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Research article

Nilpotent graphs of Lie superalgebras: structure and graph-theoretic properties

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Abstract: We introduce the nilpotent graph $\Gamma_N(L)$ of a finite-dimensional Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over a field $\mathbb{F}$, with vertices consisting of non-nilpotent elements and edges connecting pairs that generate nilpotent subsuperalgebras. We prove that the nilpotentizer $\mathscr{N}(L)$ coincides with the hypercenter $Z^*(L)$ when $\text{char}(\mathbb{F}) = 0$. For the triangular Lie superalgebra $\mathfrak{t}(2, \mathbb{F}_q)$, we show that $\Gamma_N(L)$ is the disjoint union of $q+1$ complete graphs $K_{q(q-1)}$, each having $q(q-1)$ vertices. We characterize the bipartiteness of $\Gamma_N(L)$, demonstrating that it is bipartite if and only if the odd component $L_{\bar{1}} \subseteq \mathscr{N}(L)$. We also analyze connectivity, diameter, clique number, and chromatic number for Lie superalgebras such as $\mathfrak{sl}(1|1, \mathbb{F}_q)$ and investigate direct sums and complement graphs. SageMath algorithms are provided to compute $\Gamma_N(L)$ and its invariants, linking the algebraic structure of Lie superalgebras to graph theory and emphasizing the role of $\mathbb{Z}_2$-grading in topological properties. Open problems on higher-dimensional structures and spectral properties are proposed.

Keywords: Lie superalgebra; nilpotent graph; nilpotentizer; graph invariants; triangular matrices
Mathematics Subject Classification: 17B05, 05C25, 17B20, 68W30

1. Introduction

Lie superalgebras, introduced by Kac [1], generalize Lie algebras by incorporating a $\mathbb{Z}_2$-grading, decomposing as $L = L_{\bar{0}} \oplus L_{\bar{1}}$, where $L_{\bar{0}}$ and $L_{\bar{1}}$ are the even and odd components, respectively. These structures are fundamental in theoretical physics, particularly in supersymmetry, where they unify bosonic and fermionic symmetries [2]. In mathematics, Lie superalgebras are central to representation theory, algebraic geometry, and deformation theory, with extensive studies on the classification of simple Lie superalgebras and their cohomological properties [3–5]. The study of algebraic structures through graph-theoretic lenses has gained significant traction, revealing deep connections between algebraic invariants and topological properties. For instance, in the context of Lie algebras, recent work has introduced the nilpotent graph $\Gamma_{\mathfrak{N}}(L)$ for finite-dimensional Lie algebras over a field $\mathbb{F}$, where vertices are non-nilpotent elements, and edges connect pairs, generating nilpotent subalgebras [6]. This construction, analogous to nilpotent graphs for finite groups [7], allows for the analysis of solvability and nilpotency via graph invariants like connectivity and clique number. Bhowal et al. [8] further extended solvable graphs to Lie algebras, exploring subalgebra generation by element pairs. These developments highlight how graph theory can uncover structural insights in non-abelian algebraic systems. Building on these ideas, our work extends the nilpotent graph framework to Lie superalgebras, accounting for the $\mathbb{Z}_2$-grading that introduces unique challenges, such as graded brackets and odd-odd commutators. Recent contributions in Lie superalgebra theory, including multi-component superintegrable hierarchies based on $\mathfrak{sl}(2N, 1)$ [9] and conformal superalgebras of rank (2+1) [10], underscore the relevance of graded structures in integrable systems and representation theory. Additionally, studies on deformations and extensions in generalized Lie theory [11] provide foundational tools for our graph constructions. These connections emphasize the broader applicability of our results to areas such as quantum field theory and integrable systems [2, 3]. A growing trend in modern algebra involves studying algebraic structures through graph-theoretic tools. Graphs associated with groups, rings, and Lie algebras reveal structural properties via topological invariants such as connectivity, diameter, clique number, and chromatic number [12, 13]. Notable algebraic graphs include the commuting graph [14], non-commuting graph [15], and nilpotent graph [16]. The nilpotent graph of a finite group, introduced by Gutierrez et al. [7], has vertices as non-nilpotent elements, with adjacency defined by pairs generating a nilpotent subgroup. Bhowal et al. [8] extended this concept to Lie algebras, defining solvable graphs based on subalgebra structures generated by element pairs. Motivated by these developments, we propose a graph-theoretic model for Lie superalgebras that reflects their graded structure. We define the *nilpotent graph* $\Gamma_N(L)$ of a finite-dimensional Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over a field $\mathbb{F}$. The vertex set is $L \setminus \mathcal{N}(L)$, where

$$\mathcal{N}(L) = \{x \in L \mid \langle x, y \rangle \text{ is nilpotent for all } y \in L\},$$

and vertices $x, y \in L \setminus \mathcal{N}(L)$, $x \neq y$, are adjacent if the subsuperalgebra $\langle x, y \rangle$ is nilpotent. We also consider the *modified nilpotent graph* $\Gamma(L)$, which excludes universal vertices (those adjacent to all others), aligning with recent advancements in algebraic graph theory [6, 16]. This construction captures element interactions in L through nilpotency, with complexity arising from the $\mathbb{Z}_2$-grading. Our objective is to analyze $\Gamma_N(L)$ using graph-theoretic invariants and relate these to the algebraic structure of L.

1.1. Related work

The interplay between graph theory and algebra traces back to Cayley graphs and extends to various algebraically motivated graphs, such as commuting graphs [14], non-commuting graphs [15], and nilpotent graphs for finite groups [7]. Nath et al. [16] refined nilpotent graphs to study group solvability, while Bhowal et al. [8] explored solvable graphs for Lie algebras. More recently, Ceballos et al. [6] introduced the nilpotent graph for finite-dimensional Lie algebras, proving results on connectivity and diameter that parallel our findings for the superalgebra case, such as diameter bounds for solvable structures. For Lie superalgebras, graph-based studies are less developed. Their complex structure, governed by graded Lie brackets and identities, demands novel approaches to define and interpret algebraic graphs. The classification of simple Lie superalgebras [1] and their representation theory [3, 4] provide a foundation, but graph-theoretic investigations remain largely unexplored, underscoring the novelty of our contributions. Recent works on superintegrable hierarchies [9] and conformal superalgebras [10] highlight applications in nonlinear dynamics and theoretical physics, motivating our graded graph constructions. Furthermore, foundational studies in generalized Lie theory, such as those on Hom-Lie superalgebras [11], offer tools for analyzing graded nilpotency.

1.2. Contributions

This paper develops the nilpotent graph framework for Lie superalgebras, offering theoretical and computational insights. The main contributions are:

- **Characterization of nilpotentizers:** We prove that for fields of characteristic zero, the nilpotentizer equals the hypercenter, that is $\mathscr{N}(L) = Z^*(L)$.
- **Closed neighborhoods:** We characterize solvable Lie superalgebras where the closed neighborhood
$$N[x] = \{x\} \cup \{y \in L \setminus \mathscr{N}(L) \mid \langle x, y\rangle \text{ is nilpotent}\}$$
generates a nilpotent subsuperalgebra.
- **Connectivity and diameter:** We investigate the connectivity of $\Gamma(L)$ for families like $\mathfrak{sl}(m|n, \mathbb{F}_q)$ and $\mathfrak{t}(m|n, \mathbb{F}_q)$, providing upper bounds on the diameter.
- **Topological invariants:** We derive results on bipartiteness, chromatic number, and domination number, emphasizing the influence of $\mathbb{Z}_2$-grading on graph structure.
- **Direct sums:** We analyze the relationship between $\Gamma_N(L_1 \oplus L_2)$ and the graphs $\Gamma_N(L_1)$, $\Gamma_N(L_2)$, with implications for disconnected components.
- **Computational tools:** We provide a SageMath implementation to construct $\Gamma_N(L)$ and compute its graph invariants [18].

Our novel results include a diameter bound of 2 for solvable Lie superalgebras with large nilpotentizers, bipartiteness conditions dependent on $L_{\bar{1}}$, and clique number bounds for triangular Lie superalgebras $\mathfrak{t}(m|n, \mathbb{F}_q)$. We include examples for $\mathfrak{sl}(2|1, \mathbb{F}_q)$ and $\mathfrak{t}(3|2, \mathbb{F}_q)$, and study the complement graph $\Gamma_N(L)^c$. We conclude with open problems on higher-dimensional structures and spectral graph theory in the superalgebra setting.

2. Preliminaries

2.1. Basic concepts from graph theory

A simple undirected graph $\Gamma = (V, E)$ consists of a finite vertex set V and an edge set $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$, with no loops or multiple edges. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of vertices adjacent to v. A subset $C \subseteq V$ is a *clique* if every pair of distinct vertices in C is adjacent. The *clique number* $\omega(\Gamma)$ is the size of the largest clique. A graph is *connected* if there exists a path between every pair of vertices. The *diameter* of a connected graph is the maximum distance (in edges) between any two vertices. The *chromatic number* $\chi(\Gamma)$ is the minimum number of colors needed to color V such that adjacent vertices have different colors. A graph $\Gamma = (V, E)$ is called *bipartite* if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 to a vertex in V_2. A subset $D \subseteq V$ is a *dominating set* if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. The *domination number* $\gamma(\Gamma)$ is the minimum cardinality of a dominating set. The *complement graph* Γ^c has the same vertex set as Γ, with an edge $\{u, v\}$ in Γ^c if and only if it is absent in Γ [12, 13].

2.2. Lie superalgebras

A *Lie superalgebra* L over a field $\mathbb{F}$ is a $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ equipped with a bilinear Lie bracket $[\cdot, \cdot] : L \times L \to L$, satisfying for all homogeneous elements $x, y, z \in L$:

(1) *Graded anticommutativity*:

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

where $|x| \in \mathbb{Z}_2$ denotes the degree of x.

(2) *Graded Jacobi identity*:

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

A *subsuperalgebra* is a graded subspace closed under the bracket. An *ideal* $I \subseteq L$ is a graded subspace satisfying $[I, L] \subseteq I$. The *lower central series* is defined by

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1,$$

and L is *nilpotent* if $L^k = \{0\}$ for some k. The *derived series* is

$$L^{(0)} = L, \quad L^{(k+1)} = [L^{(k)}, L^{(k)}],$$

and L is *solvable* if $L^{(n)} = \{0\}$ for some $n \in \mathbb{N}$. The *center* of L is

$$Z(L) = \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}.$$

The *upper central series* $\{Z_i(L)\}$ is defined by

$$Z_0(L) = \{0\}, \quad Z_i(L)/Z_{i-1}(L) = Z(L/Z_{i-1}(L)), \quad i \geq 1,$$

with the *hypercenter* $Z^*(L) = \bigcup_{i \geq 0} Z_i(L)$. The *nilradical* $\mathcal{N}(L)$ is the largest nilpotent ideal of L. For $x \in L$, the *adjoint map* $\mathrm{ad}_x : L \to L$ is defined by $\mathrm{ad}_x(y) = [x, y]$. An element $x \in L$ is an *Engel element* if $(\mathrm{ad}_y)^n(x) = 0$ for some $n \in \mathbb{N}$, for all $y \in L$. The *Engel subalgebra* of x is

$$E_L(x) = \{y \in L \mid (\mathrm{ad}_x)^n(y) = 0 \text{ for some } n \in \mathbb{N}\},$$

forming a graded subsuperalgebra [1].

3. Nilpotentizer of Lie superalgebras

Definition 3.1. Let L be a Lie superalgebra over a field $\mathbb{F}$. The *graded nilpotentizer* of an element $x \in L$ is

$$\text{nil}_L(x) = \{y \in L \mid \langle x, y\rangle \text{ is a nilpotent graded subsuperalgebra}\},$$

and the *graded nilpotentizer* of L is

$$\text{nil}(L) = \{x \in L \mid \langle x, y\rangle \text{ is nilpotent for all } y \in L\}.$$

Theorem 3.2. *Let L be a finite-dimensional Lie superalgebra over $\mathbb{F}$. Then*

(1) $Z^*(L) \subseteq \text{nil}(L)$*;*
(2) If $\text{char}(\mathbb{F}) = 0$*, then* $\text{nil}(L) = Z^*(L)$*;*
(3) If L is completely solvable, then $\text{nil}(L) \subseteq N(L)$*;*
(4) If L is classical simple or semisimple over an infinite field of characteristic $p > 3$ or 0, then $\text{nil}(L) = \{0\}$.

Proof. (1) For $x \in Z^*(L)$, the hypercenter, $[x, L] \subseteq Z_{k-1}(L)$, which is central. Thus, $\langle x, y\rangle$ has derived series terminating at zero, so $x \in \text{nil}(L)$.
(2) In characteristic zero, Ado's theorem [19] embeds $L/Z^*(L)$ in $\mathfrak{gl}(m|n, \mathbb{F})$. If $x \in \text{nil}(L)$, then $\langle x, y\rangle$ is nilpotent for all y, implying x lies in a nilpotent ideal, hence in $Z^*(L)$.
(3) For completely solvable L, the nilradical $N(L)$ contains all nilpotent elements. Because $\langle x, y\rangle$ is nilpotent for $x \in \text{nil}(L)$, $x \in N(L)$.
(4) For classical simple or semisimple L (e.g., $\mathfrak{sl}(m|n)$), the center is trivial, and no non-zero element generates nilpotent subsuperalgebras with all others [1].

□

Remark 3.3. For $L = \mathfrak{sl}(1|1)$, the nilpotent graph $\Gamma_N(L)$ over finite fields $\mathbb{F}_q$ has the following structure:

(1) Over $\mathbb{F}_2$, $\Gamma_N(L)$ has 3 components, each isomorphic to K_2, see Figure 1.

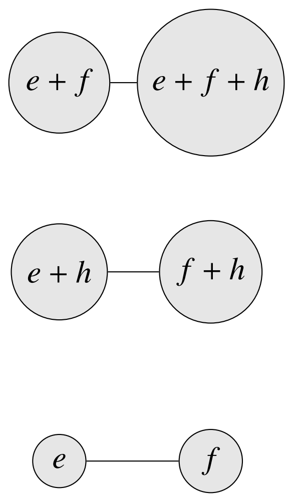

Figure 1. Nilpotent graph $\Gamma_N(\mathfrak{sl}(1|1), \mathbb{F}_2)$, with three K_2 components.

(2) Over $\mathbb{F}_3$, $\Gamma_N(L)$ has 4 components, each isomorphic to K_6, see Figure 2.

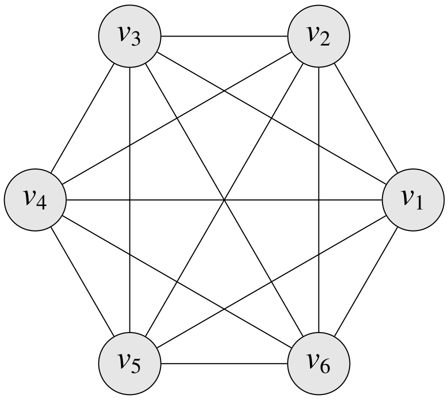

Figure 2. One K_6 component of $\Gamma_N(\mathfrak{sl}(1|1), \mathbb{F}_3)$, with 4 such components.

(3) Over $\mathbb{F}_4$, $\Gamma_N(L)$ has 5 components, each isomorphic to K_{12}, see Figure 3.

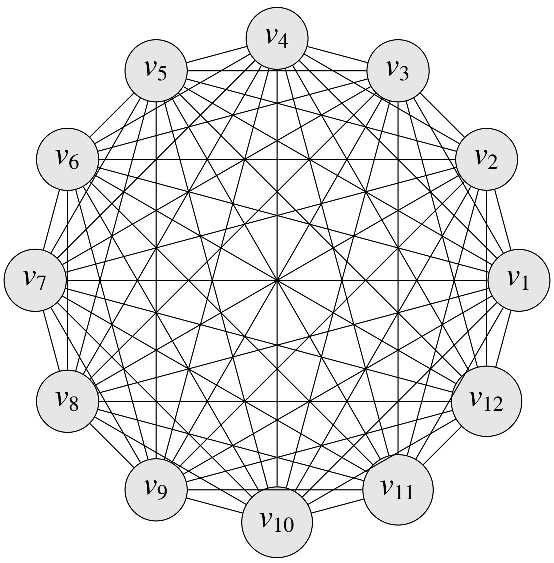

Figure 3. One K_{12} component of $\Gamma_N(\mathfrak{sl}(1|1), \mathbb{F}_4)$, with 5 such components.

(4) Over $\mathbb{F}_5$, $\Gamma_N(L)$ has 6 components, each isomorphic to K_{20}, see Figure 4.

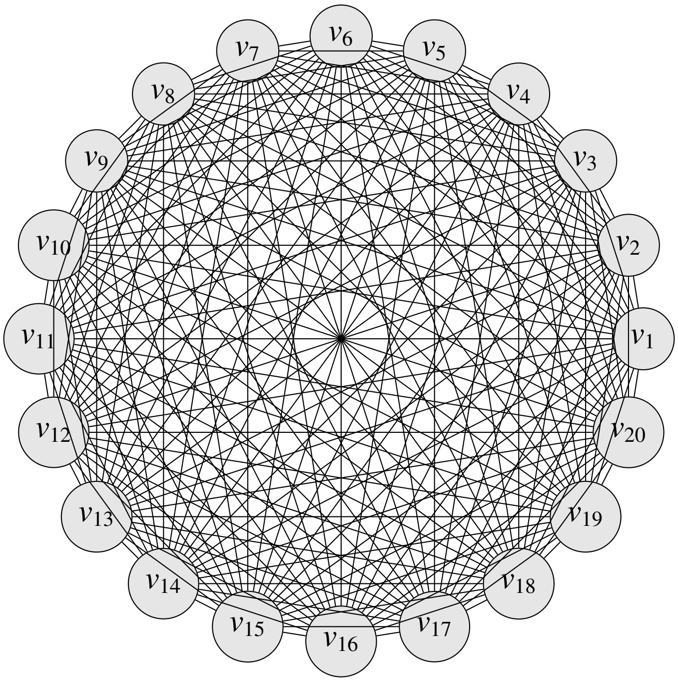

Figure 4. One K_{20} component of $\Gamma_N(\mathfrak{sl}(1|1), \mathbb{F}_5)$, with 6 such components.

4. Nilpotent graphs of triangular Lie algebras

Definition 4.1. Let $L = \mathfrak{t}(2, \mathbb{F}_q)$, the Lie algebra of 2×2 upper triangular matrices over $\mathbb{F}_q$. The *nilpotent graph* $\Gamma_N(L)$ has vertex set $L \setminus \mathrm{nil}(L)$, where vertices $x, y \in L \setminus \mathrm{nil}(L)$, $x \neq y$, are adjacent if $\langle x, y\rangle$ is a nilpotent subalgebra.

Lemma 4.2. *For* $L = \mathfrak{t}(2, \mathbb{F}_q)$*, the graded nilpotentizer satisfies* $\mathrm{nil}(L) = \mathrm{Z}(L)$.

Proof. Let $L = \mathfrak{t}(2, \mathbb{F}_q)$ with basis $\{z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}$. The center is $\mathrm{Z}(L) = \{\alpha z \mid \alpha \in \mathbb{F}_q\}$, as $[z, x] = [z, y] = 0$. For $a \in \mathrm{Z}(L)$, $\langle a, b\rangle$ is abelian for all $b \in L$, so $a \in \mathrm{nil}(L)$. Conversely, if $a \notin \mathrm{Z}(L)$, say $a = \alpha x + \beta y + \gamma z$, then $[a, z] = 0$, but $\langle a, z\rangle = \mathrm{span}\{a, z\}$ may not be nilpotent unless $a \in \mathrm{Z}(L)$. Thus, $\mathrm{nil}(L) = \mathrm{Z}(L)$. □

Theorem 4.3. *The nilpotent graph* $\Gamma_N(\mathfrak{t}(2, \mathbb{F}_q))$ *has* $q + 1$ *connected components, each isomorphic to* $K_{q(q-1)}$.

Proof. Let $L = \mathfrak{t}(2, \mathbb{F}_q)$, with $|L| = q^3$ and $|\mathrm{Z}(L)| = q$ by Lemma 4.2. The vertex set is $L \setminus \mathrm{nil}(L) = L \setminus \mathrm{Z}(L)$, with $q^3 - q$ vertices. For $h \notin \mathrm{Z}(L)$, the nilpotentizer is $\mathrm{nil}_L(h) = \mathbb{F}_q h + \mathrm{Z}(L)$, a subspace of dimension 2, so $|\mathrm{nil}_L(h)| = q^2$. The set $\mathrm{nil}_L(h) \setminus \mathrm{Z}(L)$ has $q^2 - q = q(q-1)$ elements and forms a clique, as $\langle x, y\rangle \subseteq \mathrm{nil}_L(h)$ is nilpotent for $x, y \in \mathrm{nil}_L(h) \setminus \mathrm{Z}(L)$. The number of components is:

$$\frac{|L \setminus \mathrm{Z}(L)|}{|\mathrm{nil}_L(h) \setminus \mathrm{Z}(L)|} = \frac{q^3 - q}{q(q-1)} = q + 1.$$

Each component is isomorphic to $K_{q(q-1)}$. □

For example, in $\mathrm{t}(2, \mathbb{F}_2)$, take $S = \{y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x + y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$. Compute $[y, x + y] = [y, x] = 0$, so $\langle y, x + y \rangle$ is nilpotent, forming a clique of size $2 = 2(2 - 1)$.

Corollary 4.4. *The nilpotent graph $\Gamma_N(\mathrm{t}(2, \mathbb{F}_q))$ is $(q^2 - q - 1)$-regular.*

Proof. Each vertex in a component $K_{q(q-1)}$ is adjacent to $q(q - 1) - 1 = q^2 - q - 1$ vertices, as shown in Theorem 4.3. □

Lemma 4.5. *Let L be a finite-dimensional Lie algebra over $\mathbb{F}_q$. Then*

(1) For $h \notin \mathrm{nil}(L)$, the degree is $\deg(h) = |\,\mathrm{nil}_L(h)| - |\,\mathrm{nil}(L)| - 1$;
(2) $\Gamma_N(L)$ is never a star graph.

Proof. (1) For $h \notin \mathrm{nil}(L)$, the neighbors of h in $\Gamma_N(L)$ are $\{y \in \mathrm{nil}_L(h) \setminus \mathrm{nil}(L) \mid y \neq h\}$. Thus, $\deg(h) = |\,\mathrm{nil}_L(h) \setminus \mathrm{nil}(L)| - 1 = |\,\mathrm{nil}_L(h)| - |\,\mathrm{nil}(L)| - 1$.

(2) Suppose $\Gamma_N(L)$ is a star graph with center h. Then $\deg(h) = |L \setminus \mathrm{nil}(L)| - 1$, so $|\,\mathrm{nil}_L(h)| = |L|$, implying $h \in \mathrm{nil}(L)$, a contradiction.

□

Generalization to infinite dimensions

The results for finite-dimensional triangular Lie algebras in this section extend naturally to infinite-dimensional settings, such as the Lie algebra of upper triangular operators on a separable Hilbert space, which arises in the study of infinite-dimensional representations and quantum mechanics [2, 3]. In this context, the nilpotent graph $\Gamma_N(L)$ for L the algebra of bounded upper triangular operators with respect to an orthonormal basis can be defined analogously, with vertices non-nilpotent operators and edges for pairs generating nilpotent subalgebras. While the finite-field case provides discrete structure, the infinite-dimensional analog reveals continuous spectral properties, linking to operator theory and functional analysis [4]. For instance, the component structure generalizes to orbits under the adjoint action, with cliques corresponding to Engel elements in the centralizer. This extension underscores the versatility of the nilpotent graph framework, applicable to unbounded operators in conformal field theory [10]. Future work could explore the diameter in terms of operator norms, supported by recent advances in infinite-dimensional Lie superalgebras [9]. More precisely, infinite-dimensional analogues of nilpotent and solvable Lie algebras, including triangular ones, are captured by the classes of *pro-nilpotent* and *pro-solvable* Lie algebras [20]. A pro-nilpotent Lie algebra L is residually nilpotent with finite-dimensional successive quotients in its lower central series, generalizing the finite termination of the series in finite dimensions. Similarly, pro-solvable algebras generalize solvable ones via the derived series. These structures admit extended triangularization theorems: pro-nilpotent algebras have strictly triangularizable adjoint operators, meaning there exists a flag of ideals $L = V_1 \supset V_2 \supset \cdots \supset \{0\}$ with $\dim(V_i/V_{i+1}) = 1$ and $[\mathrm{ad}_L(V_i), V_i] \subseteq V_{i+1}$, analogous to Engel's theorem for nilpotent algebras. For triangular superalgebras, the infinite-dimensional version $N(\infty, \mathbb{F})$ of strictly upper triangular matrices over a field $\mathbb{F}$ is a prototypical pro-nilpotent Lie algebra, with basis $\{E_{ij} \mid i < j\}$ where E_{ij} has a 1 in position (i, j) and zeros elsewhere, and brackets $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$. This algebra is residually nilpotent, with lower central series $N_k = \mathrm{span}\{E_{ij} \mid j - i \geq k\}$, and each quotient N_k/N_{k+1} is finite-dimensional (dimension infinite but graded). The nilpotent graph $\Gamma_N(N(\infty, \mathbb{F}))$ would have vertices non-nilpotent operators (those not in the center, which is trivial), and edges for pairs generating pro-nilpotent subalgebras, potentially leading to infinite cliques corresponding to root spaces with respect to a maximal torus. In the superalgebra setting, residually solvable extensions of pro-nilpotent Leibniz superalgebras provide a graded analogue [21], where the nilradical is pro-nilpotent and the extension is controlled by derivations. For example, infinite-dimensional filiform superalgebras, such as those with basis $\{e_i\}_{i\in\mathbb{N}} \cup \{f_j\}_{j\in\mathbb{N}}$ and brackets $[e_i, f_j] = e_{i+j+1}$, $|e_i| = \bar{0}$, $|f_j| = \bar{1}$, exhibit pro-nilpotency while incorporating the $\mathbb{Z}_2$-grading. The nilpotent graph here would reflect the graded structure, with bipartiteness potentially tied to the odd part lying in the pro-nilpotent radical, extending Proposition 6.1. These generalizations highlight how the nilpotent graph framework adapts to infinite dimensions, where connectivity might be analyzed via filtration topologies and diameter bounded by the nilpotency class of quotients, opening avenues for applications in quantum field theory and integrable hierarchies [9, 10].

5. Connectivity and diameter

Definition 5.1. For a finite-dimensional non-nilpotent Lie superalgebra L over $\mathbb{F}$, the *graded nilpotent graph* $\Gamma_N(L)$ has vertex set $L \setminus \mathrm{nil}(L)$, with vertices $x, y \in L \setminus \mathrm{nil}(L)$, $x \neq y$, adjacent if $\langle x, y\rangle$ is a nilpotent graded subsuperalgebra. The *modified nilpotent graph* $\Gamma(L)$ is $\Gamma_N(L)$ with universal vertices (adjacent to all others) removed.

Theorem 5.2. *For $L = \mathfrak{t}(m|n, \mathbb{F}_q)$, the modified nilpotent graph $\Gamma(L)$ is connected with diameter at most 2.*

Proof. Let $L = \mathfrak{t}(m|n, \mathbb{F}_q)$, the Lie superalgebra of upper triangular $(m|n)\times(m|n)$ supermatrices over $\mathbb{F}_q$. The center is

$$\mathcal{Z}(L) = \{\alpha I_{m|n} \mid \alpha \in \mathbb{F}_q\},$$

because $[I_{m|n}, x] = 0$ for all $x \in L$. Thus, $|\mathcal{Z}(L)| = q$. As $\mathfrak{t}(m|n)$ is solvable, we have $\mathrm{nil}(L) = \mathcal{Z}(L)$ (see Lemma 4.2). The dimension of L is $m^2 + n^2 + mn$, so $|L| = q^{m^2+n^2+mn}$, and the vertex set $L \setminus \mathrm{nil}(L)$ has $q^{m^2+n^2+mn} - q$ elements.

Case 1. No universal vertices. Take $x \in L \setminus \mathcal{Z}(L)$. Choose $y \in L$ with a non-zero off-diagonal entry (e.g., $E_{i,j}$ for $i < j$). Then

$$[x, y] \notin \mathcal{Z}(L),$$

because it is strictly upper triangular. Hence, $\langle x, y\rangle$ is not nilpotent, and x is not adjacent to all vertices. Therefore, $\Gamma(L) = \Gamma_N(L)$.

Case 2. Connectivity. Let $L' = [L, L]$, the derived algebra, consisting of strictly upper triangular matrices. Choose $z \in L' \setminus \mathcal{Z}(L)$, e.g., a matrix with a single non-zero entry $E_{i,j}$, $i < j$. For any $x \in L \setminus \mathcal{Z}(L)$, we have

$$[x, z] \in \mathcal{Z}(L) \quad \text{because} \quad z \in L'.$$

Thus, $\langle x, z\rangle \subseteq \mathbb{F}x + \mathbb{F}z + \mathcal{Z}(L)$, and

$$[\langle x, z\rangle, \langle x, z\rangle] \subseteq \mathcal{Z}(L),$$

so $\langle x, z\rangle$ is nilpotent by [22]. Similarly, $\langle z, y\rangle$ is nilpotent for all $y \in L \setminus \mathcal{Z}(L)$. Hence, every pair $x, y \in L \setminus \mathcal{Z}(L)$ is connected via a path $x \to z \to y$, so the graph is connected with diameter at most 2. □

For example, consider $\mathfrak{t}(2, \mathbb{F}_2)$, with basis:

$$\left\{ z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then $|L| = 2^3 = 8$, $|\mathcal{Z}(L)| = 2$, and the vertex set has 6 elements. Choosing $z = y$, the connectivity path construction holds as illustrated in Figure 5.

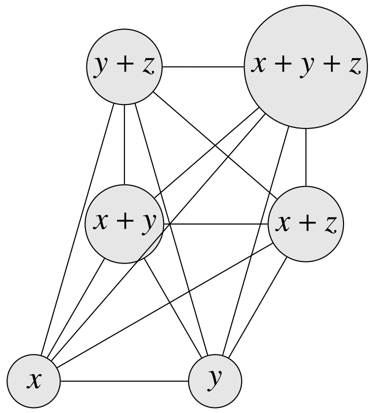

Figure 5. Nilpotent graph $\Gamma(\mathfrak{t}(2, \mathbb{F}_2))$, a complete graph K_6 with diameter 1.

Theorem 5.3. *For $L = \mathfrak{sl}(1|1, \mathbb{F}_q)$, the modified nilpotent graph $\Gamma(L)$ has $q + 1$ components, each isomorphic to $K_{q(q-1)}$, with diameter 1.*

Proof. Let $L = \mathfrak{sl}(1|1, \mathbb{F}_q)$, with basis $\{e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$. The center is $Z(L) = \{\alpha h \mid \alpha \in \mathbb{F}_q\}$, as $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$, so $|Z(L)| = q$. Because L is simple, $\mathrm{nil}(L) = Z(L)$. The dimension is 3, so $|L| = q^3$, and the vertex set has $q^3 - q$ elements.

Case 1. Component structure. For $x \notin Z(L)$, the nilpotentizer is $\mathrm{nil}_L(x) = \mathbb{F}x + Z(L)$, since $\langle x, y \rangle$ is nilpotent only if $[x, y] \in Z(L)$. This has dimension 2, so $|\mathrm{nil}_L(x)| = q^2$, and $\mathrm{nil}_L(x) \setminus Z(L)$ has $q^2 - q = q(q - 1)$ elements. Because $\mathrm{nil}_L(x)$ is abelian, these vertices form a clique $K_{q(q-1)}$. Distinct nilpotentizers intersect only in $Z(L)$, as $\mathbb{F}x + Z(L) = \mathbb{F}y + Z(L)$ implies $x = \beta y + \alpha h$, with $\alpha h \in Z(L)$. The number of components is:

$$\frac{|L \setminus Z(L)|}{|\mathrm{nil}_L(x) \setminus Z(L)|} = \frac{q^3 - q}{q(q - 1)} = q + 1.$$

Case 2. No universal vertices. For $x \notin Z(L)$, there exists $y \notin \mathrm{nil}_L(x)$ such that $[x, y] \notin Z(L)$, so $\langle x, y \rangle$ is not nilpotent. Thus, $\Gamma(L) = \Gamma_N(L)$.

Case 3. Diameter. Each component is a complete graph $K_{q(q-1)}$, so the diameter is 1. □

For example, in $\mathfrak{sl}(1|1, \mathbb{F}_3)$, $|L| = 27$, $|Z(L)| = 3$, with 4 components of K_6, as shown in Remark 3.3.

Theorem 5.4. *Let L be a finite-dimensional solvable Lie superalgebra over $\mathbb{F}_q$ with nilpotency class c and* $\dim(\mathrm{nil}(L)) \geq \dim(L) - 1$. *Then $\Gamma(L)$ is connected with diameter at most 2.*

Proof. Suppose $\dim(\mathrm{nil}(L)) \geq \dim(L) - 1$, so $\dim(L/\mathrm{nil}(L)) \leq 1$. The vertex set is $L \setminus \mathrm{nil}(L)$.

Case 1. No universal vertices. If $L = \mathbb{F}x + \mathrm{nil}(L)$, then for $x, y \in L \setminus \mathrm{nil}(L)$, we have

$$\langle x, y\rangle \subseteq \mathbb{F}x + \mathbb{F}y + \mathrm{nil}(L).$$

Because $[\langle x, y\rangle, \langle x, y\rangle] \subseteq \mathrm{nil}(L)$, and $\mathrm{nil}(L)$ is nilpotent, it follows that $\langle x, y\rangle$ is nilpotent. Thus, $\Gamma_N(L)$ is a complete graph with diameter 1, and $\Gamma(L) = \Gamma_N(L)$.

Case 2. Universal vertices exist. If $u \in L \setminus \mathrm{nil}(L)$ is universal, then $\langle u, v\rangle$ is nilpotent for all $v \in L \setminus \mathrm{nil}(L)$. Removing universal vertices may disconnect $\Gamma(L)$. Within each component, choose $z \in L' \setminus \mathrm{nil}(L)$. For $x, y \in L \setminus \mathrm{nil}(L)$, we compute:

$$[x, z] \in \mathrm{nil}(L) \quad \Rightarrow \quad \langle x, z\rangle \subseteq \mathbb{F}x + \mathbb{F}z + \mathrm{nil}(L),$$

which is nilpotent. Similarly, $\langle z, y\rangle$ is nilpotent. Thus, there is a path $x \to z \to y$, and the diameter is at most 2. □

Remark 5.5. The condition $\dim(\mathrm{nil}(L)) \geq \dim(L) - 1$ ensures a high-dimensional nilpotentizer, facilitating connectivity. This extends group-theoretic results from [14] to the graded setting of Lie superalgebras.

6. Some topological properties

Proposition 6.1. *For a solvable Lie superalgebra L over $\mathbb{F}_q$, the graded nilpotent graph $\Gamma_N(L)$ is bipartite if and only if $L_{\bar{1}} \subseteq \mathrm{nil}(L)$.*

Proof. Suppose $L_{\bar{1}} \subseteq \mathrm{nil}(L)$. The vertex set of $\Gamma_N(L)$ is $L_{\bar{0}} \setminus \mathrm{nil}(L)$, as odd elements are in $\mathrm{nil}(L)$. For $x, y \in L_{\bar{0}} \setminus \mathrm{nil}(L)$, $\langle x, y\rangle$ is nilpotent if $[x, y] \in Z(L)$, where $[x, y] \in L_{\bar{0}}$. Partition the vertex set into cosets of $L_{\bar{0}}/Z(L)$. If x, y are in the same coset, say $x = y + z$, $z \in Z(L)$, then $[x, y] = 0$, so no edge exists. Edges may exist between different cosets, but cycles require alternating cosets, which cannot form odd cycles due to the even grading [12]. Thus, $\Gamma_N(L)$ is bipartite. Conversely, suppose $x \in L_{\bar{1}} \setminus \mathrm{nil}(L)$. For $L = \mathfrak{sl}(1|1, \mathbb{F}_q)$, with basis $\{e, f, h\}$, where $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in L_{\bar{1}}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in L_{\bar{1}}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in L_{\bar{0}}$, and $Z(L) = \{\alpha h \mid \alpha \in \mathbb{F}_q\}$. Let $x = e$, $y = f$, $z = e + f$. We compute

$$[e, f] = h \in Z(L), \quad [f, e + f] = [f, e] + [f, f] = h + 0 = h \in Z(L),$$
$$[e + f, e] = [e, e] + [f, e] = 0 + h = h \in Z(L).$$

Thus, $e \sim f$, $f \sim e + f$, $e + f \sim e$, forming a triangle, so $\Gamma_N(L)$ is not bipartite. □

Remark 6.2. This result highlights the role of the odd component $L_{\bar{1}}$ in determining graph properties (see Figure 6), a novel aspect in the context of Lie superalgebras compared to group-theoretic graphs [14].

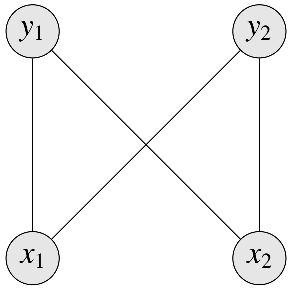

Figure 6. A bipartite graph example for $\Gamma_N(L)$ when $L_{\bar{1}} \subseteq \mathrm{nil}(L)$.

Theorem 6.3. *For $L = \mathfrak{t}(2, \mathbb{F}_q)$, the clique number of the graded nilpotent graph is $\omega(\Gamma_N(L)) = q(q-1)$.*

Proof. For $L = \mathfrak{t}(2, \mathbb{F}_q)$, the Lie algebra of 2×2 upper triangular matrices over $\mathbb{F}_q$, with basis $\{z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}$, the center is $\mathrm{Z}(L) = \{\alpha z \mid \alpha \in \mathbb{F}_q\}$. A clique in $\Gamma_N(L)$ is a set $S \subseteq L \setminus \mathrm{Z}(L)$ where $\langle x, y \rangle$ is nilpotent for all $x, y \in S$. From Theorem 4.3, $\Gamma_N(L)$ has $q+1$ components, each a clique $K_{q(q-1)}$. For $h \notin \mathrm{Z}(L)$, $\mathrm{nil}_L(h) = \mathbb{F}h + \mathrm{Z}(L)$, with $|\,\mathrm{nil}_L(h) \setminus \mathrm{Z}(L)| = q^2 - q = q(q-1)$, forming a maximal clique. Thus, $\omega(\Gamma_N(L)) = q(q-1)$. □

Theorem 6.4. *For a reductive Lie superalgebra L over $\mathbb{F}$ of characteristic zero,* $\mathrm{nil}(L) = \mathrm{Z}(L)$*, and the modified nilpotent graph $\Gamma(L)$ is connected if $L/\,\mathrm{Z}(L)$ is simple.*

Proof. For a reductive Lie superalgebra L, we have $L = [L, L] \oplus \mathrm{Z}(L)$, where $[L, L]$ is semisimple [4].

Case 1. Nilpotentizer. If $x \in \mathrm{Z}(L)$, then $[x, y] = 0$ for all y, so $\langle x, y \rangle$ is abelian, hence $x \in \mathrm{nil}(L)$. Conversely, if $x \in \mathrm{nil}(L)$, write $x = s + z$, with $s \in [L, L]$, $z \in \mathrm{Z}(L)$. If $s \neq 0$, there exists $y \in [L, L]$ such that $[s, y] \neq 0$, forming a non-nilpotent subalgebra (e.g., isomorphic to $\mathfrak{sl}(2)$). Thus, $s = 0$, and $x \in \mathrm{Z}(L)$, so $\mathrm{nil}(L) = \mathrm{Z}(L)$.

Case 2. Connectivity. If $L/\,\mathrm{Z}(L)$ is simple, let $x, y \in L \setminus \mathrm{Z}(L)$. Because $[L, L]$ is semisimple and $L/\,\mathrm{Z}(L) \cong [L, L]$, choose $w \in [L, L] \setminus \mathrm{Z}(L)$ such that $[x, w] \in \mathrm{Z}(L)$, possible due to the simplicity of $[L, L]$. Thus, $\langle x, w \rangle \subseteq \mathbb{F}x + \mathbb{F}w + \mathrm{Z}(L)$ is nilpotent, as $[\langle x, w \rangle, \langle x, w \rangle] \subseteq \mathrm{Z}(L)$. Similarly, $\langle w, y \rangle$ is nilpotent. Hence, there is a path $x \to w \to y$, and $\Gamma(L)$ is connected. □

Theorem 6.5. *For $L = \mathfrak{sl}(1|1, \mathbb{F}_q)$, the chromatic number of the graded nilpotent graph is $\chi(\Gamma_N(L)) = q + 1$.*

Proof. From Theorem 5.3, $\Gamma_N(L)$ has $q+1$ components, each a clique $K_{q(q-1)}$. Because the components are disconnected, each requires a distinct color, and the chromatic number of a complete graph K_n is n. Thus, $\chi(\Gamma_N(L)) = q + 1$. □

7. Direct sums

Theorem 7.1. *Let L_1, L_2 be non-nilpotent finite-dimensional Lie superalgebras over a field $\mathbb{F}$. Then the graded nilpotent graph $\Gamma_N(L_1 \oplus L_2)$ is connected with diameter at most 3.*

Proof. Consider the Lie superalgebra $L = L_1 \oplus L_2$, with nilpotentizer $\mathrm{nil}(L) = \mathrm{nil}(L_1) \oplus \mathrm{nil}(L_2)$. The vertex set of the graded nilpotent graph $\Gamma_N(L)$ consists of all elements in $L \setminus \mathrm{nil}(L)$. To show that $\Gamma_N(L)$

is connected with diameter at most 3, we construct paths between any two vertices $(x_1, y_1), (x_2, y_2) \in L\backslash\mathrm{nil}(L)$. Because L_1 and L_2 are non-nilpotent, we first verify that $\Gamma_N(L) = \Gamma(L)$ (the modified nilpotent graph with universal vertices removed). A vertex $(x, y) \in L \setminus \mathrm{nil}(L)$ is universal if $\langle (x, y), (a, b)\rangle$ is nilpotent for all $(a, b) \in L \setminus \mathrm{nil}(L)$. If $x \notin \mathrm{nil}(L_1)$, choose $a \in L_1 \setminus \mathrm{nil}(L_1)$ such that $\langle x, a\rangle$ is not nilpotent (possible since L_1 is non-nilpotent). Then $\langle (x, y), (a, b)\rangle = \langle x, a\rangle \oplus \langle y, b\rangle$ is not nilpotent, so no universal vertices exist, and $\Gamma_N(L) = \Gamma(L)$. We proceed by considering two cases based on the components of the vertices. First, suppose $x_1 \notin \mathrm{nil}(L_1)$. Because L_1 is non-nilpotent, there exists $x_2 \in L_1 \setminus \mathrm{nil}(L_1)$ such that the subsuperalgebra $\langle x_1, x_2\rangle \subseteq L_1$ is nilpotent, for example, by choosing $x_2 \in \mathrm{nil}_{L_1}(x_1) \setminus \mathrm{nil}(L_1)$, if non-empty, or another element forming a nilpotent subalgebra. Construct a path from (x_1, y_1) to (x_2, y_2). The subsuperalgebra generated by (x_1, y_1) and $(x_1, 0)$ is $\langle (x_1, y_1), (x_1, 0)\rangle = \langle x_1\rangle \oplus \langle y_1\rangle$. Because $\langle x_1\rangle$ is abelian, this subsuperalgebra is nilpotent, so $(x_1, y_1) \to (x_1, 0)$ is an edge. Next, the subsuperalgebra generated by $(x_1, 0)$ and (x_2, y_2) is $\langle (x_1, 0), (x_2, y_2)\rangle = \langle x_1, x_2\rangle \oplus \langle y_2\rangle$. Because $\langle x_1, x_2\rangle$ is nilpotent and $\langle y_2\rangle$ is abelian, this subsuperalgebra is nilpotent, so $(x_1, 0) \to (x_2, y_2)$ is an edge. Thus, the path $(x_1, y_1) \to (x_1, 0) \to (x_2, y_2)$ has length 2. Now, suppose $x_1 \in \mathrm{nil}(L_1)$, so $y_1 \notin \mathrm{nil}(L_2)$ (since $(x_1, y_1) \notin \mathrm{nil}(L)$). Because L_1 is non-nilpotent, there exists $z \in L_1 \setminus \mathrm{nil}(L_1)$ such that $\langle z, x_2\rangle \subseteq L_1$ is nilpotent. Construct a path from (x_1, y_1) to (x_2, y_2). The subsuperalgebra generated by (x_1, y_1) and $(z, 0)$ is $\langle (x_1, y_1), (z, 0)\rangle = \langle x_1, z\rangle \oplus \langle y_1\rangle$. Because $x_1 \in \mathrm{nil}(L_1)$, $\langle x_1\rangle$ is abelian, and the subsuperalgebra is nilpotent, so $(x_1, y_1) \to (z, 0)$ is an edge. Next, the subsuperalgebra generated by $(z, 0)$ and (x_2, y_2) is $\langle (z, 0), (x_2, y_2)\rangle = \langle z, x_2\rangle \oplus \langle y_2\rangle$, which is nilpotent because $\langle z, x_2\rangle$ is nilpotent. Thus, $(z, 0) \to (x_2, y_2)$ is an edge, giving the path $(x_1, y_1) \to (z, 0) \to (x_2, y_2)$ of length 2. If $x_2 \notin \mathrm{nil}(L_1)$, the path may require an additional step via another vertex, but the diameter remains at most 3 due to the finite-dimensionality of L. The case where $y_1 \notin \mathrm{nil}(L_2)$ is symmetric, using vertices like $(0, y_1)$. Thus, $\Gamma_N(L)$ is connected with diameter at most 3. □

For example, consider $L = \mathfrak{sl}(1|1, \mathbb{F}_2) \oplus \mathfrak{sl}(1|1, \mathbb{F}_2)$, with bases $\{e_1, f_1, h_1\}$ and $\{e_2, f_2, h_2\}$, where $[e_i, f_i] = h_i$, $[h_i, e_i] = 2e_i$, $[h_i, f_i] = -2f_i$. For vertices (e_1, e_2) and (f_1, f_2), the path $(e_1, e_2) \to (e_1, 0) \to (f_1, f_2)$ is valid, as $\langle e_1\rangle$ and $\langle e_1, f_1\rangle$ are nilpotent in $\mathfrak{sl}(1|1, \mathbb{F}_2)$.

Corollary 7.2. *If L_1 is a non-nilpotent finite-dimensional Lie superalgebra and L is a nilpotent Lie superalgebra over $\mathbb{F}$, then the vertex connectivity of the graded nilpotent graph satisfies $\kappa(L_1 \oplus L) = \kappa(L_1)$.*

Proof. Because L is nilpotent, the nilpotentizer of the direct sum is $\mathrm{nil}(L_1 \oplus L) = \mathrm{nil}(L_1) \oplus L$. The vertex set of $\Gamma_N(L_1 \oplus L)$ is $(L_1 \setminus \mathrm{nil}(L_1)) \times \{0\}$, as any $(x, y) \in L_1 \oplus L$ with $y \neq 0$ lies in $\mathrm{nil}(L_1 \oplus L)$. Define a map $\phi : L_1 \setminus \mathrm{nil}(L_1) \to (L_1 \oplus L) \setminus \mathrm{nil}(L_1 \oplus L)$ by $\phi(z) = (z, 0)$. For vertices $z, w \in L_1 \setminus \mathrm{nil}(L_1)$, the subsuperalgebra $\langle \phi(z), \phi(w)\rangle = \langle (z, 0), (w, 0)\rangle = \langle z, w\rangle \oplus \{0\}$ is nilpotent if and only if $\langle z, w\rangle$ is nilpotent in L_1. Thus, ϕ is a graph isomorphism from $\Gamma_N(L_1)$ to $\Gamma_N(L_1 \oplus L)$. Because vertex connectivity is preserved under graph isomorphism, we have $\kappa(L_1 \oplus L) = \kappa(L_1)$. □

8. Nilpotent graph complement

Theorem 8.1. *For $L = \mathfrak{t}(m|n, \mathbb{F}_q)$, the non-nilpotent graph $\Gamma_N(L)^c$ is connected with diameter at most 2.*

Proof. Let $L = \mathfrak{t}(m|n, \mathbb{F}_q)$, the Lie superalgebra of upper triangular $(m|n) \times (m|n)$ supermatrices over a finite field $\mathbb{F}_q$. The non-nilpotent graph $\Gamma_N(L)^c$ has vertex set $L\backslash Z(L)$, where the center is $Z(L) = \{\alpha I_{m|n} \mid$

$\alpha \in \mathbb{F}_q\}$, and two vertices $x, y \in L \setminus Z(L)$ are adjacent if the subsuperalgebra $\langle x, y\rangle$ is not nilpotent, equivalently, if their Lie bracket $[x, y] \notin Z(L)$. To prove that $\Gamma_N(L)^c$ is connected with diameter at most 2, consider any two vertices $x, y \in L \setminus Z(L)$. If $[x, y] \notin Z(L)$, then $\langle x, y\rangle$ is not nilpotent, as its derived series does not terminate in $Z(L)$. Thus, there is an edge $x \sim y$ in $\Gamma_N(L)^c$, forming a path of length 1. If $[x, y] \in Z(L)$, we construct a path of length 2. Let $L' = [L, L]$ be the derived algebra, consisting of strictly upper triangular supermatrices, which spans all off-diagonal entries. Choose an element $z \in L' \setminus Z(L)$, for example, a supermatrix with a single non-zero entry in an off-diagonal position distinct from those in x and y. Because L' is large (dimension $m^2 + n^2 + mn - m - n$), such a z exists such that $[x, z] \notin Z(L)$ and $[z, y] \notin Z(L)$. The subsuperalgebra $\langle x, z\rangle$ has $[x, z] \notin Z(L)$, so it is not nilpotent, giving an edge $x \sim z$. Similarly, $\langle z, y\rangle$ is not nilpotent, giving an edge $z \sim y$. Thus, the path $x \to z \to y$ exists in $\Gamma_N(L)^c$, with length 2. Because either a one-step or two-step path connects any pair of vertices, the graph $\Gamma_N(L)^c$ is connected with diameter at most 2. □

Example 8.2. For $L = \mathfrak{t}(2, \mathbb{F}_2)$, with basis $\{z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}$, the non-nilpotent graph $\Gamma_N(L)^c$ has vertex set $L \setminus Z(L)$, where $Z(L) = \{\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_2\}$, so there are $2^3 - 2 = 6$ vertices. Consider vertices $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Their Lie bracket is $[x, w] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin Z(L)$, so $x \sim w$ in $\Gamma_N(L)^c$. All non-central elements are connected via non-nilpotent subsuperalgebras, forming a single connected component, specifically a complete graph K_6, see Figure 7.

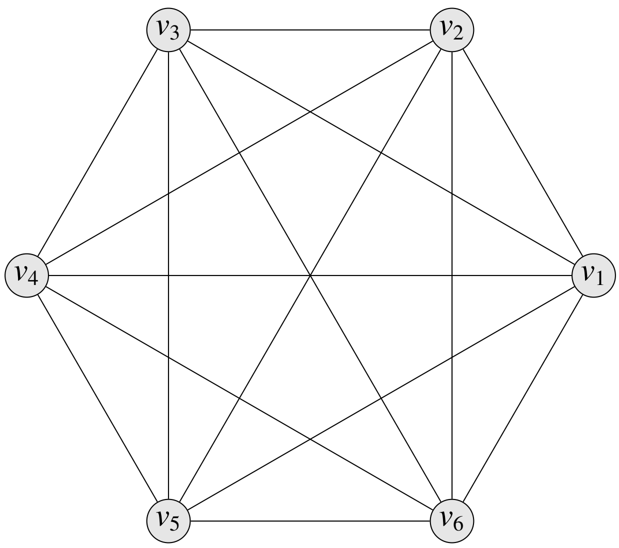

Figure 7. Non-nilpotent graph $\Gamma_N(\mathfrak{t}(2, \mathbb{F}_2))^c$, a complete graph K_6.

9. Strongly self-centralizing subalgebras

Definition 9.1. A subsuperalgebra U of L is *strongly self-centralizing* if

$$C_L(x) = U \quad \text{for all } 0 \neq x \in U,$$

where $C_L(x) = \{y \in L \mid [x, y] = 0\}$.

Example 9.2. Let L have basis $\{x, y\}$, with $x \in L_{\bar{0}}$, $y \in L_{\bar{1}}$, and Lie brackets defined by $[x, y] = y$, $[x, x] = [y, y] = 0$. Set $U = \mathbb{F}x$. For $0 \neq \alpha x \in U$, we compute:

$$C_L(\alpha x) = \{ax + by \mid [\alpha x, ax + by] = by = 0\} = \mathbb{F}x = U.$$

Thus, U is strongly self-centralizing.

Example 9.3. Let L have basis $\{e, f, g\}$, with $e, f \in L_{\bar{0}}$, $g \in L_{\bar{1}}$, and brackets:

$$[e, f] = 0, \quad [e, g] = e, \quad [f, g] = f, \quad [g, g] = 0.$$

Set $U = \mathbb{F}e + \mathbb{F}f$. For $0 \neq x = ae + bf \in U$, compute:

$$C_L(x) = \{ue + vf + wg \mid [ae + bf, ue + vf + wg] = (aw)e + (bw)f = 0\}.$$

If $a \neq 0$, then $w = 0$, so $C_L(x) = \mathbb{F}e + \mathbb{F}f = U$. Similarly for $b \neq 0$. Hence, U is strongly self-centralizing.

Example 9.4. Let $L = \mathfrak{sl}(n|n, \mathbb{F}_q) \oplus V$, where $V = \mathbb{F}g$ is a one-dimensional odd subspace satisfying $[g, g] = 0$ and $[\mathfrak{sl}(n|n, \mathbb{F}_q), g] = 0$. Thus, V lies in the center $Z(L)$. Let $U \subseteq \mathfrak{sl}(n|n, \mathbb{F}_q)$ be the subalgebra of diagonal matrices, where $q = p^a$ and $p \nmid n$. For any nonzero $x \in U$, we have

$$C_L(x) = \{\, y \in \mathfrak{sl}(n|n) \mid [x, y] = 0 \,\} \oplus V = C_{\mathfrak{sl}(n|n)}(x) \oplus \mathbb{F}g = U \oplus \mathbb{F}g.$$

Hence, although U is abelian, it is *not* strongly self-centralizing in L, because the odd central element g also belongs to the centralizer $C_L(x)$.

Lemma 9.5. *Let U be a nilpotent subsuperalgebra of L with $U \nsubseteq \mathrm{nil}(L)$. Then $U \setminus (U \cap \mathrm{nil}(L))$ is a clique in $\Gamma_N(L)$. If N is a nilpotent ideal, and L is finite-dimensional over a finite field, then $|N \setminus (N \cap \mathrm{nil}(L))| \leq \omega(\Gamma_N(L))$. In particular, $|\mathrm{N}(L) \setminus (\mathrm{N}(L) \cap \mathrm{nil}(L))| \leq \omega(\Gamma_N(L))$.*

Proof. For $x, y \in U \setminus (U \cap \mathrm{nil}(L))$, because U is nilpotent, $\langle x, y\rangle \subseteq U$ is nilpotent, so $x \sim y$ in $\Gamma_N(L)$. Thus, $U \setminus (U \cap \mathrm{nil}(L))$ is a clique. For a nilpotent ideal N, $N \setminus (N \cap \mathrm{nil}(L))$ is a clique, so its size is at most the clique number $\omega(\Gamma_N(L))$. Because $\mathrm{N}(L)$ is the largest nilpotent ideal, the bound holds. □

Theorem 9.6. *Let L be a semisimple Lie superalgebra over a finite field with a strongly self-centralizing subsuperalgebra U. Then*

(1) $\mathrm{nil}_L(x) = U$ *for all* $0 \neq x \in U$.
(2) $\Gamma_N(L)$ *is disconnected.*
(3) $|\mathrm{N}(L)| - 1 \leq \omega(\Gamma_N(L))$.

Proof. Because L is semisimple, $\mathrm{nil}(L) = \mathrm{Z}(L) = 0$.

(1) For $0 \neq x \in U$, because $U = C_L(x)$, we have $U \subseteq \mathrm{nil}_L(x)$. If $y \in \mathrm{nil}_L(x)$, then $\langle x, y\rangle$ is nilpotent, so $[x, y] = 0$, implying $y \in C_L(x) = U$. Thus, $\mathrm{nil}_L(x) = U$.
(2) For $y \in L \setminus U$, there exists $x \in U$ such that $[x, y] \neq 0$, so $\langle x, y\rangle$ is not nilpotent. Thus, $U \setminus \{0\}$ forms a disconnected component in $\Gamma_N(L)$.

(3) Because $\mathcal{N}(L) = 0$ (as L is semisimple), we have $|\mathcal{N}(L)| - 1 = -1 \leq \omega(\Gamma_N(L))$, which holds trivially.

□

Definition 9.7. The *non-nilpotent graph* $\Gamma_N(L)^c$ of a finite-dimensional non-nilpotent Lie superalgebra L has vertices $L \setminus \mathrm{nil}(L)$, with x, y adjacent if $\langle x, y\rangle$ is not nilpotent.

Example 9.8. For $L = \mathfrak{t}(2, \mathbb{F}_2) \oplus \mathbb{F}g$, where $g \in L_{\bar{1}}$, $[g, g] = 0$, and $\mathfrak{t}(2, \mathbb{F}_2)$ acts trivially on $\mathbb{F}g$, the non-nilpotent graph $\Gamma_N(L)^c$ is connected. The vertex set is $L \setminus \mathrm{Z}(L)$, with $\mathrm{Z}(L) = \{\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_2\}$. For $x = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0)$, $y = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, g)$, compute $[x, y] \notin \mathrm{Z}(L)$, so $x \sim y$. All non-central elements are connected via non-nilpotent pairs, forming one component.

10. Conclusions

In this paper, we introduced the nilpotent graph $\Gamma_N(L)$ for finite-dimensional Lie superalgebras over an arbitrary field, extending the recent nilpotent graph framework from ordinary Lie algebras to the $\mathbb{Z}_2$-graded setting. The graded structure introduces fundamentally new phenomena that have no analogue in the non-graded case, most notably the crucial role played by the odd component $L_{\bar{1}}$ in determining global graph-theoretic properties.

Our main results can be summarised as follows:

- In characteristic zero, the nilpotentizer $\mathcal{N}(L)$ coincides with the hypercenter $Z^*(L)$, providing a complete algebraic characterisation of the isolated vertices of $\Gamma_N(L)$.
- For the basic triangular superalgebra $\mathfrak{t}(2, \mathbb{F}_q)$, the graph $\Gamma_N(L)$ consists of exactly $q + 1$ disjoint complete graphs, each isomorphic to $K_{q(q-1)}$, yielding explicit values for all major invariants (regularity, clique number, chromatic number, etc.).
- For the simple Lie superalgebra $\mathfrak{sl}(1|1, \mathbb{F}_q)$, the structure is identical in form ($q + 1$ components, each $K_{q(q-1)}$), illustrating that simplicity does not force connectivity of the nilpotent graph, in sharp contrast to the ordinary Lie algebra case.
- Bipartiteness of $\Gamma_N(L)$ is completely determined by the position of the odd part: when L is solvable, $\Gamma_N(L)$ is bipartite if and only if $L_{\bar{1}} \subseteq \mathcal{N}(L)$. The presence of odd elements outside the nilpotentizer inevitably creates odd cycles (typically triangles).
- For solvable triangular superalgebras $\mathfrak{t}(m|n, \mathbb{F}_q)$, the modified nilpotent graph $\Gamma(L)$ is connected with diameter at most 2, while the complement (non-nilpotent graph) is also connected with diameter at most 2.
- Direct sums of non-nilpotent Lie superalgebras yield nilpotent graphs of diameter at most 3, and when one factor is nilpotent the graph is isomorphic to the nilpotent graph of the non-nilpotent factor.
- Strong self-centralising subalgebras force disconnection of $\Gamma_N(L)$ in the semisimple case, and the spectral and combinatorial properties of these graphs are intimately tied to classical invariants of Lie superalgebras (nilradical, hypercenter, derived length, grading).

We provided efficient SageMath algorithms for constructing $\Gamma_N(L)$ and computing its invariants for concrete finite-dimensional examples, making the theory computationally accessible and independent of manual bracket calculations.

The $\mathbb{Z}_2$-grading dramatically enriches the interplay between algebraic and graph-theoretic structure: odd elements act as natural "cycle generators", while the even part tends to produce large cliques aligned with centraliser/coset geometry. This graded dichotomy offers a new lens for studying nilpotency and solvability that is invisible in the ordinary (non-super) setting.

Several challenging questions remain open, most prominently:

(1) Does there exist a non-solvable Lie superalgebra for which $\Gamma_N(L)$ is bipartite?
(2) What are the possible clique numbers and chromatic numbers for the classical simple series $\mathfrak{psl}(n|n)$, $\mathfrak{osp}(m|2n)$ over finite fields?
(3) How do spectral properties (eigenvalues of the adjacency or Laplacian matrix) of $\Gamma_N(L)$ relate to representation-theoretic data of L?

These problems, together with the infinite-dimensional and quantum deformations of the construction, suggest that nilpotent graphs of Lie superalgebras constitute a rich and largely unexplored bridge between graded Lie theory, combinatorial graph theory, and computational algebra.

Author contributions

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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Appendix. Computational tools

Algorithm 1: Construct the modified nilpotent graph $\Gamma(L)$.

Input: A finite-dimensional Lie superalgebra L over $\mathbb{F}_q$
Output: The modified nilpotent graph $\Gamma(L)$

```
elements ← basis elements of L;
nilpotentizer ← {x ∈ elements | bracket(x, y).is_nilpotent() for all y ∈ elements};
nodes ← elements \ nilpotentizer;
universal ← {x ∈ nodes | bracket(x, y).is_nilpotent() for all y ∈ nodes};
nodes ← nodes \ universal;
G ← empty graph;
G.add_vertices(nodes);
foreach x, y ∈ nodes, x ≠ y do
    S ← L.subalgebra([bracket(x, y)]);
    if S.is_nilpotent() then
        G.add_edge(x, y);
return G.
```

Algorithm 2: Compute the chromatic number of $\Gamma_N(L)$.

Input: The nilpotent graph $\Gamma_N(L)$
Output: The chromatic number $\chi(\Gamma_N(L))$

```
components ← Γ_N(L).connected_components();
chi ← len(components);
foreach C ∈ components do
    if C.is_clique() then
        chi ← max(chi, |C|);
return chi.
```

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