B_y)} |n|^{\frac{p(x)}{p_\varphi(x)} - 1} \, d\mu \leq C \, \mu(B_y).$$

By the classical Jensen's inequality we have

$$\int_{E_\varphi \cap \varphi^{-1}(B_y)} \left(\frac{p(x)}{p_\varphi(x)} - 1 \right) d\mu \leq \log^{-1}(n) \, \mu(E_\varphi \cap \varphi^{-1}(B_y)) \log\left(\frac{C \, \mu(B_y)}{\mu(E_\varphi \cap \varphi^{-1}(B_y))} \right)$$

so, taking $n \to +\infty$

$$\int_{E_\varphi \cap \varphi^{-1}(B_y)} \left(\frac{p(x)}{p_\varphi(x)} - 1 \right) d\mu = 0.$$

Therefore, $\mu(E_\varphi \cap \varphi^{-1}(B_y)) = 0$, which is in contradiction with (3.3). Consequently, $\mu(E_\varphi) = 0$. In addition, $u_\varphi \in L^{+\infty}(X)$ follows from (3.2) with $B = E \in \mathcal{B}_X$. □

3.2. Continuity for C_φ

In this section, we extend the classical continuity result in standard L^p spaces to function spaces with variable integrability $L^{p(\cdot)}(X)$. The proof strategy is inspired by the argument given by Cruz-Uribe and Fiorenza on the boundedness of the maximal operator in $L^{p(\cdot)}(X)$ when X is a Euclidean domain (see [9, Theorem 3.16]). For this purpose, we adapt [9, Lemmas 3.26 and 3.24] to the setting of metric measure spaces with Ahlfors regularity; see also [15].

Lemma 1. *Let $\varphi : X \to X$ be a non-singular Borel map. Then,*

(i) If $p(\cdot) \in LH_0(X)$ and μ is lower Ahlfors Q-regular, then there exists a positive constant C such that

$$\mu(B)^{p(\varphi(x)) - p_B^+} \leq C, \quad \textit{for all } \; B \in \mathcal{B}_0 \textit{ and } x \in \varphi^{-1}(B).$$

(ii) If $p(\cdot) \in LH_\infty(X)$ and μ is upper Ahlfors Q-regular, then there are positive constants C_1, C_2 such that for every function f with $0 \le f \le 1$ on $E \in \mathcal{B}$, we have

$$\int_{\varphi^{-1}(E)} |f(\varphi(x))|^{p_\infty}\, d\mu \le C_1 \int_{\varphi^{-1}(E)} |f(\varphi(x))|^{p(x)}\, d\mu + C_2.$$

Similarly,

$$\int_{\varphi^{-1}(E)} |f(\varphi(x))|^{p(x)}\, d\mu \le C_1 \int_{\varphi^{-1}(E)} |f(\varphi(x))|^{p_\infty}\, d\mu + C_2.$$

Proof. (i) Since $p(\varphi(x)) - p_B^+ \le 0$ for $x \in \varphi^{-1}(B)$, it suffices to check (i) for balls $B := B(v, r)$ with $r \le 1/2$. For $y_0 \in B$ such that

$$p_B^+ \le p(y_0) + \frac{1}{\log(1/2r)},$$

from $p(\cdot) \in LH(X)$, it follows that

$$p_B^+ - p(\varphi(x)) \le |p(\varphi(x)) - p(y_0)| + 1 \le \frac{C_0}{-\log d(\varphi(x), y_0)} \le \frac{C_0}{\log(1/2r)}.$$

So,

$$\log(2r)^{p(\varphi(x)) - p_B^+} \le C_0, \quad x \in \varphi^{-1}(B).$$

Therefore, since μ is lower Ahlfors Q-regular and $p(\varphi(x)) - p_B^+ \le 0$, we have

$$\mu(B)^{p(\varphi(x)) - p_B^+} \le (Cr^Q)^{p(\varphi(x)) - p_B^+} \le C\, r^{Q[p(\varphi(x)) - p_B^+]} \le C_Q.$$

(ii) For $x_0 \in X$, define the function $h_{x_0} : X \to \mathbb{R}$ given by $h_{x_0}(x) := (e + d(x, x_0))^{-r}, \quad x \in X$. First, we show that

$$h_{x_0}^{p^-} \in L^1(X) \text{ provided that } rp^- \in (q, +\infty).$$

Indeed, for the base point $x_0 \in X$, we consider the countable collection $\{C_j : j \in \mathbb{N}\}$ where,

$$C_j := B(x_0, 2^j) \setminus B(x_0, 2^j - 1) \in \mathcal{B}_0, \quad \text{for each } j \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
\int_X |h_{x_0}(x)|^{p^-}\, d\mu &= \sum_{j=1}^{+\infty} \int_{C_j} |h_{x_0}(x)|^{p^-}\, d\mu \\
&= \sum_{j=1}^{+\infty} \int_{B_{2^j} \setminus B_{2^{j-1}}} \left(\frac{1}{e + d(x, x_0)}\right)^{rp^-} d\mu \\
&\le \sum_{j=1}^{+\infty} \int_{B_{2^j} \setminus B_{2^{j-1}}} 2^{-jrp^-} d\mu \\
&\le \sum_{j=1}^{+\infty} \mu(B_{2^j})\, 2^{-jrp^-} \\
&\le 4^q \sum_{j=1}^{+\infty} 2^{(q - rp^-)j} < +\infty \quad (rp^- \in (q, +\infty)).
\end{aligned}$$

On the other hand, decompose $\varphi^{-1}(E) := F_1 \cup F_2$ where

$$F_1 := \{x : f(\varphi(x)) \leq h_{x_0}(x)\}, \quad F_2 := \{x : f(\varphi(x)) > h_{x_0}(x)\}.$$

On F_1 we have,

$$\int_{F_1} f(\varphi(x))^{p_\infty} \, d\mu \leq \int_{F_1} f(\varphi(x))^{p^-} \, d\mu \leq \int_{F_1} h_{x_0}(x)^{p^-} \, d\mu.$$

By the LH_∞-regularity,

$$h_{x_0}(x)^{-|p(x)-p_\infty|} \leq \exp\left(r \log(e + d(x, x_0))|p(x) - p_\infty|\right) \leq \exp\left(rC_\infty\right).$$

Finally, since $f(\varphi(x)) \leq 1$, we get

$$\int_{F_1} f(\varphi(x))^{p_\infty} \, d\mu \leq \int_{F_1} f(\varphi(x))^{p(x)} \, h_{x_0}(x)^{-|p(x)-p_\infty|} \, d\mu$$

$$\leq \exp\left(rC_\infty\right) \int_{F_1} f(\varphi(x))^{p(x)} \, d\mu.$$

□

With the previous lemma, the main result of this work in response to (Q.1) is the following:

Theorem 2. *Assume that μ is a Q-Ahlfors regular measure on X, let $\varphi : X \to X$ be a non-singular map. Then:*

($C1$) *Let $p(\cdot) \in \mathcal{P}^{log}_{\varphi^+}(X)$. If the map u_φ is essentially bounded on X, then the operator C_φ maps $L^{p(\cdot)}(X)$ into itself.*

($C2$) *Let $p(\cdot) \in \mathcal{P}^{log}_{\varphi^-}(X)$. If the operator C_φ maps $L^{p(\cdot)}(X)$ into itself, then there exists a positive constant C such that $\mu(\varphi^{-1}(B)) \leq C\,\mu(B)$ for every ball $B \in \mathcal{B}_0$, i.e.,*

$$\mathcal{U}(\varphi) < +\infty.$$

($C3$) *Assume $p(\cdot) \in \mathcal{P}^{log}_{\varphi}(X)$. If, in addition, μ is a doubling measure on X, then ($C1$) and ($C2$) are equivalent.*

Proof. ($C1$). Suppose that $u_\varphi \in L^\infty(X)$. Since $p^+ < +\infty$, by density it is sufficient to show that

$$C_\varphi : L^\infty_c(X) \cap L^{p(\cdot)}(X) \to L^{p(\cdot)}(X).$$

Indeed, let $f \in L^\infty_c(X) \cap L^{p(\cdot)}(X)$ with $\|f\|_{p(\cdot)} \leq 1$ and consider the decomposition $|f| = f_1 + f_2$ where $f_1 := |f|\chi_{\{|f|>1\}}$ and $f_2 := |f|\chi_{\{|f|\leq 1\}}$. We will divide the test into two steps:

Step 1. Let us see that there exists a constant $C > 0$ independent of f_1 such that

$$\int_X |(f_1 \circ \varphi)(x)|^{p(x)} \, d\mu \leq C. \tag{3.4}$$

Since $f_1 \in L^\infty(\mathbb{R})$ and X is separable, it follows that the open cover of X

$$\{B(x, 5r_Q) : x \in X\},$$

with $K := \operatorname{ess\,sup}_{x\in X}(f_1\circ\varphi)(x) > 1$ and $r_Q := K^{-\frac{1}{Q}}$, admits a pairwise disjoint countable open sub-cover $\{B^j_{5r_Q} : j \in \mathbb{N}\}$ such that

$$X \subset \bigcup_{j\in\mathbb{N}} B^j_{r_Q}.$$

From the μ-regularity, $\mu(B^j_{r_Q}) \le C_Q\ K^{-1}$ for all $j \in \mathbb{N}$. In addition, since $p(\cdot) \in LH_0(X)$, by using Lemma 1 there exists $M > 0$ such that for $B \in \mathcal{B}_0$ and $x \in \varphi^{-1}(B)$, we have $\mu(B)^{p_\varphi(x)-p^+(B)} \le M$; also, by the fact that $K \in (1, +\infty)$ taking $r(\cdot) := p(\cdot) - p_\varphi(\cdot)$ for $x \in X$, we get

$$\begin{aligned}
|(f_1\circ\varphi)(x)|^{p(x)} &= |(f_1\circ\varphi)(x)|^{r(x)}\ |(f_1\circ\varphi)(x)|^{p_\varphi(x)}\\
&\le \left(|(f_1\circ\varphi)(x)|^{r(x)}\ \mathcal{X}_{\{r(x)\ge 0\}} + 1\right)\ |(f_1\circ\varphi)(x)|^{p_\varphi(x)}\\
&\le \left(K^{r(x)}\mathcal{X}_{\{r(x)\ge 0\}} + 1\right)\ |(f_1\circ\varphi)(x)|^{p_\varphi(x)}\\
&= \left(\mu(B^j_{r_Q})^{-r(x)}\mathcal{X}_{\{r(x)\ge 0\}} + 1\right)\ |f_1\circ\varphi)(x)|^{p_\varphi(x)}.
\end{aligned}$$

So,

$$|(f_1\circ\varphi)(x)|^{p(x)} \le \left(\mu(B^j_{r_Q})^{-r(x)}\mathcal{X}_{\{r(x)\ge 0\}} + 1\right)\ |(f_1\circ\varphi)(x)|^{p_\varphi(x)},\ \text{ for }\ x \in X.$$

From $[\varphi]_{p^+} \ge 1$,

$$p^+_\varphi(B^j_{r_Q}) \le p^+(B^j_{r_Q}),\ \ \text{for all}\ \ j \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
\int_X |(f_1\circ\varphi)(x)|^{p(x)}\,d\mu &= \sum_j \int_{\varphi^{-1}(B^j_{r_Q})} |(f_1\circ\varphi)(x)|^{p(x)}\,d\mu\\
&\le \sum_j \int_{\varphi^{-1}(B^j_{r_Q})} (K^{p(x)-p_\varphi(x)} + 1)|(f_1\circ\varphi)(x)|^{p_\varphi(x)}\,d\mu\\
&\le \sum_j \int_{\varphi^{-1}(B^j_{r_Q})} (\mu(B^j_{r_Q})^{p_\varphi(x)-p^+_{B^j}} + 1)|(f_1\circ\varphi)(x)|^{p_\varphi(x)}\,d\mu\\
&\le (M+1)\sum_j \int_{\varphi^{-1}(B^j_{r_Q})} |(f_1\circ\varphi)(x)|^{p_\varphi(x)}\,d\mu\\
&\le (M+1)\|u_\varphi\|_\infty.
\end{aligned}$$

Therefore, the inequality (3.4) is obtained with $C_1 := (M+1)\mathcal{U}(\varphi)$. Now, we estimate the size of $C_\varphi f_2$.

Step 2. There exists a constant $C_2 > 0$ independent of f_2 such that

$$\int_X |(f_2\circ\varphi)(x)|^{p(x)}\,d\mu \le C_2 + \int_X h_{x_0}(x)^{p_-}\,d\mu. \tag{3.5}$$

Since $p(\cdot) \in LH_\infty(X)$, we get $f_2 \in L^{p_\infty}(X)$ and $u_\varphi \in L^\infty(X)$ implies also that $C_\varphi f_2 \in L^{p_\infty}(X)$. Hence, for some $C > 0$ independent of f_2, we obtain

$$\begin{aligned}
\int_X |(f_2\circ\varphi)(x)|^{p(x)}\,d\mu &\le C\ \int_X |(f_2\circ\varphi)(x)|^{p_\infty}\,d\mu + \int_X h_{x_0}(x)^{p_-}\,d\mu\\
&\le C\ \|u_\varphi\|_\infty \int_X |f_2(x)|^{p_\infty}\,d\mu + \int_X h_{x_0}(x)^{p_-}\,dx
\end{aligned}$$

$$\leq \; C_1 + \int_X h_{x_0}(x)^{p_-}\, d\mu,$$

this last integral is finite by Lemma 1. Therefore, from (3.4) and (3.5), we obtain

$$\|f\circ\varphi\|_{p(\cdot)} \leq C_{\varphi,p(\cdot)}\,\|f\|_{p(\cdot)}, \quad \text{for all } \; f\in L^\infty_c(X)\cap L^{p(\cdot)}(X).$$

$(C2)$. Suppose that

$$C_\varphi : L^{p(\cdot)}(X) \to L^{p(\cdot)}(X).$$

By the inequality (2.1) and the relation given by (1.2), it is sufficient to consider the case when $B \in \mathcal{B}_0$ is such that

$$\|\mathcal{X}_{\varphi^{-1}(B)}\|_{p(\cdot)}, \|\mathcal{X}_B\|_{p(\cdot)} \leq 1.$$

In this case, since $p(\cdot) \in LH_0(X)$ and $[\varphi]_{p^-} \leq 1$, applying Lemma 1, we can find a constant $C > 0$ such that,

$$\mu(\varphi^{-1}(B)) \leq \|\mathcal{X}_{\varphi^{-1}(B)}\|^{p^-_\varphi}_{p(\cdot)} \leq C\,\|\mathcal{X}_B\|^{p^-_\varphi}_{p(\cdot)} \leq C\,\mu(B)^{\frac{p^-_\varphi}{p^+}}$$

$$\leq C\,\mu(B)^{\frac{1}{p^+_B}(p^-_B - p^+_B)}\mu(B) = C\,\mu(B)^{\frac{1}{p^+}(p^-_B - p(x))}\,\mu(B)^{\frac{1}{p^+}(p(x)-p^+_B)}\,\mu(B)$$

$$\leq C'\,\mu(B).$$

Finally, in order to obtain $(C3)$ by assuming the doubling property of μ, note that

$$\frac{\mu(\varphi^{-1}(B))}{\mu(B)} = \frac{1}{\mu(B)}\int_B u_\varphi(x)\,d\mu(x), \quad \text{for every ball } \; B.$$

Hence, by differentiation, it is easy to see that $\mathcal{U}(\varphi)$ is finite if and only if u_φ is in $L^\infty(X)$. □

Remark 1. *According to the proof of the above theorem:*

- *Another hypothesis to obtain $(C1)$ is $p_\varphi(\cdot) \geq p(\cdot)$ a.e. in X and $p(\cdot) \in LH_\infty(X)$. Note that in our result, we replace $p_\varphi(\cdot) \geq p(\cdot)$ by hypothesis $p(\cdot) \in LH_0(X)$ and $[\varphi]_{p^+} \leq 1$, which also replaces the embedding $L^{p_\varphi(\cdot)}(X) \hookrightarrow L^{p(\cdot)}(X)$ proposed in [2, Theorem 3.4].*

- *In the proof of the above theorem, note that the condition (1.4) applies to a uniform Vitali cover of the space. In this sense, control over the inductor map φ and the exponent $p(\cdot)$ can be relaxed as shown in Example 1. In fact, note that for a suitable φ (e.g., $\varphi(x) := Ax$, $x \in \mathbb{R}^+$), it suffices to note that*

$$\int_{\mathbb{R}^+} (f_1\circ\varphi)(x)^{p(x)}\,dx \approx \int_{\varphi^{-1}(I_{q_{n_j}})} (f_1\circ\varphi)(x)^{p(x)}\,dx, \quad \text{as } \; j \to +\infty.$$

3.3. Compactness for C_φ

We start this section by showing that $L^{p(\cdot)}(X)$ does not support non-trivial compact composition operators C_φ. In addition, we approach recent results related to weak compactness in variable Lebesgue spaces and provide some properties for C_φ.

Lemma 2. *Let μ be a Q-Ahlfors regular measure on X, $\varphi : X \to X$ be a non-singular Borel map, and $p(\cdot) \in LH_0(X)$ such that $[\varphi]_{p^-} = 1$. There exists a positive constant C_0 such that, given $A \in \mathcal{B}_0$ with $\mu(A) > 0$, if for any ball B with $A \cap B \neq \emptyset$ and $\mu(B) < 1$, then*

$$\mu(B)^{1-\frac{p(x)}{p_\varphi(x)}} \geq C_0, \quad \textit{for all} \quad x \in \varphi^{-1}(A \cap B).$$

Proof. On one hand,

$$\sup_{x\in\varphi^{-1}(A\cap B)} p_\varphi(x) \leq \sup_{x\in A\cap B} p(x) \leq \sup_{x\in B} p(x) = p_B^+.$$

On the other hand, since $[\varphi]_{p^-} = 1$

$$\inf_{x\in\varphi^{-1}(A\cap B)} p(x) \geq \inf_{x\in\varphi^{-1}(B)} p(x) \geq \inf_{x\in B} p(x) = p_B^-.$$

Therefore, for $x \in \varphi^{-1}(A \cap B)$, it follows that $\mu(B)^{\frac{p(x)}{p_\varphi(x)}} \leq \mu(B)^{\frac{p^-(B)}{p^+(B)}}$. Hence, since $p(\cdot) \in LH_0(X)$ by [15, Lemma 3.6], there exists $C_0 > 0$ such that

$$\mu(B)^{1-\frac{p(x)}{p_\varphi(x)}} \geq \mu(B)^{1-\frac{p_B^-}{p_B^+}} = \mu(B)^{\frac{p_B^+-p_B^-}{p_B^+}} \geq C_0.$$

□

Theorem 3. *Let μ be a Ahlfors Q-regular and doubling measure on X. The non-trivial bounded composition operator C_φ is not compact on $L^{p(\cdot)}(X)$.*

Proof. Note initially that the space (X, d, μ) does not contain atoms. Indeed, if $E \in \mathcal{B}_X$ is an atom, then $\mu(E) > 0$, so there exists $a \in E$ such that $\mu(E \cap B(a, r)) > 0$ for all $r > 0$. Hence, if

$$\mu(E \cap B(a, r)) < \mu(E), \quad \text{for some} \quad r > 0,$$

then the atomicity of E implies that $\mu(E \cap B(a, r)) = 0$, a contradiction. On the other hand, if $\mu(E \cap B(a, r)) = \mu(E)$ for all $r > 0$, then by using the Q-Ahlfors property of μ, we obtain $0 < \mu(E) \leq C(2r)^Q$, thus $\mu(E) = 0$ as $r \to 0^+$ which is also a contradiction. Besides, suppose that C_φ is compact on all $L^{p(\cdot)}(X)$ and we consider for $\epsilon \in (0, +\infty)$ the set

$$U_\epsilon := \{x \in X : u_\varphi(x) > \epsilon\}.$$

If $\mu(U_\epsilon) > 0$ for some $\epsilon > 0$, then from the non-atomicity of μ it follows that there exists a decreasing sequence $\{U_n\}$ such that

$$\{U_n\} \subset U_\epsilon \;\text{ with }\; 0 < \mu(U_n) < 1/n \;\text{ for any, }\; n \in \mathbb{N}.$$

Let's construct a bounded sequence in $L^{p(\cdot)}$-norm that is not equi-integrable in $L^{p(\cdot)}$; given $\delta > 0$, since $\varphi : X \to X$ is non-singular, choose $\rho > 0$ such that

$$\mu(\varphi^{-1}(S)) < \delta, \;\text{ whenever }\; \mu(S) < \rho.$$

For this $\rho > 0$, there exists $N > 0$ which $1/N \leq \rho$. Now, since $\mu(U_n) > 0$ for all $n \in \mathbb{N}$ by [17, Lemma 3.3.31] we can fix $x_n \in U_n$ satisfying $\mu(U_n \cap B(x_n, \delta)) > 0$. Consider the set $A_\delta \in \mathcal{B}_0$, given by

$$A_\delta := \varphi^{-1}(U_N \cap B(x_N, \delta)),$$

it is clear that $\mu(A_\delta) < \delta$. Moreover, by using Lemma 2, the function $f_N : X \to \mathbb{R}$ given by

$$f_N := \mu(B(x_N, \delta))^{-\frac{1}{p(\cdot)}} \chi_{U_N \cap B(x_N,\delta)}(\cdot) \in B_{L^{p(\cdot)}},$$

is such that

$$\begin{aligned}\int_{A_\delta} (C_\varphi f_N)(x)^{p(x)}\, d\mu &\geq \int_{\varphi^{-1}(U_N \cap B(x_N,\delta))} \left(\frac{1}{\mu(B(x_N,\delta))}\right)^{\frac{p(x)}{p_\varphi(x)}} d\mu \\ &\geq \mu(B(x_N,\delta))^{\frac{p_B^+ - p_B^-}{p_B^+}} \int_{U_N \cap B(x_N,\delta)} \frac{u_\varphi(x)}{\mu(B(x_N,\delta))}\, d\mu(x) \\ &\geq \epsilon\, C_0\, \frac{\mu(U_N \cap B(x_N,\delta))}{\mu(B(x_N,\delta))}.\end{aligned}$$

By differentiation,

$$\lim_{r\to 0^+} \frac{\mu(U_N \cap B(x_N, r))}{\mu(B(x_N, r))} = 1.$$

Therefore, taking $\delta \in (0, +\infty)$ small enough, we can suppose that we have the following lower bound:

$$\frac{\mu(U_N \cap B(x_N,\delta))}{\mu(B(x_N,\delta))} \geq \frac{1}{2}.$$

Thus,

$$\int_{A_\delta} (C_\varphi f_N)(x)^{p(x)}\, d\mu \geq \epsilon\, C_0/2, \quad \text{for } \delta \text{ small enough.}$$

By switching to a subsequence if necessary, we have that the sequence $\{C_\varphi f_n\}$ is not equi-integrable in $L^{p(\cdot)}(X)$. Thus, by virtue of [14, Theorem 1], this contradicts the compactness of C_φ. Consequently, $\mu(U_\epsilon) = 0$ for all $\epsilon \in (0, +\infty)$ or equivalently $u_\varphi(x) = 0$ a.e. in $x \in X$. This implies that,

$$\begin{aligned}\int_{\varphi^{-1}(B)} |C_\varphi f|^{p(x)}\, d\mu &\leq \sum_{j=1,2} \int_{\varphi^{-1}(B)} |C_\varphi f_j|^{p^\pm}\, d\mu \\ &\leq \sum_{j=1,2} \int_B |f_j(x)|^{p^\pm}\, u_\varphi(x)\, d\mu = 0,\end{aligned}$$

that is, $C_\varphi f = 0$ on $\varphi^{-1}(B)$ for all $B \in \mathcal{B}_0$ and it is enough for to get $C_\varphi = 0$. □

4. Some properties for T_φ

4.1. $L^{p(\cdot)}$-boundedness of T_φ

In this second part, we provide a complete characterization of the continuity for composition operators in the framework of variable Lebesgue spaces.

Theorem 4. *Let μ with doubling property on X, $p(\cdot) \in \mathcal{P}(X)$, and $\varphi : X \to X$ be a non-singular Borel measurable map. Then, the composition operator T_φ maps space $L^{p(\cdot)}(X)$ into $L^{p_\varphi(\cdot)}(X)$ if and only if the function $x \mapsto u_\varphi(x)^{1/p(x)}$ is essentially bounded, that is, $u_\varphi(\cdot)^{1/p(\cdot)} \in L^\infty(X)$. Moreover,*

$$\left\|T_\varphi\right\| = \operatorname*{ess\,sup}_{x\in X} \left\{u_\varphi(x)^{\frac{1}{p(x)}}\right\}.$$

Remark 2. *Theorem 4 generalizes the well-known result in the framework of Lebesgue spaces with constant exponent. More precisely, when $p(\cdot) = p \geq 1$, then both the induced exponent and the space induced by the measurable map φ remain invariant, that is,*

$$p_{\varphi}(\cdot) = p \text{ and } L^{p_{\varphi}(\cdot)}(X) = L^{p}(X).$$

Remark 3. *Since $f \in L^{p(\cdot)}(X)$ if and only if $f^{p(\cdot)-1} \in L^{p'(\cdot)}(X)$ (the same is true for $p'(\cdot)$), by Theorem 4, it follows that T_{φ} maps $L^{p(\cdot)}(X)$ into $L^{p_{\varphi}(\cdot)}(X)$ if and only if T_{φ} maps $L^{p'(\cdot)}(X)$ into $L^{p'_{\varphi}(\cdot)}(X)$. An interesting question is whether this is true for the operator C_{φ} acting on $L^{p'(\cdot)}$ and $L^{p(\cdot)}$.*

Remark 4. *In the case that*

$$L^{p_{\varphi}(\cdot)}(X) \hookrightarrow L^{p(\cdot)}(X). \tag{4.1}$$

Theorem 4 provided a sufficient condition for the operator T_{φ} to map $L^{p(\cdot)}(X)$ into itself. In fact, the embedding (4.1) provided a class of maps $\varphi : X \to X$ induced composition operators on $L^{p(\cdot)}(X)$.

Remark 5. *Note that from Remark 4, the embedding condition (4.1) can be modified by assuming that the variable exponent decays to infinity; for example, assume $p_{\varphi}(x) \geq p(x)$ a.e. in $x \in X$ (e.g., see* [9, Theorem 2.45]*) and $p(\cdot) \in LH_{\infty}(X)$.*

Remark 6. *If the function $x \mapsto u_{\varphi}(x)$ is bounded, then for any $p(\cdot) \in \mathcal{P}(X)$ the function $x \to u_{\varphi}(x)^{1/p(x)}$ is also bounded. So, by Theorem 4, we obtain the following weak inequality: there exists $C > 0$ depending of φ such that for each $t > 0$ and $f \in L^{p(\cdot)}(X)$,*

$$\left\| t\, \mathcal{X}_{\{x:(f\circ\varphi)(x)>t\}} \right\|_{p_{\varphi}(\cdot)} \leq C\, \|f \circ \varphi\|_{p_{\varphi}(\cdot)} \leq C\, \|f\|_{p(\cdot)}.$$

Proof. (Proof of Theorem 4) Suppose that T_{φ} maps $L^{p(\cdot)}(X)$ into $L^{p_{\varphi}(\cdot)}(X)$. From *closed graph theorem* there exists $C > 0$ such that $\|T_{\varphi}\| \leq C$. Let us show that $u_{\varphi}(\cdot)^{1/p(\cdot)} \in L^{\infty}(X)$; note first that $u_{\varphi}(\cdot)^{1/p(\cdot)} \in L^{1}_{loc}(X)$, so let $B \in \mathcal{B}_0$ and define the function $f : X \to \mathbb{R}$ by

$$f(x) = \mu(B)^{-\frac{1}{p(x)}}\, \mathcal{X}_B(x), \quad x \in \mathbb{R}. \tag{4.2}$$

Hence, it is clear that $\rho(f/\lambda) \leq 1$ for all $\lambda \geq 1$ so $f \in L^{p(\cdot)}(X)$ and $\|f\|_{p(\cdot)} = 1$. This implies $f \circ \varphi \in L^{p_{\varphi}(\cdot)}(X)$ and $\|f \circ \varphi\|_{p_{\varphi}(\cdot)} \leq C$, that is, there exists $\lambda_0 > 0$, which $\lambda_0 < C$ and

$$\int_{\varphi^{-1}(B)} \lambda_0^{-p_{\varphi}(x)}\, \mu(B)^{-\frac{p_{\varphi}(x)}{p_{\varphi}(x)}}\, d\mu \leq 1.$$

Consequently,

$$\begin{aligned} \frac{1}{\mu(B)} \int_B C^{-p(x)}\, u_{\varphi}(x)\, d\mu &= \frac{1}{\mu(B)} \int_{\varphi^{-1}(B)} C^{-p_{\varphi}(x)}\, d\mu \\ &\leq \int_{\varphi^{-1}(B)} \lambda_0^{-p_{\varphi}(x)}\, \mu(B)^{-\frac{p_{\varphi}(x)}{p_{\varphi}(x)}}\, d\mu \\ &\leq 1, \end{aligned}$$

by differentiation, it follows that the map $x \mapsto C^{-p(x)} u_{\varphi}(x)$ is essentially bounded. Reciprocally, denote by M_{φ} the multiplication operator with symbol $u_{\varphi}(\cdot)^{1/p(\cdot)}$ so if $u_{\varphi}(\cdot)^{1/p(\cdot)} \in L^{\infty}(X)$, then for each $f \in L^{p(\cdot)}(X)$,

$$\left|(M_{u_{\varphi}} f)(x)\right| \leq \operatorname*{ess\,sup}_{y} \left\{ u_{\varphi}(y)^{1/p(y)} \right\}\, |f(x)|, \quad \text{a.e. in } x \in X.$$

Since $L^{p(\cdot)}(X)$ is a lattice,

$$\|M_\varphi f\|_{p(\cdot)} \leq \operatorname*{ess\,sup}_y \left\{u_\varphi(y)^{1/p(y)}\right\} \|f\|_{p(\cdot)}.$$

Hence,

$$\begin{aligned}
\|f \circ \varphi\|_{p_\varphi(\cdot)} &= \inf\left\{\lambda > 0 : \int_X \lambda^{-p(\varphi(x))} \, |f(\varphi(x))|^{p(\varphi(x))} \, d\mu \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : \int_X \lambda^{-p(x)} \, |f(x)|^{p(x)} \, u_\varphi(x) \, d\mu \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : \int_X \lambda^{-p(x)} \, |(M_{u_\varphi} f)(x)|^{p(x)} \, d\mu \leq 1\right\} \\
&= \|M_{u_\varphi} f\|_{p(\cdot)} \\
&\leq \operatorname*{ess\,sup}_y \left\{u_\varphi(y)^{1/p(y)}\right\} \|f\|_{p(\cdot)}
\end{aligned}$$

that is, T_φ maps $L^{p(\cdot)}(X)$ into $L^{p_\varphi(\cdot)}(X)$ and $\|T\| \leq \operatorname{ess\,sup}_y \left\{u_\varphi(y)^{1/p(y)}\right\}$, computing with the normalized functions given in the equality of the norm (4.2). □

4.2. Compactness for T_φ

In the study of the compactness of C_φ over $L^{p(\cdot)}$, the presence of non-atomic sets was a consequence of the regular Ahlfors structure of the space, which was necessary to have at least one continuous composition operator. For the case of the operator T_φ, continuity only requires that the space admits a doubling measure, which is a weaker hypothesis than the Ahlfors regularity. Therefore, to guarantee no atomic sets, we assume that the space is connected.

Lemma 3. *Every connected doubling metric measure space (X, d, μ) does not contain atoms.*

Proof. Assume that $E \in \mathcal{B}_0$ is an atom, then $\mu(E) > 0$, which implies $\mu(E \cap B(w, r)) > 0$ for some $w \in E$ and all $r > 0$. Hence, in the case $\mu(E \cap B(w, r)) = \mu(E)$ by using [4, Lemma 3.7], there are constants $C, \sigma > 0$ such that

$$\mu(E) \leq \mu(B(w, r)) \leq C \, r^\sigma \, R^{-\sigma} \, \mu(B(w, R)), \; R > r > 0,$$

from here that, $\mu(E) = 0$ as $r \to 0^+$, which is a contradiction. On another case, $\mu(E \cap B(w, r)) < \mu(E)$, but the atomicity of E implies that $\mu(E \cap B(w, r)) = 0$, which is a contradiction. □

As a consequence of Lemma 3, the following result is well-know in the framework of non-atomic metric measure spaces (see, [2, Theorem 4.2] and [10, Theorems 5.2 and 5.3]).

Theorem 5. *Assume that (X, d, μ) is a metric measure space with doubling measure, $r(\cdot) \in \mathcal{P}(X)$ such that $1 \leq p^- \leq p^+ < +\infty$, and $\varphi : X \to X$ Borel non-singular map. If X is connected, then the space $L^{r(\cdot)}(X)$ does not admit compact composition operators T_φ.*

4.3. Weak compactness of T_φ on $L^{p(\cdot)}([0,1])$

In the case $1 < p^- \le p^+ < +\infty$, it is well-know that $L^{p(\cdot)}$ is a reflexive space, so every composition operator on $L^{p(\cdot)}$ is weakly compact. The non-reflective case ($p^- = 1$) is different and has been explored in [29] for the constant exponent case. In the following theorem, we provide some results in this direction.

Denote by λ the Lebesgue measure on $[0,1]$ or $\mathbb{R}$ and $\Omega_1 := p^{-1}(\{1\})$ for an exponent $p(\cdot)$.

Theorem 6. *Let $r(\cdot) \in \mathcal{P}([0,1])$ with $\lambda(\Omega_1) = 0$ and $r^+ < \infty$. Let $\varphi : [0,1] \to [0,1]$ be a non-singular map such that*

$$T_\varphi : L^{r(\cdot)}([0,1]) \to L^{r_\varphi(\cdot)}([0,1]).$$

Then, the following properties hold:

(w.1) *T_φ maps relatively weakly compact subsets into relatively weakly compact subsets.*

(w.2) *Let $v_\varphi(\cdot) := r(\cdot)^{1/p(\cdot)}$. The operator T_φ is relatively weakly compact if and only if the multiplication operator M_{v_φ} is relatively weakly compact.*

(w.3) *Let $r^- = 1$. If*

$$M := \inf_{z\in\Omega\setminus\Omega_1} u_\varphi(z) > 0,$$

then the operator $T_\varphi : L^{r(\cdot)}([0,1]) \to L^{r_\varphi(\cdot)}([0,1])$ is not weakly compact.

Proof. To prove $(w1)$, assume that $M > 0$; then since $r^- = 1$, we may choose a sequence $(z_n) \subset [0,1] \setminus \Omega_1$ satisfying $r(z_n) \to 1$ when $n \to +\infty$. So, denote by

$$B := B_{L^{r(\cdot)}} = \{f \in L^{r(\cdot)}([0,1]) : \|f\|_{r(\cdot)} \le 1\}.$$

Let us reason by contradiction that if T_φ is weakly compact, then the subset $T_\varphi(B)$ is relatively weakly compact in $L^{r_\varphi(\cdot)}([0,1])$. Hence, by [18, Theorem 4.3] we have

$$\lim_{\lambda\to 0^+} \sup_{f\in B} \lambda^{-1} \int_{[0,1]} |\lambda|^{r_\varphi(z)}\, |(T_\varphi f)(z)|^{r_\varphi(z)}\, dz = 0. \tag{4.3}$$

Since,

$$\lambda^{-1} \int_{[0,1]} |\lambda|^{r_\varphi(z)}\, |(T_\varphi f)(z)|^{r_\varphi(z)}\, dz = \lambda^{-1} \int_{[0,1]} |\lambda|^{r(z)}\, |f(z)|^{r(z)}\, u_\varphi(z)\, dz, \tag{4.4}$$

by (4.3) we get

$$\lim_{\lambda\to 0^+} \sup_{f\in B} \lambda^{-1} \int_{[0,1]} |\lambda|^{r(z)}\, |f(z)|^{r(z)}\, u_\varphi(z)\, dz = 0. \tag{4.5}$$

Now, let $\mathcal{I}'_0 := \{[a,b] \in \mathcal{I}_0 : [a,b] \subset [0,1]\}$ and we consider the functions $f : \Omega \to \mathbb{R}$ given by

$$f(x) := f_{ab}(x) := (b-a)^{-1/r(x)}\, \mathcal{X}_{[a,b]}(x), \quad x \in [0,1].$$

So, it is clear that $\{f_{ab} : [a,b] \in \mathcal{I}'_0\} \subset B$ and thus from (4.5), given $\epsilon > 0$, there exists $\lambda_0 > 0$ small such that

$$\frac{1}{b-a} \int_a^b \lambda_0^{r(z)-1}\;\; u_\varphi(z)\, dz < \epsilon, \quad \forall\ [a,b] \in \mathcal{I}'_0$$

by differentiation,

$$\lambda_0^{r(z)-1}\, u_\varphi(z) < \epsilon, \text{ a.e. in } z \in [0,1] \setminus \Omega_1.$$

In particular, taking $z = z_n$ and $n \to +\infty$, we obtained $M < \liminf u_\varphi(z_n) = 0$ because ϵ is taken arbitrarily so $M < 0$, which is a contradiction. The property ($w2$) follows from (4.4), [18, Theorem 4.3] and from the fact $\mathcal{U}(\varphi) < \infty$. Finally, the property $w3$ follows from 4.4 and [18, Theorem 4.3]. □

Remark 7. *The proof of Theorem* 6 *can be easily extended to $L^{r(\cdot)}(\mathbb{R})$ by applying results recently obtained in* [19, Section 5] *under the restriction that Ω_1 be a null-set.*

The case $\lambda(\Omega_1) > 0$ ***and*** $r^- = 1$**:** Suppose that $\varphi^{-1}(\Omega_1) \subset \Omega_1$, let us choose $z_0 \in \mathbb{R}$ such that $p(z_0) = 1$ and for each $n \in \mathbb{N}$ define $I_n := (z_0 - 1/2^n, z_0 + 1/2^n)$ so $I_n \cap \Omega_1 \neq \emptyset$ for each $n \in \mathbb{N}$. We show that $T_\varphi(B)$ is not relatively weakly compact provided that $M > 0$; define the sequence of measurable functions $\{f_n\}$ by

$$f_n(x) := \frac{1}{\lambda(I_n)}\, \mathcal{X}_{I_n \cap \Omega_1}(x), \quad x \in \mathbb{R}.$$

It is clear that $f_n \in B$ for every $n \in \mathbb{N}$. However, from $u_\varphi(x) \geq M$ a.e. $x \in \mathbb{R}$, we obtain

$$\int_{\varphi^{-1}(I_n)\cap\Omega_1} (f_n \circ \varphi)(x)\, dx \geq \int_{I_n\cap\Omega_1} f_n(x)\, u_\varphi(x)\, dx \geq M\, \frac{\lambda(I_n \cap \Omega_1)}{\lambda(I_n)}.$$

Taking a subsequence if necessary, let us make $\lambda(I_n) \to 0$, this means $\lambda(\varphi^{-1}(I_n)) \to 0$ and by differentiation

$$\frac{\lambda(I_n \cap \Omega_1)}{\lambda(I_n)} \to 1, \quad n \to +\infty.$$

Therefore,

$$\liminf \int_{\varphi^{-1}(I_n)\cap\Omega_1} (f_n \circ \varphi)(x)\, dx \geq M > 0, \quad \mu(\varphi^{-1}(I_n)) \to 0.$$

Applying [19, Proposition 5.11], we yields the assertion for $T_\varphi(B)$.

5. Conclusions

Lebesgue spaces with variable integrability have proven to be an excellent framework for partial differential equations with non-standard growth. In particular, eigenvalue problems have been analyzed in Sobolev spaces with a constant exponent, where the composition operator has revealed connections with eigenvalue estimates. In this direction, one can not only study the applicability of the composition operator on Sobolev spaces of variable integrability in an n-dimensional Euclidean domain but also extend these techniques when defining such spaces on a complete Riemannian manifold. Therefore, in this work, we outline a possible direction of study within the framework of non-standard function spaces.

Author contributions

Javier Henríquez-Amador and Carlos Álvarez: Writing-original draft, formal analysis and commenting; Eiver Rodríguez and John Millán: Commenting and review. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors thank the anonymous referees for the useful suggestions to improve this article. Carlos F. Álvarez and Jhon Millán were partially supported by inner project BASEX-PD/2024-02 of University of Sinu, Cartagena.

Conflict of interest

We declare there are no conflicts of interest associated with this work

References

1. S. C. Arora, G. Datt, S. Verma, Composition operators on Lorentz spaces, *Bull. Austral. Math. Soc.*, **76** (2007), 205–214. https://doi.org/10.1017/S0004972700039599
2. D. Bajaj, G. Datt, Composition operators on variable exponent Lebesgue spaces, *Anal. Math.*, **50** (2024), 345–366. https://doi.org/10.1007/s10476-024-00015-y
3. P. Bevanda, S. Sosnowski, S. Hirche, Koopman operator dynamical models: Learning, analysis and control, *Annu. Rev. Control*, **52** (2021), 197–212. https://doi.org/10.1016/j.arcontrol.2021.09.002
4. A. Björn, J. Björn, *Nonlinear potential theory on metric spaces*, European Mathematical Society, 2011. https://doi.org/10.4171/099
5. G. Bourdaud, Changes of variable in Besov spaces. II, *Forum Math.*, **12** (2000), 545–563. https://doi.org/10.1515/form.2000.018
6. G. Bourdaud, M. Moussai, Continuity of composition operators in Sobolev spaces, *Ann. I. H. Poincaré C Anal. Non Linéaire*, **36** (2019), 2053–2063. https://doi.org/10.1016/J.ANIHPC.2019.07.002
7. C. C. Cowen, B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. https://doi.org/10.1201/9781315139920
8. N. Črnjarić-Žic, S. Maćešić, I. Mezić, Koopman operator spectrum for random dynamical systems, *J. Nonlinear Sci.*, **30** (2020), 2007–2056. https://doi.org/10.1007/s00332-019-09582-z
9. D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-0348-0548-3
10. G. Datt, D. S. Bajaj, A. Fiorenza, Weighted composition operators on variable exponent Lebesgue spaces, *Adv. Oper. Theory*, **9** (2024), 68. https://doi.org/10.1007/s43036-024-00366-1
11. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

12. L. Diening, P. Hästö, A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, in FSDONA04 Proceedings, Milovy, Czech Republic, **66** (2004), 38–58. Available from: `https://www.problemsolving.fi/pp/opFinal.pdf`.

13. V. Gol'dshtein, A. Ukhlov, About homeomorphisms that induce composition operators on Sobolev spaces, *Complex Var. Elliptic*, **55** (2010), 833–845. https://doi.org/10.1080/17476930903394705

14. P. Górka, A. Macios, Almost everything you need to know about relatively compact sets in variable Lebesgue spaces, *J. Funct. Anal.*, **269** (2015), 1925–1949. https://doi.org/10.1016/j.jfa.2015.06.024

15. P. Harjulehto, P. Hästö, M. Pere, Variable exponent Lebesgue spaces on metric spaces: The Hardy-Littlewood maximal operator, *Real Anal. Exch.*, **30** (2004), 87–104.

16. R. Hataya, Y. Kawahara, Glocal hypergradient estimation with Koopman operator, *arXiv preprint*, 2024. https://doi.org/10.48550/arXiv.2402.02741

17. J. Heinonen, P. Koskela, N. Shanmugalingam, J. T. Tyson, *Sobolev spaces on metric measure spaces*, Cambridge University Press, **27** (2015). https://doi.org/10.1017/CBO9781316135914

18. F. L. Hernández, C. Ruiz, M. Sanchiz, Weak compactness in variable exponent spaces, *J. Funct. Anal.*, **281** (2021), 109087. https://doi.org/10.1016/j.jfa.2021.109087

19. F. L. Hernández, C. Ruiz, M. Sanchiz, Weak compactness and representation in variable exponent Lebesgue spaces on infinite measure spaces, *RACSAM Rev. R. Acad. A*, **116** (2022), 152. https://doi.org/10.1007/s13398-022-01298-2

20. M. Ikeda, I. Ishikawa, K. Taniguchi, Boundedness of composition operators on higher order Besov spaces in one dimension, *Math. Ann.*, **388** (2024), 4487–4510. https://doi.org/10.1007/s00208-023-02637-3

21. A. Karapetyants, J. E. Restrepo, Composition operators on holomorphic variable exponent spaces, *Math. Method. Appl. Sci.*, **45** (2022), 8566–8577. https://doi.org/10.1002/mma.7307

22. V. M. Kokilashvili, A. Meskhi, H. Rafeiro, S. G. Samko, *Integral operators in non-standard function spaces*, Springer, **248** (2016). https://doi.org/10.1007/978-3-319-21018-6

23. B. O. Koopman, Hamiltonian systems and transformation in Hilbert space, *Proc. Natl. Acad. Sci.*, **17** (1931), 315–318. https://doi.org/10.1073/pnas.17.5.315

24. B. O. Koopman, J. von Neumann, Dynamical systems of continuous spectra, *Proc. Natl. Acad. Sci.*, **18** (1932), 255–263. https://doi.org/10.1073/pnas.18.3.255

25. L. Maligranda, *Hidegoro Nakano (1909–1974)—On the centenary of his birth*, In: International Symposium on Banach and Function Spaces, Yokohama Publishers, 2011, 99–171.

26. A. Morovatpoor, A. Abkar, Boundedness and compactness of composition operators on variable exponent Bergman spaces, *Mediterr. J. Math.*, **17** (2020), 9. https://doi.org/10.1007/s00009-019-1441-8

27. P. Rosenthal, *Composition operators and classical function theory*, Composition operators on function spaces, Bull. Amer. Math. Soc. (N.S.), 1995, 150–153. https://doi.org/10.1090/S0273-0979-1995-00562-8

28. R. K. Singh, J. S. Manhas, *Composition operators on function spaces*, North-Holland Mathematics Studies, North-Holland, Amsterdam, **179** (1993). https://doi.org/10.1016/s0304-0208(08)x7086-0

29. H. Takagi, Compact weighted composition operators on L^p, *Proc. Amer. Math. Soc.*, **116** (1992), 505–511. https://doi.org/10.2307/2159759

30. S. K. Vodopyanov, Mappings of homogeneous groups and imbeddings of functional spaces, *Siberian Math. J.*, **30** (1989), 685–698. https://doi.org/10.1007/BF00971258

31. S. K. Vodopyanov, Composition operators on Sobolev spaces, *Contemp. Math.*, **382** (2005), 401.

32. C. Zhang, E. Zuazua, A quantitative analysis of Koopman operator methods for system identification and predictions, *Comptes Rendus. Mécanique*, **351** (2023), 1–31. https://doi.org/10.5802/crmeca.138