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Research article

Optimal investment strategy for an investor with partial information under exchange rate risk based on Malliavin calculus

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Abstract: This paper investigates the optimal investment decision for an investor with partial information under the criterion of maximizing the expected utility of terminal wealth. The domestic and foreign stock prices, as well as the exchange rate, are modeled as jump-diffusion processes with stochastic coefficients. By employing Malliavin calculus, we derive a sufficient and necessary condition for the optimal investment strategy in cross-border transactions. In some special cases, a closed-form expression is obtained. Finally, a numerical example is provided to illustrate the impacts of parameters ρ_1, ρ_2, and σ_R on the optimal investment strategy.

Keywords: optimal investment; partial information; exchange rate; jump-diffusion model; Malliavin calculus
Mathematics Subject Classification: 91G80, 93E20

1. Introduction

The volatility of the exchange rate directly impacts investment returns and has become a critical factor in cross-border trade. Moreover, the market information can be fully observed in the assumptions of traditional asset allocation models. However, there exists information asymmetry in the practical financial market, that is, the economic agents' investment strategies only depend on partial information. In recent years, numerous related works have adopted diffusion models to describe the price processes of risky assets and the exchange rate processes. In reality, the prices of assets and exchange rates are often affected by unexpected events, such as financial crises and natural disasters. These events cause severe fluctuations in the prices. In this setting, how to construct a strategy for cross-border trade under partial information in a jump-diffusion market has become one of the significant issues in the field of optimal investment.

In the seminal paper, Merton [1] first proposed the jump-diffusion model, which incorporates the continuous fluctuations and discrete jumps of asset prices into a unified framework; so far, the jump-diffusion process remains an indispensable tool in modern financial modeling [1,2]. However, the jump process implies non-smooth (non-differentiable) shocks upon the model, which lead to the classical dynamic programming approaches being ineffective. By contrast, Malliavin calculus can characterize the discontinuous fluctuation properties via "stochastic derivatives". Since its theoretical foundations have been established by Malliavin (1978) [3], it has been widely applied in the field of finance and actuarial science. For example, Di Nunno et al. (2006) [4] investigated optimal investment strategies for an investor under insider information in a jump-diffusion market. They derived the explicit expressions for the optimal strategy using forward integration and Malliavin calculus. Peng and Chen (2022) [5] constructed a financial model that incorporated inflation-indexed bonds under the mean-variance criterion. Using Malliavin calculus, they obtained the relationship for optimal investment strategies and provided closed-form solutions for optimal investments in some special cases. Chen et al. (2022) [6] applied Malliavin calculus and forward stochastic calculus to address the optimal investment and risk control problem for insurers under uncertain time horizons and partial information, thus, providing explicit expressions for the optimal strategies. For more details, also see the following relevant literature: Kieu et al. (2013) [7], and Pamen et al. (2014) [8].

The exchange rate plays a critical role in cross-border trade. The cause of exchange rate risk lies in the fact that the domestic currency value fluctuates with the exchange rate fluctuations. Kiyota and Urata (2004) [9] investigated the impact of exchange rates and their volatility on foreign investment decisions from an empirical perspective. Guo et al. (2018) [10] developed a framework that incorporated inflation risk and exchange rate risk. Assuming that exchange rates and stock prices follow geometric Brownian motion, using the dynamic programming theory, they derived an explicit expression for the optimal investment-reinsurance strategy under the criterion of maximizing the expected utility of terminal wealth. Fei et al. (2021) [11] explored the intertemporal asset allocation problem for multinational corporations under the exchange rate risk. They obtained analytic solutions for the optimal consumption and investment strategies under the criterion of maximizing the expected utility of intertemporal consumption and terminal wealth by solving the Hamilton-Jacobi-Bellman (HJB) equation. Wang et al. (2023) [12] examined the robust optimal investment problem under the exchange rate and default risks.

The aforementioned works focus on markets where information is fully observable. In particular, investors can know all key state variables. Nevertheless, in practice, the information owned by agents is only partial information about the financial market, that is, investors can only get access to information such as stock prices and are unable to directly observe the instantaneous rate of return or driving Brownian motion of risky assets. The optimal problem under partial information makes the optimal decision more realistic and complicated. The existing works on partial information mostly adopt a Kalman filter approach to transform unobserved variables to fully observed ones [13–15]. Pikovsky and Karatzas [16] investigated the optimal portfolio problem by incorporating a Kalman filter approach. Subsequently, Baltas et al. [17] applied the Kalman filter approach to the optimal investment and reinsurance and provided the explicit expression for the optimal strategy. Liang and Song [18] applied the HJB approaches to explore the optimal investment and reinsurance strategies with partial information. Di Nunno and Øksendal (2008) [19] sought the optimal portfolio policy under partial information by Malliavin calculus in the Lévy market. Peng and Hu (2013) [20] extended the model in Di Nunno and Øksendal (2008) [19] to a jump-diffusion framework and investigated the optimal investment and insurance strategy by the same approach. Moreover, Peng et al. (2018) [21] and Chen et al. (2022) [6] studied the optimal investment and risk control strategy for an insurer with partial information using the forward integral and Malliavin calculus.

Most of the existing works on exchange rates focus on the case where full information about the market is known and ignore the matter of partial information. This results in a failure to be consistent with the practical financial market. Meanwhile, the stochastic control approach is widely adopted for the optimal investment strategy in the existing literature. However, few works on exchange rates use the Malliavin calculus. Compared with Di Nunno et al. (2008) [22] , which focused on an optimal investment strategy in a single market, our work extends the analysis to a multi-market framework that includes both domestic and foreign markets, and we incorporate the jump-diffusion model of the exchange rate. In contrast to Wang et al. (2023) [12], we adopt a more general jump-diffusion model to formulate the exchange rate process instead of the classical diffusion model. Furthermore, our research is based on a partial information framework, which is more consistent with the practical financial market. Inspired by [12, 22], in the present paper, the models of risky assets and exchange rates are both assumed to be jump-diffusion processes with stochastic coefficients. We derive a sufficient and necessary condition of the optimal asset allocation for an investor with partial information by applying the Malliavin calculus under the criterion of maximizing the expected utility of terminal wealth. The explicit solutions are provided in some special cases.

The remainder of this paper is organized as follows: Section 2 formulates the optimization problem by incorporating the exchange rate risk and jump-diffusion processes; Section 3 applies Malliavin calculus to derive a necessary and sufficient condition for the optimal asset allocation; Section 4 provides explicit expressions for the optimal strategy in some special cases and some numerical examples are given and Section 5 concludes the paper.

2. Modeling of investment decisions

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T],P})$ be a complete filtered probability space, and let $\mathcal{F}_t$ satisfy the usual conditions, where $T \geq 0$ is a constant that represents the terminal transaction time; all stochastic processes are defined on this probability space. Moreover, we assume that the transactions are to be continuous without cost and taxes.

For convenience, we suppose that the financial market consists of a risk-free bond $B(t)$ and a domestic stock $S^d(t)$. The price dynamics of the risk-free bond $B(t)$ follow the stochastic differential equation:

$$dB(t) = r_d(t)B(t)dt, \tag{2.1}$$

where $r_d(t)$ denotes the domestic risk-free rate and is a deterministic and bounded function with respect to t.

The domestic stock price $S^d(t)$ is modeled by the following:

$$dS^d(t) = S^d(t^-)\left(\mu_d(t)dt + \sigma_d(t)dW_1(t) + \int_{\mathbb{R}_0} \theta_d(t,z)\tilde{N}_d(dt,dz)\right). \tag{2.2}$$

Here, $\mu_d(t)$ and $\sigma_d(t)$ are the drift rate and volatility of the domestic stock, respectively. $W_1(t)$ is a standard Brownian motion, $\theta_d(t,z)$ is a positive jump amplitude, and $\tilde{N}_d(dt,dz)$ is a compensated Poisson measure given by $\tilde{N}_d(dt,dz) = N_d(dt,dz) - \lambda_d(t)dt$. Moreover, we assume that the stochastic processes $\mu_d(t)$, $\sigma_d(t)$, and $\theta_d(t,z)$ are bounded, càglàd, and adapted to the filtration $\mathcal{F}_t$, for all $t \in (0,T]$. The condition $\theta_d(t,z) > -1$ is required in order to ensure that the domestic stock price takes a positive value.

Moreover, the foreign stock is allowed to be invested in. We suppose that the foreign stock price $S^f(t)$ is given by the following:

$$dS^f(t) = S^f(t)\left(\mu_f(t)dt + \sigma_f(t)dW_2(t) + \int_{\mathbb{R}_0} \theta_f(t,z)\tilde{N}_f(dz,dt)\right), \tag{2.3}$$

where $\mu_f(t)$, $\sigma_f(t)$, and $\theta_f(t,z)$ are the drift rate, volatility, and jump amplitude of the foreign stock, respectively. And $\widetilde{N}_f(dt,dz)$ represents the compensated Poisson measure of the foreign stock. $W_2(t)$ is a standard Brownian motion independent of W_1, and the coefficients $\mu_f(t)$, $\sigma_f(t)$, and $\theta_f(t,z)$ are assumed to be bounded, càglàd, and adapted to the filtration $\mathcal{F}_t$, for all $t \in (0,T]$. The restriction $\theta_f(t,z) > -1$ is assumed to hold $dt \times \nu_f(dz)$-almost surely for all $t \in (0,T]$.

In cross-border investment, the exchange rate is a critical factor to be considered. Exchange rate fluctuations will affect the value of the domestic currency and subsequently influence the investment returns. We assume that the exchange rate $R(t)$ evolves as follows:

$$dR(t)=R(t)\left(\mu_R(t)dt+\rho_1\sigma_R(t)dW_1(t)+\rho_2\sigma_R(t)dW_2(t)+\sqrt{1-\rho_1^2-\rho_2^2}\sigma_R(t)dW_3(t)+\int_{\mathbb{R}_0}\theta_R(t,z)\tilde{N}_R(dz,dt)\right). \tag{2.4}$$

Here, $\sigma_R(t)$ denotes the exchange rate volatility, and $\widetilde{N}_R(dt,dz)$ is a compensated Poisson measure. ρ_1 and $\rho_2 \in [0,1]$, ρ_1 and ρ_2 denote the correlation between the exchange rate and domestic stock and the exchange rate and foreign stock, respectively. Furthermore, the stochastic processes $\mu_R(t)$, $\sigma_R(t)$, and $\theta_R(t,z)$ are assumed to be bounded, càglàd, and adapted to the filtration $\mathcal{F}_t$, for any $t \in (0,T]$. In the model of the exchange rate, the same restriction $\theta_R(t,z)$ holds. Conversely, the transformation of foreign stock prices into domestic currency is given by the following:

$$S(t) = S^f(t)R(t). \tag{2.5}$$

Applying Itô's lemma to (2.5) gives rise to the following:

$$dS(t) = S^f(t)dR(t) + R(t)dS^f(t) + d\langle S^f, R\rangle_t, \tag{2.6}$$

where

$$d\left\langle S^f, R\right\rangle_t = S^f(t)R(t)\rho_2\sigma_f(t)\sigma_R(t)dt = S(t)\rho_2\sigma_f(t)\sigma_R(t)dt. \tag{2.7}$$

Substituting (2.3) and (2.4) into their quadratic covariation $d\left\langle S^f, R\right\rangle_t$, and plugging (2.7) into (2.6) yields

$$dS(t) = S(t)\left(\left(\mu_R(t) + \mu_f(t) + \rho_2\sigma_f(t)\sigma_R(t)\right)dt + \rho_1\sigma_R(t)dW_1(t) + \left(\rho_2\sigma_R(t) + \sigma_f(t)\right)dW_2(t)\right.$$

$$+\sqrt{1-\rho_1^2-\rho_2^2}\sigma_R(t)dW_3(t)+\int_{R_0}\theta_f(t,z)\tilde{N}_f(dz,dt)+\int_{R_0}\theta_R(t,z)\tilde{N}_R(dz,dt)\Bigg). \tag{2.8}$$

Let $\omega_d(t)$ and $\omega_f(t)$ denote the amounts of money invested in domestic and foreign stocks, respectively, at time t. We call $\omega(t)=(\omega_d(t),\omega_f(t))'$, $t\in(0,T]$ an investment strategy. We assume that the information flow $\mathcal{F}_t=\sigma(W_1(t),W_2(t),W_3(t),\tilde{N}_R,\tilde{N}_f)$ represents all available market information up to time t. All model parameters in this paper are assumed to be $\mathcal{F}_t$-adapted processes. However, in practice, at time t, investors cannot get access to the entire market information ($\mathcal{F}_t$); instead, they may only observe historical asset prices and exchange rates, but not the instantaneous rate of return and the underlying volatility, etc. In this situation, the information available to investors is called to be partial information, denoted by $\mathcal{G}_t$, with $\mathcal{G}_t\subseteq\mathcal{F}_t$. Investors need to make decisions based on the partial information $\mathcal{G}_t$ instead of the complete information $\mathcal{F}_t$.

Definition 2.1. *The $\omega(t)=\left(\omega_d(t),\omega_f(t)\right)$ is said to be an admissible strategy if the properties are satisfied as follows:*

(1) the process $\omega(t)$ is càdlàg and adapted to the filtration $\mathcal{G}=(\mathcal{G}_t)_{0\leq t\leq T}$;
(2) $0\leqslant\omega_d(t)<\infty$ and $0\leqslant\omega_f(t)<\infty$, for all $t\in[0,T]$; and
(3)

$$\begin{aligned}
&E\int_0^T\Bigg[|\omega_d(t)|\,|\mu_d(t)-r_d(t)|+\omega_d^2(t)\sigma_d^2(t)+\left|\omega_f(t)\right|\left|\mu_R(t)+\mu_f(t)-r_d(t)+\rho_2\sigma_f(t)\sigma_R(t)\right|\\
&+\rho_1^2\omega_f^2(t)\sigma_R^2(t)+\omega_f^2(t)\left(\rho_2\sigma_R(t)+\sigma_f(t)\right)^2+\omega_f^2(t)\left(1-\rho_1^2-\rho_2^2\right)\sigma_R^2(t)\Bigg]dt\\
&+E\int_0^T\Bigg[\int_{\mathbb{R}_0}\omega_d^2(t)\theta_d^2(t,z)\nu_d(dz)+\int_{\mathbb{R}_0}\omega_f^2(t)\theta_f^2(t,z)\nu_f(dz)+\int_{\mathbb{R}_0}\omega_f^2(t)\theta_R^2(t,z)\nu_R(dz)\Bigg]dt<\infty.
\end{aligned}$$

The set of all admissible strategies is denoted by $\mathcal{A}$. Under the expected utility of terminal wealth criterion, the optimization problem for the investor is formulated as follows:

$$u(\omega):=\sup_{\omega(t)\in\mathcal{A}}E[U(V_\omega(T))\mid V(t_0)=v_0], \tag{2.9}$$

where $E[\cdot]$ denotes the expectation operator, and $U(\cdot)$ is a monotonic increasing and concave utility function (i.e., $U'(\cdot)>0,U''(\cdot)<0$. $V_\omega(t)$ represents the wealth process depending on the strategy ω at time t). It can be expressed as follows:

$$\begin{aligned}
dV_\omega(t)=&\frac{\omega_d(t)V_\omega(t^-)}{S^d(t)}dS^d(t)+\frac{\omega_f(t)V_\omega(t^-)}{S(t)}dS(t)+\frac{\left(1-\omega_d(t)-\omega_f(t)\right)V_\omega(t^-)}{B(t)}dB(t)\\
=&V_\omega(t^-)\Bigg\{\omega_d(t)\left(\mu_d(t)dt+\sigma_d(t)dW_1(t)+\int_{R_0}\theta_d(t,z)\tilde{N}_d(dz,dt)\right)\\
&+\omega_f(t)\Big(\left(\mu_R(t)+\mu_f(t)+\rho_2\sigma_f(t)\sigma_R(t)\right)dt+\rho_1\sigma_R(t)dW_1(t)+\left(\rho_2\sigma_R(t)+\sigma_f(t)\right)dW_2(t)\\
&+\sqrt{1-\rho_1^2-\rho_2^2}\sigma_R(t)dW_3(t)+\int_{R_0}\theta_f(t,z)\tilde{N}_f(dt,dz)+\int_{R_0}\theta_R(t,z)\tilde{N}_R(dt,dz)\Big)\\
&+\left(1-\omega_d(t)-\omega_f(t)\right)r_d(t)dt\Bigg\}
\end{aligned} \tag{2.10}$$

with $V_\omega(0)=v_0$.

3. Main result

In this section, we derive the necessary and sufficient conditions for the optimal investment strategy for an investor with partial information under the utility maximization criterion by Malliavin calculus.

Theorem 3.1. *An admissible strategy $\omega^*(t)$ is called the optimal investment strategy if it satisfies $E[U(V(T))]\big|_{\omega=\omega^*} \geq E[U(V(T))]\big|_{\omega\neq\omega^*}$, where $\omega^*(t) = \left(\omega_d^*(t), \omega_f^*(t)\right)$. The optimality condition for $\omega^*(t)$ is given by the following equations:*

$$\begin{aligned} &E\left[\left(\mu_d(s) - r_d(s) - \omega_d^*(s)\sigma_d^2(s) - \omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)\right) F_{\omega^*}(T) \mid \mathcal{G}_s\right] \\ &+ E\left[\int_{\mathbb{R}_0} \frac{\theta_d(s,z)D_{s,z}^d\left(F_{\omega*}(T)\right) - \omega_d^*(s)\theta_d^2(s,z)F_{\omega^*}(T)}{1+\omega_d^*(s)\theta_d(s,z)}\nu(dz) \mid \mathcal{G}_s\right] = 0, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &E\Bigg[\Big(\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s) - \omega_f^*(s)\Big(\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s)\Big) \\ &- \omega_d^*(s)\rho_1\sigma_d(s)\sigma_R(s)\Big)F_{\omega^*}(T) + \Big(\rho_2\sigma_R(s) + \sigma_f(s)\Big)D_s^2\Big(F_{\omega^*}(T)\Big) \\ &+ \sqrt{1-\rho_1^2-\rho_2^2}\,\sigma_R(s)D_s^3\Big(F_{\omega^*}(T)\Big) \,\Bigg|\, \mathcal{G}_s\Bigg] + E\left[\int_{\mathbb{R}_0} \frac{\theta_f(s,z)D_{s,z}^f\left(F_{\omega^*}(T)\right) - \omega_f^*(s)\theta_f^2(s,z)F_{\omega^*}(T)}{1+\omega_f^*(s)\theta_f(s,z)}\nu_f(dz)\right. \\ &\left.+ \int_{\mathbb{R}_0} \frac{\theta_R(s,z)D_{s,z}^R\left(F_{\omega^*}(T)\right) - \omega_f^*(s)\theta_R^2(s,z)F_{\omega^*}(T)}{1+\omega_f^*(s)\theta_R(s,z)}\nu_R(dz) \,\middle|\, \mathcal{G}_s\right] = 0, \end{aligned} \tag{3.2}$$

where $F_{\omega^*}(T) = U'(V_{\omega^*}(T))V_{\omega^*}(T)$, for all $\omega \in \mathcal{A}$.

Conversely, if a strategy $\omega^*(t)$ satisfies (3.1) *and* (3.2)*, then it holds that $E[U(V(T))]\big|_{\omega=\omega^*} \geq E[U(V(T))]\big|_{\omega\neq\omega^*}$.*

Moreover, the following conditions should be satisfied:

$$\int_{\mathbb{R}_0} \frac{\theta_d(s,z)D_{s,z}^d\left(F_\omega(T)\right) - \omega_d(s)\theta_d^2(s,z)F_\omega(T)}{1+\omega_d(s)\theta_d(s,z)}\nu_d(dz),$$

and

$$\int_{\mathbb{R}_0}\frac{\theta_f(s,z)D_{s,z}^f(F_\omega(T))-\omega_f(s)\theta_f^2(s,z)F_\omega(T)}{1+\omega_f(s)\theta_f(s,z)}\nu_f(dz)+\int_{\mathbb{R}_0}\frac{\theta_R(s,z)D_{s,z}^R(F_\omega(T))-\omega_f(s)\theta_R^2(s,z)F_\omega(T)}{1+\omega_f(s)\theta_R(s,z)}\nu_R(dz)$$

are uniformly integrable.

Proof. Simultaneously dividing both sides of (2.10) by $V(t^-)$ and using Itô's formula, we have the following:

$$\begin{aligned} V_\omega(T) = v_0 \exp\Bigg\{ \int_0^T \Bigg[& r_d(s) + \omega_d(s)\left(\mu_d(s) - r_d(s)\right) + \omega_f(s)\left(\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s)\right) \\ & - \frac{1}{2}\omega_d^2(s)\sigma_d^2(s) - \frac{1}{2}\omega_f^2(s)\left(\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s)\right) \end{aligned}$$

$$
\begin{aligned}
&- \omega_d(s)\omega_f(s)\rho_1\sigma_d(s)\sigma_R(s)\Big] ds + \int_0^T \Big[\omega_d(s)\sigma_d(s) + \omega_f(s)\rho_1\sigma_R(s)\Big] dW_1(s) \\
&+ \int_0^T \omega_f(s)\Big[\rho_2\sigma_R(s) + \sigma_f(s)\Big] dW_2(s) + \int_0^T \omega_f(s)\sqrt{1-\rho_1^2-\rho_2^2}\,\sigma_R(s)dW_3(s) \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln(1+\omega_d(s)\theta_d(s,z)) - \omega_d(s)\theta_d(s,z)\Big]\nu_d(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_f(s,z)\Big) - \omega_f(s)\theta_f(s,z)\Big]\nu_f(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_R(s,z)\Big) - \omega_f(s)\theta_R(s,z)\Big]\nu_R(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln(1+\omega_d(s)\theta_d(s,z))\Big]\tilde{N}_d(ds,dz) + \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_f(s,z)\Big)\Big]\widetilde{N}_f(ds,dz) \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_R(s,z)\Big)\Big]\tilde{N}_R(ds,dz)\Big\}. \qquad (3.3)
\end{aligned}
$$

Let

$$
\begin{aligned}
J_\omega(t) = &\int_0^T \Big[r_d(s) + \omega_d(s)(\mu_d(s) - r_d(s)) + \omega_f(s)\Big(\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s)\Big) \\
&- \frac{1}{2}\omega_d^2(s)\sigma_d^2(s) - \frac{1}{2}\omega_f^2(s)\Big(\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s)\Big) \\
&- \omega_d(s)\omega_f(s)\rho_1\sigma_d(s)\sigma_R(s)\Big] ds + \int_0^T \Big[\omega_d(s)\sigma_d(s) + \omega_f(s)\rho_1\sigma_R(s)\Big] dW_1(s) \\
&+ \int_0^T \omega_f(s)\Big[\rho_2\sigma_R(s) + \sigma_f(s)\Big] dW_2(s) + \int_0^T \omega_f(s)\sqrt{1-\rho_1^2-\rho_2^2}\,\sigma_R(s)dW_3(s) \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln(1+\omega_d(s)\theta_d(s,z)) - \omega_d(s)\theta_d(s,z)\Big]\nu_d(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_f(s,z)\Big) - \omega_f(s)\theta_f(s,z)\Big]\nu_f(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_R(s,z)\Big) - \omega_f(s)\theta_R(s,z)\Big]\nu_R(dz)ds \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln(1+\omega_d(s)\theta_d(s,z))\Big]\tilde{N}_d(ds,dz) + \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_f(s,z)\Big)\Big]\widetilde{N}_f(ds,dz) \\
&+ \int_0^T \int_{\mathbb{R}_0} \Big[\ln\Big(1+\omega_f(s)\theta_R(s,z)\Big)\Big]\tilde{N}_R(ds,dz). \qquad (3.4)
\end{aligned}
$$

Let $\omega^*(t)$ be the optimal strategy. For all bounded $(\beta_1,\cdot),(\cdot,\beta_2) \in \mathcal{A}$, $y \in (-\delta,\delta)$, and $\delta > 0$, the following inequalities hold:

$$
E\left[U(V_{\omega_1}(T))\right] \le E\left[U(V_{\omega^*}(T))\right], \quad \text{and} \quad E\left[U(V_{\omega_2}(T))\right] \le E\left[U(V_{\omega^*}(T))\right],
$$

where $\omega_1 := (\omega_d^* + y\beta_1, \omega_f^*) \in \mathcal{A}$ and $\omega_2 := (\omega_d^*, \omega_f^* + y\beta_2) \in \mathcal{A}$. Let

$$g_1(y) := E\left[U\left(V_{\omega_1}(T)\right)\right]. \tag{3.5}$$

By the first-order optimality condition, $y = 0$ is a local maximum point of the function $g_1(y)$, which means that

$$\frac{d}{dy}g_1(y)\bigg|_{y=0} = E\left[U'\left(V_{\omega^*}(T)\right)V_{\omega^*}(T)\frac{dJ_{\omega_1}(T)}{dy}\bigg|_{y=0}\right] = 0, \tag{3.6}$$

where

$$\begin{aligned}\frac{dJ_{\omega_1}(T)}{dy}|_{y=0} = &\int_0^T \Big[\beta_1(s)\left(\mu_d(s) - r_d(s)\right) - \beta_1(s)\omega_d^*(s)\sigma_d^2(s) - \beta_1(s)\omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)\Big]ds \\ &+ \int_0^T \beta_1(s)\sigma_d(s)dW_1(s) - \int_0^T\int_{\mathbb{R}_0}\left[\frac{\beta_1(s)\omega_d^*(s)\theta_d^2(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\right]\nu_d(dz)ds \\ &+ \int_0^T\int_{\mathbb{R}_0}\frac{\beta_1(s)\theta_d(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\tilde{N}_d(ds,dz).\end{aligned} \tag{3.7}$$

For a given initial time $t > 0$ and $h > 0$, such that $t + h \le T, \beta_1$ is defined as follows:

$$\beta_1(s) = \alpha\mathbf{1}_{(t,t+h]}(s), \quad 0 \le s \le T, \tag{3.8}$$

where $\mathbf{1}_{(t,t+h]}(s)$ denotes the indicator function of the interval $(t, t+h]$, and the bounded random variable α is adapted to $\mathcal{G}_t$.

By plugging $\beta_1(s)$ into (3.7), we obtain the following:

$$\begin{aligned}0 = E\bigg[U'\left(V_{\omega^*}(T)\right)V_{\omega^*}(T)\cdot\alpha\bigg\{&\int_t^{t+h}\Big[\mu_d(s) - r_d(s) - \omega_d^*(s)\sigma_d^2(s) - \omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)\Big]ds \\ + \int_t^{t+h}\sigma_d(s)dW_1(s) - &\int_t^{t+h}\int_{\mathbb{R}_0}\left[\frac{\omega_d^*(s)\theta_d^2(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\right]\nu_d(dz)ds \\ + \int_t^{t+h}\int_{\mathbb{R}_0}&\frac{\theta_d(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\tilde{N}_d(ds,dz)\bigg\}\bigg].\end{aligned} \tag{3.9}$$

(3.9) holds for all α. For convenience, we denote by the following:

$$F_\omega(T) = U'(V_{\omega^*}(T))(V_{\omega^*}(T)), \tag{3.10}$$

where $\omega \in \mathcal{A}$, and $U' = \frac{d}{dx}U(x)$. Then, (3.9) can be rewritten as follows:

$$E\left[F_{\omega^*}(T)\left(J_{\omega_1}(t+h) - J_{\omega_1}(t)\right)\cdot\alpha\right] = 0. \tag{3.11}$$

By the duality formulas of the Malliavin calculus [22], the following are stated:

$$E\left[F\cdot\int_0^T\varphi(t)dB(t)\right] = E\left[\int_0^T\varphi(t)D_tFdt\right], \tag{3.12}$$

and

$$E\left[F\cdot\int_0^T\int_{\mathbb{R}_0}\psi(t,z)d\tilde{N}(t,z)\right] = E\left[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)D_{t,z}F\,\nu(dz)dt\right]. \tag{3.13}$$

This leads to the following results:

$$E\left[F_{\omega^*}(T)\int_t^{t+h}\alpha\sigma_d(s)dW_1(s)\right]=E\left[\int_t^{t+h}\alpha\sigma_d(s)D_s^1\left(F_{\omega^*}(T)\right)ds\right], \tag{3.14}$$

and

$$E\left[F_{\omega^*}(T)\int_t^{t+h}\int_{\mathbb{R}_0}\frac{\alpha\theta_d(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\tilde{N}_d(ds,dz)\right]=E\left[\int_t^{t+h}\int_{\mathbb{R}_0}\frac{\alpha\theta_d(s,z)D_{s,z}^d\left(F_{\omega^*}(T)\right)}{1+\omega_d^*(s)\theta_d(s,z)}\nu_d(dz)ds\right]. \tag{3.15}$$

Since the strategy $\beta_1(s)=\alpha\mathbf{1}_{(t,t+h]}(s)$ is adapted to the filtration $\{\mathcal{G}_s\}$, it follows that

$$\begin{aligned}E\Bigg[\int_t^{t+h}&\left\{\left(\mu_d(s)-r_d(s)-\omega_d^*(s)\sigma_d^2(s)-\omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)\right)F_{\omega^*}(T)\ +\sigma_d(s)D_s^1\left(F_{\omega^*}(T)\right)\right.\\&\left.-\int_{\mathbb{R}_0}\frac{\omega_d^*(s)\theta_d^2(s,z)F_{\omega^*}(T)}{1+\omega_d^*(s)\theta_d(s,z)}\nu_d(dz)\ +\int_{\mathbb{R}_0}\frac{\theta_d(s,z)D_{s,z}^d\left(F_{\omega^*}(T)\right)}{1+\omega_d^*(s)\theta_d(s,z)}\nu_d(dz)\right\}ds\,\Bigg|\,\mathcal{G}_t\Bigg]=0.\end{aligned} \tag{3.16}$$

Furthermore, from the assumptions we made concerning the theorem and Fubini's theorem, we can derive the following:

$$\begin{aligned}\int_t^{t+h}E\Bigg[&\left\{\left(\mu_d(s)-r_d(s)-\omega_d^*(s)\sigma_d^2(s)-\omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)\right)F_{\omega^*}(T)\ +\sigma_d(s)D_s^1\left(F_{\omega^*}(T)\right)\right.\\&\left.-\int_{\mathbb{R}_0}\frac{\omega_d^*(s)\theta_d^2(s,z)F_{\omega^*}(T)}{1+\omega_d^*(s)\theta_d(s,z)}\nu_d(dz)\ +\int_{\mathbb{R}_0}\frac{\theta_d(s,z)D_{s,z}^d\left(F_{\omega^*}(T)\right)}{1+\omega_d^*(s)\theta_d(s,z)}\nu_d(dz)\right\}\Bigg|\,\mathcal{G}_t\Bigg]ds=0.\end{aligned}$$

The above equation always holds for any h, thus, we have the following:

$$\begin{aligned}&E\left[\left(\mu_d(s)-r_d(s)-\omega_d^*(s)\sigma_d^2(s)-\omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)\right)F_{\omega^*}(T)\mid\mathcal{G}_s\right]\\&+E\left[\int_{\mathbb{R}_0}\frac{\theta_d(s,z)D_{s,z}^d\left(F_{\omega^*}(T)\right)-\omega_d^*(s)\theta_d^2(s,z)F_{\omega^*}(T)}{1+\omega_d^*(s)\theta_d(s,z)}\nu(dz)\mid\mathcal{G}_s\right]=0.\end{aligned} \tag{3.17}$$

Thus far, we derived Equation (3.1) in Theorem 3.1.

Subsequently, we prove (3.2) using an approach analogous to that for (3.1).

Let

$$g_2(y):=E\left[U\left(V_{\omega_2}(T)\right)\right]. \tag{3.18}$$

From the first-order optimal condition, we conclude that

$$\left.\frac{d}{dy}g_2(y)\right|_{y=0}=E\left[U'\left(V_{\omega^*}(T)\right)V_{\omega^*}(T)\left.\frac{dJ_{\omega_2}(T)}{dy}\right|_{y=0}\right]=0, \tag{3.19}$$

where

$$\frac{dJ_{\omega_2}(T)}{dy}\bigg|_{y=0} = \int_0^T \Big[\beta_2(s)\left(\mu_R(s)+\mu_f(s)-r_d(s)+\rho_2\sigma_f(s)\sigma_R(s)\right)$$
$$-\beta_2(s)\omega_f^*(s)\left(\sigma_R^2(s)+\sigma_f^2(s)+2\rho_2\sigma_R(s)\sigma_f(s)\right)-\beta_2(s)\omega_d^*(s)\rho_1\sigma_d(s)\sigma_R(s)\ \Big]ds$$
$$+\int_0^T\beta_2(s)\left(\rho_2\sigma_R(s)+\sigma_f(s)\right)dW_2(s)+\int_0^T\beta_2(s)\sqrt{1-\rho_1^2-\rho_2^2}\sigma_R(s)dW_3(s)$$
$$-\int_0^T\int_{\mathbb{R}_0}\left[\frac{\beta_2(s)\omega_f^*(s)\theta_f^2(s,z)}{1+\omega_f^*(s)\theta_f(s,z)}\right]\nu_f(dz)ds-\int_0^T\int_{\mathbb{R}_0}\left[\frac{\beta_2(s)\omega_f^*(s)\theta_R^2(s,z)}{1+\omega_f^*(s)\theta_R(s,z)}\right]\nu_R(dz)ds$$
$$+\int_0^T\int_{\mathbb{R}_0}\frac{\beta_2(s)\theta_f(s,z)}{1+\omega_f^*(s)\theta_f(s,z)}\tilde{N}_f(ds,dz)+\int_0^T\int_{\mathbb{R}_0}\frac{\beta_2(s)\theta_R(s,z)}{1+\omega_f^*(s)\theta_R(s,z)}\tilde{N}_R(ds,dz). \tag{3.20}$$

After some simplified calculations, we can obtain the following result:

$$E\Big[\Big(\mu_R(s)+\mu_f(s)-r_d(s)+\rho_2\sigma_f(s)\sigma_R(s)-\omega_f^*(s)\left(\sigma_R^2(s)+\sigma_f^2(s)+2\rho_2\sigma_R(s)\sigma_f(s)\right)$$
$$-\omega_d^*(s)\rho_1\sigma_d(s)\sigma_R(s))F_{\omega^*}(T)+\left(\rho_2\sigma_R(s)+\sigma_f(s)\right)D_s^2\left(F_{\omega^*}(T)\right)+\sqrt{1-\rho_1^2-\rho_2^2}\sigma_R(s)D_s^3\left(F_{\omega^*}(T)\right)|\mathcal{G}_s\Big]$$
$$+E\left[\int_{\mathbb{R}_0}\frac{\theta_f(s,z)D_{s,z}^f\left(F_{\omega^*}(T)\right)-\omega_f^*(s)\theta_f^2(s,z)F_{\omega^*}(T)}{1+\omega_f^*(s)\theta_f(s,z)}\nu_f(dz)\right.$$
$$\left.+\int_{\mathbb{R}_0}\frac{\theta_R(s,z)D_{s,z}^R\left(F_{\omega^*}(T)\right)-\omega_f^*(s)\theta_R^2(s,z)F_{\omega^*}(T)}{1+\omega_f^*(s)\theta_R(s,z)}\nu_R(dz)\right|\mathcal{G}_s\Bigg]=0. \tag{3.21}$$

Conversely, suppose that (3.1) and (3.2) are satisfied. We denote the following:

$$E\left[U\left(V_{\left(\omega_d^*+y\beta_1,\omega_f^*+y\beta_2\right)}(T)\right)\right]=E\left[U\left(V_{\omega^*}(T;y)\right)\right]. \tag{3.22}$$

Its derivative with respect to y is given by the following:

$$\frac{\partial}{\partial y}E[U(V_{\omega^*}(T;y))]=E\left[\frac{\partial}{\partial y}U(V_{\omega^*}(T;y))\right]=E\left[U'(V_{\omega^*}(T;y))\cdot\frac{\partial V_{\omega^*}(T;y)}{\partial y}\right]. \tag{3.23}$$

Let $y=0$ yield the following:

$$\frac{\partial}{\partial y}E[U(V_{\omega^*}(T;y))]\bigg|_{y=0}=E\left[U'(V_{\omega^*}(T))\cdot\frac{\partial V_{\omega^*}(T;y)}{\partial y}\bigg|_{y=0}\right], \tag{3.24}$$

where

$$
\begin{aligned}
\frac{\partial V_{\omega^*}(T;y)}{\partial y}\bigg|_{y=0} =\ & V_{\omega^*}(T)\bigg\{\int_0^T \Big\{\beta_1\left(\mu_d(s)-r_d(s)\right)+\beta_2\left(\mu_R(s)+\mu_f(s)-r_d(s)+\rho_2\sigma_f(s)\sigma_R(s)\right)\\
&-\omega_d^*(s)\beta_1\sigma_d^2(s)-\omega_f^*(s)\beta_2\left(\sigma_R^2(s)+\sigma_f^2(s)+2\rho_2\sigma_R(s)\sigma_f(s)\right)\\
&-\beta_1\omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s)-\beta_2\omega_d^*(s)\rho_1\sigma_d(s)\sigma_R(s)\ \Big\}\,ds\\
&+\int_0^T\left[\beta_1\sigma_d(s)+\beta_2\rho_1\sigma_R(s)\right]dW_1(s)+\int_0^T\beta_2\left(\rho_2\sigma_R(s)+\sigma_f(s)\right)dW_2(s)\\
&+\int_0^T\beta_2\sqrt{1-\rho_1^2-\rho_2^2}\,\sigma_R(s)dW_3(s)-\int_0^T\int_{\mathbb{R}_0}\frac{\beta_1\omega_d^*(s)\theta_d^2(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\nu_d(dz)ds\\
&-\int_0^T\int_{\mathbb{R}_0}\frac{\beta_2\omega_f^*(s)\theta_f(s,z)}{1+\omega_f^*(s)\theta_f(s,z)}\nu_f(dz)ds-\int_0^T\int_{\mathbb{R}_0}\frac{\beta_2\omega_f^*(s)\theta_R^2(s,z)}{1+\omega_f^*(s)\theta_R(s,z)}\nu_R(dz)ds\\
&+\int_0^T\int_{\mathbb{R}_0}\frac{\beta_1\theta_d(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\tilde{N}_d(ds,dz)+\int_0^T\int_{\mathbb{R}_0}\frac{\beta_2\theta_f^2(s,z)}{1+\omega_f^*(s)\theta_f(s,z)}\tilde{N}_f(ds,dz)\\
&+\int_0^T\int_{\mathbb{R}_0}\frac{\beta_2\theta_R(s,z)}{1+\omega_f^*(s)\theta_R(s,z)}\tilde{N}_R(ds,dz)=V_{\omega^*}(T)\left(\frac{dJ_{\omega_1}(T)}{dy}\bigg|_{y=0}+\frac{dJ_{\omega_2}(T)}{dy}\bigg|_{y=0}\right).
\end{aligned}
\tag{3.25}
$$

Therefore, (3.24) can be rewritten as follows:

$$
\begin{aligned}
\frac{\partial}{\partial y}E[U(V_{\omega^*}(T;y))]\bigg|_{y=0} &= E\left[U'(V_{\omega^*}(T))\cdot V_{\omega^*}(T)\cdot\left(\frac{dJ_{\omega_1}(T)}{dy}\bigg|_{y=0}+\frac{dJ_{\omega_2}(T)}{dy}\bigg|_{y=0}\right)\right]\\
&= E\left[U'(V_{\omega^*}(T))V_{\omega^*}(T)\frac{dJ_{\omega_1}(T)}{dy}\bigg|_{y=0}+U'(V_{\omega^*}(T))V_{\omega^*}(T)\frac{dJ_{\omega_2}(T)}{dy}\bigg|_{y=0}\right]\\
&= E\left[U'(V_{\omega^*}(T))V_{\omega^*}(T)\frac{dJ_{\omega_1}(T)}{dy}\bigg|_{y=0}\right]+E\left[U'(V_{\omega^*}(T))V_{\omega^*}(T)\frac{dJ_{\omega_2}(T)}{dy}\bigg|_{y=0}\right].
\end{aligned}
\tag{3.26}
$$

Substituting (3.6) and (3.19) (which follow from (3.17) and (3.21)) into (3.26) gives rise to the following:

$$
\frac{\partial}{\partial y}E[U(V_{\omega^*}(T;y))]\bigg|_{y=0}=0. \tag{3.27}
$$

From the optimal problem (2.9), we can see that $U(V_{\omega^*}(T;y))$ is a concave function. It can be shown that $y=0$ is the local optimum point; thus, ω^* satisfies (3.1) and (3.2) is the optimal strategy. □

4. Special cases and numerical analysis

4.1. Special cases

In this section, we discuss some special cases and explore the explicit expression for the optimal strategy. We adopt the logarithmic utility function for the optimal problem (2.9). Let

$$
U(x)=\log x,\quad x>0. \tag{4.1}
$$

For this utility function, we have the following:

$$F_\pi(T) = U'\left(V_\omega(T)\right) V_\omega(T) = 1. \tag{4.2}$$

Consequently,

$$D_s^1\left(F_\pi(T)\right) = D_{s,z}^d\left(F_\pi(T)\right) = D_s^2\left(F_\pi(T)\right) = D_s^3\left(F_\pi(T)\right) = D_{s,z}^f\left(F_\pi(T)\right) = D_{s,z}^R\left(F_\pi(T)\right) = 0. \tag{4.3}$$

Simultaneously, (3.1) and (3.2) reduce to the following forms:

$$\begin{cases} E\left[\left\{\mu_d(s) - r_d(s) - \omega_d^*(s)\sigma_d^2(s) - \omega_f^*(s)\rho_1\sigma_d(s)\sigma_R(s) - \int_{\mathbb{R}_0} \frac{\omega_d^*(s)\theta_d^2(s,z)}{1+\omega_d^*(s)\theta_d(s,z)}\nu(dz) \mid \mathcal{G}_s\right] = 0, \\ E\left[\left\{\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s) - \omega_f^*(s)\left(\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s)\right)\right.\right. \\ \left. -\omega_d^*(s)\rho_1\sigma_d(s)\sigma_R(s) - \int_{\mathbb{R}_0} \frac{\omega_f^*(s)\theta_f^2(s,z)}{1+\omega_f^*(s)\theta_f(s,z)}\nu_f(dz) - \int_{\mathbb{R}_0} \frac{\omega_f^*(s)\theta_R^2(s,z)}{1+\omega_f^*(s)\theta_R(s,z)}\nu_R(dz) \mid \mathcal{G}_s\right] = 0. \end{cases} \tag{4.4}$$

Furthermore, we consider the model without jumps(i.e., $\theta_i(s,z) = 0$ for $i = d, f, R$). (4.4) reduces to the following:

$$\begin{cases} E\left[\mu_d(s) - r_d(s) \mid \mathcal{G}_s\right] - \omega_d^*(s)E\left[\sigma_d^2(s) \mid \mathcal{G}_s\right] - \omega_f^*(s)E\left[\rho_1\sigma_d(s)\sigma_R(s) \mid \mathcal{G}_s\right] = 0, \\ E\left[\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s) \mid \mathcal{G}_s\right] - \omega_f^*(s)E\left[\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s) \mid \mathcal{G}_s\right] \\ -\, \omega_f^*(s)E\left[\rho_1\sigma_d(s)\sigma_R(s) \mid \mathcal{G}_s\right] = 0. \end{cases} \tag{4.5}$$

Solving the above equation system yields the following:

$$\omega_d^*(s) = \frac{C_1(s)B_2(s) - C_2(s)B_1(s)}{A_1(s)B_2(s) - A_2(s)B_1(s)}, \tag{4.6}$$

$$\omega_f^*(s) = \frac{C_1(s)A_2(s) - C_2(s)A_1(s)}{A_2(s)B_1(s) - A_1(s)B_2(s)}, \tag{4.7}$$

where

$$C_1(s) = E\left[\mu_d(s) - r_d(s) \mid \mathcal{G}_s\right], \quad A_1(s) = E\left[\sigma_d^2(s) \mid \mathcal{G}_s\right], \quad B_1(s) = E\left[\rho_1\sigma_d(s)\sigma_R(s) \mid \mathcal{G}_s\right],$$

$$C_2(s) = E\left[\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s) \mid \mathcal{G}_s\right], \quad A_2(s) = E\left[\rho_1\sigma_d(s)\sigma_R(s) \mid \mathcal{G}_s\right],$$

and $B_2(s) = E\left[\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s) \mid \mathcal{G}_s\right]$.

The (4.6) and (4.7) show that domestic stock strategies are correlated with the coefficients of foreign stock price processes, and similarly, foreign stock strategies are influenced by the coefficients of domestic price processes.

Moreover, we assume that $\rho_1 = 0$. The (4.6) and (4.7) take the following form

$$\omega_d^*(s) = \frac{E\left[\mu_d(s) - r_d(s) \mid \mathcal{G}_s\right]}{E\left[\sigma_d^2(s) \mid \mathcal{G}_s\right]}, \tag{4.8}$$

and

$$\omega_f^*(s) = \frac{E\left[\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s) \mid \mathcal{G}_s\right]}{E\left[\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s) \mid \mathcal{G}_s\right]}, \text{ respectively} \tag{4.9}$$

Equations (4.8) and (4.9) indicate that when $\rho_1 = 0$, the domestic stock strategy does not depend on the coefficients of foreign stock price and exchange rate processes. However, the foreign stock strategy is not only related to its own coefficients but also to the exchange rate risk. This is in line with reality in cross-border transactions.

From the above arguments, we summarize the following theorem.

Theorem 4.1. *Let the utility function be $U(x) = \log x, x > 0$. If $\mathcal{G}_s = \mathcal{F}_s$, $\theta_i(s,z) = 0$ for $i = d, f, R$, then the explicit expressions of the optimal investment strategy are given by the following:*

$$\omega_d^*(s) = \frac{C_1(s)B_2(s) - C_2(s)B_1(s)}{A_1(s)B_2(s) - A_2(s)B_1(s)}, \tag{4.10}$$

$$\omega_f^*(s) = \frac{C_1(s)A_2(s) - C_2(s)A_1(s)}{A_2(s)B_1(s) - A_1(s)B_2(s)}, \tag{4.11}$$

Here, $C_1(s) = \mu_d(s) - r_d(s)$, $A_1(s) = \sigma_d^2(s)$, $B_1(s) = \rho_1\sigma_d(s)\sigma_R(s)$, $C_2(s) = \mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s)$, $A_2(s) = \rho_1\sigma_d(s)\sigma_R(s)$, and $B_2(s) = \sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s)$. Furthermore, when $\rho_1 = 0$, Equations (4.10) *and* (4.11) *simplify to the following:*

$$\omega_d^*(s) = \frac{\mu_d(s) - r_d(s)}{\sigma_d^2(s)},$$

and

$$\omega_f^*(s) = \frac{\mu_R(s) + \mu_f(s) - r_d(s) + \rho_2\sigma_f(s)\sigma_R(s)}{\sigma_R^2(s) + \sigma_f^2(s) + 2\rho_2\sigma_R(s)\sigma_f(s)}, \text{ respectively}$$

4.2. Numerical analysis

In the previous subsection, explicit expressions for the optimal investment strategy given by (4.10) and (4.11) were derived. In this subsection, we provide numerical examples to illustrate the impact of the correlation coefficient ρ_1, ρ_2 and the exchange rate volatility σ_R on the optimal strategy under the full information framework. For simplicity, we assume the following basic parameters: $r_d = 0.05$, $\mu_d = 0.10$, $\sigma_d = 0.15$, $\mu_f = 0.12$, $\sigma_f = 0.20$, $\mu_R = 0.03$.

Figure 1 illustrates that investments in foreign stock decrease as the exchange rate volatility increases, while investments in domestic stock are almost unaffected by the volatility. This indicates that as the exchange rate risk grows, investors will reduce their investment in foreign stock. Meanwhile, we can also observe that the change of ρ_2 has almost no impact on the investment in domestic stock, while the investment in foreign stock decreases as ρ_2 increases. This demonstrates that when the correlation between foreign stock and the exchange rate is relatively high, investors will reduce their holdings of foreign stock.

As shown in Figure 2, when $\rho_1 = 0.5$, that is, the correlation between domestic stock and the exchange rate is high, the investment in domestic stock will initially increase and then decrease; however, the trend of this change is not significant. Additionally, the investment in foreign stock will decrease as the volatility of the exchange rate increases. This implies that the systematic risks between foreign stock and the exchange rate increase, which leads investors to reduce their investments in foreign stock.

From Figures 1 and 2, it can be observed that when the exchange rate risk is at a low level, foreign stock investments either exceed or approach domestic stock investments. Investors tend to allocate more to foreign stocks in order to seek higher returns. Additionally, changes in the parameter ρ_1 give rise to slight variations in domestic stock investments. However, it would have a relatively smaller impact on the trend changes of foreign stocks.

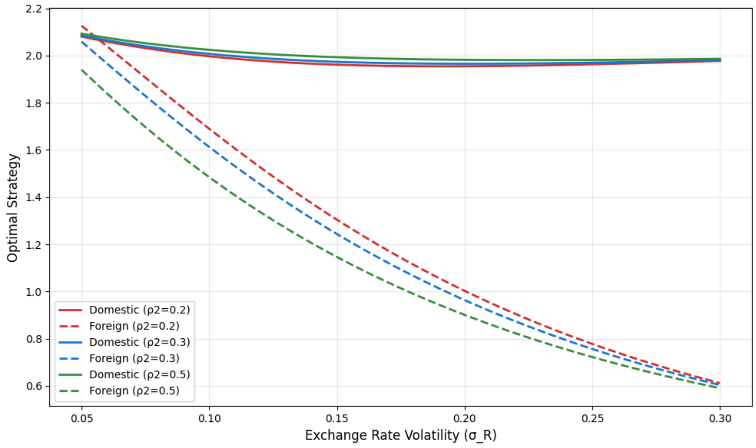

Figure 1. $\rho_1 = 0.2, \omega_d^*, \omega_f^*$with different values of ρ_2.

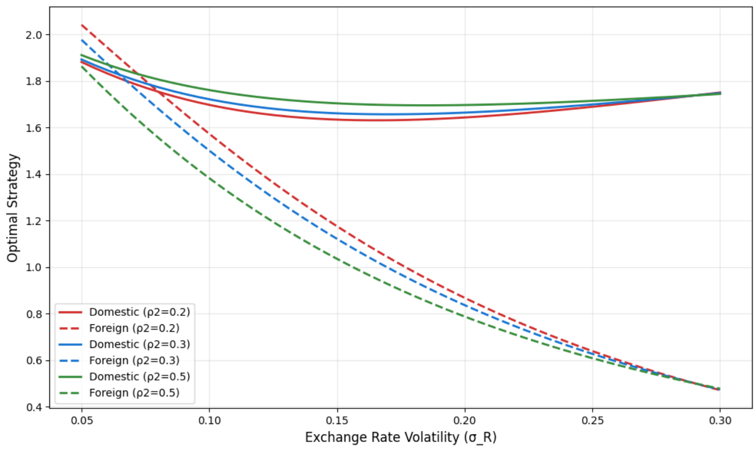

Figure 2. $\rho_1 = 0.5, \omega_d^*, \omega_f^*$with different values of ρ_2.

5. Conclusions

We investigated the optimal investment strategies for investors with partial information under the expected utility of terminal wealth criterion. The financial market consists of a risk-free asset, a domestic stock, a foreign stock, and an exchange rate, where the processes of the domestic stock, foreign stock, and exchange rate are all described by a jump-diffusion model with stochastic coefficients, which is different from that of Wang et al. (2023) [12]. Using Malliavin calculus, we derived the necessary and sufficient conditions for the optimal investment strategies. In some special cases, we provided an explicit expression for the optimal investment strategy. Finally, through numerical examples, we demonstrated the effects of the correlation coefficient ρ_1, ρ_2 and the exchange rate volatility on the optimal investment strategies. A higher exchange rate volatility led to decreased foreign investments, and a higher correlation between foreign stocks and the exchange rate also resulted in a reduction of foreign investments. However, changes in the parameters σ_R, ρ_2, and ρ_1 have little impact on domestic stock investments.

Author contributions

Hongwei Liu: Writing–review & editing, Supervision, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis; Tianjing Kan: Writing–review & editing, Methodology, Investigation, Formal analysis.

Use of Generative-AI tools declaration

We declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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