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Research article

Convergence of interval fuzzy number sequences

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Abstract: There are many opinions about multiplication and division formulas in interval numbers, but there is a common weakness in these formulas: That the division between two equal interval numbers does not produce the identity. Similar to the interval number sequence, many concepts about the convergence of the interval sequence are offered by various authors but they cannot prove that the basic properties of the convergence of the real number sequence can be generalized to the properties of convergence of the interval number sequence. Here, we used the algebra for interval numbers from the author, as contained in Mashadi et al. (2023), that is, the algebra for interval numbers using midpoints, which guarantees the existence of the inverse of any interval number. In this article, we showed that some basic properties of the sequence of real numbers can be generalized to the sequence of interval numbers. In addition to the convergence properties of the interval number sequence, the convergence of the interval number sequence with positive, negative, and fractional powers were also shown. Based on the definition of convergence of interval sequences given along with various basic theorems for convergence given in this paper, it was expected that all theorems related to the convergence of real number sequences can be generalized to interval number sequences; for example, the properties of tail sequences, the Monotone Convergence Theorem, the Existence of Monotone Subsequences, Subsequences and the Bolzano-Weierstrass Theorem, and the Cauchy criterion for the convergence of interval number sequences.

Keywords: convergence of interval number; limit of interval number; power of interval number
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1. Introduction

The development of interval numbers is not as fast as the development of fuzzy numbers. In particular, the development of interval numbers in the form of convergence of interval numbers can be seen in [1–3]. However, interval numbers in their application in various fields also experienced developments, as seen in [4–6]. There are several opinions regarding the arithmetic operations that apply to interval numbers, especially regarding multiplication and division operations. In [7–9], minimum, maximum, and constant are used in multiplication between two interval numbers, whereas in [10–12], multiplication between two interval numbers was defined only by using maximum and minimum. It is stated in [13–15] that, suppose there is an interval number $\tilde{a}_I = [\underline{a}, \overline{a}]$, then the inverse of $\tilde{a}_I$ cannot be determined if $\underline{a} < 0$ and $\overline{a} = 0$. Based on [16], state that division between two interval numbers is undefined if $\underline{a} = 0$ and $\overline{a} = 0$ for any interval number $\tilde{a}_I = [\underline{a}, \overline{a}]$, while [17, 18] division between two interval numbers is undefined if $0 \in \tilde{a}_I$. Another opinion on the formula for interval number division is given in [19], but the result of division between two equal interval numbers does not produce an identity $\tilde{\iota}_I = [1,1]$. To address this problem, another formula is used. This formula, which is based on a study in [20–22], introduces a new multiplication, and together with the general inverse form of an interval number shown in [23], we have that the division between two equal interval numbers is the identity.

Next, the researchers in [24] introduce interval number series and convergence of interval number series, showing the form of interval numbers with positive integer powers and convergence of the number series. This is continued in [25], showing that the set of all interval numbers is a metric space and defines the sequence of null spaces and the sequence of finite interval numbers. The researchers in [26] showed the convergence of interval number line using H-difference. In the study, some properties that apply to the interval number sequence have been shown, such as the addition property of two interval number sequences and the multiplication property between a constant and interval number sequence. Seeing that many properties apply to the sequence of real numbers, we are interested in further research to show whether these properties also apply to the sequence of interval numbers. These properties include the difference between two interval number lines and the multiplication and division between two interval number lines. The convergence of interval number series with positive, negative, and fractional integer powers will also be shown. In [23], the general form given for interval numbers with positive integer powers has shortcomings, namely that the general properties of exponents of interval numbers do not apply. Thus, in this case, a new formula for interval numbers with positive integer powers will be given using the multiplication formula shown in [27]. In order to show that these properties hold, some other properties are needed such as the properties of the sequence of real numbers shown in [28], the properties of midpoints, and the properties of interval numbers with powers of positive integers and fractions.

2. Preliminaries

There are many ways given by authors to determine the algebraic operations of an interval number.

For example, for two interval numbers $\tilde{a}_I = [\underline{a}, \overline{a}]$ and $\tilde{b}_I = [\underline{b}, \overline{b}]$ with each of $\underline{a} \leq \overline{a}$ and $\underline{b} \leq \overline{b}$ and defined **IR** $= \{\tilde{a}_I = [\underline{a}, \overline{a}]$ with $a_l \leq a_r$ and $\underline{a}, \overline{a}$ **R**$\}$ one form of algebraic operation that is quite widely used by writers [29, 30] is defined as follows:

i. $\tilde{a}_I \oplus \tilde{b}_I = [\underline{a} + \underline{b},\ \overline{a} + \overline{b}]$,

ii. $\tilde{a}_I \ominus \tilde{b}_I = [\underline{a} - \overline{b},\ \overline{a} - \underline{b}]$,

iii. $\tilde{a}_I \otimes \tilde{b}_I = [min\{\underline{a}.\underline{b},\ \underline{a}.\overline{b},\ \overline{a}.\underline{b},\ \overline{a}.\overline{b}\},\ max\{\underline{a}.\underline{b}, \underline{a}.\overline{b},\ \overline{a}.\underline{b},\ \overline{a}.\overline{b}\}]$, (1)

iv. $\alpha\tilde{a}_I = \begin{cases} [\alpha\underline{a},\ \alpha\overline{a}], & if\ \alpha \geq 0 \\ [\alpha a_r,\ \alpha\overline{a}], & if\ \alpha < 0 \end{cases}$,

v. $\frac{\tilde{a}_I}{\tilde{b}_I} = [\underline{a}, \overline{a}] \otimes \left[\frac{1}{\overline{b}}, \frac{1}{\underline{b}}\right]$.

While [7] defines division by

$$(\text{v*})\ \frac{\tilde{a}_I}{\tilde{b}_I} = \left[min\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}, max\left\{\frac{\underline{a}}{\underline{b}}, \frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}\right\}\right]. \quad (2)$$

In the algebraic definitions, if $0 \in \tilde{a}_I = [\underline{a}, \overline{a}]$ in the sense $\underline{a}, \leq 0 \leq \overline{a}$, then $\frac{1}{\tilde{a}_I} = \left[\frac{1}{\overline{a}}, \frac{1}{a_l}\right]$ is not necessarily $\frac{1}{\tilde{a}_I} \in I$**R**. Thus, $\frac{\tilde{a}_I}{\tilde{a}_I}$ is also not the same as $\tilde{\imath}_I = [1,1]$. As for defining division by using (vi), $\frac{\tilde{a}_I}{\tilde{a}_I}$ is not the same as $\tilde{\imath}_I = [1,1]$.

Then, for $\tilde{a}_I = [\underline{a}, \overline{a}]$, other authors [4–6] define midpoint and width (or half-width) from $\tilde{a}_I$ by $m(\tilde{a}_I) = \frac{\underline{a}, + \overline{a}}{2}$, $w(\tilde{a}_I) = \frac{\overline{a} - \underline{a}}{2}$ and $dual(\tilde{a}_I) = dual\,[\underline{a}, \overline{a}]$ so that we get $\tilde{a}_I - dual\,(\tilde{a}_I) = \tilde{0}_I = [0,0]$, but this not applicable since $\tilde{a}_I - \tilde{a}_I = \tilde{0}_I$. They also claim that $\tilde{a}_I \otimes \frac{1}{dual(\tilde{a}_I)} = [\underline{a}, \overline{a}] \otimes \frac{1}{[\underline{a}, \overline{a}]} = [1,1]$, but this does not apply to $\underline{a} = 0$ or $\overline{a} = 0$. On the other hand, they also stated that two intervals $\tilde{a}_I = [\underline{a}, \overline{a}]$ and $\tilde{b}_I = [\underline{b}, \overline{b}]$ is equal if $m(\tilde{a}_I) = m(\tilde{b}_I)$. If this concept is used, it means that $\tilde{a}_I = [-k, k]$ and $\tilde{b}_I = [-l, l]$ are two equal interval numbers, even though $k \neq l$. The form of algebraic operations they offer is:

i. $\tilde{a}_I \oplus \tilde{b}_I = [m(\tilde{a}_I) + m(\tilde{b}_I) - k, m(\tilde{a}_I) + m(\tilde{b}_I) + k]$.

ii. $\tilde{a}_I \ominus \tilde{b}_I = [m(\tilde{a}_I) - m(\tilde{b}_I) - k, m(\tilde{a}_I) - m(\tilde{b}_I) + k]$,

with $k = \left\{\frac{(\overline{a} + \overline{b}) - (\underline{a} + \underline{b})}{2}\right\}$. (3)

iii. $\tilde{a}_I \otimes \tilde{b}_I = [m(\tilde{a}_I)m(\tilde{b}_I) - k, m(\tilde{a}_I)\, m(\tilde{b}_I) + k]$, with $k = min\{m(\tilde{a}_I)m(\tilde{b}_I) - \alpha,\ \beta - m(\tilde{a}_I)\, m(\tilde{b}_I)\}$, and

$$\alpha = \min\{\underline{a}.\underline{b}, \underline{a}\,\overline{b}, \overline{a}.\underline{b},\ \overline{a}.\overline{b}\},\quad \beta = \max\{\underline{a}.\underline{b}, \underline{a}\,.\overline{b}, \overline{a}.\underline{b},\ \overline{a}\,.\overline{b}\}.$$

iv. $1{:}\tilde{a}_I = \frac{1}{\tilde{a}_I} = \frac{1}{[\underline{a},\overline{a}]} = \left[\frac{1}{m(\tilde{a}_I)} - \delta, \frac{1}{m(\tilde{a})} + \delta\right]$,

with $\delta = \min\left\{\frac{1}{a_r}\left(\frac{\overline{a}-\underline{a}}{\underline{a}+\overline{a}}\right), \frac{1}{\underline{a}}\left(\frac{\overline{a}-\underline{a}}{\underline{a}+\overline{a}}\right)\right\}$ and $0 \notin \tilde{a}_I$.

It is clear from the formulas they offer that they will not be able to calculate $\frac{1}{\tilde{a}_I}$ if $0 \in \tilde{a}_I = [\underline{a},\ \overline{a}]$, meaning we must look for some alternatives to determine $\frac{1}{\tilde{a}_I}$ for any $\tilde{a}_I \neq \tilde{0}_I = [0,0]$. On the other hand, the formula (i′) from [4–6] is the same as formula (i). Likewise, the formula in (ii′) is also the same as (ii). It is mathematically certain that the application of interval numbers and interval matrices is used only for linear programming problems or solving systems of linear equations in the form of interval numbers, whose values are positive. Then, mathematical problems, as mentioned above, do not arise but will arise if used in applications that use interval matrices whose values are negative.

Especially for the convergence of interval number lines, there are also many definitions given by authors, which is quite interesting. For instance, the researchers in [26,28] indicate that the convergence of interval number lines uses the concept of H-difference, that is for $\tilde{a}_I \oplus \tilde{b}_I \in \boldsymbol{IR}$, if there is $\tilde{c}_I \in \boldsymbol{IR}$ such that $\tilde{a}_I = \tilde{b}_I \oplus \tilde{c}_I$, we say that $\tilde{c}_I$ is the H-difference of $\tilde{a}_I$ and $\tilde{b}_I$, denoted $\tilde{c}_I = \tilde{a}_I -_H \tilde{b}_I$. Based on this definition, the researchers in [26,28] define convergent for interval sequence as follows:

Definition 2.1. Given an interval number sequence $(\tilde{a}_I)_n = [\underline{a}_n, \overline{a}_n] \in \boldsymbol{R}, n = 1,2,3,\cdots$ and $\tilde{a}_I = [\underline{a}, \overline{a}] \in \boldsymbol{R}$. We say that interval number $\tilde{a}_I$ is the limit of interval number sequencer $\{(\tilde{a}_I)_n\}$, denoted by $\lim_{n\to\infty} (\tilde{a}_I)_n = \tilde{a}_I$, if there is a positive integer N such that $(\tilde{a}_I)_n -_H \tilde{a}_I,\ \tilde{a}_I -_H (\tilde{a}_I)_n$ exist and $|(\tilde{a}_I)_n -_H \tilde{a}_I| \le \varepsilon$ for any $n > N$.

Based on the concept of limit above, the researchers in [26] also show the convergence of the interval sequence, but the algebra used is the concept of min-max, which is as follows:

Theorem 2.2. Suppose there is an interval number $\tilde{a}_I = [\underline{a}, \overline{a}]$ and $(\tilde{a}_I)_n = [\underline{a}_n, \overline{a}_n]$ with $n \in \mathbb{N}$. Then, the interval number sequence $\{(\tilde{a}_I)_n\}$ converges to $\tilde{a}_I$ or $\lim(\tilde{a}_I)_n = \tilde{a}_I$ if and only if $\lim(\underline{a}_n) = \underline{a}$ and $\lim(\overline{a}_n) = \overline{a}$. Then, there is a positive integer K such that $(\tilde{a}_I)_n - \tilde{a}_I, \tilde{a}_I - (\tilde{a}_I)_n$ for every $n \ge K$.

Proof. ($\Rightarrow$) Because $\lim(\tilde{a}_I)_n = \tilde{a}_I$ then for every $\tilde{\varepsilon}_I = [\underline{\varepsilon}, \overline{\varepsilon}] \ge 0$ there is a positive integer K such that $(\tilde{a}_I)_n - \tilde{a}_I, \tilde{a}_I - (\tilde{a}_I)_n$ exists for every $n \ge K$. Next, applies $|(\tilde{a}_I)_n - \tilde{a}_I| = [min\{|\underline{a}_n - \underline{a}|, |\overline{a}_n - \overline{a}|\},\ max\{|\underline{a}_n - \underline{a}|, |\overline{a}_n - \overline{a}|\}] \le [\underline{\varepsilon}, \overline{\varepsilon}]$ for every $n \ge K$. Then, $|\underline{a}_n - \underline{a}| \le \underline{\varepsilon}$ dan $|\overline{a}_n - \overline{a}| \le \overline{\varepsilon}$, so $\lim(\underline{a}_n) = \underline{a}$ and $\lim(\overline{a}_n) = \overline{a}$.

Based on Definition 2.1 and Theorem 2.2, there would still be a problem in algebraic properties of the sequence of real numbers. For example, if $(\tilde{a}_I)_n$ converges to $\tilde{a}_I$, it is still undetermined whether $\sqrt{(\tilde{a}_I)_n}$ coverges to $\sqrt{\tilde{a}_I}$. This condition will also be problematic if we use the various concepts of interval sequence convergence given [4,21,31].

On the other hand, referring to Definition 2.1, the author proves only that $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} \pm \lim_{n\to\infty}\left\{\left(\tilde{b}_I\right)_n\right\}$ and the uniqueness of the limit as well as the property of $\lim_{n\to\infty}\{(k.\tilde{a}_I)_n\} =$

$k.\tilde{a}_I$. The author does not show any other traits. Thus, if this concept is applied, and if $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} = \tilde{a}_I$, it does not necessarily imply $\lim_{n\to\infty}\left\{\sqrt{(\tilde{a}_I)_n}\right\} = \sqrt{\tilde{a}_I}$ or $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} = |\tilde{a}_I|$. The general concept for interval number sequences that is given by [1,2] is as follows:

Definition 2.3. A sequence $\{(\tilde{a}_I)_n\}$ of interval numbers is said to be convergent to the interval $\tilde{a}_I$ if for each $\varepsilon > 0$ there exists a positive interger k_o such that $d((\tilde{a}_I)_n, \tilde{a}_I) < \varepsilon$ for all $n \geq k_o$ and we denote it by $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} = \tilde{a}_I$.

However, since the algebra for multiplication, division, and inverse used by [1,2] also does not express $\frac{\tilde{a}_I}{\tilde{a}_I} = \tilde{\imath}_I$, then they also cannot prove that if $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} = \tilde{a}_I$ and $\lim_{n\to\infty}\left\{\left(\tilde{b}_I\right)_n\right\} = \tilde{b}_I \neq \tilde{0}_i$ then $\lim_{n\to\infty}\left\{\frac{(\tilde{a}_I)_n}{\left(\tilde{b}_I\right)_n}\right\} = \frac{\tilde{a}_I}{\tilde{b}_I}$, since they use max-min algebra. They mostly discuss superior limits and inferior limits, and based on Definitions 2.1 and 2.3 if $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} = \tilde{a}_I$ and $\lim_{n\to\infty}\left\{\left(\tilde{b}_I\right)_n\right\} = \tilde{b}_I \neq \tilde{0}_i$ does not necessarily imply $\lim_{n\to\infty}\left\{\frac{(\tilde{a}_I)_n}{\left(\tilde{b}_I\right)_n}\right\} = \frac{\tilde{a}_I}{\tilde{b}_I}$, and $\lim_{n\to\infty}\{(\tilde{a}_I)_n\} \times \left\{\left(\tilde{b}_I\right)_n\right\} = \lim_{n\to\infty}\{(\tilde{a}_I)_n\} \times \lim_{n\to\infty}\left\{\left(\tilde{b}_I\right)_n\right\}$.

Based on the above conditions, we feel the need to determine a concept of algebraic operations for interval numbers, especially for multiplication and division, so that it applies $\tilde{a}_I \otimes \frac{1}{\tilde{a}_I} = \tilde{\imath}_I = [1,1]$ and other properties of an interval number operation. The methodology used is to adopt the pattern of defining the case of the membership of fuzzy triangular number and trapezoidal fuzzy number as in [24,32,33], which is applied to define the membership of an interval number. Thus, $\tilde{a}_I = \left[\underline{a}, \overline{a}\right]$ based on the value of $m(\tilde{a}_I)$.

3. Material and methods

3.1. Arithmetic operations of interval numbers

The main point that makes the accuracy and correctness of the basic properties of interval sequence is the arithmetic of interval numbers used. Thus, in this paper, the algebra of interval numbers that will be used is the arithmetic of interval numbers from the author , as contained in [23,25]. That is:

The general form of interval numbers is $\tilde{a}_I = \left[\underline{a}, \overline{a}\right] \in IR$ with $IR = \left\{\tilde{a}_I = \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\right\}$. Suppose there are two interval numbers $\tilde{a}_I = \left[\underline{a}, \overline{a}\right]$ and $\tilde{b}_I = \left[\underline{b}, \overline{b}\right]$ with $\tilde{a}_I, \tilde{b}_I \in IR^*$ and defined $IR^* = \left\{\tilde{a}_I = \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R}\right\}$. In [23, 25], the operations that apply to numbers in an interval are as follows:

i. $\tilde{a}_I \oplus \tilde{b}_I = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$,

ii. $\tilde{a}_I \ominus \tilde{b}_I = \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right]$,

iii. $k\tilde{a}_I = \begin{cases} \left[k\underline{a}, k\overline{a}\right], & k \geq 0, \\ \left[k\overline{a}, k\underline{a}\right], & k < 0, \end{cases}$ (4)

iv. $\tilde{a}_I \otimes \tilde{b}_I = [\underline{a} \cdot m(\tilde{b}_I) + \underline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I).m(\tilde{b}_I), \overline{a} \cdot m(\tilde{b}_I) + \overline{b} \cdot m(\tilde{a}_I) -$

$m(\tilde{a}_I).m(\tilde{b}_I)\big]$,

with $m(\tilde{a}_I) = \frac{\underline{a}+\overline{a}}{2}$.

v. $\frac{\tilde{a}_I}{\tilde{b}_I} = \tilde{a}_I \otimes \frac{1}{\tilde{b}_I}$,

with $\frac{1}{\tilde{b}_I} = \left[\frac{2\cdot m(\tilde{b}_I)-\underline{b}}{m(\tilde{b}_I)^2}, \frac{2\cdot m(\tilde{b}_I)-\overline{b}}{m(\tilde{b}_I)^2}\right]$.

Based on the arithmetic (iv) and (v) above, then for every non-degenerate interval $\tilde{a}_I = [\underline{a}, \overline{a}]$ with $\underline{a} < \overline{a}$, it is obtained that:

$$\frac{\tilde{a}_I}{\tilde{a}_I} = \frac{[\underline{a},\overline{a}]}{[\underline{a},\overline{a}]} = [1,1] \otimes \left(\frac{[\underline{a},\overline{a}]}{[\underline{a},\overline{a}]}\right) = [1,1] \otimes \left([\underline{a},\overline{a}] \otimes \frac{1}{[\underline{a},\overline{a}]}\right)$$
$$= \frac{[1,1] \otimes [\underline{a},\overline{a}]}{[\underline{a},\overline{a}]} = [1,1].$$

3.2. Power of interval number

The second part that determines the validity of the properties of the interval sequence is how the power of the interval numbers is defined (a topic that arguably has never been discussed by other authors). In this paper, the power process to be used is as presented in [25], which is as follows.

Theorem 3.1. Suppose there is an interval number $\tilde{a}_i$ with n be a positive integer, then the following applies

$$(\tilde{a}_I)^n = \big[(n\underline{a}) \cdot m((\tilde{a}_I)^{n-1}) - (n-1) \cdot m((\tilde{a}_I)^n), (n\overline{a}) \cdot m((\tilde{a}_I)^{n-1}) - (n-1) \cdot m((\tilde{a}_I)^n)\big]. \quad (5)$$

Proof. We use the induction method. Let there be a set where $P(n)$ is true for $n \in \mathbb{N}$. If $n = 1$, then

$$\tilde{a} = \big[(1 \cdot \underline{a}) \cdot m(\tilde{a})^{1-1} - (1-1) \cdot m(\tilde{a})^{1-1}, (1 \cdot \overline{a}) \cdot m(\tilde{a})^{1-1} - (1-1) \cdot m(\tilde{a})^{1-1}\big] = [\underline{a}, \overline{a}]. \quad (6)$$

Therefore, $P(1)$ is true and $1 \in S$. Next, we assume that $P(k)$ is true and wish to infer from this assumption that $P(k+1)$ is also true. If $k \in S$ then

$$\tilde{a}^k = \big[(k\underline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k), (k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\big]. \quad (7)$$

If we multiply to both sides, then

$$\tilde{a}^k \otimes \tilde{a} = \big[(k\underline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k), (k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\big] \otimes \tilde{a}$$
$$= \Big[\Big((k\underline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\Big) m(\tilde{a}) + \underline{a} \cdot m(\tilde{a}^k) - m(\tilde{a}^k) \cdot m(\tilde{a}), \quad (8)$$

$$\left((k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\right) m(\tilde{a}) + \overline{a} \cdot m(\tilde{a}^k) - m(\tilde{a}^k) \cdot m(\tilde{a})\Big]$$

$$= \left[\left(k\underline{a}\right) \cdot m(\tilde{a})^k - (k-1) \cdot m(\tilde{a})^{k+1} + \underline{a} \cdot m(\tilde{a})^k - m(\tilde{a})^{k+1}, (k\overline{a}) \cdot m(\tilde{a})^k \right.$$

$$\left. - (k-1) \cdot m(\tilde{a})^{k+1} + \overline{a} \cdot m(\tilde{a})^k - m(\tilde{a})^{k+1}\right]$$

$$= \left[\left(\underline{a} \cdot m(\tilde{a})^k\right)(k+1) - m(\tilde{a})^{k+1}(k-1+1), (\overline{a} \cdot m(\tilde{a})^k)(k+1) \right.$$

$$\left. - m(\tilde{a})^{k+1}(k-1+1)\right]$$

$$= \left[(k+1)\left(\underline{a} \cdot m(\tilde{a})^k\right) - ((k+1)-1) m(\tilde{a})^{k+1}, (k+1)(\overline{a} \cdot m(\tilde{a})^k) \right.$$

$$\left. - ((k+1)-1) m(\tilde{a})^{k+1}\right]$$

$$= \tilde{a}^{k+1}.$$

Therefore, $P(k+1)$ is true and $(k+1) \in S$. Based on the principle of mathematical induction, this formula holds for all $n \in \mathbb{N}$.

Theorem 3.2. Suppose there is an interval number $\tilde{x}_I$ with a and b as positive integers, then the following applies

$$(\tilde{x}_I)^{\frac{a}{b}} = \left[\frac{a\underline{x} + (b-a)m(\tilde{x}_I)}{b \cdot m(\tilde{x}_I)^{1-\frac{a}{b}}}, \frac{a\overline{x} + (b-a)m(\tilde{x}_I)}{b \cdot m(\tilde{x}_I)^{1-\frac{a}{b}}}\right]. \tag{9}$$

Proof. We use the induction method. Let S be a set where $P(n)$ is true for $n \in \mathbb{Q}^+$. If $n = 1$, then

$$\begin{aligned} \tilde{x} &= \left[\frac{1 \cdot \underline{x} + (1-1)m(\tilde{x})}{1 \cdot m(\tilde{x})^{1-1}}, \frac{1 \cdot \overline{x} + (1-1)m(\tilde{x})}{1 \cdot m(\tilde{x})^{1-1}}\right] \\ \tilde{x} &= [\underline{x}, \overline{x}]. \end{aligned} \tag{10}$$

Therefore, $P(1)$ is true and $1 \in S$. Next, we assume that $P(k)$ is true and wish to infer from this assumption that $P(k+1)$ and $P(k-1)$ are also true. If $k \in S$ where $k = \frac{c}{d}$, then

$$\begin{aligned} \tilde{x}^{\frac{c}{d}} &= \left[\frac{c\underline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}, \frac{c\overline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right], \\ \tilde{x} &= [\underline{x}, \overline{x}]. \end{aligned} \tag{11}$$

If we multiply $\tilde{x}$ to both sides, then

$$\tilde{x}^{\frac{c}{d}} \otimes \tilde{x} = \left[\frac{c\underline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}, \frac{c\overline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right] \otimes \tilde{x}$$

$$= \left[\left(\frac{c\underline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)m(\tilde{x}) + \underline{x} \cdot m\left(\tilde{x}^{\frac{c}{d}}\right) - m\left(\tilde{x}^{\frac{c}{d}}\right) \cdot m(\tilde{x}),\right.$$

$$\left.\left(\frac{c\overline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)m(\tilde{x}) + \overline{x} \cdot m\left(\tilde{x}^{\frac{c}{d}}\right) - m\left(\tilde{x}^{\frac{c}{d}}\right) \cdot m(\tilde{x})\right]$$

$$= \left[\left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)c\underline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \underline{x}d \cdot m(\tilde{x})^{\frac{c}{d}}m(\tilde{x})^{1-\frac{c}{d}} - d\right.$$

$$\cdot m(\tilde{x})^{\frac{c}{d}+1}m(\tilde{x})^{1-\frac{c}{d}}, \left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)c\overline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \overline{x}d$$

$$\left.\cdot m(\tilde{x})^{\frac{c}{d}}m(\tilde{x})^{1-\frac{c}{d}} - d \cdot m(\tilde{x})^{\frac{c}{d}+1}m(\tilde{x})^{1-\frac{c}{d}}\right] \tag{12}$$

$$= \left[\left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)c\underline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \underline{x}d \cdot m(\tilde{x}) - d \cdot m(\tilde{x})^2,\right.$$

$$\left.\left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)c\overline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \overline{x}d \cdot m(\tilde{x}) - d \cdot m(\tilde{x})^2\right]$$

$$= \left[\left(\frac{(c+d) \cdot \underline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{-\frac{c}{d}}}\right), \left(\frac{(c+d) \cdot \overline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{-\frac{c}{d}}}\right)\right]$$

$$= \left[\left(\frac{(c+d) \cdot \underline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\left(\frac{c+d}{d}\right)}}\right), \left(\frac{(c+d) \cdot \overline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\left(\frac{c+d}{d}\right)}}\right)\right]$$

$$= \tilde{x}^{\frac{c+d}{d}}$$

$$= \tilde{x}^{\frac{c}{d}+1}.$$

Therefore, $P(k+1)$ is true and $(k+1) \in S$. Based on the principle of mathematical induction, this formula holds for all $n \in \mathbb{N}$.

Theorem 3.3. Suppose there is an interval number $\tilde{x}_I$ with a, b, c , and d being a positive integer, then the following hold :

1. $(\tilde{a}_I)^p \otimes (\tilde{a}_I)^q = (\tilde{a}_I)^{p+q}$, (13)

2. $(\tilde{a}_I)^{\frac{p}{q}} \otimes (\tilde{a}_I)^{\frac{r}{s}} = (\tilde{a}_I)^{\frac{p}{q}+\frac{r}{s}}$,

3. $((\tilde{a}_I)^p)^{\frac{1}{p}} = \tilde{a}_I$,

4. $\left((\tilde{a}_I)^{\frac{p}{q}}\right)^{\frac{q}{q}} = \tilde{a}_I$,

5. $\left((\tilde{a}_I)^{\frac{p}{q}}\right)^{\frac{r}{r}} = \left((\tilde{a}_I)^{\frac{p}{q}}\right)^{\frac{r}{s}}$.

Proof.

1. $$\begin{aligned}(\tilde{a}_I)^p \otimes (\tilde{a}_I)^q &= \big[(p\underline{a}).m((\tilde{a}_I)^{p-1}) - (p-1).m((\tilde{a}_I)^p), (p\overline{a}),\\ &\quad m((\tilde{a}_I)^{p-1}) - (p-1).m((\tilde{a}_I)^p)\big] \otimes\\ &\quad \big[(q\underline{a}).m((\tilde{a}_I)^{q-1}) - (q-1).m((\tilde{a}_I)^q),\\ &\quad (q\overline{a}).m((\tilde{a}_I)^{q-1}) - (q-1).m((\tilde{a}_I)^q)\big]\\ &= \Big[\Big((p\underline{a}) \cdot m((\tilde{a}_I)^{p-1}) - (p-1) \cdot m((\tilde{a}_I)^p)\Big) m((\tilde{a}_I)^q) +\\ &\quad \Big((q\underline{a}).m((\tilde{a}_I)^{q-1}) - (q-1).m((\tilde{a}_I)^q)\Big) m((\tilde{a}_I)^p) -\\ &\quad m((\tilde{a}_I)^p).m((\tilde{a}_I)^q),\\ &\quad \big((p\overline{a}) \cdot m((\tilde{a}_I)^{p-1}) - (p-1) \cdot m((\tilde{a}_I)^p)\big) m((\tilde{a}_I)^q) +\\ &\quad \Big((q\underline{a}).m((\tilde{a}_I)^{q-1}) - (q-1) \cdot m((\tilde{a}_I)^q)\Big) m((\tilde{a}_I)^p) -\\ &\quad m((\tilde{a}_I)^p) m((\tilde{a}_I)^q)\Big]\\ &= \big[(p\underline{a}) \cdot m((\tilde{a}_I)^{p-1}) \cdot m((\tilde{a}_I)^q) - (p-1) \cdot m((\tilde{a}_I)^p) \cdot m((\tilde{a}_I)^q) +\\ &\quad (q\underline{a}) \cdot m((\tilde{a}_I)^{q-1}) \cdot m((\tilde{a}_I)^p) - (q-1) \cdot m((\tilde{a}_I)^q) \cdot m((\tilde{a}_I)^p) -\\ &\quad m((\tilde{a}_I)^p) m((\tilde{a}_I)^q), (p\overline{a}).m((\tilde{a}_I)^{p-1}).m((\tilde{a}_I)^q) -\\ &\quad (p-1).m((\tilde{a}_I)^p) \cdot m((\tilde{a}_I)^q) + (q\overline{a}) \cdot m((\tilde{a}_I)^{q-1}) \cdot m((\tilde{a}_I)^p) -\\ &\quad (q-1) \cdot m((\tilde{a}_I)^q) \cdot m((\tilde{a}_I)^p) - m((\tilde{a}_I)^p) m((\tilde{a}_I)^q)\big]\end{aligned} \tag{14}$$

$$= \Big[\big(p\underline{a}\big)\cdot m\big((\tilde{a}_I)\big)^{p-1+q} - (p-1)\cdot m\big((\tilde{a}_I)\big)^{p+q} + \big(q\underline{a}\big)\cdot m\big((\tilde{a}_I)\big)^{p-1+q}$$

$$-(q-1)\cdot m\big((\tilde{a}_I)\big)^{p+q} - m\big((\tilde{a}_I)\big)^{p+q},$$

$$(p\overline{a})\cdot m\big((\tilde{a}_I)\big)^{p-1+q} - (p-1)\cdot m\big((\tilde{a}_I)\big)^{p+q} +$$

$$(q\overline{a})\cdot m\big((\tilde{a}_I)\big)^{p-1+q} - (q-1)\cdot m\big((\tilde{a}_I)\big)^{p+q} - m\big((\tilde{a}_I)\big)^{p+q}\Big]$$

$$= \Big[(p+q)\cdot \underline{a}m\big((\tilde{a}_I)\big)^{p-1+q} - (p+q-1)\cdot m\big((\tilde{a}_I)\big)^{p+q},$$

$$(p+q)\cdot \overline{a}m\big((\tilde{a}_I)\big)^{p-1+q} - (p+q-1)\cdot m\big((\tilde{a}_I)\big)^{p+q}\Big]$$

$$= \tilde{a}^{p+q}.$$

2. $$(\tilde{a}_I)^{\frac{p}{q}} \otimes (\tilde{a}_I)^{\frac{r}{s}} = \left[\frac{p.\underline{a} + (q-p)m(\tilde{a}_I)}{q\cdot m\big((\tilde{a}_I)\big)^{1-\frac{p}{q}}}, \frac{p.\overline{a} + (q-p)m(\tilde{a}_I)}{q\cdot m\big((\tilde{a}_I)\big)^{1-\frac{p}{q}}}\right]$$

$$\otimes\left[\frac{r.\underline{a} + (s-d)m(\tilde{a}_I)}{s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{r}{s}}}, \frac{r.\overline{a} + (r-s)m(\tilde{a}_I)}{s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{r}{s}}}\right]$$

$$= \Bigg[\left(\frac{p.\underline{a} + (q-p)m(\tilde{a}_I)}{q\cdot m\big((\tilde{a}_I)\big)^{1-\frac{p}{q}}}\right) m\left((\tilde{a}_I)^{\frac{r}{s}}\right) +$$

$$\left(\frac{r.\underline{a} + (s-r)m(\tilde{a}_I)}{s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{r}{s}}}\right) m\left((\tilde{a}_I)^{\frac{p}{q}}\right) - m\left((\tilde{a}_I)^{\frac{p}{q}}\right) m\left((\tilde{a}_I)^{\frac{r}{s}}\right), \tag{15}$$

$$\left(\frac{p.\overline{a} + (p-q)m(\tilde{a}_I)}{b\cdot m\big((\tilde{a}_I)\big)^{1-\frac{p}{q}}}\right) m\left((\tilde{a}_I)^{\frac{r}{s}}\right) +$$

$$\left(\frac{r.\overline{a} + (s-r)m(\tilde{a}_I)}{s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{r}{s}}}\right). m\left((\tilde{a}_I)^{\frac{p}{q}}\right) - m\left((\tilde{a}_I)^{\frac{p}{q}}\right) m\left((\tilde{a}_I)^{\frac{r}{s}}\right)\Bigg]$$

$$\Bigg[\frac{p.\underline{a}\cdot m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{q\cdot m(\tilde{a}_I)} + \frac{(q-p)m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{q} + \frac{r.\underline{a}\cdot m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{s\cdot m(\tilde{a}_I)} +$$

$$\frac{(s-r)m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{s}-m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}},$$

$$\frac{p.\overline{a}\cdot m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{q\cdot m(\tilde{a}_I)}+$$

$$\frac{(q-p)m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{q}+\frac{r.\overline{a}\cdot m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{s\cdot m(\tilde{a}_I)}+$$

$$\left.\frac{(s-r)m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}}{s}-m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}\right]$$

$$=\left[m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}\left(\frac{p.\underline{a}}{q\cdot m(\tilde{a}_I)}+\frac{(q-p)}{q}+\frac{r.\underline{a}}{s\cdot m(\tilde{a}_I)}+\frac{(s-r)}{s}-1\right)\right.$$

$$m\big((\tilde{a}_I)\big)^{\frac{p}{q}+\frac{r}{s}}\left(\frac{p.\overline{a}}{q\cdot m(\tilde{a}_I)}+\frac{(q-p)}{q}+\frac{s.\overline{a}}{s\cdot m(\tilde{a}_I)}+\frac{(s-r)}{s}-1\right)$$

$$\left.+\frac{(s-r)}{s}-1\right]$$

$$=\left[m(\tilde{a}_I)^{\frac{ps+qr}{qs}}\left(\frac{\underline{a}}{m(\tilde{a}_I)}\left(\frac{p}{q}+\frac{r}{s}\right)+\frac{qs-ps-qr}{qs}\right),\right.$$

$$\left.m(\tilde{a}_I)^{\frac{ps+qr}{qs}}\left(\frac{\overline{a}}{m(\tilde{a}_I)}\left(\frac{p}{q}+\frac{r}{s}\right)+\frac{qs-ps-qr}{qs}\right)\right]$$

$$=\left[m(\tilde{a}_I)^{\frac{ps+qr}{qs}}\left(\frac{\underline{a}}{m(\tilde{a}_I)}\left(\frac{ps+qr}{qs}\right)+\frac{(qs-ps-qr)m(\tilde{a}_I)}{bd\cdot m(\tilde{a}_I)}\right),\right.$$

$$\left.m\big((\tilde{a}_I)\big)^{\frac{ps+qr}{qs}}\left(\frac{\overline{a}}{m(\tilde{a}_I)}\left(\frac{ps+qr}{qs}\right)+\frac{(qs-ps-qr)m(\tilde{a}_I)}{qs\cdot m(\tilde{a}_I)}\right)\right]$$

$$=\left[\frac{\underline{a}(ps+qr)+(qs-ps-qr)m(\tilde{x})}{qs\cdot m\big((\tilde{a}_I)\big)^{1-\frac{ps+qr}{qs}}},\frac{\overline{a}(ps+qr)+(qs-ps-qr)m(\tilde{a}_I)}{qs\cdot m\big((\tilde{a}_I)\big)^{1-\frac{ps+qr}{qs}}}\right]$$

$$=(\tilde{a}_I)^{\frac{ps+qr}{qs}}$$

$$=(\tilde{a}_I)^{\frac{p}{q}+\frac{r}{s}}.$$

3. $$((\tilde{a}_I)^n)^{\frac{1}{n}}=\Big((n\underline{a}).m((\tilde{a}_I)^{n-1})-(n-1).m((\tilde{a}_I)^n),(n\overline{a}).m((\tilde{a}_I)^{n-1})-(n-1).m((\tilde{a}_I)^n)\Big)^{\frac{1}{n}} \tag{16}$$

$$= \left[\frac{1}{n \cdot m((\tilde{a}_I)^n)^{1-\frac{1}{n}}} \left(\left(n\underline{a} \right) \cdot m((\tilde{a}_I)^{n-1}) - (n-1) \cdot m((\tilde{a}_I)^n) + \right. \right.$$

$$\left. (n-1) \cdot m((\tilde{a}_I)^n) \right), \frac{1}{n \cdot m((\tilde{a}_I)^n)^{1-\frac{1}{n}}} \left((n\overline{a}) \cdot m((\tilde{a}_I)^{n-1}) - (n-1) \cdot \right.$$

$$\left. \left. m((\tilde{a}_I)^n) + ((n-1) \cdot m((\tilde{a}_I)^n)) \right) \right]$$

$$= \left[\frac{n\underline{a} \cdot m((\tilde{a}_I))^{n-1}}{n \cdot m((\tilde{a}_I))^{n\left(1-\frac{1}{n}\right)}}, \frac{n\overline{a} \cdot m((\tilde{a}_I))^{n-1}}{n \cdot m((\tilde{a}_I))^{n\left(1-\frac{1}{n}\right)}} \right]$$

$$= [\underline{a}, \overline{a}].$$

4. $\left((\tilde{a}_I)^{\frac{p}{q}} \right)^{\frac{q}{p}} = \left[\frac{p.\underline{a} + (q-p).m(\tilde{a}_I)}{q \cdot m((\tilde{a}_I))^{1-\frac{p}{q}}}, \frac{p.\overline{a} + (q-p)m(\tilde{a}_I)}{q \cdot m((\tilde{a}_I))^{1-\frac{p}{q}}} \right]^{\frac{q}{p}}$

$$\left[\frac{1}{p \cdot m((\tilde{a}_I))^{\frac{p}{q}\left(1-\frac{q}{p}\right)}} \left(\frac{q\left(p.\underline{a} + (q-p)m(\tilde{a}_I)\right)}{q \cdot m((\tilde{a}_I))^{1-\frac{p}{q}}} + (p-q)m((\tilde{a}_I))^{\frac{p}{q}} \right), \right.$$

$$\left. \frac{1}{p \cdot m((\tilde{a}_I))^{\frac{p}{q}\left(1-\frac{q}{p}\right)}} \left(\frac{q.\left(p\overline{a} + (q-p)m(\tilde{a}_I)\right)}{q \cdot m((\tilde{a}_I))^{1-\frac{p}{q}}} + (p-q)m((\tilde{a}_I))^{\frac{p}{q}} \right) \right]$$

$$= \left[\frac{1}{p \cdot m((\tilde{a}_I))^{\frac{p}{q}-1}} \left(\frac{p.\underline{a} + (q-p)m(\tilde{a}_I)}{m((\tilde{a}_I))^{1-\frac{p}{q}}} - \frac{(q-p)m((\tilde{a}_I))^{\frac{p}{q}} \cdot m((\tilde{a}_I))^{1-\frac{p}{q}}}{m((\tilde{a}_I))^{1-\frac{p}{q}}} \right), \right. \quad (17)$$

$$\left. \frac{1}{p \cdot m((\tilde{a}_I))^{\frac{p}{q}-1}} \left(\frac{p.\overline{a} + (q-p)m(\tilde{a}_I)}{m((\tilde{a}_I))^{1-\frac{p}{q}}} - \frac{(q-p)m((\tilde{a}_I))^{\frac{p}{q}} \cdot m((\tilde{a}_I))^{1-\frac{p}{q}}}{m((\tilde{a}_I))^{1-\frac{p}{q}}} \right) \right]$$

$$= \left[\left(\frac{p\underline{a} + (q-p)m(\tilde{a}_I)}{p.m((\tilde{a}_I))^{\frac{p}{q}-1+1-\frac{p}{q}}} - \frac{(c-b)m(\tilde{a})^{\frac{b}{c}+1-\frac{b}{c}}}{p.m((\tilde{a}_I))^{\frac{p}{q}-1+1-\frac{p}{q}}} \right), \right.$$

$$\left(\frac{p.\overline{a}+(q-p)m(\tilde{a}_I)}{p.m((\tilde{a}_I))^{\frac{p}{q}-1+1-\frac{p}{q}}}-\frac{(c-b)m(\tilde{a})^{\frac{b}{c}+1-\frac{b}{c}}}{p.m((\tilde{a}_I))^{\frac{p}{q}-1+1-\frac{p}{q}}}\right)\Bigg]$$

$$=[\underline{a},\overline{a}].$$

5. $$\left((\tilde{a}_I)^{\frac{r}{s}}\right)^{\frac{t}{u}}=\left[\frac{r.\underline{a}+(s-r)m(\tilde{a}_I)}{s\cdot m((\tilde{a}_I))^{1-\frac{r}{s}}},\frac{r.\overline{a}+(s-r)m(\tilde{a}_I)}{s\cdot m((\tilde{a}_I))^{1-\frac{r}{s}}}\right]^{\frac{t}{u}}$$

$$=\Bigg[\frac{1}{u\cdot m((\tilde{a}_I))^{\frac{r}{s}\left(1-\frac{t}{u}\right)}}\left(\frac{t.r.\underline{a}+t(s-r).m(\tilde{a}_I)}{s\cdot m((\tilde{a}_I))^{1-\frac{r}{s}}}+\right.$$

$$\frac{(u-t)m((\tilde{a}_I))^{\frac{r}{s}}\left(s\cdot m((\tilde{a}_I))^{1-\frac{r}{s}}\right)}{s\cdot m((\tilde{a}_I))^{1-\frac{r}{s}}},$$

$$\frac{1}{u\cdot m((\tilde{a}_I))^{\frac{r}{s}\left(1-\frac{t}{u}\right)}}\left(\frac{t.r.\overline{a}+t(s-r)m(\tilde{a}_I)}{s\cdot m((\tilde{a}_I))^{1-\frac{r}{s}}}+\right.$$

$$\left.\frac{(u-t)m\big((\tilde{a}_I)\big)^{\frac{r}{s}}\left(s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{r}{s}}\right)}{s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{r}{s}}}\right)\Bigg] \tag{18}$$

$$=\left[\left(\frac{t.r.\underline{a}+t(s-r)m(\tilde{a}_I)}{u\cdot s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{tr}{us}}}+\frac{s(u-t)m(\tilde{a}_I)}{u\cdot s\cdot m\big((\tilde{a}_I)\big)^{1-\frac{tr}{us}}}\right),\right.$$

$$\left.\left(\frac{t.r.\overline{a}+t(s-r)m(\tilde{a}_I)}{u\cdot s\cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}}+\frac{s(u-t)m(\tilde{a}_I)}{u\cdot s\cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}}\right)\right]$$

$$=\left[\left(\frac{t.r.\underline{a}+m(\tilde{a})(t.s-t.r+u.s-t.s)}{u\cdot s\cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}}\right),\left(\frac{t.r.\overline{a}+m(\tilde{a}_I)(t.s-t.r+u.s-t.s)}{u\cdot s\cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}}\right)\right]$$

$$=\left[\left(\frac{t.r.\underline{a}+m(\tilde{a}_I)(r.u-t.r+s.u-r.u)}{u\cdot s\cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}}\right),\left(\frac{t.r.\overline{a}+m(\tilde{a}_I)(r.u-t.r+s.u-r.u)}{u\cdot s\cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}}\right)\right]$$

$$= \left[\left(\frac{t.r.\underline{a} + r(u-t)m(\tilde{a}_I)}{u \cdot s \cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}} + \frac{u(s-r)m(\tilde{a}_I)}{u \cdot s \cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}} \right), \right.$$

$$\left. \left(\frac{t.r.\overline{a} + r(u-t)m(\tilde{a}_I)}{u \cdot s \cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}} + \frac{u(s-r)m(\tilde{a}_I)}{u \cdot s \cdot m((\tilde{a}_I))^{1-\frac{tr}{us}}} \right) \right]$$

$$= \left[\frac{1}{s \cdot m((\tilde{a}_I))^{\frac{t}{u}\left(1-\frac{r}{s}\right)}} \left(\frac{t.r.\underline{a} + r(u-t)m(\tilde{a}_I)}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}} + \frac{(s-r)m((\tilde{a}_I))^{\frac{t}{u}}\left(u \cdot m((\tilde{a}_I))^{1-\frac{t}{u}}\right)}{u \cdot m((\tilde{a}_I))^{1-\frac{t}{u}}} \right), \right.$$

$$\frac{1}{s \cdot m((\tilde{a}_I))^{\frac{t}{u}\left(1-\frac{r}{s}\right)}} \left(\frac{t.r.\overline{a} + r(u-t)m(\tilde{a}_I)}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}} \right.$$

$$\left. \left. + \frac{(s-r)m((\tilde{a}_I))^{\frac{t}{u}}\left(u \cdot m((\tilde{a}_I))^{1-\frac{t}{u}}\right)}{u \cdot m((\tilde{a}_I))^{1-\frac{t}{u}}} \right) \right]$$

$$= \left[\frac{t.\underline{a} + (u-t)m(\tilde{a}_I)}{u \cdot m((\tilde{a}_I))^{1-\frac{t}{u}}}, \frac{t.\overline{a} + (u-t)m((\tilde{a}_I))}{u \cdot m((\tilde{a}_I))^{1-\frac{t}{u}}} \right]^{\frac{r}{s}}$$

$$= \left((\tilde{a}_I)^{\frac{t}{u}} \right)^{\frac{r}{s}}.$$

4. Result and discussions

4.1. Midpoint

Referring to the process of proving various area sizes in [20,32,33] and the concept of the midpoint in [23,25], it is necessary to first prove properties of the midpoint of these intervals. Before that, the definition of the concept of convergence of a sequence of intervals is be presented as follows:

Definition 4.1. The sequence of intervals $(\tilde{a}_I)_n = [\underline{a}_n, \overline{a}_n]$ is said to converge to the interval number $\tilde{a}_I = [\underline{a}, \overline{a}]$ if for every $\tilde{\varepsilon}_I = [0, \varepsilon]$, $\varepsilon > 0$ there are $k(\varepsilon) \in$ **N,** such that for every $n \geq k(\varepsilon)$, the condition $\left|[\underline{a}_n, \overline{a}_n] \ominus [\underline{a}, \overline{a}]\right| \leq \tilde{\varepsilon}_I$.

If a sequence of interval numbers $(\tilde{a}_I)_n = [\underline{a}_n, \overline{a}_n]$ converges to the interval number $\tilde{a}_I = [\underline{a}, \overline{a}]$, denoted by $\lim_{n\to\infty} \{(\tilde{a}_I)_n\} = \tilde{a}_I$ or $\lim_{n\to\infty} [\underline{a}_n, \overline{a}_n] = [\underline{a}, \overline{a}]$, it can be shown that $\lim_{n\to\infty} \underline{a}_n = \underline{a}$ and $\lim_{n\to\infty} \overline{a}_n = \overline{a}$. Observe that $\lim_{n\to\infty} \underline{a}_n = \underline{a}$ and $\lim_{n\to\infty} \overline{a}_n = \overline{a}$ aligns exactly with the concept of limits in real numbers, with $|\tilde{a}_I| = \left|[\underline{a}, \overline{a}]\right| = max\left\{|\underline{a}|, |\overline{a}|\right\}$ and based on the Definition 4.1 above, it is clear that the limit of convergent interval number sequence is unique. Futhermore, based on Definition 4.1, the following theorem can also be proven:

Theorem 4.2. For any interval numbers $\tilde{a}_I = [\underline{a}, \overline{a}]$ and $\tilde{b}_I = [\underline{b}, \overline{b}]$ the following hold:

$$
\begin{aligned}
&\text{a. } m(\tilde{a}_I \otimes \tilde{b}_I) = m(\tilde{a}_I) \cdot m(\tilde{b}_I),\\
&\text{b. } m\left(\frac{1}{\tilde{a}_I}\right) = \frac{1}{m(\tilde{a}_I)},\\
&\text{c. } m((\tilde{a}_I)^n) = (m(\tilde{a}_I))^n,\\
&\text{d. } m\left((\tilde{a}_I)^{\frac{p}{q}}\right) = (m(\tilde{a}_I))^{\frac{p}{q}},\\
&\text{e. } \lim(m((\tilde{a}_i)_n)) = m(\tilde{a}_i),\\
&\text{f. } \lim\left(\frac{1}{m(\tilde{a}_I)_n}\right) = \frac{1}{m(\tilde{a}_I)}.
\end{aligned} \tag{19}
$$

Proof.

a.
$$
\begin{aligned}
m(\tilde{a}_I \otimes \tilde{b}_I) &= \frac{\underline{a} \cdot m(\tilde{b}_I) + \underline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I).m(\tilde{b}_I)}{2} + \\
&\quad \frac{\overline{a} \cdot m(\tilde{b}_I) + \overline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I).m(\tilde{b}_I)}{2}\\
&= \frac{m(\tilde{b}_I)(\underline{a} + \overline{a}) + m(\tilde{a}_I)(\underline{b} + \overline{b}) - 2m(\tilde{a}_I)m(\tilde{b}_I)}{2}\\
&= m(\tilde{b}_I).m(\tilde{a}_I) + m(\tilde{a}_I)m(\tilde{b}_I) - m(\tilde{a}_I).m(\tilde{b}_I)\\
&= m(\tilde{a}_I)m(\tilde{b}_I).
\end{aligned} \tag{20}
$$

b.
$$
\begin{aligned}
m\left(\frac{1}{\tilde{a}_I}\right) &= \frac{1}{2}\left(\frac{2 \cdot m(\tilde{a}_I) - \underline{a}}{m(\tilde{a}_I)^2} + \frac{2 \cdot m(\tilde{a}_I) - \overline{a}}{m(\tilde{a}_I)^2}\right)\\
&= \frac{2m(\tilde{a}_I)}{2m(\tilde{a}_I)^2} - \frac{\underline{a}}{2m(\tilde{a}_I)^2} + \frac{2m(\tilde{a}_I)}{2m(\tilde{a}_I)^2} - \frac{\overline{a}}{2m(\tilde{a}_I)^2}\\
&= \frac{2}{m(\tilde{a}_I)} - \frac{m(\tilde{a}_I)}{m(\tilde{a}_I)^2}\\
&= \frac{1}{m(\tilde{a}_I)}.
\end{aligned}
$$

c. $m((\tilde{a}_I)^n) = \frac{m(\tilde{a}_I \cdot \tilde{a}_I \cdots (\tilde{a}_I)}{n\, factor}$, based on (1) we have

$$= \underbrace{\frac{m(\tilde{a}_I) \cdot m(\tilde{a}_I) \cdots m(\tilde{a}_I)}{n\, factor}}$$

$$= \left(m(\tilde{a}_I)\right)^n.$$

d. $m\left((\tilde{a}_I)^{\frac{p}{q}}\right) = \frac{p.\underline{a} + (p-q)m(\tilde{a}_I) + (q-p)m(\tilde{a}_I)}{2q \cdot m(\tilde{a}_I)^{1-\frac{p}{q}}}$

$$= \frac{p\left(\underline{a} + \overline{a}\right) + 2(p-q)m\left((\tilde{a}_I)\right)}{2p \cdot m(\tilde{a}_I)^{1-\frac{p}{q}}}$$

$$= \frac{p.m(\tilde{a}_I)}{q \cdot \left(m(\tilde{a}_I)\right)^{1-\frac{p}{q}}} + \frac{2q.m(\tilde{a}_I)}{2q \cdot \left(m(\tilde{a}_I)\right)^{1-\frac{p}{q}}} - \frac{2p.m(\tilde{a}_I)}{2q \cdot \left(m(\tilde{a}_I)\right)^{1-\frac{p}{q}}}$$

$$= \frac{p}{q} m(\tilde{a}_I)^{\frac{p}{q}} + m(\tilde{a}_I)^{\frac{p}{q}} - \frac{p}{q} m(\tilde{a}_I)^{\frac{p}{q}}$$

$$= \left(m(\tilde{a}_I)\right)^{\frac{p}{q}}.$$

e. $\lim\left(m(\tilde{a}_n)\right) = \lim\left(\frac{\underline{a}_n + \overline{a}_n}{2}\right) = \lim\left(\frac{1}{2}\right)\lim\left(\underline{a}_n + \overline{a}_n\right) = \frac{\underline{a} + \overline{a}}{2} = m(\tilde{a}_I).$

f. $\lim\left(\frac{1}{m(\tilde{a}_n)}\right) = \lim\left(\frac{2}{\underline{a}_n + \overline{a}_n}\right) = \lim\left(\frac{1}{\underline{a}_n + \overline{a}_n}\right)\lim(2) = \frac{2}{\underline{a} + \overline{a}} = \frac{1}{m(\tilde{a}_I)}.$

4.2. Convergence of interval sequence

By using Definition 4.1 and Theorem 4.2, the convergence of the following sequence of intervals can be proven.

Theorem 4.3. Let $\lim\{(\tilde{a}_I)_n\}$ and $\lim\left\{(\tilde{b}_I)_n\right\}$ exist, then the following applies:

$$\lim\left\{(\tilde{a}_I)_n \ominus \left(\tilde{b}_I\right)_n\right\} = \lim\{(\tilde{a}_I)_n\} \ominus \lim\left\{\left(\tilde{b}_I\right)_n\right\},$$
$$\lim\left\{(\tilde{a}_I)_n \ominus \left(\tilde{b}_I\right)_n\right\} = \lim\{(\tilde{a}_I)_n\} \ominus \lim\left\{\left(\tilde{b}_I\right)_n\right\}. \quad (21)$$

Proof. b). Here, we provide the proof for part b), while part a) can be done in a similar way. Let $\lim\{(\tilde{a}_I)_n\} = \tilde{a}_I$ and $\lim\left\{(\tilde{b}_I)_n\right\} = \tilde{b}_I$, thus

$$\lim \underline{a}_n = \underline{a} \text{ and } \lim \overline{a}_n = \overline{a},$$
$$\lim \underline{b}_n = \underline{b} \text{ and } \lim \overline{b}_n = \overline{b}. \tag{22}$$

Then, note that

$$(\tilde{a}_I)_n \ominus (\tilde{b}_I)_n = \left[\underline{a_n \ominus b_n}, \overline{a_n \ominus b_n}\right] = [\underline{a}_n, \overline{a}_n] \ominus [\underline{b}_n, \overline{b}_n] = [\underline{a}_n - \overline{b}_n, \overline{a}_n - \underline{b}_n]$$
$$\tilde{a}_I \ominus \tilde{b}_I = [\underline{a \ominus b}, \overline{a \ominus b}] = [\underline{a}, \overline{a}] \ominus [\underline{b}, \overline{b}] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]. \tag{23}$$

Furthermore, because

$$\lim \left(\underline{a_n \ominus b_n}\right) = \lim(\underline{a}_n - \overline{b}_n) = \lim(\underline{a}_n) - \lim(\overline{b}_n) = \underline{a} - \overline{b} = \underline{a \ominus b},$$
$$\lim(\overline{a_n \ominus b_n}) = \lim(\overline{a}_n - \underline{b}_n) = \lim(\overline{a}_n) - \lim(\underline{b}_n) = \overline{a} - \underline{b} = \overline{a \ominus b}. \tag{24}$$

So, based on the theorem, it is obtained that

$$\lim((\tilde{a}_I)_n \ominus (\tilde{b}_I)_n) = [\underline{a \ominus b}, \overline{a \ominus b}] = \tilde{a} \ominus \tilde{b} = \lim((\tilde{a}_I)_n) \ominus \lim((\tilde{b}_I)_n). \tag{25}$$

For $\lim((\tilde{a}_I)_n \oplus (\tilde{b}_I)_n) = \lim((\tilde{a}_I)_n) \oplus \lim((\tilde{b}_I)_n)$ can be proven similarly.

Theorem 4.4. Let $\lim\{(\tilde{a}_I)_n\}$ and $\lim\{\tilde{b}_{I_n}\}$ exist, then the following hold

$$\lim\left\{(\tilde{a}_I)_n \otimes (\tilde{b}_I)_n\right\} = \lim\{(\tilde{a}_I)_n\} \otimes \lim\left\{(\tilde{b}_I)_n\right\}. \tag{26}$$

Proof. Let $\lim\{(\tilde{a}_I)_n\} = \tilde{a}_I$ and $\lim\{(\tilde{b}_I)_n\} = \tilde{b}_I$, then

$$\lim \underline{a}_n = \underline{a} \text{ and } \lim \overline{a}_n = \overline{a},$$
$$\lim \underline{b}_n = \underline{b} \text{ and } \lim \overline{b}_n = \overline{b}. \tag{27}$$

Then, note that

$$(\tilde{a}_I)_n \otimes (\tilde{b}_I)_n = \left[\underline{a}_n \cdot m(\tilde{b}_n) + \underline{b}_n \cdot m((\tilde{a}_I)_n) - m((\tilde{a}_I)_n) m\left((\tilde{b}_I)_n\right)\right.$$
$$\left.\overline{a}_n \cdot m\left((\tilde{b}_I)_n\right) + \overline{b}_n \cdot m((\tilde{a}_I)_n) - m((\tilde{a}_I)_n). m((\tilde{b}_I)_n)\right], \tag{28}$$

$$\tilde{a}_I \otimes \tilde{b}_I = \left[\underline{a} \cdot m(\tilde{b}_I) + \underline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I).m(\tilde{b}_I), \overline{a} \cdot m(\tilde{b}_I) + \overline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I).m(\tilde{b}_I)\right].$$

Furthermore,

$$\begin{aligned} \lim\left(\underline{a}_n \cdot m\left((\tilde{b}_I)_n\right) + \underline{b}_n \cdot m((\tilde{a}_I)_n) - m((\tilde{a}_I)_n)m\left((\tilde{b}_I)_n\right)\right) &= \underline{a} \cdot m(\tilde{b}_I) + \underline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I)m(\tilde{b}_I), \\ \lim\left(\overline{a}_n \cdot m\left((\tilde{b}_I)_n\right) + \overline{b}_n \cdot m((\tilde{a}_I)_n) - m((\tilde{a}_I)_n).m\left((\tilde{b}_I)_n\right)\right) &= \overline{a} \cdot m(\tilde{b}_I) + \overline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I)m(\tilde{b}_I). \end{aligned} \tag{29}$$

Thus, it is obtained that

$$\lim\left((\tilde{a}_I)_n \otimes (\tilde{b}_I)_n\right) = \left[\underline{a} \cdot m(\tilde{b}_I) + \underline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I).m(\tilde{b}_I), \overline{a} \cdot m(\tilde{b}_I) + \overline{b} \cdot m(\tilde{a}_I) - m(\tilde{a}_I)m(\tilde{b}_I)\right] = \tilde{a}_I \otimes \tilde{b}_I = \lim\,(\tilde{a}_n) \otimes \lim\left((\tilde{b}_I)_n\right). \tag{30}$$

Interval number sequences $(\tilde{a}_I)_n = \left[\underline{a}_n, \overline{a}_n\right]$ are called bounded if there is a real number $M \in \mathbb{R}$ such that $|(\tilde{a}_I)_n| = \left|\left[\underline{a}_n, \overline{a}_n\right]\right| = maks\left\{\left|\underline{a}_n,\right|, |\overline{a}_n|\right\} \leq M,$ for all $n \in \mathbb{N}$. Thus, we have the same property as in the real number sequence, namely every convergent interval number sequence is bounded. Apart from that, it will also be straightforward to show that if $\lim(\tilde{a}_I)_n = \tilde{a}_I$, then $\lim|(\tilde{a}_I)_n| = |\tilde{a}_I|$ and if $(\tilde{a}_I)_n \geq 0$ for all n, then $\lim\sqrt{(\tilde{a}_I)_n} = \sqrt{\tilde{a}_I}$. This means that if $(\tilde{a}_I)_n$ to $\tilde{a}_I$, then the sequence $|(\tilde{a}_I)_n|$ converges to $|\tilde{a}_I|$ and the sequence $\sqrt{(\tilde{a}_I)_n}$ converges to $\sqrt{\tilde{a}_I}$. We also have that if $(\tilde{a}_I)_n \geq 0$ for all n and there is $\tilde{l}_I = \left[\underline{l}_n, \overline{l}_n\right]$ with $\left|\tilde{l}_I\right| \leq 1$, such that $\lim\left|\frac{(\tilde{a}_I)_{n+1}}{(\tilde{a}_I)_n}\right| = \tilde{l}_I$, then $(\tilde{a}_I)_n$ converges and $\lim\,(\tilde{a}_I)_n = \tilde{0}_I = [0,0]$.

Definition 4.6. An interval number sequence $\{(\tilde{a}_I)_n\}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$, such that for all natural numbers $n, m \geq N(\varepsilon)$, the terms $(\tilde{a}_I)_n, (\tilde{a}_I)_m$ satisfy $|(\tilde{a}_I)_n - (\tilde{a}_I)_m| < \varepsilon$.

It should be noted that $|(\tilde{a}_I)_n - (\tilde{a}_I)_m| = \left|\left[\underline{a}_n - \overline{a}_m, \overline{a}_n - \underline{a}_m\right]\right| = maks\,\langle\left|\underline{a}_n - \overline{a}_m\right|, \left|\overline{a}_n - \underline{a}_m\right|\rangle$. Thus, based on Definition 4.6, this will imply that if $\{(\tilde{a}_I)_n\}$ is a convergent sequence of real numbers, then $\{(\tilde{a}_I)_n\}$ is a Cauchy sequence of interval number, so that a Cauchy sequence of interval number is bounded. The m-tail sequence for the interval number sequence is defined by $\{(\tilde{a}_I)_{m+n};\ n \in \mathbb{N}\} = \{(\tilde{a}_I)_{m+1}, (\tilde{a}_I)_{m+2}, (\tilde{a}_I)_{m+3}, \cdots\}$, so the m-tail $\{(\tilde{a}_I)_{m+n}; n \in \mathbb{N}\}$ of $\{(\tilde{a}_I)_n\}$ converges if and only if $\{(\tilde{a}_I)_n\}$ converges. Moreover, it will also be true that $\lim\ \{(\tilde{a}_I)_{m+n}; n \in \mathbb{N}\} = \lim\{(\tilde{a}_I)_n\}$.

Theorem 4.7. Let $\lim\{(\tilde{a}_I)_n\} = \tilde{a}_I$, then

$$\lim\left(\frac{1}{(\tilde{a}_I)_n}\right) = \frac{1}{\tilde{a}_I}. \tag{31}$$

Proof.

$$\lim\left(\frac{1}{(\tilde{a}_I)_n}\right) = \lim\left(\left[\frac{2\cdot m((\tilde{a}_I)_n) - \underline{a}_n}{m((\tilde{a}_I)_n)^2}, \frac{2\cdot m((\tilde{a}_I)_n) - \overline{a}_n}{m((\tilde{a}_I)_n)^2}\right]\right). \tag{32}$$

Furthermore,

$$\lim\left(\frac{2\cdot m((\tilde{a}_I)_n) - \underline{a}_n}{m((\tilde{a}_I)_n)^2}\right) = \lim\left(\frac{1}{m((\tilde{a}_I)_n)^2}\right)\lim\left(2\cdot m((\tilde{a}_I)_n) - \underline{a}_n\right)$$

$$= \left(\frac{1}{m\left((\tilde{a}_I)\right)^2}\right)\left(2\cdot m\left((\tilde{a}_I)\right) - \underline{a}\right).$$

$$\lim\left(\frac{2\cdot m((\tilde{a}_I)_n) - \overline{a}_n}{m((\tilde{a}_I)_n)^2}\right) = \lim\left(\frac{1}{m((\tilde{a}_I)_n)^2}\right)\lim(2\cdot m((\tilde{a}_I)_n) - \overline{a}_n)$$

$$= \left(\frac{1}{m\left((\tilde{a}_I)\right)^2}\right)(2\cdot m(\tilde{a}_I) - \overline{a}). \tag{33}$$

Thus, it is obtained that

$$\lim\left(\frac{1}{(\tilde{a}_I)_n}\right) = \lim\left(\left[\frac{2\cdot m((\tilde{a}_I)_n) - \underline{a}_n}{m((\tilde{a}_I)_n)^2}, \frac{2\cdot m((\tilde{a}_I)_n) - \overline{a}_n}{m((\tilde{a}_I)_n)^2}\right]\right) \tag{34}$$

$$= \left[\frac{2\cdot m(\tilde{a}_I) - \underline{a}}{m(\tilde{a}_I)^2}, \frac{2\cdot m(\tilde{a}_I) - \overline{a}}{m(\tilde{a}_I)^2}\right] = \frac{1}{\tilde{a}_I}.$$

Theorem 4.8. Given $\lim\{(\tilde{a}_I)_n\}$ and $\lim\{(\tilde{b}_I)_n\}$, then

$$\lim\left(\frac{(\tilde{a}_I)_n}{(\tilde{b}_I)_n}\right) = \frac{\lim((\tilde{a}_I)_n)}{\lim((\tilde{b}_I)_n)}. \tag{35}$$

Proof. It is clear from the following relation that

$$\lim\left(\frac{(\tilde{a}_I)_n}{(\tilde{b}_I)_n}\right) = \lim\left((\tilde{a}_I)_n \otimes \frac{1}{(\tilde{b}_I)_n}\right) = \lim((\tilde{a}_I)_n) \otimes \lim\left(\frac{1}{(\tilde{b}_I)_n}\right) = \tilde{a}_I \otimes \frac{1}{\tilde{b}_I} = \frac{\lim((\tilde{a}_I)_n)}{\lim((\tilde{b}_I)_n)}. \tag{36}$$

Theorem 4.9. If $\lim\{(\tilde{a}_I)_n\}$ exist with $k \in \mathbb{Q}$ and $k \neq 0$, then we have

$$\lim(((\tilde{a}_I)_n)^k) = (\lim(\tilde{a}_I)_n)^k. \tag{37}$$

Proof.

Case 1. Let k be a positive integer. Suppose $\lim(\tilde{a}_I)_n = \tilde{a}_I$, then using Theorem 4.2, we have

$$\lim\left({(\tilde{a}_I)_n}^k\right) = \lim\left(\left[\left(k\underline{a}_n\right)\cdot m(((\tilde{a}_I)_n)^{k-1}) - (k-1)\cdot m(((\tilde{a}_I)_n)^k), (k\overline{a}_n)\right.\right.$$

$$\left.\left.\cdot m(((\tilde{a}_I)_n)^{k-1}) - (k-1)\cdot m(((\tilde{a}_I)_n)^k)\right]\right)$$

$$= \left[\lim\left(\left(k\underline{a}_n\right)\cdot m((\tilde{a}_I)_n)^{k-1} - (k-1)\cdot m((\tilde{a}_I)_n)^k\right), \lim\left((k\overline{a}_n)\right.\right.$$

$$\left.\left.\cdot m((\tilde{a}_I)_n)^{k-1} - (k-1)\cdot m((\tilde{a}_I)_n)^k\right)\right]$$

$$= \left[\lim\left(\left(k\underline{a}_n\right)\cdot m((\tilde{a}_I)_n)^{k-1}\right)\right.$$

$$- \lim\left((k-1)\cdot m((\tilde{a}_I)_n)^k\right), \lim\left((k\overline{a}_n)\cdot m(\tilde{a}_n)^{k-1}\right) \tag{38}$$

$$\left.- \lim\left((k-1)\cdot m((\tilde{a}_I)_n)^k\right)\right]$$

$$= \left[\left(k\underline{a}\right)\cdot m(\tilde{a}_I)^{k-1} - (k-1)\cdot m(\tilde{a}_I)^k, (k\overline{a})\cdot m(\tilde{a}_I)^{k-1} - (k-1)\right.$$

$$\left.\cdot m(\tilde{a}_I)^k\right]$$

$$= (\tilde{a}_I)^k$$

$$= (\lim\{(\tilde{a}_I)_n\})^k.$$

Case 2. Let k be a negative integer, suppose $k = -m$. Using Theorem 4.2, we have

$$\begin{aligned}\lim(((\tilde{a}_I)_n)^k) &= \lim(((\tilde{a}_I)_n)^{-m}) \\ &= \lim\left(\frac{1}{((\tilde{a}_I)_n)^m}\right) \\ &= \frac{1}{\lim(((\tilde{a}_I)_n)^m)} \\ &= \frac{1}{(\lim(\tilde{a}_I)_n)^m} \\ &= (\lim((\tilde{a}_I)_n))^{-m} \\ &= (\lim(\tilde{a}_I)_n)^k.\end{aligned} \tag{39}$$

Case 3. Let k be a negative integer, suppose $k = \frac{x}{y}$, we have

$$\lim\left(((\tilde{a}_I)_n)^{\frac{x}{y}}\right) = \lim\left(\left[\frac{x\underline{a}_n + (y-x)m((\tilde{a}_I)_n)}{y\cdot m((\tilde{a}_I)_n)^{1-\frac{x}{y}}}, \frac{x\overline{a}_n + (y-x)m((\tilde{a}_I)_n)}{y\cdot m((\tilde{a}_I)_n)^{1-\frac{x}{y}}}\right]\right). \tag{40}$$

Let $\lim(\tilde{a}_I)_n = \tilde{a}_I$, then

$$\lim\left(\frac{x\underline{a}_n+(y-x)m((\tilde{a}_I)_n)}{y\cdot(m(\tilde{a}_I)_n)^{1-\frac{x}{y}}}\right)=\frac{\lim\left(x\underline{a}_n+(y-x)m((\tilde{a}_I)_n)\right)}{\lim\left(y\cdot(m(\tilde{a}_I)_n)^{1-\frac{x}{y}}\right)}$$

$$=\frac{x\underline{a}+(y-x)m((\tilde{a}_I))}{y\cdot\left(m(\tilde{a}_I)\right)^{1-\frac{x}{y}}}.$$

$$\lim\left(\frac{x\overline{a}_n+(y-x)m((\tilde{a}_I)_n)}{y\cdot(m(\tilde{a}_I)_n)^{1-\frac{x}{y}}}\right)=\frac{\lim\left(x\overline{a}_n+(y-x)m(\tilde{a}_n)\right)}{\lim\left(y\cdot(m(\tilde{a}_I)_n)^{1-\frac{x}{y}}\right)} \tag{41}$$

$$=\frac{x\overline{a}+(y-x)m(\tilde{a}_I)}{y\cdot\left(m(\tilde{a}_I)\right)^{1-\frac{x}{y}}}.$$

Thus, it is obtained that

$$\lim\left((\tilde{a}_I)_n{}^{\frac{x}{y}}\right)=\left[\frac{x\underline{a}+(y-x)m(\tilde{a}_I)}{y\cdot\left(m(\tilde{a}_I)\right)^{1-\frac{x}{y}}},\frac{x\overline{a}+(y-x)m(\tilde{a}_I)}{y\cdot\left(m(\tilde{a}_I)\right)^{1-\frac{x}{y}}}\right]$$

$$=(\tilde{a}_I)^{\frac{x}{y}}=(\lim\,(\tilde{a}_I)_n)^{\frac{x}{y}}.$$

om the various theorems in Subsections 4.1 and 4.2, it is evident that by using the algebra offered, we can prove various fundamental properties of the limit of interval sequences, making it a fundamental basis for the application of the concept of interval number sequences in various conditions or other properties. Furthermore, by proving the existence of the convergence of this interval number sequence with its various basic properties, all numerical analysis problems, dynamic systems, and various other applications that use interval sequences can be solved validly.

5. Conclusions

Some properties of the sequence of real numbers also apply to the sequence of interval numbers, including operations like addition, subtraction, multiplication, and division between two interval number sequences. Another property that applies is the sequence of interval numbers with positive, negative, and fractional exponents. Due to the numerous differences in opinion regarding some operations in interval numbers, certain properties such as multiplication, division, and the sequence of interval numbers raised to a power apply only when using specific formulas. To demonstrate that these properties of the interval number sequence hold, the properties of the midpoint have also been shown in this article.

Author contributions

Mashadi: Conceptualization, software, formal analysis, investigation, resources, writing-original draft preparation, visualization, supervision, funding acquisition; Rasi Adishamita: Conceptualization, methodology, software, formal analysis, investigation, data curation, writing-original draft preparation, visualization; Sukono: Conceptualization, methodology, validation, writing-original draft preparation, project administration, funding acquisition; Igif Gimin Prihanto: Methodology, software, validation, formal analysis, investigation, writing-review and editing; Nurnadiah Zamri: Software, resources, data curation, writing-review and editing; Moch Panji Agung Saputra: Resources, writing-review and editing, visualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tool in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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