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On ideal matrices whose entries are the generalized $k-$Horadam numbers

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Abstract: Ideal matrices, which generalize circulant and $r-$circulant matrices, play a key role in Ajtai's construction of collision-resistant hash functions. In this paper, we study ideal matrices whose entries are the generalized $k-$Horadam numbers, which represent a generalization of second-order sequences and include many well-known sequences such as Fibonacci, Lucas, and Pell numbers as special cases. We derive two explicit formulas for calculating the eigenvalues and determinants of these matrices. Additionally, we obtain upper bounds for the spectral norm and the Frobenius norm of ideal matrices with generalized $k-$Horadam number entries. These results not only extend existing findings on ideal matrices but also highlight the versatility and applicability of generalized $k-$Horadam numbers in matrix theory and related fields.

Keywords: ideal matrix; generalized $k-$Horadam numbers; spectral norms; eigenvalues
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1. Introduction

In recent years, circulant matrices and $r-$circulant matrices have been two of the most interesting research areas in the fields of computational mathematics and applied mathematics. Many scholars have done a lot of meaningful work on this. It is well known that these matrices have a wide range of applications in coding theory, signal processing, image processing, digital image disposal, and linear forecasting in [4, 20, 21, 26]. Further, ideal matrices are more general circulant matrices and $r-$circulant matrices. Therefore, they not only play an important role in the above aspects but also have significance in the study of ideal lattices [12, 18].

Many generalizations of the circulant matrices with special entries have been introduced and studied [13, 16, 17, 23, 24]. For example, Shen, Cen, and Hao [17] computed the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. In 2012, Yazlik and Taskara [23] defined circulant matrices whose entries are the generalized $k-$Horadam numbers and calculated the spectral norms, eigenvalues, and determinants of the matrices. The generalized $k-$Horadam sequence is a

generalization of the special second-order sequences or polynomials. Inspired by these studies, we aim to extend this line of research by studying ideal matrices whose entries are the generalized $k-$Horadam numbers. We compute the eigenvalues and determinants of these matrices and establish novel upper bounds for the spectral norm and the Frobenius norm of ideal matrices with generalized $k-$Horadam number entries. This work not only generalizes existing results on circulant and $r-$circulant matrices but also contributes to the broader understanding of ideal matrices and their algebraic properties.

Horadam sequences, introduced by A. F. Horadam [8] in 1965, have since found applications across a wide range of fields, including geometry, combinatorics, approximation theory, statistics, cryptography, and physics, due to their ability to model various natural and theoretical phenomena. The study of these sequences has become particularly prominent in matrix theory, where they play a crucial role in areas such as coding theory, signal processing, image processing, and digital image analysis. In recent years, many scholars have made significant contributions to the study of Horadam sequences. For example, W. M. Abd-Elhameed et al. [1] explored a Horadam-type of generalized numbers involving four parameters and presented several new identities related to them. Prasad et al. [27] extended the k-Horadam numbers to third order and studied their properties. W. M. Abd-Elhameed [2] established new connection formulae for Fibonacci and Lucas polynomials, extending known results and deriving novel applications. Frontczak [7] explored Horadam identities using binomial coefficients and gave many other known identities. G. Y. Şentürk et al. [15] investigated the algebraic properties of Horadam numbers and provided matrix representations.

In [22], the generalized $k-$Horadam sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$ was defined as follows:

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \quad H_{k,0} = a, \quad H_{k,1} = b \ (a, b \in \mathbb{R}), \tag{1.1}$$

where $k \in \mathbb{R}^+$, $f(k), g(k)$ are scalar-valued polynomials, and $f(k)^2 + 4g(k) > 0$. Clearly, if $f(k) = g(k) = 1, a = 0$ and $b = 1$, the well-known Fibonacci number is obtained. If $f(k) = g(k) = 1, a = 2$ and $b = 1$, we obtain the famous Lucas number. If $f(x) = 2x, g(x) = -1, a = 1$ and $b = x$, then we have the Chebyshev polynomial. Note that for Chebyshev polynomials, the condition $f(k)^2 + 4g(k) > 0$ is satisfied only when $x^2 > 1$. For $x \in [-1, 1]$, the characteristic roots of the recurrence relation are complex. This is consistent with the oscillatory nature of Chebyshev polynomials in this interval, as they can be expressed in terms of trigonometric functions that remain real for $x \in [-1, 1]$. That is to say, if we take an appropriate value with $f(k), g(k), a, b$ in (1.1), then we can get the familiar second-order sequences.

Let α and β be the roots of the characteristic equation $x^2 - f(k)x - g(k) = 0$ of the sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$. Then the Binet formula of this sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$ has the form

$$H_{k,n} = \frac{X\alpha^n - Y\beta^n}{\alpha - \beta}, \tag{1.2}$$

where $X = b - a\beta$, $Y = b - a\alpha$.

Many scholars have also studied some properties of $r-$circulant matrices with generalized $k-$Horadam numbers [11, 19, 25]. An $r-$circulant matrix is not only a generalization of a classical circulant matrix but is also a special case of an ideal matrix. For example, Yazlik and Taskara [25] obtained upper and lower bounds for the spectral norm of an $r-$circulant matrix whose entries are the generalized $k-$Horadam numbers. In 2018, by using the algebra methods and the properties of the $r-$circulant matrix, Shi [19] gave a new estimate for the spectral norms of an $r-$circulant matrix

involving the generalized $k-$Horadam numbers. Let us take any matrix $A = [a_{i,j}] \in M_{m,n}(\mathbb{C})$; the well-known spectral norm of A is given by

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i(A^*A)}, \tag{1.3}$$

where $\lambda_i(A^*A)$ are the eigenvalues of A^*A, A^* is the conjugate transpose of A. The Frobenius (or Euclidean) norm of A is given by

$$\|A\|_F = (\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}} = \sqrt{\text{trace}(A^*A)}. \tag{1.4}$$

In this paper, we first give the basic properties of ideal matrices and the definition of the ideal matrix with the generalized $k-$Horadam numbers. And then we calculate its eigenvalues and determinants. Finally, we will obtain upper bound estimates of the spectral norm and the Frobenius norm for the ideal matrix. The results to be derived here will turn out to be the most general statements concerning an $r-$circulant matrix and such matrices having as elements the terms of a second order sequence.

2. Some basic properties of ideal matrix

We now introduce a generalization of circulant matrices and $r-$circulant matrices—the so-called ideal matrices, according to the lattice-based cryptosystem theory.

Definition 2.1. *[27] Suppose $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0 \in \mathbb{Z}[x]$, $G_p \in \mathbb{Z}^{n \times n}$ is a square matrix given by*

$$G = G_p = \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ & & & a_1 \\ & I_{n-1} & & \vdots \\ & & & a_{n-1} \end{pmatrix}, \tag{2.1}$$

where I_{n-1} is the $(n-1) \times (n-1)$ unit matrix. Let $h(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_{n-1}x^{n-1} \in \mathbb{C}[x]$. Denote the vector h by

$$h = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{pmatrix}. \tag{2.2}$$

Then an ideal matrix generated by a vector h is defined as

$$G(h) = G_p(h) = [h, Gh, G^2h, \ldots, G^{n-1}h]_{n \times n} \tag{2.3}$$

and each column vector is $G^i h (0 \le i \le n-1)$.

Obviously, $p(x)$ is the characteristic polynomial of G. If $p(x) = x^n - 1$, i.e., $a_1 = a_2 = \cdots = a_{n-1} = 0, a_0 = 1$. So

$$G = G_p = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ & I_{n-1} & & \vdots \\ & & & 0 \end{pmatrix}$$

in (2.1). Then the ideal matrix $G(h)$ generated by h is given by

$$G(h) = G_p(h) = [h, Gh, G^2h, \ldots, G^{n-1}h]_{n \times n}$$

$$= \begin{pmatrix} h_0 & h_{n-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & \cdots & h_3 & h_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{n-2} & h_{n-3} & \cdots & h_0 & h_{n-1} \\ h_{n-1} & h_{n-2} & \cdots & h_1 & h_0 \end{pmatrix},$$

which is known as the classical circulant matrix.

If we let $p(x) = x^n - r$, $r \in \mathbb{C}$, i.e., $a_1 = a_2 = \cdots = a_{n-1} = 0, a_0 = r$. Then

$$G(h) = G_p(h) = [h, Gh, G^2h, \ldots, G^{n-1}h]_{n \times n}$$

$$= \begin{pmatrix} h_0 & rh_{n-1} & rh_{n-2} & \cdots & rh_1 \\ h_1 & h_0 & rh_{n-1} & \cdots & rh_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{n-2} & h_{n-3} & h_{n-4} & \cdots & rh_{n-1} \\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_0 \end{pmatrix}$$

is the $r-$circulant matrix. The $r-$circulant matrix in this paper applies the factor r to the main diagonal entries, unlike the classical $r-$circulant matrix, where the factor r is applied to entries below the main diagonal. This approach simplifies the expression of $G(h)$ by directly using column vectors instead of row vectors with a transpose operation.

Therefore, an ideal matrix is a more general form of a circulant matrix and an $r-$circulant matrix. We know that both circulant matrices and $r-$circulant matrices are special cases of Toeplitz matrices (see [5]), but ideal matrices are not Toeplitz matrices, generally.

Next, we give some basic properties for an ideal matrix $G(h)$.

Lemma 2.1. *For any* $h = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{n-1} \end{pmatrix} \in \mathbb{R}^n$, *we have* $G(h) = h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1}G^{n-1}$, *where I is*

a $n \times n$ *matrix, G is defined as in Eq (2.1).*

Proof. Let $e_1, e_2, \ldots, e_n$ be the n column unit vectors in $\mathbb{R}^n$, i.e.,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

It is easy to check that $h = h_0e_1 + h_1e_2 + \cdots + h_{n-1}e_n$ and $G(g + h) = G(g) + G(h)$ for any $g = (g_0 \ g_1 \ \cdots \ g_{n-1})^T \in \mathbb{R}^n$. Thus, $G(h) = h_0G(e_1) + h_1G(e_2) + \cdots + h_{n-1}G(e_n)$. We claim that $G(e_k) = G^{k-1}(1 \le k \le n)$. We prove it by induction on k. And we know $Ge_i = e_{i+1}(1 \le i \le n - 1)$. If

$k = 1, G(e_1) = I_n$ is trivial. Assume it is true for $k = n-1$, that is $G(e_{n-1}) = (e_{n-1}, Ge_{n-1}, \ldots, G^{n-1}e_{n-1}) = G^{n-2}$. We consider the case of $k = n$,

$$
\begin{aligned}
G(e_n) &= (e_n, Ge_n, \ldots, G^{n-1}e_n) \\
&= (Ge_{n-1}, G^2 e_{n-1}, \ldots, G^n e_{n-1}) \\
&= G(e_{n-1}, Ge_{n-1}, \ldots, G^{n-1}e_{n-1}) \\
&= G \cdot G^{n-2} = G^{n-1}.
\end{aligned}
$$

So $G(h) = h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1}G^{n-1}$. We complete the proof of Lemma 2.1. □

We always assume that $p(x) \in \mathbb{Z}[x]$ is a separable polynomial and $\omega_1, \omega_2, \ldots, \omega_n$ are distinct zeros of p(x). The Vandermonde matrix V generated by $\{\omega_1, \omega_2, \ldots, \omega_n\}$ is

$$
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\omega_1 & \omega_2 & \cdots & \omega_n \\
\cdots & \cdots & \cdots & \cdots \\
\omega_1^{n-1} & \omega_2^{n-1} & \cdots & \omega_n^{n-1}
\end{pmatrix}, \Delta \text{ and } \det(V) \neq 0.
$$

Lemma 2.2. *Let* $l(x) = l_0 + l_1 x + \cdots + l_{n-1}x^{n-1} \in \mathbb{R}[x]$, *then we have*

$$
G(l) = V^{-1} diag\{l(\omega_1), l(\omega_2), \ldots, l(\omega_n)\}V,
$$

where $diag\{l(\omega_1), l(\omega_2), \ldots, l(\omega_n)\}$ *is the diagonal matrix.*

Proof. By Theorem 3.2.5 of [6], for G, we have

$$
G = V^{-1} diag\{\omega_1, \omega_2, \ldots, \omega_n\}V. \tag{2.4}
$$

By Lemma 2.1, it follows that

$$
G(l) = V^{-1} diag\{l(\omega_1), l(\omega_2), \ldots, l(\omega_n)\}V.
$$

□

Lemma 2.3. *Let* $g, h \in \mathbb{R}^n$ *be two column vectors and* $G(g)$ *be the ideal matrix generated by g, then we have*

 (i) $G(g)G(h) = G(h)G(g)$;
 (ii) $G(g)G(h) = G(G(g)h)$;
(iii) $\det(G(g)) = \prod_{i=1}^n g(\omega_i)$;
 (iv) $G(g)$ *is an invertible matrix if and only if p(x) and g(x) are coprime, i.e.,* $\gcd(p(x), g(x)) = 1$.

Proof. (i) By Lemma 2.1, we have

$$
G(g) = g_0 I + g_1 G + g_2 G^2 + \cdots + g_{n-1}G^{n-1}
$$

and

$$
G(h) = h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1}G^{n-1}.
$$

We know $g_i, h_i \in \mathbb{R}(0 \le i \le n-1)$, $g_i h_i = h_i g_i$. Then

$$
\begin{aligned}
G(g)G(h) &= (g_0 I + g_1 G + g_2 G^2 + \cdots + g_{n-1} G^{n-1})(h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1} G^{n-1}) \\
&= (h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1} G^{n-1})(g_0 I + g_1 G + g_2 G^2 + \cdots + g_{n-1} G^{n-1}) \\
&= G(h)G(g).
\end{aligned}
$$

(ii) By Lemma 2.1, we have

$$
G(g)h = g_0 I h + g_1 G h + g_2 G^2 h + \cdots + g_{n-1} G^{n-1} h,
$$

and

$$
\begin{aligned}
G(G(g)h) =& (g_0 I h_0 + g_1 G h_0 + g_2 G^2 h_0 + \cdots + g_{n-1} G^{n-1} h_0) I \\
&+ (g_0 I h_1 + g_1 G h_1 + g_2 G^2 h_1 + \cdots + g_{n-1} G^{n-1} h_1) G \\
&+ \cdots \\
&+ (g_0 I h_{n-1} + g_1 G h_{n-1} + g_2 G^2 h_{n-1} + \cdots + g_{n-1} G^{n-1} h_{n-}) G \\
=& (g_0 I + g_1 G + g_2 G^2 + \cdots + g_{n-1} G^{n-1})(h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1} G^{n-1}) \\
=& G(g)G(h).
\end{aligned}
$$

(iii) By Lemma 2.2, we have

$$
G(g) = V^{-1} diag\{g(\omega_1), g(\omega_2), \ldots, g(\omega_n)\} V,
$$

then

$$
\begin{aligned}
\det(G(g)) &= \det(diag\{g(\omega_1), g(\omega_2), \ldots, g(\omega_n)\}) \\
&= \prod_{i=1}^{n} g(\omega_i).
\end{aligned}
$$

(iv) It is clear that $p(x) = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_n)$. $G(g)$ is an invertible matrix $\iff g(\omega_i) \ne 0 (1 \le i \le n) \iff \gcd(p(x), g(x)) = 1$. $\qquad \square$

Finally, we give the definition of the ideal matrix with the generalized $k-$Horadam numbers.

Definition 2.2. *Let*

$$
h = \begin{pmatrix} H_{k,0} \\ H_{k,1} \\ \vdots \\ H_{k,n-1} \end{pmatrix}
$$

in Definition 2.1. That is $h(x) = H_{k,0} + H_{k,1} x + H_{k,2} x^2 + \cdots + H_{k,n-1} x^{n-1}$, where $H_{k,i}(0 \le i \le n-1)$ is ith generalized $k-$Horadam number. Then $G(h) = [h, Gh, G^2 h, \ldots, G^{n-1} h]_{n \times n} = h(G) = H_{k,0} I + H_{k,1} G + H_{k,2} G^2 + \cdots + H_{k,n-1} G^{n-1}$ is the ideal matrix with the generalized $k-$Horadam numbers.

Similarly, we can give the definitions of ideal matrices with the Fibonacci and Lucas numbers, respectively. Throughout this paper, the ideal matrix with the generalized $k-$Horadam numbers will be denoted by $K = G(h) = H_{k,0} I + H_{k,1} G + H_{k,2} G^2 + \cdots + H_{k,n-1} G^{n-1}$.

3. Eigenvalues and determinants of K

In this section, we consider the eigenvalues and determinants of ideal matrices with the generalized $k-$Horadam numbers. Our results generalize the results in [11, 17, 25].

Lemma 3.1. *Assume $\omega_1, \omega_2, \ldots, \omega_n$ are the roots of $p(x)$. Then $h(\omega_1), h(\omega_2), \ldots, h(\omega_n)$ are all eigenvalues of the ideal matrix $G(h)$. That is,*

$$\lambda_j(G(h)) = h(\omega_j) = \sum_{i=0}^{n-1} h_i \omega_j^{\ i}, \ \ j = 1, 2, \ldots, n.$$

Proof. Because $p(x)$ is the characteristic polynomial of G, $\omega_1, \omega_2, \ldots, \omega_n$ are all eigenvalues of G. And $G(h) = h(G) = h_0 I + h_1 G + h_2 G^2 + \cdots + h_{n-1} G^{n-1}$, $h(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_{n-1} x^{n-1}$ is any polynomial with degree greater than 0 in $\mathbb{C}$. So $h(\omega_1), h(\omega_2), \ldots, h(\omega_n)$ are all eigenvalues of $G(h)$. $\square$

The method of first studying the eigenvalues of G and the expression of $G(h)$ to subsequently derive the eigenvalues of $G(h)$ is similarly reflected in the approaches of references [3, 9, 10]. In [9], the authors first obtained the eigenvalues of a generalized permutation matrix U, they then used these results to determine the eigenvalues of a generalized weighted circulant matrix C generated by U, where $C = \sum_{r=0}^{k} c_r U^r$. In [3], the eigenvalues of a (k, n)-circulant like matrix $\mathbb{A}(k, n) = \sum_{i=0}^{m-1} a_i \mathbb{Q}^i(k, n)$ were computed based on similar principles. In [10], the eigenvalues of both Q-circulant matrices and Q-circulant matrices whose entries are Horadam numbers were derived. Next, we will also calculate the eigenvalues of the ideal matrix with the generalized $k-$Horadam numbers.

Theorem 3.1. *Let $K = G(h)$ be the ideal matrix with the generalized $k-$Horadam numbers defined in Definition 2.2. Then the eigenvalues of K are given by*

$$\lambda_j(K) = \frac{\omega_j^{\ n} H_{k,n} + g(k)\omega_j^{\ n+1} H_{k,n-1} + (af(k) - b)\omega_j - H_{k,0}}{g(k)\omega_j^{\ 2} + f(k)\omega_j - 1},$$

where $H_{k,n}$ is the nth generalized $k-$Horadam number and ω_j are the roots of $p(x)$, $j = 1, 2, \ldots, n$.

Proof. By Lemma 3.1,

$$\lambda_j(K) = \sum_{i=0}^{n-1} H_{k,i} \omega_j^{\ i}$$

$$= H_{k,0} + H_{k,1}\omega_j + H_{k,2}\omega_j^{\ 2} + \cdots + H_{k,n-1}\omega_j^{\ n-1}.$$

And by Binet's formula (1.2), we have

$$\lambda_j(K) = \sum_{i=0}^{n-1} \frac{X\alpha^i - Y\beta^i}{\alpha - \beta} \omega_j^{\ i}$$

$$= \frac{1}{\alpha - \beta}(X\frac{(\alpha\omega_j)^n - 1}{\alpha\omega_j - 1} - Y\frac{(\beta\omega_j)^n - 1}{\beta\omega_j - 1})$$

$$= \frac{1}{\alpha - \beta} \frac{X(\alpha^n\omega_j^{\ n} - 1)(\beta\omega_j - 1) - Y(\beta^n\omega_j^{\ n} - 1)(\alpha\omega_j - 1)}{(\alpha\omega_j - 1)(\beta\omega_j - 1)}$$

$$= \frac{1}{\alpha - \beta} \frac{X(\alpha^n \beta \omega_j{}^{n+1} - \alpha^n \omega_j{}^n - \beta \omega_j + 1) - Y(\alpha \beta^n \omega_j{}^{n+1} - \beta^n \omega_j{}^n - \alpha \omega_j + 1)}{\alpha \beta \omega_j{}^2 - (\alpha + \beta)\omega_j + 1}.$$

Also, $\alpha\beta = -g(k)$, $\alpha + \beta = f(k)$, then

$$\lambda_j(K) = \frac{1}{-g(k)\omega_j{}^2 - f(k)\omega_j + 1}$$
$$\times \frac{-g(k)X\alpha^{n-1}\omega_j{}^{n+1} - X\alpha^n \omega_j{}^n - X\beta\omega_j + X + g(k)Y\beta^{n-1}\omega_j{}^{n+1} + Y\beta^n\omega_j{}^n + Y\alpha\omega_j - Y}{\alpha - \beta}$$

$$= \frac{1}{-g(k)\omega_j{}^2 - f(k)\omega_j + 1}$$
$$\times \left(\frac{-g(k)\omega_j{}^{n+1}(X\alpha^{n-1} - Y\beta^{n-1})}{\alpha - \beta} - \frac{\omega_j{}^n(X\alpha^n - Y\beta^n)}{\alpha - \beta} - \frac{\omega_j(X\beta - Y\alpha)}{\alpha - \beta} + \frac{X - Y}{\alpha - \beta} \right)$$

$$= \frac{1}{-g(k)\omega_j{}^2 - f(k)\omega_j + 1}\left(-g(k)\omega_j{}^{n+1}H_{k,n-1} - \omega_j{}^n H_{k,n} - \omega_j\frac{X\beta - Y\alpha}{\alpha - \beta} + H_{k,0}\right).$$

Moreover, $X = b - a\beta$, $Y = b - a\alpha$, we obtain

$$\lambda_j(K) = \frac{\omega_j{}^n H_{k,n} + g(k)\omega_j{}^{n+1}H_{k,n-1} + (af(k) - b)\omega_j - H_{k,0}}{g(k)\omega_j{}^2 + f(k)\omega_j - 1}.$$

$\square$

Theorem 3.2. *The determinant of $K = G(h) = [h, Gh, G^2h, \ldots, G^{n-1}h]$ is formulated by*

$$\det(K) = \prod_{j=1}^{n} \frac{\omega_j{}^n H_{k,n} + g(k)\omega_j{}^{n+1}H_{k,n-1} + (af(k) - b)\omega_j - H_{k,0}}{g(k)\omega_j{}^2 + f(k)\omega_j - 1},$$

where $H_{k,n}$ is the nth generalized $k-$Horadam number and ω_j are the roots of $p(x)$, $j = 1, 2, \ldots, n$.

Proof. It is clear that $\det(K) = \prod_{j=1}^{n} \lambda_j(K)$. By Theorem 3.1, which completes the proof. $\square$

Remark 3.1. *If we take $p(x) = x^n - r$ in Theorems 3.1 and 3.2, then K is an $r-$circulant matrix with the generalized $k-$Horadam numbers. The result is reduced to Theorems 7 and 8 in [25], respectively.*

Corollary 3.1. *If we take*

$$h = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{n-1} \end{pmatrix}$$

in Theorem 3.1, where $\{F_i\}_{i \in \mathbb{N}}$ is the Fibonacci sequence. Then K is the ideal matrix with the Fibonacci numbers. In this case, the eigenvalues and determinants of K are

$$\lambda_j(K) = \frac{\omega_j{}^n F_n + \omega_j{}^{n+1}F_{n-1} - \omega_j}{\omega_j{}^2 + \omega_j - 1} \quad (1 \le j \le n)$$

and

$$\det(K) = \prod_{j=1}^{n} \frac{\omega_j{}^n F_n + \omega_j{}^{n+1}F_{n-1} - \omega_j}{\omega_j{}^2 + \omega_j - 1},$$

where ω_j are the roots of $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0 \in \mathbb{Z}[x]$, $j = 1, 2, \ldots, n$.

Proof. Because $\{F_i\}_{i\in\mathbb{N}}$ is the Fibonacci sequence, that is

$$f(k) = g(k) = 1, \ a = 0 \text{ and } b = 1$$

in (1.1). Then by Theorems 3.1 and 3.2, the proof is completed. $\qquad\square$

In Reference [14], the author derived the eigenvalues of a k-circulant matrix with Fibonacci numbers (see Theorem 4) and considered the cases where $\psi\omega^{-j} = \frac{1}{\alpha}$ or $\psi\omega^{-j} = \frac{1}{\beta}$.

Corollary 3.2. *If we take*

$$h = \begin{pmatrix} L_0 \\ L_1 \\ \vdots \\ L_{n-1} \end{pmatrix}$$

in Theorem 3.1, where $\{L_i\}_{i\in\mathbb{N}}$ is the Lucas sequence. Then K is the ideal matrix with the Lucas numbers. In this case, the eigenvalues and determinants of K are

$$\lambda_j(K) = \frac{\omega_j^{\,n} L_n + \omega_j^{\,n+1} L_{n-1} + \omega_j - 2}{\omega_j^{\,2} + \omega_j - 1} \quad (1 \le j \le n)$$

and

$$\det(K) = \prod_{j=1}^{n} \frac{\omega_j^{\,n} L_n + \omega_j^{\,n+1} L_{n-1} + \omega_j - 2}{\omega_j^{\,2} + \omega_j - 1},$$

where ω_j are the roots of $p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0 \in \mathbb{Z}[x]$, $j = 1, 2, \ldots, n$.

Proof. Because $\{L_i\}_{i\in\mathbb{N}}$ is the Lucas sequence, that is

$$f(k) = g(k) = 1, \ a = 2 \text{ and } b = 1$$

in (1.1). Then by Theorems 3.1 and 3.2, the proof is completed. $\qquad\square$

4. Norms of K

We start this section by giving the following lemma.

Lemma 4.1. *Suppose that $\{H_{k,i}\}_{i\in\mathbb{N}}$ is a generalized $k-$Horadam sequence defined in (1.1). N is an arbitrary constant. The following conclusions hold.*

(i) If $Nf(k) + N^2 g(k) \ne 1$, then

$$\sum_{i=0}^{n-1} N^i H_{k,i} = \frac{a - N(af(k) - b) - N^n H_{k,n} - g(k)N^{n+1} H_{k,n-1}}{1 - Nf(k) - N^2 g(k)}.$$

(ii) If $Nf(k) + N^2 g(k) = 1$, then

$$\sum_{i=0}^{n-1} N^i H_{k,i} = \frac{a + nbN + naN^2 g(k) - N^n H_{k,n}}{1 + N^2 g(k)}.$$

Proof. (*i*) According to (1.1), we have

$$H_{k,-1} = \frac{H_{k,1} - f(k)H_{k,0}}{g(k)} = \frac{b - af(k)}{g(k)} \tag{4.1}$$

and

$$H_{k,-2} = \frac{H_{k,0} - f(k)H_{k,1}}{g(k)} = \frac{a}{g(k)} - \frac{f(k)(b - af(k))}{g^2(k)}. \tag{4.2}$$

Shifting the summation index, we obtain

$$\sum_{i=0}^{n-1} N^i H_{k,i} = \sum_{i=0}^{n-1} N^i (f(k)H_{k,i-1} + g(k)H_{k,i-2})$$

$$= f(k) \sum_{i=0}^{n-1} N^i H_{k,i-1} + g(k) \sum_{i=0}^{n-1} N^i H_{k,i-2}$$

$$= f(k) \left(\sum_{i=0}^{n-1} N^{i+1} H_{k,i} + H_{k,-1} - N^n H_{k,n-1} \right)$$

$$+ g(k) \left(\sum_{i=0}^{n-1} N^{i+2} H_{k,i} + H_{k,-2} + N H_{k,-1} - N^n H_{k,n-2} - N^{n+1} H_{k,n-1} \right).$$

Then by direct calculation and (4.1), (4.2), we have

$$(1 - Nf(k) - N^2 g(k)) \sum_{i=0}^{n-1} N^i H_{k,i}$$

$$= f(k)H_{k,-1} - f(k)N^n H_{k,n-1} + g(k)H_{k,-2} + g(k)N H_{k,-1} - g(k)N^n H_{k,n-2} - g(k)N^{n+1} H_{k,n-1}$$

$$= f(k)\frac{b - af(k)}{g(k)} - f(k)N^n H_{k,n-1} + a - \frac{f(k)(b - af(k))}{g(k)}$$

$$+ N(b - af(k)) - g(k)N^n H_{k,n-2} - g(k)N^{n+1} H_{k,n-1}$$

$$= a - N(af(k) - b) - N^n H_{k,n} - g(k)N^{n+1} H_{k,n-1}.$$

Because $Nf(k) + N^2 g(k) \neq 1$, we obtain that

$$\sum_{i=0}^{n-1} N^i H_{k,i} = \frac{a - N(af(k) - b) - N^n H_{k,n} - g(k)N^{n+1} H_{k,n-1}}{1 - Nf(k) - N^2 g(k)}.$$

(*ii*) If $Nf(k) + N^2 g(k) = 1$, then $1 + N^2 g(k) \neq 0$. Because by the definition of $k-$Horadam numbers, we know $f^2(k) + 4g(k) > 0$. And $f(k) = \frac{1 - N^2 g(k)}{N}$. Hence, we have

$$\frac{(1 - N^2 g(k))^2 + 4N^2 g(k)}{N^2} > 0,$$

that is

$$(1 + N^2 g(k))^2 > 0.$$

Therefore, $1 + N^2 g(k) \neq 0$.

In addition, we first demonstrate that

$$H_{k,i+2} + Ng(k)H_{k,i+1} = \frac{1}{N}(H_{k,i+1} + Ng(k)H_{k,i}).$$

According to (1.1),

$$
\begin{aligned}
H_{k,i+2} + Ng(k)H_{k,i+1} &= f(k)H_{k,i+1} + g(k)H_{k,i} + Ng(k)H_{k,i+1} \\
&= (f(k) + Ng(k))H_{k,i+1} + g(k)H_{k,i} \\
&= \frac{1}{N}H_{k,i+1} + g(k)H_{k,i} \\
&= \frac{1}{N}(H_{k,i+1} + Ng(k)H_{k,i}).
\end{aligned}
$$

So we have

$$
\begin{aligned}
H_{k,i+2} + Ng(k)H_{k,i+1} &= \frac{1}{N^2}(H_{k,i} + Ng(k)H_{k,i-1}) \\
&= \cdots \\
&= \frac{1}{N^{i+1}}(H_{k,1} + Ng(k)H_{k,0}) \\
&= \frac{1}{N^{i+1}}(b + Ng(k)a).
\end{aligned}
$$

One can obtain that

$$N^{i+2}(H_{k,i+2} + Ng(k)H_{k,i+1}) = N(b + Ng(k)a).$$

Evaluating summation from 0 to $n - 1$, we have

$$\sum_{i=0}^{n-1}(N^{i+2}H_{k,i+2} + N^{i+3}g(k)H_{k,i+1}) = nN(b + Ng(k)a). \tag{4.3}$$

Shifting the summation index in (4.3), we obtain

$$
\begin{aligned}
&nN(b + Ng(k)a) \\
&= \sum_{i=0}^{n-1} N^{i+2}H_{k,i+2} + N^2 g(k) \sum_{i=0}^{n-1} N^{i+1}H_{k,i+1} \\
&= \sum_{i=0}^{n-1} N^i H_{k,i} - H_{k,0} - NH_{k,1} + N^n H_{k,n} + N^{n+1}H_{k,n+1} + N^2 g(k)(\sum_{i=0}^{n-1} N^i H_{k,i} - H_{k,0} + N^n H_{k,n}) \\
&= (1 + N^2 g(k)) \sum_{i=0}^{n-1} N^i H_{k,i} - a - Nb + N^n H_{k,n} + N^{n+1}H_{k,n+1} - N^2 g(k)a + N^{n+2}g(k)H_{k,n}.
\end{aligned}
$$

Since

$$N^{n+1}H_{k,n+1} + N^{n+2}g(k)H_{k,n} = N(b + Ng(k)a).$$

Then, we have

$$(1 + N^2 g(k)) \sum_{i=0}^{n-1} N^i H_{k,i} = nNb + nN^2 g(k)a + a - N^n H_{k,n}.$$

Therefore,

$$\sum_{i=0}^{n-1} N^i H_{k,i} = \frac{a + nNb + nN^2 g(k)a - N^n H_{k,n}}{1 + N^2 g(k)}.$$

$\square$

Theorem 4.1. *Let* $K = G(h) = [h, Gh, G^2 h, \ldots, G^{n-1} h]$ *be an* $n \times n$ *ideal matrix with the generalized* k*−Horadam numbers. Then we have the following upper bound estimates for the spectral norm of K.*

(i) If $N f(k) + N^2 g(k) \neq 1$*, then*

$$\|K\|_2 \leq \frac{a - N(af(k) - b) - N^n H_{k,n} - g(k)N^{n+1} H_{k,n-1}}{1 - N f(k) - N^2 g(k)}.$$

(ii) If $N f(k) + N^2 g(k) = 1$*, then*

$$\|K\|_2 \leq \frac{a + nbN + naN^2 g(k) - N^n H_{k,n}}{1 + N^2 g(k)},$$

where $N = \sqrt{\frac{1 + \sum_{i=0}^{n-1} a_i^2 + \sqrt{(1 + \sum_{i=0}^{n-1} a_i^2)^2 - 4a_0^2}}{2}}$, $a_0, a_1, \ldots, a_{n-1}$ *are the coefficients of the polynomial* $p(x)$.

Proof. By (2.1),

$$G = \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ & & & a_1 \\ & I_{n-1} & & \vdots \\ & & & a_{n-1} \end{pmatrix}.$$

G^* is the conjugate transpose of G, so G^*G has the form

$$G^*G = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a_0{}^2 + a_1{}^2 + \cdots + a_{n-1}{}^2 \end{pmatrix}.$$

One can obtain that

$$
|\lambda I - G^*G| = \begin{vmatrix} \lambda - 1 & 0 & \cdots & 0 & -a_1 \\ 0 & \lambda - 1 & \cdots & 0 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda - 1 & -a_{n-1} \\ -a_1 & -a_2 & \cdots & -a_{n-1} & \lambda - (a_0{}^2 + a_1{}^2 + \cdots + a_{n-1}{}^2) \end{vmatrix}
$$

$$
= \begin{vmatrix} \lambda - 1 & 0 & \cdots & 0 & -a_1 \\ 0 & \lambda - 1 & \cdots & 0 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda - 1 & -a_{n-1} \\ 0 & 0 & \cdots & 0 & \lambda - \sum_{i=0}^{n-1} a_i{}^2 - \frac{1}{\lambda-1}(\sum_{i=1}^{n-1} a_i{}^2) \end{vmatrix}
$$

$$
= (\lambda - 1)^{n-1}(\lambda - \sum_{i=0}^{n-1} a_i{}^2 - \frac{1}{\lambda-1}(\sum_{i=1}^{n-1} a_i{}^2))
$$

$$
= (\lambda - 1)^{n-2}(\lambda^2 - \lambda(1 + \sum_{i=0}^{n-1} a_i{}^2) + a_0{}^2).
$$

Let $M = 1 + \sum_{i=0}^{n-1} a_i{}^2$. So the eigenvalues of G^*G are

$$
\lambda_1(G^*G) = \lambda_2(G^*G) = \cdots = \lambda_{n-2}(G^*G) = 1, \quad \lambda_{n-1,n}(G^*G) = \frac{M \pm \sqrt{M^2 - 4a_0{}^2}}{2},
$$

where $M^2 - 4a_0{}^2 \geq 0$. Because $a_i \in \mathbb{Z}$, $M^2 - 4a_0{}^2 \geq (1 + a_0{}^2)^2 - 4a_0{}^2 = (a_0{}^2 - 1)^2 \geq 0$. We know $K = G(h) = H_{k,0}I + H_{k,1}G + H_{k,2}G^2 + \cdots + H_{k,n-1}G^{n-1}$, so

$$
\|K\|_2 = \|\sum_{i=0}^{n-1} H_{k,i}G^i\|_2 \leq \sum_{i=0}^{n-1} H_{k,i}\|G\|_2{}^i.
$$

Because of $M \geq 2$, and $\lambda_{n-1}(G^*G) = \frac{M + \sqrt{M^2-4a_0{}^2}}{2} \geq 1$. Then

$$
\|G\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(G^*G)}
$$

$$
= \sqrt{\frac{M + \sqrt{M^2 - 4a_0{}^2}}{2}}.
$$

Let $N = \sqrt{\frac{M + \sqrt{M^2-4a_0{}^2}}{2}}$, we have

$$
\|K\|_2 \leq \sum_{i=0}^{n-1} H_{k,i}\|G\|_2{}^i = \sum_{i=0}^{n-1} N^i H_{k,i},
$$

by Lemma 4.1, which completes the proof. □

Theorem 4.2. *We have the following upper bound estimates for the Frobenius norm of K.*

(i) If $Sf(k) + S^2g(k) \neq 1$, then

$$\|K\|_F \leq \frac{a - S(af(k) - b) - S^n H_{k,n} - g(k)S^{n+1}H_{k,n-1}}{1 - Sf(k) - S^2g(k)}.$$

(ii) If $Sf(k) + S^2g(k) = 1$, then

$$\|K\|_F \leq \frac{a + nbS + naS^2g(k) - S^n H_{k,n}}{1 + S^2g(k)}.$$

Here $S = \sqrt{n - 1 + \sum_{i=0}^{n-1} a_i^2}$, $a_0, a_1, \ldots, a_{n-1}$ are the coefficients of the polynomial $p(x)$.

Proof. From the proof of Theorem 4.1, we know trace$(G^*G) = n - 1 + \sum_{i=0}^{n-1} a_i^2$. So

$$\|G\|_F = \sqrt{\text{trace}(G^*G)} = \sqrt{n - 1 + \sum_{i=0}^{n-1} a_i^2}.$$

And $K = G(h) = H_{k,0}I + H_{k,1}G + H_{k,2}G^2 + \cdots + H_{k,n-1}G^{n-1}$, then

$$\|K\|_F = \|\sum_{i=0}^{n-1} H_{k,i}G^i\|_F \leq \sum_{i=0}^{n-1} H_{k,i}\|G\|_F^i.$$

Let $S = \sqrt{n - 1 + \sum_{i=0}^{n-1} a_i^2}$, we have

$$\|K\|_F \leq \sum_{i=0}^{n-1} H_{k,i}\|G\|_F^i = \sum_{i=0}^{n-1} S^i H_{k,i},$$

where S is a constant. By Lemma 4.1, the proof is completed. $\qquad\square$

5. Conclusions

This paper investigates ideal matrices with generalized k-Horadam numbers and develops explicit formulas for calculating the eigenvalues and determinants of these matrices and establishes upper bounds for their spectral and Frobenius norms. These findings contribute to the understanding of ideal matrices and their applications in matrix theory and related fields. Future research could aim to establish connections between ideal matrices with generalized k-Horadam numbers and ideal lattices, provide new insights into the fundamental theory of ideal lattices, and explore their specific properties and potential applications in cryptography.

Author contributions

Man Chen: Wrote the main manuscript text; Huaifeng Chen: Checked and revised the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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