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Research article

Analytical solution of the systems of nonlinear fractional partial differential equations using conformable Laplace transform iterative method

Nisar Gul[1], **Saima Noor**[2,3,*], **Abdulkafi Mohammed Saeed**[4], **Musaad S. Aldhabani**[5] **and Roman Ullah**[6,*]

[1] School of Mathematics and Statistics, Central South University, Changsha 410083, China

[2] Department of Basic Sciences, General Administration of Preparatory Year, King Faisal University, P.O. Box 400, Al Ahsa 31982, Saudi Arabia

[3] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al Ahsa 31982, Saudi Arabia

[4] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; abdulkafi.ahmed@qu.edu.sa

[5] Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia; maldhabani@ut.edu.sa

[6] Department of General Studies, Higher Colleges of Technology, Dubai Women Campus, UAE

* **Correspondence:** Email: snoor@kfu.edu.sa, romanullah@yahoo.com.

Abstract: In this study, we presented the conformable Laplace transform iterative method to find the approximate solution of the systems of nonlinear temporal-fractional differential equations in the sense of the conformable derivative. The advantage of the suggested approach was to compute the solution without discretization and restrictive assumptions. Three distinct examples were provided to show the applicability and efficacy of the proposed approach. To examine the exact and approximate solutions, we utilized the 2D and 3D graphics. Furthermore, the outcomes produced in this study were consistent with the exact solutions; hence, this strategy efficiently and effectively determined exact and approximate solutions to nonlinear temporal-fractional differential equations.

Keywords: temporal-fractional differential equations; conformable Laplace transform; iterative method; conformable derivative; numerical experiments
Mathematics Subject Classification: 35J05, 35J10

1. Introduction

The theory of fractional calculus has attracted a lot of interest in a variety of scientific and engineering domains because of its wide range of applications and importance in characterizing the intricate dynamic behavior of real-world problems like biological populations, diffusion systems, fluid flow, and traffic [1, 2]. Powerful modeling tools emerge through fractional-order equations that researchers utilize to represent complex systems which contain memory and hereditary effects based on multiple studies. Fractional-order models enabled to study polytropic gas unsteady flow dynamics with successful results [3]. The fractional dynamics research field gained foundational insights from Institute Technical Dissertation about framework development for fractional control systems and from Bira et al. [4] who employed Lie group methods to solve time-fractional evolution equations [5]. The algorithm proposed delivers effective numerical solutions along with precision calculations to fractional partial integro-differential problems [6]. Building upon existing breakthroughs in fractional partial differential equations the paper examines chaotic soliton perturbations through advanced analytical methods which produce new solutions and decay mechanisms of nonlinear system behavior [7, 8]. The fractional state is superior to the classical order in many ways, making it easier to govern the modeling of nature without sacrificing inherited characteristics or memory effort. Many nonlinear problems, such as those involving electrical circuits, damping laws, electromagnetic waves, signal processing, and rheology, can be effectively simulated using the fractional operator, a potent mathematical tool [9, 10]. In various fields of science and engineering [11–14], fractional differential equations (FDEs) have been extensively used to model different phenomena. Different approaches have been developed in [15–17] to solve linear/nonlinear FDEs to demonstrate the superiority and effectiveness of fractional calculus. The use of fractional-order differential equations (FODEs) allows for the solution of many physical problems that cannot be solved using integer-order differential equations (DEs) [18, 19]. The principal argument is that fractional-order derivatives permit the rate of change to be adjusted by the requirements of certain circumstances. Riesz, Letnikov, Hadamard, Grunwald, Caputo, Riemann-Louville, and the conformable derivative (CD) [20–22] are several forms of fractional derivative definitions that have been established and utilized in a variety of applications and natural circumstances. Compared to previous definitions, the CD is more straightforward to calculate [26]. The basic characteristics of CD are presented in [24, 25]. In contrast to the preceding ones, this fractional formulation satisfies all the rules of an ordinary derivative, particularly the quotient, product, and chain rules. It can be easily and quickly modified to solve FODEs exactly and approximately. It simplifies well-known transforms such as the natural and Laplace transforms, which are utilized to explain different FODEs [23, 27].

Lately, various strategies have been developed for determining FDEs, such as the Adomian and modified decomposition approach [28], the differential transform method [29, 30], and He's variational iterative technique (VIT) [31]. Abdulaziz [32] examined the homotopy perturbation technique (HPT) for the system of FDEs. In [33], the author examined the variational iteration method (VIM) to a system of autonomous differential equations. Recent advances in nonlinear dynamics and fractional calculus have led to the development of innovative methods for analyzing complex physical and mathematical systems. Kai and Yin [34] investigated the linear structures and soliton molecules in the Sharma-Tasso-Olver-Burgers equation, shedding light on soliton dynamics. He and Kai [35] extended this work by exploring wave structures and chaotic behaviors in Kudryashov's equation with third-order dispersion. Meanwhile, Xie et al. [36] analyzed resonance and attraction domains in asymmetric Duffing systems with fractional damping, highlighting the role of nonlinearity in multi-degree-of-freedom systems. Zhu et al. [37] contributed an analytical approach to nonlinear models using a modified Schrödinger's equation and logarithmic transformations, emphasizing the versatility of fractional techniques. In algebraic systems, Shuangjian and Apurba [38] explored cohomology and deformations of generalized Reynolds operators on Leibniz algebras. Lastly, Guo et al. [39] proposed a fixed-time safe tracking control strategy for uncertain high-order nonlinear pure-feedback systems, leveraging unified transformation functions.

In recent years, significant progress has been made in the study of nonlinear fractional differential equations due to their wide-ranging applications in physics, engineering, and other scientific domains. Fractional calculus, with its nonlocal properties and memory effects, provides a robust framework for modeling complex systems more effectively than traditional integer-order models. Several advanced analytical and numerical techniques have been developed to explore the solutions of these equations, unveiling novel soliton phenomena, wave dynamics, and diffusive processes.

For instance, Alderremy et al. [40] utilized the Laplace residual power series method to construct fractional series solutions for nonlinear fractional reaction-diffusion Brusselator models, showcasing the versatility of fractional methods in addressing chemical oscillations. Similarly, Yasmin et al. [41] explored soliton solutions of the perturbed Gerdjikov-Ivanov equation via Bäcklund transformation, demonstrating the intricate interplay of nonlinearity and dispersion.

The fractional view analysis by Al-Sawalha et al. [42] provided insights into the Kersten-Krasil'shchik coupled Korteweg-de Vries-modified Korteweg-de Vries (KdV-mKdV) systems using non-singular kernel derivatives, while Elsayed and Nonlaopon [43] proposed a novel approach for analyzing the fractional-order Navier-Stokes equations. Furthermore, Naeem et al. [44] extended this exploration to fuzzy fractional-order solitary wave solutions for the KdV equation, employing the Caputo-Fabrizio derivative.

Expanding on soliton dynamics, Alqhtani et al. [45] presented analytical solutions to the (3 + 1)-modified fractional Zakharov-Kuznetsov equation relevant in electrical engineering, whereas the fractional-order local Poisson equation in fractal porous media was analyzed by the same authors [46]. Recent advancements also include studies on fractional generalized Burger-Fisher equations [47], fractional stochastic Kraenkel-Manna-Merle systems in ferromagnetic materials [48], and fractional nonlinear damped Burger's-type equations in fluids and plasmas [49], all employing innovative methods like residual power series transforms and Aboodh transform techniques. In this work, we present the conformable Laplace transform iterative method (CLTIM) to find the approximate solution of the systems of nonlinear temporal-fractional differential equations in the sense of the conformable derivative. The precision of the proposed method has been verified by comparing it with the Homotopy analysis method (HAM) in terms of error for different problems. The results show that our improved method is uncomplicated, precise, and applicable. This paper is summarized as follows: Section 2 discusses some basic definitions and properties. Section 3 presents the methodology of the suggested approach. Sections 4 and 5, respectively, present a few numerical examples and concluding remarks.

2. Preliminaries

The fundamental concepts and characteristics of the fractional calculus theory that will be applied in this work are defined in this section.

Definition 2.1. *Assume a function* $\Theta : [0, \infty) \to R$ *as well the conformable derivative of* Θ *of order* Φ *stated in the following manner [50]:*

$$T_{\vartheta}^{\Phi}\Theta(\vartheta) = \lim_{\epsilon \to 0} \frac{\Theta(\vartheta + \epsilon\, \vartheta^{1-\Phi}) - \Theta(\vartheta)}{\epsilon}, \tag{2.1}$$

for $\vartheta > 0$, *and* $\Phi \in (0, 1]$. *If* Θ *is* $\Phi-$ *differentiable in some* $(0, p)$, $p > 0$, *and* $\lim_{\epsilon \to 0^+}(T^{\Phi}\Theta)(\vartheta)-$ *occurs subsequently, it comes out as follows:*

$$(T^{\Phi}\Theta)(0) = \lim_{\epsilon \to 0^+}(T^{\Phi}\Theta)(\vartheta). \tag{2.2}$$

Theorem 2.1. *Let* $\Phi \in (0, 1]$ *and* f, g *be* Φ*- differentiable at a point* $\Im > 0$. *Then,*

$$\begin{aligned} &(i)\ T^{\Phi}(af + bg) = a(T^{\Phi}f) + b(T^{\Phi}g),\ a, b \in R,\\ &(ii)\ T^{\Phi}(\Im^{p}) = p\Im^{p-\Phi},\ p \in R,\\ &(iii)\ T^{\Phi}(f(\Im)) = 0,\ f(\Im) = \lambda,\\ &(iv)\ T^{\Phi}(fg) = f(T^{\Phi}g) + g(T^{\Phi}f),\\ &(iv)\ T^{\Phi}(f/g) = \frac{g(T^{\Phi}f) - f(T^{\Phi}g)}{g^2}. \end{aligned} \tag{2.3}$$

Moreover, if $f(\Im)$ is differentiable, then $T^{\Phi}(f(\Im)) = \Im^{1-\Phi}\frac{d}{d\Im}f(\Im)$.

Definition 2.2. *Let* $g : [0, \infty) \to R$ *be a real-valued function and* $0 < \Phi \leq 1$. *Then, the conformable Laplace transform* $\mathscr{L}_{\Phi}$ *of* g *is defined as [51]*

$$\mathscr{L}_{\Phi}g(\vartheta) = G_{\Phi}(s) = \int_0^{\infty} g(\vartheta)e^{(-s(\frac{\vartheta^{\Phi}}{\Phi}))}d_{\Phi}\vartheta = \int_0^{\infty} g(\vartheta)e^{-s(\frac{\vartheta^{\Phi}}{\Phi})}\vartheta^{\Phi-1}d\vartheta, \tag{2.4}$$

provided the integral exists.

Theorem 2.2 (See [52, 53]). *Let* $g : [0, \infty) \to R$ *be a function such that* $L_{\Phi}g(\vartheta) = G_{\Phi}(s)$ *exists. Then,*

$$\mathscr{L}_{\Phi}\{g(\vartheta)\}(s) = G_{\Phi}(s) = \mathscr{L}\{g((\Phi\vartheta)^{\frac{1}{\Phi}})\}(s), \quad 0 < \Phi \leq 1. \tag{2.5}$$

and $\mathscr{L}$ *represents the usual Laplace transform.*

Theorem 2.3 (See [52, 53]). *Let* $g : [0, \infty) \to R$, $f : [0, \infty) \to R$, *and let* $\alpha, \beta, \nu \in R$ *and* $0 < \Phi \leq 1$. *Then,*

$$(i)\ \ \mathscr{L}_{\Phi}\{\alpha f(\vartheta) + \beta g(\vartheta)\} = \alpha F_{\Phi}(s) + \beta G_{\Phi}(s), \quad s > 0.$$

$$(ii)\ \ \mathscr{L}_{\Phi}\{e^{-a(\frac{\vartheta^{\Phi}}{\Phi})}f(\vartheta)\}(s) = F_{\Phi}(s + \nu), \quad s > |\nu|.$$

$$(iii)\ \ \mathscr{L}_{\Phi}\{I^{f}(\vartheta)\}(s) = \frac{F_{\Phi}(s)}{s}, \quad s > 0.$$

$$(iv)\ \mathscr{L}_{\Phi}\{(\frac{\vartheta^{n\Phi}}{\alpha^{n}})f(\vartheta)\}(s) = (-1)^{n}(\frac{d^{n}}{ds^{n}})F_{\Phi}(s), \quad s > 0.$$

$$(v)\ \mathscr{L}_{\Phi}\{(f * g)(\vartheta)\} = F_{\Phi}(s)G_{\Phi}(s), \quad s > 0,$$

where $F_\Phi(s)$ and $G_\Phi(s)$ are the conformable Laplace transform of the functions f and g, respectively, $f * g$ is the convolution product of f and g, and $I^\Phi f(\vartheta)$ is the conformable integral.

Theorem 2.4 (See [54]). *Let $\nu, q, \varrho \in R$, and $0 < \Phi \leq 1$. Then,*

$$(i)\quad \mathscr{L}_\Phi\{\varrho\}(s) = \frac{\varrho}{s},\quad s > 0.$$

$$(ii)\quad \mathscr{L}_\Phi\{\eta^q\}(s) = \Phi^{\frac{q}{\Phi}} \frac{\Gamma(1+\frac{q}{\Phi})}{s^{1+\frac{q}{\Phi}}},\quad s > 0.$$

$$(iii)\quad \mathscr{L}_\Phi\{e^{\nu\frac{\eta^\Phi}{\Phi}}\}(s) = \frac{1}{s-\nu},\quad s > 0.$$

$$(iv)\quad \mathscr{L}_\Phi\{sin\nu\frac{\eta^\Phi}{\Phi}\}(s) = \frac{\nu}{s^2+\nu^2},\quad s > 0.$$

$$(v)\quad \mathscr{L}_\Phi\{cos\nu\frac{\eta^\Phi}{\Phi}\}(s) = \frac{s}{s^2+\nu^2},\quad s > 0.$$

$$(vi)\quad \mathscr{L}_\Phi\{sinh\nu\frac{\eta^\Phi}{\Phi}\}(s) = \frac{\nu}{s^2-\nu^2},\quad s > |\nu|.$$

$$(vii)\quad \mathscr{L}_\Phi\{cosh\nu\frac{\eta^\Phi}{\Phi}\}(s) = \frac{s}{s^2-\nu^2},\quad s > |\nu|.$$

3. Methodology of the proposed technique

We consider a general inhomogeneous temporal fractional differential equation with the initial condition [55] to present the basic concept of the proposed method, for the systems of nonlinear temporal-fractional differential equations

$$\begin{gathered} T_\vartheta^\Phi \Theta(\Im,\vartheta) + H\Theta(\Im,\vartheta) + S\Theta(\Im,\vartheta) = g(\Im,\vartheta),\quad 0 < \Phi \leq 1, \\ \Theta(\Im,0) = P(\Im), \end{gathered} \tag{3.1}$$

where T_ϑ^Φ is the conformable derivative, $H(\Theta(\Im,\vartheta))$ represents the linear operator, $S(\Theta(\Im,\vartheta))$ denotes nonlinear operators, $g(\Im,\vartheta)$ is a source term, and $P(\Im)$ is a function of $\Im$.

Taking both sides of CLT of Eq (3.1),

$$\mathscr{L}_\Phi[T_\vartheta^\Phi \Theta(\Im,\vartheta) + H(\Theta(\Im,\vartheta)) + S(\Theta(\Im,\vartheta))] = \mathscr{L}_\Phi[g(\Im,\vartheta)]. \tag{3.2}$$

Now, simplifying Eq (3.2),

$$\mathscr{L}_\Phi[\Theta(\Im,\vartheta)] = \frac{1}{s}\mathscr{L}_\Phi[g(\Im,\vartheta)] + \frac{1}{s}\Theta(\Im,0) - \frac{1}{s}\mathscr{L}_\Phi[H\Theta(\Im,\vartheta)] - \frac{1}{s}\mathscr{L}_\Phi[S\Theta(\Im,\vartheta)]. \tag{3.3}$$

Now, taking the inverse CLT on both sides,

$$\Theta(\Im,\vartheta) = \mathscr{L}_\Phi^{-1}\{\frac{1}{s}\Theta(\Im,0) + \frac{1}{s}\mathscr{L}_\Phi[g(\Im,\vartheta)]\} - \mathscr{L}_\Phi^{-1}\{\frac{1}{s}\mathscr{L}_\Phi[H\Theta(\Im,\vartheta)]\} - \mathscr{L}_\Phi^{-1}\{\frac{1}{s}\mathscr{L}_\Phi[S\Theta(\Im,\vartheta)]\}. \tag{3.4}$$

Assume that

$$E(\Theta(\Im,\vartheta)) = \mathscr{L}_\Phi^{-1}\{\frac{1}{s}\Theta(\Im,0) + \frac{1}{s}\mathscr{L}_\Phi[g(\Im,\vartheta)]\}, \tag{3.5}$$

$$F(\Theta(\Im,\vartheta)) = \mathscr{L}_{\Phi}^{-1}\{\frac{1}{s}\mathscr{L}_{\Phi}[H\Theta(\Im,\vartheta)]\}, \tag{3.6}$$

$$G(\Theta(\Im,\vartheta)) = \mathscr{L}_{\Phi}^{-1}\{\frac{1}{s}\mathscr{L}_{\Phi}[S\,\Theta(\Im,\vartheta)]\}. \tag{3.7}$$

By using these results, Eq (3.4) can be written as

$$\Theta(\Im,\vartheta) = E(\Theta(\Im,\vartheta)) + F(\Theta(\Im,\vartheta)) + G(\Theta(\Im,\vartheta)). \tag{3.8}$$

In Eq (3.8), $E(\Theta(\wp,\vartheta))$ is a known function, while F and G are given linear and nonlinear operators of $\Theta(\eta,\vartheta)$. The solution to Eq (3.8) can be represented in the following expansion form:

$$\Theta(\Im,\vartheta) = \sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta). \tag{3.9}$$

Assume that

$$F\Big(\sum_{k=1}^{\infty}\Theta_k(\Im,\vartheta)\Big) = \sum_{k=1}^{\infty}F(\Theta_k(\Im,\vartheta)). \tag{3.10}$$

The nonlinear operator G can be decomposed as follows [56]:

$$G\Big(\sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta)\Big) = G(\Theta_0(\Im,\vartheta)) + \sum_{k=0}^{\infty}\Big[G\Big(\sum_{m=0}^{k}\Theta_m(\Im,\vartheta)\Big) - G\Big(\sum_{m=0}^{k-1}\Theta_m(\Im,\vartheta)\Big)\Big]. \tag{3.11}$$

Substituting Eqs (3.9)–(3.11) in Eq (3.8),

$$\begin{aligned}\sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta) =& E(\Theta(\Im,\vartheta)) + \sum_{k=0}^{\infty}F(\Theta_k(\Im,\vartheta))\\ &+G(\Theta_0(\Im,\vartheta)) + \sum_{k=0}^{\infty}\Big[G\Big(\sum_{m=0}^{k}\Theta_m(\Im,\vartheta)\Big) - G\Big(\sum_{m=0}^{k-1}\Theta_m(\Im,\vartheta)\Big)\Big].\end{aligned} \tag{3.12}$$

By utilizing (3.12) to define the recurrence relation,

$$\Theta_0(\Im,\vartheta) = E(\Theta(\Im,\vartheta)), \tag{3.13}$$

$$\Theta_1(\Im,\vartheta) = F(\Theta_0(\Im,\vartheta)) + G(\Theta_0(\Im,\vartheta)), \tag{3.14}$$

$$\begin{aligned}\Theta_{i+1}(\Im,\vartheta) =& F(\Theta_i(\Im,\vartheta)) + G(\Theta_0(\Im,\vartheta) + \Theta_1(\Im,\vartheta) + \cdots\\ &+ \Theta_i(\Im,\vartheta)) - G(\Theta_0(\Im,\vartheta) + \Theta_1(\Im,\vartheta) + \cdots + \Theta_{i-1}(\Im,\vartheta)).\end{aligned} \tag{3.15}$$

Finally, we get the following results,

$$\begin{aligned}&\Big(\Theta_1(\Im,\vartheta) + \Theta_2(\Im,\vartheta) + \cdots + \Theta_{i+1}(\Im,\vartheta)\Big)\\ =\ &F\Big(\Theta_0(\Im,\vartheta) + \Theta_1(\Im,\vartheta) + \cdots + \Theta_i(\Im,\vartheta)\Big)\\ &G\Big(\Theta_0(\Im,\vartheta) + \Theta_1(\Im,\vartheta) + \cdots + \Theta_i(\Im,\vartheta)\Big).\end{aligned} \tag{3.16}$$

particularly

$$\sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta) = E\Big(\sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta)\Big) + F\Big(\sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta)\Big) + G\Big(\sum_{k=0}^{\infty}\Theta_k(\Im,\vartheta)\Big). \tag{3.17}$$

Equation (3.1) has the i-th approximate solution given by

$$\Theta_i(\Im,\vartheta) = \Theta_0(\Im,\vartheta) + \Theta_1(\Im,\vartheta) + \cdots + \Theta_{i-1}(\Im,\vartheta). \tag{3.18}$$

4. Numerical experiments and discussion

This section presents some numerical examples of two-dimensional fractional-order partial differential equations (PDEs) to demonstrate the proposed technique's simplicity and efficiency in obtaining approximate series solutions for the suggested technique. The outcomes of the simulations show that the suggested technique is very efficient and entirely consistent with the difficulty of fractional-order PDEs.

Example 4.1. *Consider the following system of nonlinear temporal-fractional differential equations [56]:*

$$\begin{aligned}
T_{\vartheta}^{\Phi}u(\Im,\wp,\vartheta) + \frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta) - \frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta) &= -u(\Im,\wp,\vartheta),\\
T_{\vartheta}^{\Phi}v(\Im,\wp,\vartheta) + \frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta) + \frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta) &= v(\Im,\wp,\vartheta), \quad 0<\Phi\leq 1,\\
T_{\vartheta}^{\Phi}w(\Im,\wp,\vartheta) + \frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta) + \frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta) &= w(\Im,\wp,\vartheta),
\end{aligned} \tag{4.1}$$

with the initial conditions

$$u(\Im,\wp,0) = e^{\Im+\wp}, \quad v(\Im,\wp,0) = e^{\Im-\wp}, \quad w(\Im,\wp,0) = e^{-\Im+\wp}. \tag{4.2}$$

Taking CLT on both sides of Eq (4.1),

$$\begin{aligned}
\mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}u(\Im,\wp,\vartheta)\Big] &= \mathscr{L}_{\Phi}\Big[-\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta) + \frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta) - u(\Im,\wp,\vartheta)\Big],\\
\mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}v(\Im,\wp,\vartheta)\Big] &= \mathscr{L}_{\Phi}\Big[-\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta) - \frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta) + v(\Im,\wp,\vartheta)\Big],\\
\mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}w(\Im,\wp,\vartheta)\Big] &= \mathscr{L}_{\Phi}\Big[-\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta) - \frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta) + w(\Im,\wp,\vartheta)\Big].
\end{aligned} \tag{4.3}$$

Applying the process described in Section 3, we get the following outcome:

$$\begin{aligned}
\mathscr{L}_{\Phi}[u(\Im,\wp,\vartheta)] =& \frac{1}{s}(e^{\Im+\wp}) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)\Big)\\
&+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\Big) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(u(\Im,\wp,\vartheta)\Big),\\
\mathscr{L}_{\Phi}[v(\Im,\wp,\vartheta)] =& \frac{1}{s}(e^{\Im-\wp}) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)\Big)
\end{aligned}$$

$$-\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\Big)+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(v(\Im,\wp,\vartheta)\Big),$$

$$\begin{aligned}\mathscr{L}_{\Phi}[w(\Im,\wp,\vartheta)] =& \frac{1}{s}(e^{-\Im+\wp})-\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\Big)\\ &+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\Big)+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(w(\Im,\wp,\vartheta)\Big).\end{aligned} \tag{4.4}$$

Now, taking inverse CLT on both sides,

$$\begin{aligned}u(\Im,\wp,\vartheta) =& \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(e^{\Im+\wp})\Big]-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)\Big)\Big]\\ &+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\Big)\Big]-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(u(\Im,\wp,\vartheta)\Big)\Big],\\ v(\Im,\wp,\vartheta) =& \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(e^{\Im-\wp})\Big]-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)\Big)\Big]\\ &-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\Big)\Big]+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(v(\Im,\wp,\vartheta)\Big)\Big],\\ w(\Im,\wp,\vartheta) =& \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(e^{-\Im+\wp})\Big]-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\Big)\Big]\\ &+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\Big)\Big]+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(w(\Im,\wp,\vartheta)\Big)\Big].\end{aligned} \tag{4.5}$$

Using the procedures described in Section 3, we get the following result:

$$u,v,w(\Im,\wp,\vartheta)=\sum_{k=0}^{\infty}u_k,v_k,w_k(\Im,\wp,\vartheta). \tag{4.6}$$

Using Eq (4.6) in (4.5),

$$\begin{aligned}\sum_{k=0}^{\infty}u_k(\Im,\wp,\vartheta) =& \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(e^{\Im+\wp})\Big]-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty}v_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty}w_k(\Im,\wp,\vartheta)\Big)\Big\}\Big]\\ &+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty}v_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty}w_k(\Im,\wp,\vartheta)\Big)\Big\}\Big]\\ &-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty}u_k(\Im,\wp,\vartheta)\Big)\Big],\\ \sum_{k=0}^{\infty}v_k(\Im,\wp,\vartheta) =& \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(e^{\Im+\wp})\Big]-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty}u_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty}w_k(\Im,\wp,\vartheta)\Big)\Big\}\Big]\\ &+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty}u_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty}w_k(\Im,\wp,\vartheta)\Big)\Big\}\Big]\\ &-\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty}v_k(\Im,\wp,\vartheta)\Big)\Big],\end{aligned}$$

$$\sum_{k=0}^{\infty} w_k(\Im,\wp,\vartheta) = \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(e^{\Im+\wp})\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} u_k(\Im,\wp,\vartheta)\Big)\frac{\partial}{\partial\wp}\Big(\sum_{k=0}^{\infty} v_k(\Im,\wp,\vartheta)\Big)\Big\}\Big]$$
$$+ \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty} u_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} v_k(\Im,\wp,\vartheta)\Big)\Big\}\Big]$$
$$- \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty} w_k(\Im,\wp,\vartheta)\Big)\Big]. \tag{4.7}$$

The following are the outcomes of Eq (4.7) when the iteration procedure described in Section 3 is applied:

$$u_0(\Im,\wp,\vartheta) = e^{\Im+\wp}, \quad v_0(\Im,\wp,\vartheta) = e^{\Im-\wp}, \quad w_0(\Im,\wp,\vartheta) = e^{-\Im+\wp},$$
$$u_1(\Im,\wp,\vartheta) = -\frac{\vartheta^{1\Phi}}{1!\Phi^1}e^{\Im+\wp}, \quad v_1(\Im,\wp,\vartheta) = \frac{\vartheta^{1\Phi}}{1!\Phi^1}e^{\Im-\wp}, \quad w_1(\Im,\wp,\vartheta) = \frac{\vartheta^{1\Phi}}{1!\Phi^1}e^{-\Im+\wp},$$
$$u_2(\Im,\wp,\vartheta) = \frac{\vartheta^{2\Phi}}{2!\Phi^2}e^{\Im+\wp}, \quad v_2(\Im,\wp,\vartheta) = \frac{\vartheta^{2\Phi}}{2!\Phi^2}e^{\Im-\wp}, \quad w_2(\Im,\wp,\vartheta) = \frac{\vartheta^{2\Phi}}{2!\Phi^2}e^{-\Im+\wp},$$
$$u_3(\Im,\wp,\vartheta) = -\frac{\vartheta^{3\Phi}}{3!\Phi^3}e^{\Im+\wp}, \quad v_3(\Im,\wp,\vartheta) = \frac{\vartheta^{3\Phi}}{3!\Phi^3}e^{\Im-\wp}, \quad w_3(\Im,\wp,\vartheta) = \frac{\vartheta^{3\Phi}}{3!\Phi^3}e^{-\Im+\wp},$$
$$\vdots$$

Consequently, we get the following approximate solution to Eq (4.1):

$$\begin{aligned}
u(\Im,\wp,\vartheta) &= u_0(\Im,\wp,\vartheta) + u_1(\Im,\wp,\vartheta) + u_2(\Im,\wp,\vartheta) + u_3(\Im,\wp,\vartheta) + \cdots, \\
&= e^{\Im+\wp}\Big(1 - \frac{\vartheta^{1\Phi}}{1!\Phi^1} + \frac{\vartheta^{2\Phi}}{2!\Phi^2} - \frac{\vartheta^{3\Phi}}{3!\Phi^3} + \cdots\Big), \\
v(\Im,\wp,\vartheta) &= v_0(\Im,\wp,\vartheta) + v_1(\Im,\wp,\vartheta) + v_2(\Im,\wp,\vartheta) + v_3(\Im,\wp,\vartheta) + \cdots, \\
&= e^{\Im-\wp}\Big(1 + \frac{\vartheta^{1\Phi}}{1!\Phi^1} + \frac{\vartheta^{2\Phi}}{2!\Phi^2} + \frac{\vartheta^{3\Phi}}{3!\Phi^3} + \cdots\Big), \\
w(\Im,\wp,\vartheta) &= w_0(\Im,\wp,\vartheta) + w_1(\Im,\wp,\vartheta) + w_2(\Im,\wp,\vartheta) + w_3(\Im,\wp,\vartheta) + \cdots, \\
&= e^{-\Im+\wp}\Big(1 + \frac{\vartheta^{1\Phi}}{1!\Phi^1} + \frac{\vartheta^{2\Phi}}{2!\Phi^2} + \frac{\vartheta^{3\Phi}}{3!\Phi^3} + \cdots\Big).
\end{aligned} \tag{4.8}$$

At $\Phi = 1$, the exact solutions are $u(\Im,\wp,\vartheta) = e^{\Im+\wp-\vartheta}$, $v(\Im,\wp,\vartheta) = e^{\Im-\wp+\vartheta}$, and $w(\Im,\wp,\vartheta) = e^{-\Im+\wp+\vartheta}$, respectively. We examine the results and graphical outcomes of the precise and approximate solutions to the models described in Example 4.1. Using the error function, we investigate the accuracy and capabilities of the proposed technique. It is crucial to identify the error between the exact and approximate solutions. To illustrate the proposed method's efficiency and accuracy, we implemented the absolute error function. The $2D$ graphs of the comparison study are shown in Figure 1. They represent various values of $\Phi = 0.4,\ 0.6,\ 0.8,\ 1$ of the approximate and precise solution, respectively, generated by the recommended approach at $\wp = 1, \vartheta = 0.2$ in the interval $\Im \in [0,4]$. Figure 2 represents the $3D$ graphs of the exact and the approximate solutions attained by the suggested technique at $\Phi = 1$ in the intervals $\vartheta \in [0,1]$, $\Im \in [0,2]$ at $\wp = 1$ for Example 4.1. These graphs demonstrate how well the suggested method's approximative solution works. The close approximation and precise solution demonstrate the accuracy and efficacy of the proposed technique.

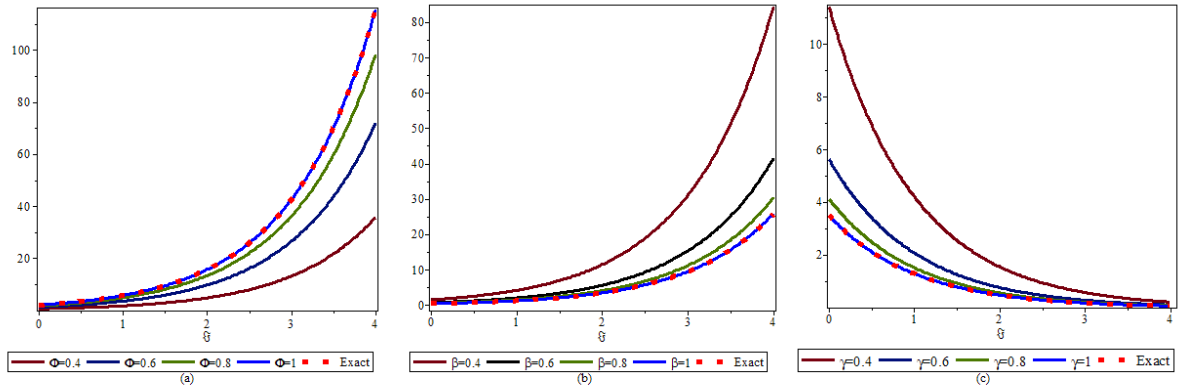

Figure 1. $(a), (b), (c)$ are the $2D$ plots of the proposed method for $u(\Im, \wp, \vartheta)$, $v(\Im, \wp, \vartheta)$, and $w(\Im, \wp, \vartheta)$ at different values of $\Phi = 0.4,\ 0.6,\ 0.8,\ 1$, respectively, at $\vartheta = 0.2$ and $\wp = 1$ for Example 4.1.

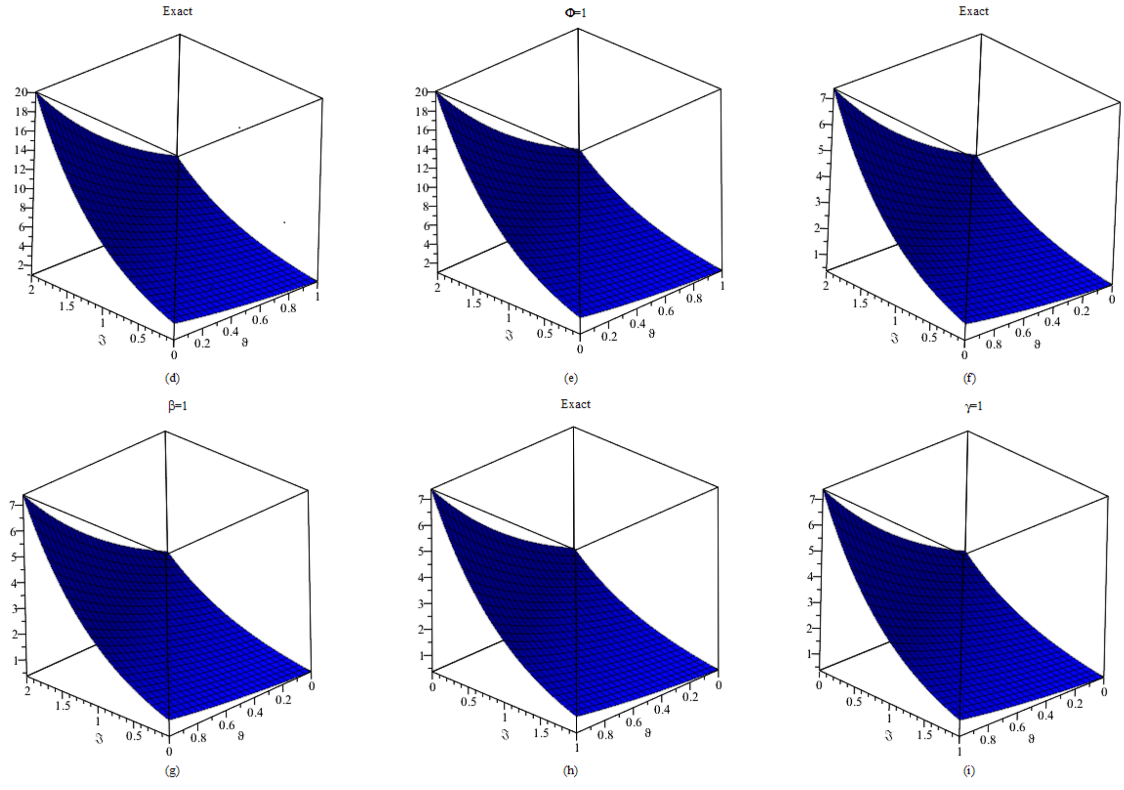

Figure 2. $(d), (e), (f), (g), (h), (i)$ are the $3D$ plots of $u(\Im, \wp, \vartheta)$, $v(\Im, \wp, \vartheta)$, and $w(\Im, \wp, \vartheta)$, of the exact and approximate solution for the proposed method of Example 4.1 at $\Phi = 1$, respectively.

Tables 1–3 show the approximate and exact solutions and absolute error at $\Phi = 0.4,\ 0.6,\ 0.8,\ 1$ values. The point-wise error of the approximate and exact solutions of Example 4.1 at $\vartheta = 0.2$ is shown in Table 4. A comparison with the solution derived using the HAM is also provided. All the mentioned Tables in Example 4.1 show that both types of solutions are highly comparable, supporting the efficacy of the developed approach.

Table 1. The approximate and exact solutions of $u(\Im, \wp, \vartheta)$ by CLTIM for Example 4.1 at different values of Φ when $\wp = 1$ and $\vartheta = 0.2$.

$\Im$	$\Phi = 0.4$	$\Phi = 0.6$	$\Phi = 0.8$	$\Phi = 1$	Exact solution	$\lvert u_{exact} - u_{CLTIM}\rvert$
0.0	0.73417084	1.44118118	1.92527293	2.22554093	2.22554092	7×10^{-9}
0.1	0.81138426	1.59275152	2.12775565	2.45960311	2.45960311	8×10^{-9}
0.2	0.89671829	1.76026266	2.35153366	2.71828183	2.71828182	9×10^{-9}
0.3	0.99102698	1.94539111	2.59884662	3.00416603	3.00416602	1×10^{-8}
0.4	1.09525420	2.14998967	2.87216971	3.32011693	3.32011692	1×10^{-8}
0.5	1.21044309	2.37610606	3.17423843	3.66929667	3.66929666	1.1×10^{-8}
0.6	1.33774650	2.62600332	3.50807600	4.05519997	4.05519996	1.2×10^{-8}
0.7	1.47843853	2.90218250	3.87702357	4.48168908	4.48168907	1.4×10^{-8}
0.8	1.63392726	3.20740770	4.28477370	4.95303243	4.95303242	1.5×10^{-8}
0.9	1.80576890	3.54473371	4.73540729	5.47394740	5.47394739	1.6×10^{-8}
1.0	1.99568327	3.91753661	5.23343442	6.04964748	6.04964746	1.9×10^{-8}

Table 2. The aproximate and exact solutions of $v(\Im, \wp, \vartheta)$ by CLTIM for Example 4.1 at different values of Φ when $\wp = 1$ and $\vartheta = 0.2$.

$\Im$	$\Phi = 0.4$	$\Phi = 0.6$	$\Phi = 0.8$	$\Phi = 1$	Exact solution	$\lvert v_{exact} - v_{CLTIM}\rvert$
0.1	1.51109629	0.76685816	0.57403341	0.49658530	0.49658530	9×10^{-10}
0.2	1.67001967	0.84750934	0.63440503	0.54881163	0.54881163	1×10^{-9}
0.3	1.84565718	0.93664267	0.70112599	0.60653065	0.60653065	1×10^{-9}
0.4	2.03976664	1.03515024	0.77486406	0.67032004	0.67032004	1.1×10^{-9}
0.5	2.25429077	1.14401795	0.85635722	0.74081821	0.74081822	1.3×10^{-9}
0.6	2.49137660	1.26433536	0.94642110	0.81873075	0.81873075	1.5×10^{-9}
0.7	2.75339696	1.39730668	1.04595707	0.90483741	0.90483741	1.5×10^{-9}
0.8	3.04297425	1.54426270	1.15596134	0.99999999	1.00000000	1.7×10^{-9}
0.9	3.36300665	1.70667423	1.27753485	1.10517091	1.10517091	2×10^{-9}
1.0	3.71669714	1.88616672	1.41189437	1.22140275	1.22140275	2×10^{-9}

Table 3. The aproximate and exact solutions of $w(\Im, \wp, \vartheta)$ by CLTIM for Example 4.1 at different values of Φ when $\wp = 1$ and $\vartheta = 0.2$.

$\Im$	$\Phi = 0.4$	$\Phi = 0.6$	$\Phi = 0.8$	$\Phi = 1$	Exact solution	$\lvert w_{exact} - w_{CLTIM}\rvert$
0.1	9.14159986	4.63922155	3.47269979	3.00416601	3.00416602	6×10^{-9}
0.2	8.27166161	4.19774125	3.14222871	2.71828182	2.71828182	5×10^{-9}
0.3	7.48450894	3.79827335	2.84320611	2.45960310	2.45960311	5×10^{-9}
0.4	6.77226374	3.43681985	2.57263928	2.22554092	2.22554092	4×10^{-9}
0.5	6.12779764	3.10976320	2.32782028	2.01375270	2.01375270	3×10^{-9}
0.6	5.54466059	2.81383011	2.10629889	1.82211879	1.82211880	2×10^{-9}
0.7	5.01701638	2.54605877	1.90585805	1.64872126	1.64872127	3×10^{-9}
0.8	4.53958414	2.30376924	1.72449168	1.49182469	1.49182469	3×10^{-9}
0.9	4.10758559	2.08453661	1.56038460	1.34985880	1.34985880	3×10^{-9}
1.0	3.71669714	1.88616672	1.41189437	1.22140275	1.22140275	2×10^{-9}

Table 4. The points-wise error of the approximate and exact solutions of Example 4.1 at $\vartheta = 0.2$ also shows the comparison with the solution obtained by the HAM [57].

$\Im$	$u(\Im,\wp,\vartheta)$	HAM [57]	$v(\Im,\wp,\vartheta)$	HAM [57]	$w(\Im,\wp,\vartheta)$	HAM [57]
0.1	8×10^{-9}	2.59×10^{-7}	9×10^{-10}	6.40×10^{-9}	6×10^{-9}	2.48×10^{-7}
0.2	9×10^{-9}	2.86×10^{-7}	1×10^{-9}	3.71×10^{-8}	5×10^{-9}	2.03×10^{-7}
0.3	1×10^{-8}	3.17×10^{-7}	1×10^{-9}	4.11×10^{-8}	5×10^{-9}	1.84×10^{-7}
0.4	1×10^{-8}	3.50×10^{-7}	1.1×10^{-9}	4.54×10^{-8}	4×10^{-9}	1.66×10^{-7}
0.5	1.1×10^{-8}	3.87×10^{-7}	1.3×10^{-9}	5.02×10^{-8}	3×10^{-9}	1.50×10^{-7}
0.6	1.2×10^{-8}	4.27×10^{-7}	1.5×10^{-9}	5.54×10^{-8}	2×10^{-9}	1.36×10^{-7}
0.7	1.4×10^{-8}	4.73×10^{-7}	1.5×10^{-9}	6.13×10^{-8}	3×10^{-9}	1.23×10^{-7}
0.8	1.5×10^{-8}	5.22×10^{-7}	1.7×10^{-9}	6.77×10^{-8}	3×10^{-9}	1.11×10^{-7}
0.9	1.6×10^{-8}	5.77×10^{-7}	2×10^{-9}	7.49×10^{-8}	3×10^{-9}	1.01×10^{-7}
1.0	1.9×10^{-8}	6.38×10^{-7}	2×10^{-9}	9.14×10^{-8}	2×10^{-9}	2.6×10^{-7}

Example 4.2. *Consider the following system of nonlinear temporal-fractional differential equations [56]:*

$$\begin{aligned} &T_{\vartheta}^{\Phi}u(\Im,\vartheta) - \frac{\partial}{\partial\Im}v(\Im,\vartheta) + v(\Im,\vartheta) + u(\Im,\vartheta) = 0,\\ &T_{\vartheta}^{\Phi}v(\Im,\vartheta) - \frac{\partial}{\partial\Im}u(\Im,\vartheta) + v(\Im,\vartheta) + u(\Im,\vartheta) = 0, \quad 0 < \Phi \leq 1, \end{aligned} \tag{4.9}$$

with the initial conditions

$$u(\Im,0) = sinh(\Im), \quad v(\Im,0) = cosh(\Im). \tag{4.10}$$

Taking CLT on both sides of Eq (4.9),

$$\begin{aligned} \mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}u(\Im,\vartheta)\Big] &= \mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}v(\Im,\vartheta) - v(\Im,\vartheta) - u(\Im,\vartheta)\Big],\\ \mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}v(\Im,\vartheta)\Big] &= \mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}u(\Im,\vartheta) - v(\Im,\vartheta) - u(\Im,\vartheta)\Big]. \end{aligned} \tag{4.11}$$

Applying the process described in Section 3, we get the following outcome:

$$\begin{aligned} \mathscr{L}_{\Phi}\Big[u(\Im,\vartheta)\Big] &= \frac{1}{s}(sinh(\Im)) + \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}v(\Im,\vartheta)\Big) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(v(\Im,\vartheta)\Big) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(u(\Im,\vartheta)\Big),\\ \mathscr{L}_{\Phi}\Big[v(\Im,\vartheta)\Big] &= \frac{1}{s}(cosh(\Im)) + \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\vartheta)\Big) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(v(\Im,\vartheta)\Big) - \frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(u(\Im,\vartheta)\Big). \end{aligned} \tag{4.12}$$

Now, taking inverse CLT on both sides of Eq (4.12),

$$\begin{aligned} u(\Im,\vartheta) = {} & \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(sinh(\Im))\Big] + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}v(\Im,\vartheta)\Big)\Big]\\ & - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(v(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(u(\Im,\vartheta)\Big)\Big], \end{aligned}$$

$$v(\Im,\vartheta) = \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(cosh(\Im))\Big] + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(v(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(u(\Im,\vartheta)\Big)\Big]. \tag{4.13}$$

Using the procedures described in Section 3, we get the following result:

$$u, v(\Im,\vartheta) = \sum_{k=0}^{\infty} u_k, v_k(\Im,\vartheta). \tag{4.14}$$

Using Eq (4.14) in (4.13),

$$\begin{aligned}
\sum_{k=0}^{\infty} u_k(\Im,\vartheta) &= \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(sinh(\Im))\Big] + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} v_k(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty} v_k(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty} u_k(\Im,\vartheta)\Big)\Big],\\
\sum_{k=0}^{\infty} v_k(\Im,\vartheta) &= \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(cosh(\Im))\Big] + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} u_k(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty} v_k(\Im,\vartheta)\Big)\Big] - \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\sum_{k=0}^{\infty} u_k(\Im,\vartheta)\Big)\Big].
\end{aligned} \tag{4.15}$$

The following are the outcomes of Eq (4.15) when the iteration procedure described in Section 3 is applied:

$$\begin{aligned}
u_0(\Im,\vartheta) &= sinh(\Im), \quad v_0(\Im,\vartheta) = cosh(\Im),\\
u_1(\Im,\vartheta) &= -\frac{\vartheta^{1\Phi}}{1!\,\Phi^1}cosh(\Im), \quad v_1(\Im,\vartheta) = -\frac{\vartheta^{1\Phi}}{1!\,\Phi^1}sinh(\Im),\\
u_2(\Im,\vartheta) &= \frac{\vartheta^{2\Phi}}{2!\,\Phi^2}sinh(\Im), \quad v_2(\Im,\vartheta) = \frac{\vartheta^{2\Phi}}{2!\,\Phi^2}cosh(\Im),\\
u_3(\Im,\vartheta) &= -\frac{\vartheta^{3\Phi}}{3!\,\Phi^3}cosh(\Im), \quad v_3(\Im,\vartheta) = -\frac{\vartheta^{3\Phi}}{3!\,\Phi^3}sinh(\Im),\\
&\vdots
\end{aligned}$$

Consequently, we get the following approximate solution to Eq (4.9):

$$\begin{aligned}
u(\Im,\vartheta) &= u_0(\Im,\vartheta) + u_1(\Im,\vartheta) + u_2(\Im,\vartheta) + u_3(\Im,\vartheta) + \cdots,\\
&= sinh(\Im) - \frac{\vartheta^{1\Phi}}{1!\,\Phi^1}cosh(\Im) + \frac{\vartheta^{2\Phi}}{2!\,\Phi^2}sinh(\Im) - \frac{\vartheta^{3\Phi}}{3!\,\Phi^3}cosh(\Im) + \cdots,\\
v(\Im,y,\vartheta) &= v_0(\Im,\vartheta) + v_1(\Im,\vartheta) + v_2(\Im,\vartheta) + v_3(\Im,\vartheta) + \cdots,\\
&= cosh(\Im) - \frac{\vartheta^{1\Phi}}{1!\,\Phi^1}sinh(\Im) + \frac{\vartheta^{2\Phi}}{2!\,\Phi^2}cosh(\Im) - \frac{\vartheta^{3\Phi}}{3!\,\Phi^3}sinh(\Im) + \cdots.
\end{aligned} \tag{4.16}$$

At $\Phi = 1$, the exact solutions are $u(\Im,\vartheta) = sinh(\Im - \vartheta)$, $v(\Im,\vartheta) = cosh(\Im - \vartheta)$, respectively. We examine the numerical and graphical outcomes of the precise and approximate solutions to the models described in Example 4.2. Using the error function, we investigate the accuracy and capabilities of the numerical technique. It is crucial to identify the error between the exact and approximate solutions. To illustrate the proposed method's efficiency and accuracy, we implemented the absolute error function. The $2D$ graphs of the comparison study are shown in Figure 3. They represent various values of $\Phi = 0.4,\ 0.6,\ 0.8,\ 1$ of the approximate and precise solution, respectively, generated by the recommended approach at $\wp = 1, \vartheta = 0.2$ in the interval $\Im \in [0,4]$. Figure 4 represents the $3D$ graphs of the exact and the approximate solutions attained by the suggested technique at $\Phi = 1$ in the intervals $\vartheta \in [0, 1]$, and $\Im \in [0, 2]$ at $\wp = 1$ for Example 4.2. These graphs demonstrate how well the suggested method's approximative solution works. The close approximation and precise solution demonstrate the accuracy and efficacy of the proposed technique. Tables 5 and 6 show the approximate and exact solutions and absolute error at $\Phi = 0.4,\ 0.6,\ 0.8,\ 1$ values. The point-wise error of the approximate and exact solutions of Example 4.2 at $\vartheta = 0.2$ is shown in Table 7. A comparison with the solution derived using the HAM is also provided. All the mentioned tables in Example 4.2 show that both types of solutions are highly comparable, supporting the efficacy of the developed approach.

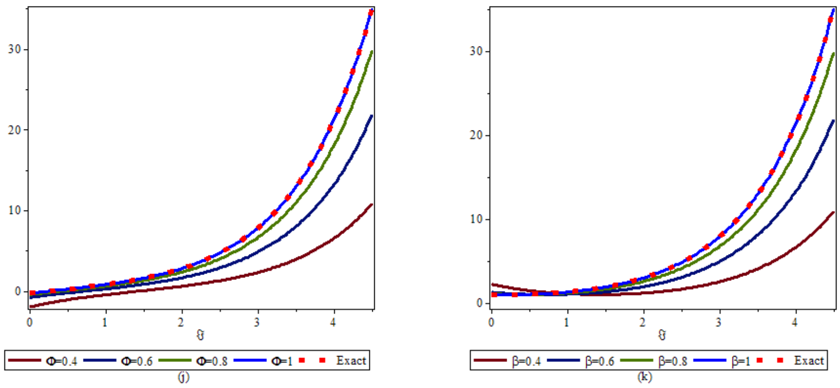

Figure 3. (j), (k) are the $2D$ plots of the proposed method for $u(\Im, \vartheta)$ and $v(\Im, \vartheta)$ at different values of $\Phi = 0.4,\ 0.6,\ 0.8,\ 1$, respectively, at $\vartheta = 0.2$ for Example 4.2.

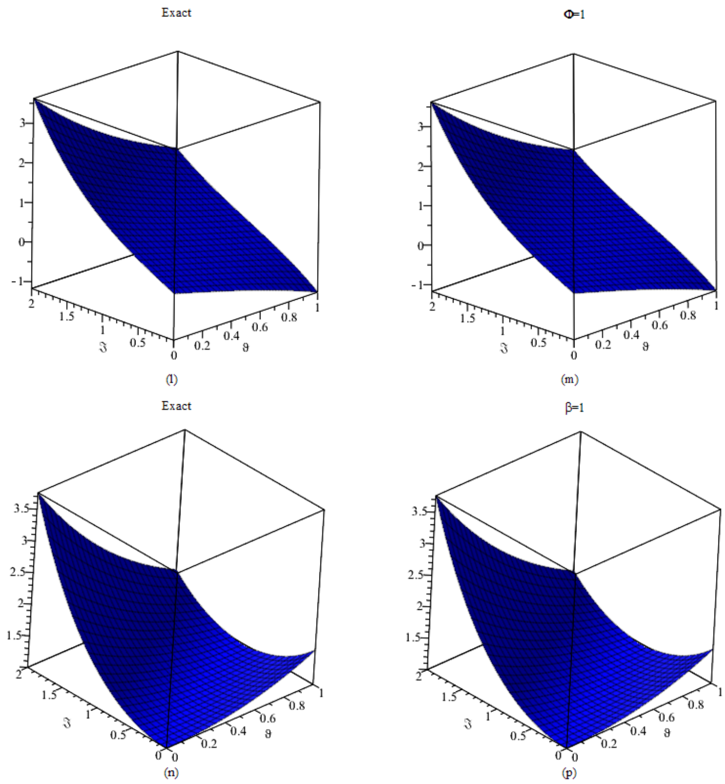

Figure 4. (l), (m), (n), (p) are the $3D$ plots of $u(\Im, \vartheta)$ and $v(\Im, \vartheta)$ of the exact and approximate solution for the proposed method of Example 4.2 at $\Phi = 1$, respectively.

Table 5. The approximate and exact solutions of $u(\Im, \vartheta)$ by CLTIM for Example 4.2 at different values of Φ when $\vartheta = 0.2$.

$\Im$	$\Phi = 0.4$	$\Phi = 0.6$	$\Phi = 0.8$	$\Phi = 1$	Exact solution	$\lvert u_{exact} - u_{CLTIM}\rvert$
0.0	-1.72330539	-0.67799290	-0.35181302	-0.20133600	-0.20133600	2.5×10^{-9}
0.1	-1.53297120	-0.56037592	-0.24738888	-0.10016675	-0.10016675	6.4×10^{-9}
0.2	-1.35797951	-0.44836738	−0.14544069	-0.00000001	0.00000000	1.543×10^{-8}
0.3	-1.19657893	-0.34084625	−0.04494812	0.10016672	0.10016675	2.44×10^{-8}
0.4	-1.04715412	-0.23673642	0.05509458	0.20133596	0.20133600	3.38×10^{-8}
0.5	-0.90820958	-0.13499593	0.15568869	0.30452025	0.30452029	4.34×10^{-8}
0.6	-0.77835470	-0.03460653	0.25784100	0.41075227	0.41075232	5.36×10^{-8}
0.7	-0.65628986	0.06543651	0.36257386	0.52109524	0.52109530	6.45×10^{-8}
0.8	-0.54079339	0.16613448	0.47093548	0.63665350	0.63665358	7.55×10^{-8}
0.9	-0.43070936	0.26849517	0.58401038	0.75858361	0.75858370	8.73×10^{-8}
1.0	-0.32493601	0.37354305	0.70293026	0.88810588	0.88810598	1.003×10^{-7}

Table 6. The aproximate and exact solutions of $v(\Im, \vartheta)$ by CLTIM for Example 4.2 at different values of Φ when $\vartheta = 0.2$.

$\Im$	$\Phi = 0.4$	$\Phi = 0.6$	$\Phi = 0.8$	$\Phi = 1$	Exact solution	$\lvert u_{exact} - u_{CLTIM}\rvert$
0.0	1.99339175	1.20817382	1.06008135	1.02006675	1.02006675	0.00000000
0.1	1.83074912	1.14630738	1.03014621	1.00500416	1.00500416	1×10^{-9}
0.2	1.68642924	1.09591357	1.01052111	1.00000000	1.00000000	1×10^{-9}
0.3	1.55898771	1.05648803	1.00100966	1.00500416	1.00500416	1×10^{-9}
0.4	1.44714905	1.02763618	1.00151664	1.02006675	1.02006675	0.00000000
0.5	1.34979395	1.00906926	1.01204714	1.04533851	1.04533851	0.00000000
0.6	1.26594803	1.00060144	1.03270655	1.08107237	1.08107237	1×10^{-9}
0.7	1.19477216	1.00214797	1.06370163	1.12762596	1.12762596	3×10^{-9}
0.8	1.13555396	1.01372434	1.10534259	1.18546522	1.18546521	3×10^{-9}
0.9	1.08770077	1.03544641	1.15804620	1.25516900	1.25516900	2×10^{-9}
1.0	1.05073365	1.06753157	1.22233992	1.33743494	1.33743494	3×10^{-9}

Table 7. The points-wise error of the approximate and exact solutions of Example 4.2 at different values of $\vartheta = 0.2$ also shows the comparison with the solution obtained by the HAM [57].

$\Im$	$u(\Im, \wp, \vartheta)$	HAM [57]	$v(\Im, \wp, \vartheta)$	HAM [57]
0.0	2.5×10^{-9}	1.21×10^{-8}	0.00000000	8.89×10^{-8}
0.1	6.4×10^{-9}	2.18×10^{-8}	1×10^{-9}	8.91×10^{-8}
0.2	1.54×10^{-8}	5.59×10^{-8}	1×10^{-9}	9.02×10^{-8}
0.3	2.44×10^{-8}	9.07×10^{-8}	1×10^{-9}	9.22×10^{-8}
0.4	3.38×10^{-8}	1.26×10^{-8}	0.00000000	9.51×10^{-8}
0.5	4.34×10^{-8}	1.63×10^{-8}	0.00000000	9.89×10^{-8}
0.6	5.36×10^{-8}	2.01×10^{-7}	1×10^{-9}	1.03×10^{-7}
0.7	6.45×10^{-8}	2.42×10^{-7}	3×10^{-9}	1.09×10^{-7}
0.8	7.55×10^{-8}	2.85×10^{-7}	3×10^{-9}	1.16×10^{-7}
0.9	8.73×10^{-8}	3.31×10^{-7}	2×10^{-9}	1.24×10^{-7}
1.0	1.003×10^{-7}	3.80×10^{-7}	3×10^{-9}	1.34×10^{-7}

Example 4.3. *Consider the following system of nonlinear temporal-fractional differential equations [56]:*

$$\begin{aligned}
&T_{\vartheta}^{\Phi}u(\Im,\wp,\vartheta)-\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)=1,\\
&T_{\vartheta}^{\Phi}v(\Im,\wp,\vartheta)-\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)=5,\quad 0<\Phi\leq 1,\\
&T_{\vartheta}^{\Phi}w(\Im,\wp,\vartheta)-\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)=5,
\end{aligned}\tag{4.17}$$

with the initial conditions

$$\begin{aligned}
u(\Im,\wp,0)&=\Im+2\wp,\\
v(\Im,\wp,0)&=\Im-2\wp,\\
w(\Im,\wp,0)&=-\Im+2\wp.
\end{aligned}\tag{4.18}$$

Taking CLT on both sides of Eq (4.23),

$$\begin{aligned}
\mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}u(\Im,\wp,\vartheta)\Big]&=\mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)+1\Big],\\
\mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}v(\Im,\wp,\vartheta)\Big]&=\mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)+5\Big],\\
\mathscr{L}_{\Phi}\Big[T_{\vartheta}^{\Phi}w(\Im,\wp,\vartheta)\Big]&=\mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)+5\Big].
\end{aligned}\tag{4.19}$$

Applying the process described in Section 3, we get the following outcome:

$$\begin{aligned}
\mathscr{L}_{\Phi}\Big[u(\Im,\wp,\vartheta)\Big]&=\frac{1}{s}(\Im+2\wp)+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)+1\Big],\\
\mathscr{L}_{\Phi}\Big[v(\Im,\wp,\vartheta)\Big]&=\frac{1}{s}(\Im-2\wp)+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)+5\Big],\\
\mathscr{L}_{\Phi}\Big[w(\Im,\wp,\vartheta)\Big]&=\frac{1}{s}(-\Im+2\wp)+\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big[\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)+5\Big].
\end{aligned}\tag{4.20}$$

Now, taking inverse CLT on both sides of Eq (4.20),

$$\begin{aligned}
u(\Im,\wp,\vartheta)&=\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(\Im+2\wp)\Big]+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}v(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}w(\Im,\wp,\vartheta)\Big)\Big],\\
v(\Im,\wp,\vartheta)&=\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(\Im-2\wp)\Big]+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}w(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}u(\Im,\wp,\vartheta)\Big)\Big],\\
w(\Im,\wp,\vartheta)&=\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(-x+2\wp)\Big]+\mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big(\frac{\partial}{\partial\Im}u(\Im,\wp,\vartheta)\frac{\partial}{\partial\wp}v(\Im,\wp,\vartheta)\Big)\Big].
\end{aligned}\tag{4.21}$$

Using the procedures described in Section 3, we get the following result:

$$u,v,w(\Im,\wp,\vartheta)=\sum_{k=0}^{\infty}u_k,v_k,w_k(\Im,\wp,\vartheta).\tag{4.22}$$

Using Eq (4.22) in (4.21),

$$\begin{aligned}
\sum_{k=0}^{\infty} u_k(\Im,\wp,\vartheta) &= \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(x+2\wp)\Big] \\
&\quad + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} v_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty} w_k(\Im,\wp,\vartheta)\Big)+1\Big\}\Big], \\
\sum_{k=0}^{\infty} v_k(\Im,\wp,\vartheta) &= \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(x-2\wp)\Big] \\
&\quad + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} w_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty} u_k(\Im,\wp,\vartheta)\Big)+5\Big\}\Big], \\
\sum_{k=0}^{\infty} w_k(\Im,\wp,\vartheta) &= \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s}(-x+2\wp)\Big] \\
&\quad + \mathscr{L}_{\Phi}^{-1}\Big[\frac{1}{s^{\Phi}}\mathscr{L}_{\Phi}\Big\{\Big(\frac{\partial}{\partial\Im}\sum_{k=0}^{\infty} u_k(\Im,\wp,\vartheta)\Big)\Big(\frac{\partial}{\partial\wp}\sum_{k=0}^{\infty} v_k(\Im,\wp,\vartheta)\Big)+5\Big\}\Big].
\end{aligned} \tag{4.23}$$

The following are the outcomes of Eq (4.23) when the iteration procedure described in Section 3 is applied:

$$\begin{aligned}
u_0(\Im,\wp,\vartheta) &= \Im+2\wp, \quad v_0(\Im,\wp,\vartheta)=\Im-2\wp, \quad w_0(\Im,\wp,\vartheta)=-\Im+2\wp, \\
u_1(\Im,\wp,\vartheta) &= \frac{3\vartheta^{1\Phi}}{1!\Phi^1}, \quad v_1(\Im,\wp,\vartheta)=\frac{3\vartheta^{1\Phi}}{1!\Phi^1}, \quad w_1(\Im,\wp,\vartheta)=\frac{3\vartheta^{1\Phi}}{1!\Phi^1}, \\
u_2(\Im,\wp,\vartheta) &= \frac{\vartheta^{1\Phi}}{1!\Phi^1}, \quad v_2(\Im,\wp,\vartheta)=\frac{5\vartheta^{1\Phi}}{1!\Phi^1}, \quad w_2(\Im,\wp,\vartheta)=\frac{5\vartheta^{1\Phi}}{1!\Phi^1}, \\
&\vdots
\end{aligned}$$

Consequently, we get the following approximate solution to Eq (4.17):

$$\begin{aligned}
u(\Im,\wp,\vartheta) &= u_0(\Im,\wp,\vartheta)+u_1(\Im,\wp,\vartheta)+u_2(\Im,\wp,\vartheta)+\cdots, \\
&= \Im+2\wp+\frac{3\vartheta^{1\Phi}}{1!\Phi^1}+\frac{\vartheta^{1\Phi}}{1!\Phi^1}+\cdots, \\
v(\Im,\wp,\vartheta) &= v_0(\Im,\wp,\vartheta)+v_1(\Im,\wp,\vartheta)+v_2(\Im,\wp,\vartheta)+\cdots, \\
&= \Im-2\wp+\frac{3\vartheta^{1\Phi}}{1!\Phi^1}+\frac{5\vartheta^{1\Phi}}{1!\Phi^1}+\cdots, \\
w(\Im,\wp,\vartheta) &= w_0(\Im,\wp,\vartheta)+w_1(\Im,\wp,\vartheta)+w_2(\Im,\wp,\vartheta)+\cdots, \\
&= -\Im+2\wp+\frac{3\vartheta^{1\Phi}}{1!\Phi^1}+\frac{5\vartheta^{1\Phi}}{1!\Phi^1}+\cdots.
\end{aligned} \tag{4.24}$$

5. Conclusions

This study has effectively extended the conformable Laplace transform iterative method to compute approximate solution of the systems of nonlinear temporal-fractional differential equations related to the conformable derivative. In addition, the method's theoretical predictions and error analysis have been examined. Three distinct examples are provided to show the applicability and efficacy of the proposed approach. The CLTIM method's validity and reliability are shown by comparing the approximate solutions to the exact solutions. To examine the exact and approximate solutions, we utilized the 2D and 3D graphs, as well as different tables to introduce precise solutions that demonstrated consistency across a range of fractional-order values. Consequently, this approach is straightforward and efficient for addressing temporal-fractional differential equations, which has the potential to be applied in a variety of scientific circumstances. As a result, we may use this method to solve more fractional-order linear and nonlinear differential equations.

Author contributions

N.G.; and S.N.; Conceptualization, A.M.S.; Visualization, S.N.; Funding, M.S.A.; Data curation, R.U.; Resources, N.G.; and S.N.; Writing-review & editing; A.M.S.; Formal analysis, M.S.A.; Project administration, R.U.; Data curation, M.S.A.; validation; Investigation, N.G.; Validation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. M. A. Herzallah, Notes on some fractional calculus operators and their properties, *J. Fract. Calc. Appl.*, **5** (2014), 1–10.
2. M. Caputo, Linear models of dissipation whose Q is almost frequency independent-II, *Geophys. J. Int.*, **5** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x
3. M. M. Al-Sawalha, R. P. Agarwal, R. Shah, O. Y. Ababneh, W. Weera, A reliable way to deal with fractional-order equations that describe the unsteady flow of a polytropic gas, *Mathematics*, **10** (2022), 2293. https://doi.org/10.3390/math10132293

4. D. Baleanu, J. A. T. Machado, A. C. Luo, *Fractional dynamics and control*, Springer, 2012. https://doi.org/10.1007/978-1-4614-0457-6

5. B. Bira, T. Raja Sekhar, D. Zeidan, Exact solutions for some time-fractional evolution equations using Lie group theory, *Math. Methods Appl. Sci.*, **41** (2018), 6717–6725. https://doi.org/10.1002/mma.5186

6. M. Al-Smadi, O. A. Arqub, Computational algorithm for solving fredholm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates, *Appl. Math. Comput.*, **342** (2019), 280–294. https://doi.org/10.1016/j.amc.2018.09.020

7. D. Baleanu, A. K. Golmankhaneh, A. K. Golmankhaneh, M. C. Baleanu, Fractional electromagnetic equations using fractional forms, *Int. J. Theor. Phys.*, **48** (2009), 3114–3123. https://doi.org/10.1007/s10773-009-0109-8

8. M. Al-Smadi, Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation, *Ain Shams Eng. J.*, **9** (2018), 2517–2525. https://doi.org/10.1016/j.asej.2017.04.006

9. Z. Altawallbeh, M. Al-Smadi, I. Komashynska, A. Ateiwi, Numerical solutions of fractional systems of two-point BVPs by using the iterative reproducing kernel algorithm, *Ukr. Math. J.*, **70** (2018), 687–701. https://doi.org/10.1007/s11253-018-1526-8

10. S. Hasan, A. El-Ajou, S. Hadid, M. Al-Smadi, S. Momani, Atangana-Baleanu fractional framework of reproducing kernel technique in solving fractional population dynamics system, *Chaos Soliton. Fract.*, **133** (2020), 109624. https://doi.org/10.1016/j.chaos.2020.109624

11. F. Riewe, Mechanics with fractional derivatives, *Phys. Rev. E*, **55** (1997), 3581. https://doi.org/10.1103/PhysRevE.55.3581

12. N. Laskin, Fractional quantum mechanics, *Phys. Rev. E*, **62** (2000), 3135. https://doi.org/10.1103/PhysRevE.62.3135

13. S. Noor, A. W. Alrowaily, M. Alqudah, R. Shah, S. A. El-Tantawy, Innovative solutions to the fractional diffusion equation using the Elzaki transform, *Math. Comput. Appl.*, **29** (2024), 75. https://doi.org/10.3390/mca29050075

14. J. Muhammad, N. Gul, R. Ali, Enhanced decision-making with advanced algebraic techniques in complex Fermatean fuzzy sets under confidence levels, *Int. J. Knowl. Innov. Stud.*, **2** (2024), 107–118. https://doi.org/10.56578/ijkis020205

15. M. Khalid, I. A. Mallah, S. Alha, A. Akgul, Exploring the Elzaki transform: unveiling solutions to reaction-diffusion equations with generalized composite fractional derivatives, *Contemp. Math.*, **5** (2024), 1426–1438. https://doi.org/10.37256/cm.5220244621

16. H. Yasmin, A. A. Alderremy, R. Shah, A. Hamid Ganie, S. Aly, Iterative solution of the fractional Wu-Zhang equation under Caputo derivative operator, *Front. Phys.*, **12** (2024), 1333990. https://doi.org/10.3389/fphy.2024.1333990

17. Y. Mahatekar, P. S. Scindia, P. Kumar, A new numerical method to solve fractional differential equations in terms of Caputo-Fabrizio derivatives, *Phys. Scr.*, **98** (2023), 024001. https://doi.org/10.1088/1402-4896/acaf1a

18. A. Khan, J. F. Gómez-Aguilar, T. S. Khan, H. Khan, Stability analysis and numerical solutions of fractional order HIV/AIDS model, *Chaos Soliton. Fract.*, **122** (2019), 119–128. https://doi.org/10.1016/j.chaos.2019.03.022

19. A. El-Ajou, O. A. Arqub, Z. A. Zhour, S. Momani, New results on fractional power series: theories and applications, *Entropy*, **15** (2013), 5305–5323. https://doi.org/10.3390/e15125305

20. A. Kilicman, Z. A. A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, *Appl. Math. Comput.*, **187** (2007), 250–265. https://doi.org/10.1016/j.amc.2006.08.122

21. A. Khan, T. Abdeljawad, J. F. Gómez-Aguilar, H. Khan, Dynamical study of fractional order mutualism parasitism food web module, *Chaos Soliton. Fract.*, **134** (2020), 109685. https://doi.org/10.1016/j.chaos.2020.109685

22. M. I. Liaqat, A. Akgül, A novel approach for solving linear and nonlinear time-fractional Schrödinger equations, *Chaos Soliton. Fract.*, **162** (2022), 112487. https://doi.org/10.1016/j.chaos.2022.112487

23. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002

24. E. Balcı, İ. Öztürk, S. Kartal, Dynamical behaviour of fractional order tumor model with Caputo and conformable fractional derivative, *Chaos Soliton. Fract.*, **123** (2019), 43–51. https://doi.org/10.1016/j.chaos.2019.03.032

25. W. Xie, C. Liu, W. Z. Wu, W. Li, C. Liu, Continuous grey model with conformable fractional derivative, *Chaos Soliton. Fract.*, **139** (2020), 110285. https://doi.org/10.1016/j.chaos.2020.110285

26. C. Ma, M. Jun, Y. Cao, T. Liu, J. Wang, Multistability analysis of a conformable fractional-order chaotic system, *Phys. Scr.*, **95** (2020), 075204. https://doi.org/10.1088/1402-4896/ab8d54

27. H. Tajadodi, A. Khan, J. F. Gómez-Aguilar, H. Khan, Optimal control problems with Atangana-Baleanu fractional derivative, *Optim. Contr. Appl. Met.*, **42** (2021), 96–109. https://doi.org/10.1002/oca.2664

28. J. S. Duan, T. Chaolu, R. Rach, Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rach-Adomian-Meyers modified decomposition method, *Appl. Math. Comput.*, **218** (2012), 8370–8392. https://doi.org/10.1016/j.amc.2012.01.063

29. A. Di Matteo, A. Pirrotta, Generalized differential transform method for nonlinear boundary value problem of fractional order, *Commun. Nonlinear Sci. Numer. Simul.*, **29** (2015), 88–101. https://doi.org/10.1016/j.cnsns.2015.04.017

30. A. Al-rabtah, V. S. Ertürk, S. Momani, Solutions of a fractional oscillator by using differential transform method, *Comput. Math. Appl.*, **59** (2010), 1356–1362. https://doi.org/10.1016/j.camwa.2009.06.036

31. S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, *Chaos Soliton. Fract.*, **27** (2006), 1119–1123. https://doi.org/10.1016/j.chaos.2005.04.113

32. O. Abdulaziz, I. Hashim, S. Momani, Solving systems of fractional differential equations by homotopy-perturbation method, *Phys. Lett. A*, **372** (2008), 451–459. https://doi.org/10.1016/j.physleta.2007.07.059

33. J. H. He, Variational iteration method for autonomous ordinary differential systems, *Appl. Math. Comput.*, **114** (2000), 115–123. https://doi.org/10.1016/S0096-3003(99)00104-6

34. Y. Kai, Z. Yin, Linear structure and soliton molecules of Sharma-Tasso-Olver-Burgers equation, *Phys. Lett. A*, **452** (2022), 128430. https://doi.org/10.1016/j.physleta.2022.128430

35. Y. He, Y. Kai, Wave structures, modulation instability analysis and chaotic behaviors to Kudryashov's equation with third-order dispersion, *Nonlinear Dyn.*, **12** (2024), 10355–10371. https://doi.org/10.1007/s11071-024-09635-3

36. J. Xie, Z. Xie, H. Xu, Z. Li, W. Shi, J. Ren, et al., Resonance and attraction domain analysis of asymmetric duffing systems with fractional damping in two degrees of freedom, *Chaos Soliton. Fract.*, **187** (2024), 115440. https://doi.org/10.1016/j.chaos.2024.115440

37. C. Zhu, S. A. Idris, M. E. M. Abdalla, S. Rezapour, S. Shateyi, B. Gunay, Analytical study of nonlinear models using a modified Schrodinger's equation and logarithmic transformation, *Results Phys.*, **55** (2023), 107183. https://doi.org/10.1016/j.rinp.2023.107183

38. S. Guo, A. Das, Cohomology and deformations of generalized reynolds operators on leibniz algebras, *Rocky Mountain J. Math.*, **54** (2024), 161–178. https://doi.org/10.1216/rmj.2024.54.161

39. C. Guo, J. Hu, J. Hao, S. Čelikovsky, X. Hu, Fixed-time safe tracking control of uncertain high-order nonlinear pure-feedback systems via unified transformation functions, *Kybernetika*, **59** (2023), 342–364. https://doi.org/10.14736/kyb-2023-3-0342

40. A. A. Alderremy, R. Shah, N. Iqbal, S. Aly, K. Nonlaopon, Fractional series solution construction for nonlinear fractional reaction-diffusion Brusselator model utilizing Laplace residual power series, *Symmetry*, **14** (2022), 1944. https://doi.org/10.3390/sym14091944

41. H. Yasmin, A. S. Alshehry, A. H. Ganie, A. M. Mahnashi, R. Shah, Perturbed Gerdjikov-Ivanov equation, soliton solutions via Backlund transformation, *Optik*, **298** (2024), 171576. https://doi.org/10.1016/j.ijleo.2023.171576

42. M. M. Al-Sawalha, A. Khan, O. Y. Ababneh, T. Botmart, Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives, *AIMS Math.*, **10** (2022), 18334–18359. https://doi.org/10.3934/math.20221010

43. E. M. Elsayed, R. Shah, K. Nonlaopon, The analysis of the fractional-order Navier-Stokes equations by a novel approach, *J. Funct. Spaces*, **2022** (2022), 8979447. https://doi.org/10.1155/2022/8979447

44. M. Naeem, H. Rezazadeh, A. A. Khammash, R. Shah, S. Zaland, Analysis of the fuzzy fractional-order solitary wave solutions for the KdV equation in the sense of Caputo-Fabrizio derivative, *J. Math.*, **2022** (2022), 3688916. https://doi.org/10.1155/2022/3688916

45. M. Alqhtani, K. M. Saad, R. Shah, W. M. Hamanah, Discovering novel soliton solutions for $(3+1)$-modified fractional Zakharov-Kuznetsov equation in electrical engineering through an analytical approach, *Opt. Quant. Electron.*, **55** (2023), 1149. https://doi.org/10.1007/s11082-023-05407-2

46. M. Alqhtani, K. M. Saad, R. Shah, W. Weera, W. M. Hamanah, Analysis of the fractional-order local Poisson equation in fractal porous media, *Symmetry*, **14** (2022), 1323. https://doi.org/10.3390/sym14071323
47. S. A. El-Tantawy, R. T. Matoog, A. W. Alrowaily, S. M. Ismaeel, On the shock wave approximation to fractional generalized Burger-Fisher equations using the residual power series transform method, *Phys. Fluids*, **36** (2024), 023105. https://doi.org/10.1063/5.0187127
48. H. Yasmin, A. S. Alshehry, A. H. Ganie, A. Shafee, R. Shah, Noise effect on soliton phenomena in fractional stochastic Kraenkel-Manna-Merle system arising in ferromagnetic materials, *Sci. Rep.*, **14** (2024), 1810. https://doi.org/10.1038/s41598-024-52211-3
49. S. Noor, W. Albalawi, R. Shah, M. M. Al-Sawalha, S. M. E. Ismaeel, S. A. El-Tantawy, On the approximations to fractional nonlinear damped Burger's-type equations that arise in fluids and plasmas using Aboodh residual power series and Aboodh transform iteration methods, *Front. Phys.*, **12** (2024), 1374481. https://doi.org/10.3389/fphy.2024.1374481
50. M. I. Liaqat, E. Okyere, The fractional series solutions for the conformable time-fractional Swift-Hohenberg equation through the conformable Shehu Daftardar-Jafari approach with comparative analysis, *J. Math.*, **2022** (2022), 3295076. https://doi.org/10.1155/2022/3295076
51. J. Younis, B. Ahmed, M. AlJazzazi, R. Al Hejaj, H. Aydi, Existence and uniqueness study of the conformable Laplace transform, *J. Math.*, **2022** (2022), 4554065. https://doi.org/10.1155/2022/4554065
52. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, **279** (2015), 57–66. https://doi.org/10.1016/j.cam.2014.10.016
53. Z. Al-Zhour, F. Alrawajeh, N. Al-Mutairi, R. Alkhasawneh, New results on the conformable fractional Sumudu transform: theories and applications, *Int. J. Anal. Appl.*, **17** (2019), 1019–1033.
54. B. Ahmed, Generalization of fractional Laplace transform for higher order and its application, *J. Innov. Appl. Math. Comput. Sci.*, **1** (2021), 79–92.
55. A. Khan, M. I. Liaqat, M. A. Alqudah, T. Abdeljawad, Analysis of the conformable temporal-fractional swift-hohenberg equation using a novel computational technique, *Fractals*, **31** (2023), 2340050. https://doi.org/10.1142/S0218348X23400509
56. M. Sultana, M. Waqar, A. H. Ali, A. A. Lupas, F. Ghanim, Z. A. Abduljabbar, Numerical investigation of systems of fractional partial differential equations by new transform iterative technique, *AIMS Math.*, **10** (2024), 26649–26670. https://doi.org/10.3934/math.20241296
57. H. Jafari, S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 1962–1969. https://doi.org/10.1016/j.cnsns.2008.06.019