AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(10): 24627–24650.
DOI: 10.3934/math.20251092
Received: 01 September 2025
Revised: 21 October 2025
Accepted: 22 October 2025
Published: 28 October 2025

Research article

Differential subordinations and superordinations for meromorphic multivalent functions involving a convolution operator

Ekram E. Ali[1,*], Teodor Bulboacă[2,*], Rabha M. El-Ashwah[3], Abeer H. Alblowy[1] and Fozaiyah A. Alhubairah[1]

[1] Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 81451, Saudi Arabia
[2] Research Centre of Applied Analysis, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
[3] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt

* **Correspondence:** Email: e.ahmad@uoh.edu.sa; teodor.bulboaca@ubbcluj.ro.

Abstract: The results of this paper deal with a few applications of the general theory of the differential subordinations and superordinations, and to obtain our results we use the general techniques of the differential subordination and superordination theory, which could be considered one of the newest techniques used in some topics of Geometric Function Theory. In our investigation we find new differential subordination and superordination results of an operator for meromorphic multivalent functions defined by convolution product with the Hurwitz-Lerch Zeta function, that is $\mathrm{K}^{s}_{d,p}(n,m)$ with $d,n,m \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{R}$. The main results are followed by a few corollaries containing some special cases, examples, and applications.

Keywords: analytic and univalent functions; Hadamard (convolution) product; differential subordination; Gaussian hypergeometric function; starlike and convex functions
Mathematics Subject Classification: 30C45, 26A33, 30C50, 30C80

1. Introduction

Let $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane, and denote by $\mathcal{H}(\mathcal{U})$ the space of all analytic functions in $\mathcal{U}$. Also, let

$$\mathcal{A}_\kappa := \left\{f \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{\kappa+1}z^{\kappa+1} + \ldots,\ z \in \mathcal{U}\right\} \quad (k \in \mathbb{N}),$$

denote $\mathcal{A} := \mathcal{A}_1$, and

$$\mathcal{H}[a,\ell] := \left\{f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_\ell z^\ell + a_{\ell+1}z^{\ell+1} + \ldots,\ z \in \mathcal{U}\right\},$$

with $a \in \mathbb{C}$ and $\ell \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Denoting $\mathcal{H} := \mathcal{H}[1, 1]$, the usually normalized class of convex functions in $\mathcal{U}$ will be denoted by

$$\mathcal{L} := \left\{ f \in \mathcal{H}(\mathcal{U}) : \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0,\ f'(0) \neq 0,\ z \in \mathcal{U} \right\},$$

and let $\mathcal{S} \subset \mathcal{A}$ the subset of $\mathcal{A}$ consisting of univalent functions in $\mathcal{U}$.

The following well-known notion of the subordination plays a crucial role in the present paper. If f and g belong to $\mathcal{H}(\mathcal{U})$, we call that f is subordinate to g (or g is superordinate to f), denoted $f(z) \prec g(z)$, if there exists a Schwarz function $\varphi \in \mathcal{H}(\mathcal{U})$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, $z \in \mathcal{U}$, such that $f = g \circ \varphi$. If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$, while if g is a univalent function in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \prec g(\mathcal{U})$ (cf., e.g., [1]; see also [2, p. 4]).

If $h \in \mathcal{H}(\mathcal{U})$ and $\mathcal{W} : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$, then a function $p \in \mathcal{H}(\mathcal{U})$ that satisfies the second-order differential subordination $\mathcal{W}\left(p(z), zp'(z), z^2 p''(z)\right) \prec h(z)$ is called a solution of this subordination. A univalent function q in $\mathcal{U}$ with the property $p(z) \prec q(z)$ for all the solutions of the differential subordination is said to be a dominant of this subordination. Moreover, a dominant $\widetilde{q}$ such that $\widetilde{q}(z) \prec q(z)$ for all other dominants q is called the best dominant. This is unique up to a rotation, while more details could be found in [2] and the references therein.

The class of meromorphic multivalent function in the punctured unit disc $\mathcal{U}^* := \mathcal{U} \setminus \{0\}$ of the form

$$f(z) = \frac{1}{z^p} + \sum_{\kappa=1-p}^{\infty} a_\kappa z^\kappa,\ z \in \mathcal{U}^* \quad (p \in \mathbb{N}), \tag{1.1}$$

will be denoted by Σ_p.

For the functions $f_j \in \Sigma_p$, $j = 1, 2$, having the form

$$f_j(z) = \frac{1}{z^p} + \sum_{\kappa=1-p}^{\infty} a_{\kappa,j} z^\kappa,\ z \in \mathcal{U}^*,$$

the Hadamard (convolution) product of f_1 and f_2 is given by (see, e.g., [3, p. 246])

$$(f_1 * f_2)(z) := \frac{1}{z^p} + \sum_{\kappa=1-p}^{\infty} a_{\kappa,1} a_{\kappa,2} z^\kappa,\ z \in \mathcal{U}^*.$$

In [4], using the convolution product of a function $f \in \Sigma_p$ with the Hurwitz-Lerch Zeta function, the authors defined the operator $\mathrm{K}^s_{p,d} : \Sigma_p \to \Sigma_p$ by

$$\mathrm{K}^s_{d,p} f(z) := \frac{1}{z^p} + \sum_{\kappa=1-p}^{\infty} \left(\frac{d}{\kappa + d + p} \right)^s a_\kappa z^\kappa,\ z \in \mathcal{U}^*$$
$$\left(d \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \mathbb{Z}_0^- := \mathbb{C} \setminus \{0, -1, -2, \ldots\},\ p \in \mathbb{N},\ s \in \mathbb{C}\right).$$

Moreover, we could easily check that for all $f \in \Sigma_p$, $t_i \in \mathcal{U}^*$ for $i \in \{1, 2, 3, \ldots, \kappa\}$, $\kappa \in \mathbb{N}$, and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, we get

$$\mathrm{K}^\kappa_{d,p} f(z) = \frac{d^\kappa}{z^{d+p}} \int_0^z \frac{1}{t_1} \int_0^{t_1} \frac{1}{t_2} \int_0^{t_2} \ldots \frac{1}{t_{\kappa-1}} \int_0^{t_{\kappa-1}} t_\kappa^{d+p-1} f(t_\kappa) \mathrm{d}t_\kappa \mathrm{d}t_{\kappa-1} \ldots \mathrm{d}t_2 \mathrm{d}t_1,\ z \in \mathcal{U}^*,$$

and
$$\mathrm{K}_{d,p}^{s+1}f(z)=\frac{d}{z^{d+p}}\int_0^z t^{d+p-1}\mathrm{K}_{d,p}^{s}f(t)\mathrm{d}t,\ z\in\mathcal{U}^*.$$
Let us define the function $\Psi_p(n,m):\mathcal{U}^*\to\mathbb{C}$ by

$$\Psi_p(n,m)(z):=\frac{1}{z^p}+\sum_{\kappa=1-p}^{\infty}\frac{(n)_{\kappa+p}}{(m)_{\kappa+p}}z^{\kappa},\ n\in\mathbb{C}^*:=\mathbb{C}\setminus\{0\},\ m\in\mathbb{C}\setminus\mathbb{Z},\ p\in\mathbb{N},$$

where $(\tau)_\kappa$ denotes the Pochhammer symbol, that is

$$(\tau)_\kappa=\frac{\Gamma(\tau+\kappa)}{\Gamma(\tau)}=\begin{cases}1, & \text{if}\quad \kappa=0,\\ \tau(\tau+1)\ldots(\tau+\kappa-1), & \text{if}\quad \kappa\in\mathbb{N}.\end{cases}$$

Consequently,
$$\Psi_p(n,m)(z)=\frac{1}{z^p}\,{}_2\Gamma_1(n,1;m;z),\ z\in\mathcal{U}^*,\quad (p\in\mathbb{N}),$$
where
$${}_2F_1(n,\nu;m;z)=\sum_{\kappa=0}^{\infty}\frac{(n)_\kappa(\nu)_\kappa}{(m)_\kappa(1)_\kappa}z^{\kappa},\ z\in\mathcal{U}\quad (n,\nu\in\mathbb{C},\ m\in\mathbb{C}\setminus\mathbb{Z}_0^-)$$
is the Gaussian hypergeometric function.

If we consider the equation

$$\mathrm{K}_{d,p}^{s}(z)*\mathcal{L}_{d,p}^{s}(z)=\frac{1}{z^p(1-z)},\ z\in\mathcal{U}^*,$$

with the unknown function $\mathcal{L}_{d,p}^{s}$, it follows immediately that

$$\mathcal{L}_{d,p}^{s}(z)=\frac{1}{z^p}+\sum_{\kappa=1-p}^{\infty}\left(\frac{\kappa+d+p}{d}\right)^{s}z^{\kappa},\ z\in\mathcal{U}^*. \tag{1.2}$$

Hence, using the operator $\mathcal{L}_{d,p}^{s}:\mathcal{U}^*\to\mathbb{C}$ given by (1.2), we define the new operator $\mathcal{L}_{d,p}^{s}(n,m):\Sigma_p\to\Sigma_p$ as the solution of the equation

$$\Psi_p(n,m)(z)=\mathcal{L}_{d,p}^{s}(z)*\mathcal{L}_{d,p}^{s}(n,m)f(z),\ z\in\mathcal{U}^*,$$

whose Taylor series expansion for $d,m\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $n\in\mathbb{C}^*$, $s\in\mathbb{C}$, and for f given by (1.1) will be

$$\mathcal{L}_{d,p}^{s}(n,m)f(z)=\frac{1}{z^p}+\sum_{\kappa=1-p}^{\infty}\left(\frac{d}{\kappa+d+p}\right)^{s}\frac{(n)_{\kappa+p}}{(m)_{\kappa+p}}a_\kappa z^{\kappa},\ z\in\mathcal{U}^*. \tag{1.3}$$

Therefore, for all $f\in\Sigma_p$ the operator $\mathcal{L}_{d,p}^{s}(n,m)$ satisfies the relations

$$z\left(\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)'=d\mathcal{L}_{d,p}^{s}(n,m)f(z)-(d+p)\mathcal{L}_{d,p}^{s+1}(n,m)f(z), \tag{1.4}$$

and
$$z\left(\mathcal{L}_{d,p}^{s}(n,m)f(z)\right)'=n\mathcal{L}_{d,p}^{s}(n+1,m;z)f(z)-(n+p)\mathcal{L}_{d,p}^{s}(n,m)f(z),\ n\in\mathbb{C}\setminus\{-1\}.$$

Remark 1. For particular values of the parameters, the operator $\mathcal{L}^s_{d,p}$ reduces to some cases studied by different authors, as follows:

(i) The operator

$$\mathcal{L}^1_{d,p}(1,1;z)f(z) =: F_d f(z) = \frac{d}{z^{d+p}} \int_0^z t^{d+p-1} f(t)\mathrm{d}t \quad (d > 0)$$

in a very similar form was studied by Miller and Mocanu in [2, p. 389].

(ii) In the article of Aqlan et al. [5], we can find the operator

$$\mathcal{L}^s_{1,p}(1,1;z)f(z) =: P^s f(z) = \frac{1}{z^p \Gamma(s)} \int_0^z \left(\log \frac{z}{t}\right)^{s-1} t^p f(t)\mathrm{d}t \quad (s > 0).$$

(iii) The special case

$$\mathcal{L}^s_{d,p}(1,1;z)f(z) =: \mathrm{J}^s_{d,p} f(z) = \frac{\alpha^s}{z^{d+p}\Gamma(s)} \int_0^z \left(\log \frac{z}{t}\right)^{s-1} t^{d+p-1} f(t)\mathrm{d}t \quad (d, s > 0)$$

was investigated by El-Ashwah and Aouf in [6].

(iv) The operator

$$\mathcal{L}^s_{d,1}(1,1;z)f(z) =: \mathcal{L}^s_d f(z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \left(\frac{d}{\kappa + 1 + d}\right)^s a_\kappa z^\kappa \quad (d \in \mathbb{C}^*, s \in \mathbb{C})$$

can be found in [7].

We recall the next notions and preliminary results, which are necessary to obtain our new findings.

Definition 1. [8, Definition 2, p. 817] Let Q be the set of the functions f that are holomorphic and univalent on $\overline{\mathcal{U}} \setminus E(f)$, where

$$E(f) := \left\{\zeta \in \partial\mathcal{U} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and satisfies $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(f)$.

The following classical subordination result is due to Hallenbeck and Ruscheweyh [9]:

Lemma 1. *[2, Theorem 3.1b, p. 71] Suppose that the function h is convex (univalent) in $\mathcal{U}$ with $h(0) = 1$, and $\varphi \in \mathcal{H}[1,1]$ such*

$$\varphi(z) + \frac{1}{\gamma} z\varphi'(z) \prec h(z),$$

with $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. It follows that

$$\varphi(z) \prec \Psi(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t)\mathrm{d}t \prec h(z),$$

where Ψ is the best dominant. Moreover, the function Ψ is convex (univalent) in $\mathcal{U}$.

The next two lemmas deal with linear operators that preserve the subordinations and superordinations, and the first one is a special case of [2, Theorem 3.4h, p. 132].

Lemma 2. *[10, Lemma 2.2, p. 3] Assume that the function q is univalent in $\mathcal{U}$, and $\psi \in \mathbb{C}$, $d \in \mathbb{C}^*$ such that*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\operatorname{Re}\frac{\psi}{d}\right\},\ z\in\mathcal{U}.$$

If $\lambda \in \mathcal{H}(\mathcal{U})$ and

$$\psi\lambda(z)+dz\lambda'(z)\prec\psi q(z)+dzq'(z),$$

then $\lambda(z) \prec q(z)$, and the function q is the best dominant.

The following lemma could be easily derived from [8, Theorem 8, p. 822], while a more general form can be found in [10, Lemma 2.4, p. 3] as a special case of [11, Corollary 3.2, p. 290]:

Lemma 3. *Assume that q is convex (univalent) in $\mathcal{U}$ and suppose that $\delta \in \mathbb{C}$ with $\operatorname{Re}\delta > 0$. If $\lambda \in \mathcal{H}[q(0),1] \cap Q$ and $\lambda(z)+\delta z\lambda'(z)$ is univalent in $\mathcal{U}$, then*

$$q(z)+\delta zq'(z)\prec\lambda(z)+\delta z\lambda'(z)$$

implies

$$q(z)\prec\lambda(z),$$

and the function q is the best subordinant.

The next lemma due to Nunokawa is a classical one in the Geometric Function Theory, and in some cases it gives additional tools for proving different specific results, and it will be used in our proofs.

Lemma 4. *[12, 13] Let $p \in \mathcal{H}[1,1]$ with $p(z) \neq 0$ for all $z \in \mathcal{U}$. If there exists $z_0 \in \mathcal{U}$ such that*

$$\left|\arg p(z)\right|<\frac{\pi}{2}\gamma,\ |z|<|z_0|\quad \text{and}\quad \left|\arg p(z_0)\right|=\frac{\pi}{2}\gamma,$$

where $0<\gamma\leq 1$, it follows that

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\gamma,$$

where

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right),\quad \text{if}\quad \arg p(z_0)=\frac{\pi}{2}\gamma,$$
$$k\leq-\frac{1}{2}\left(a+\frac{1}{a}\right),\quad \text{if}\quad \arg p(z_0)=-\frac{\pi}{2}\gamma,$$

and $\left(p(z_0)\right)^{\frac{1}{\gamma}}=\pm ia$, $a>0$.

For a, b, and c real or complex numbers, with $c \notin \mathbb{Z}_0^-$, the well-known Gaussian hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n,$$

and this series converges absolutely and uniformly on compacts of $\mathcal{U}$; hence, its sum is an analytic function in $\mathcal{U}$ (see, for details, [14, Chapter 14]; see also [2]). The properties of this function that we will be using in our proofs are presented in the next lemma. We would like to mention that the source of these formulas is the next one: The relation (1.5) represents the [14, Example 14.6.1, p. 305–306] (see also, [15, formula (9.1.6), p. 240] and [16, (15.3.1), p. 558]). The second equality of this lemma is trivial according to the above definition, while the identity (1.7) is that of [16, formula (15.3.4), p. 559] (see also the [15, relation (9.5.2), p. 247]).

Lemma 5. *For $a, b, c \in \mathbb{C}$ with $c \notin \mathbb{Z}_0^-$, we have the next identities:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z) \quad (\operatorname{Re} c > \operatorname{Re} b > 0), \tag{1.5}$$

$$z \in \mathbb{C} \setminus (1, +\infty);$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z),\; z \in \mathbb{C} \setminus (1, +\infty), \tag{1.6}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right),\; z \in \mathbb{C} \setminus (1, +\infty), \tag{1.7}$$

where the function log *is considered at the main branch, that is,* $\log 1 = 0$.

The results discussed in this work further deal with some subordination and superordination properties for the operator $\mathcal{L}^s_{d,p}(n,m)$ when linear combinations of this operators are subordinated to power of the Janowski-type functions, and with argument estimations of other linear combinations. Many previous papers deal with various applications of the theory of differential subordinations and superordination, and different connections of these results with special functions.

The new operator $\mathcal{L}^s_{d,p}(n,m)$ we defined in (1.3) as a solution of a "convolution equation" and connected with the Hurwitz-Lerch Zeta function, generalize many of some previous ones, as it is shown in the Remark 1. With specific techniques of the above-mentioned theory and using also Nunokawa's lemma (Lemma 4), we obtained general subordination and superordinations results, that in particular lead to some simple so-called "differential inequalities".

It is necessary to mention that the subordination and superordination results are sharp (i.e., the best possible), and the main results are followed by some special cases obtained for convenient choices of the parameters.

The results we obtained in this paper are new and original and we hope they will be useful for the specialists that work in this field of Geometric Function Theory, and the purpose is focused on solving other special problems of this area. In addition to the relevant applications shown in the recent classical books of this area (see, for example, [2, 11]), the differential subordinations and superordinations notions are appropriate to define new relevant classes using similar methods to those of this paper.

2. Main results

Throughout this paper we assume that $d, n, m \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\operatorname{Re} d \geq 0$, $s \in \mathbb{R}$, $\delta \in \mathbb{R}^+ \setminus \{0\}$, $\alpha \in (0, 1]$, and $C, D \in \mathbb{R}$ with $|C| \leq 1$, $|D| \leq 1$ such that $C \neq D$. The first subordination result we prove below is connected with the operator defined by (1.3), and it is a sharp (i.e., the best possible) one.

Theorem 1. *Let* $0<r\le 1$ *and for a given function* $f\in\Sigma_p$ *assume that*

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\neq 0,\ z\in\mathcal{U}. \tag{2.1}$$

Let the function Φ *defined by*

$$\Phi(z):=(1-\alpha)\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}+\alpha\left(z^p\mathcal{L}_{d,p}^{s}(n,m)f(z)\right)\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta-1}, \tag{2.2}$$

where the powers are all at the main branch, i.e., $\log 1=0$. *If*

$$\Phi(z)\prec\left(\frac{1+Cz}{1+Dz}\right)^{r}, \tag{2.3}$$

with $|C|\le 1$, $|D|\le 1$, $C\neq D$, *then*

$$\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}\prec p(z), \tag{2.4}$$

where

$$p(z)=\begin{cases} {}_2F_1\left(r,\frac{\delta d}{\alpha};\frac{\delta d}{\alpha}+1;-Dz\right)=(1+Dz)^{-r}\,{}_2F_1\left(r,1;\frac{\delta d}{\alpha}+1;\frac{Dz}{Dz+1}\right), & \textit{if}\quad C=0,\\ {}_2F_1\left(-r,\frac{\delta d}{\alpha};\frac{\delta d}{\alpha}+1;-Cz\right)=(1+Cz)^{r}\,{}_2F_1\left(-r,1;\frac{\delta d}{\alpha}+1;\frac{Cz}{Cz+1}\right), & \textit{if}\quad C\neq 0,\ D=0,\\ \left(\frac{C}{D}\right)^{r}\sum\limits_{j\ge 0}\frac{(-r)_j}{j!}\left(\frac{C-D}{C}\right)^{j}(1+Dz)^{-j}\,{}_2F_1\left(j,1;1+\frac{\delta d}{\alpha};\frac{Dz}{1+Dz}\right), & \textit{if}\quad DC\neq 0,\\ & \left|1-\frac{D}{C}\right|\le 1-|D|. \end{cases}$$

If $r\in(0,1)$, *and*

$$p(z)=\begin{cases} \frac{C}{D}+\left(1-\frac{C}{D}\right)(1+Dz)^{-1}\,{}_2F_1\left(1;1;\frac{\delta d}{\alpha}+1;\frac{Dz}{1+Dz}\right), & \textit{if}\quad D\neq 0,\\ 1+\frac{\delta d}{\delta d+\alpha}Cz, \quad \textit{if}\quad D=0 \end{cases}$$

for $r=1$. *Moreover, the function* p *is the best dominant of* (2.4) *and the next inequality holds*

$$\operatorname{Re}\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}>p(-1),\ z\in\mathcal{U}, \tag{2.5}$$

while the inequality (2.5) *is the best possible.*

Proof. If

$$\phi(z):=\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta},\ z\in\mathcal{U}, \tag{2.6}$$

from the assumption (2.1), it follows that the function ϕ is well-defined, $\phi\in\mathcal{H}$ and $\phi(0)=1$. Differentiating (2.6) and using the relation (1.4), we get

$$\phi(z)+\frac{z\phi'(z)}{\frac{\delta d}{\alpha}}=\Phi(z),\ z\in\mathcal{U},$$

hence the subordination (2.3) leads to

$$\phi(z)+\frac{z\phi'(z)}{\frac{\delta d}{\alpha}}=\Phi(z)\prec\left(\frac{1+Cz}{1+Dz}\right)^{r}=:q(z). \tag{2.7}$$

Since $|C|\leq 1$ and $|D|\leq 1$ it follows that the function q is analytic in $\mathcal{U}^{*}$ with $q(0)=1$, and it is easy to check that for $r\in(0,1]$ the inequality

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)&=-1+(1-r)\operatorname{Re}\frac{1}{1+Cz}+(1+r)\operatorname{Re}\frac{1}{1+Dz}\\&>-1+\frac{1-r}{1+|C|}+\frac{1+r}{1+|D|}\geq 0,\ z\in\mathcal{U},\end{aligned}$$

holds, therefore q is a convex (univalent) function in $\mathcal{U}$. Using the fact that $\operatorname{Re}(\delta d/\alpha)\geq 0$, from Lemma 1 the subordination (2.7) implies the sharp subordination

$$\phi(z)\prec p(z):=\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\int_0^z t^{\frac{\delta d}{\alpha}-1}\left(\frac{1+Ct}{1+Dt}\right)^{r}\mathrm{d}t.$$

For computing the above integral, first we will use the binomial formula

$$(1-\zeta)^{\rho}=1+\sum_{k=1}^{\infty}\frac{\rho(\rho-1)\dots(\rho-k+1)}{k!}(-1)^{k}\zeta^{k},\ |\zeta|<1\quad(\rho\in\mathbb{C}), \tag{2.8}$$

where the power is at the main branch, i.e., $\log 1=0$. The above right-hand-side series diverges for $|\zeta|>1$ excepting the case $\rho\in\mathbb{N}\cup\{0\}$, and we could split our study in the following cases according to the values of the parameters C and D.

(i) If $C=0$, with the substitution $t=uz$ we get

$$\begin{aligned}p(z)=&\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\int_0^z t^{\frac{\delta d}{\alpha}-1}(1+Dt)^{-r}\,\mathrm{d}t=\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\int_0^1 z^{\frac{\delta d}{\alpha}-1}u^{\frac{\delta d}{\alpha}-1}(1+Dzu)^{-r}\,z\mathrm{d}u\\=&\frac{\delta d}{\alpha}\int_0^1 u^{\frac{\delta d}{\alpha}-1}(1-(-Dz)u)^{-r}\,\mathrm{d}u,\end{aligned}$$

hence by taking in (1.5) $b:=\dfrac{\delta d}{\alpha}$, $a:=r$ and $c:=b+1$, from Lemma 5, we have

$$\begin{aligned}p(z)&=\frac{\delta d}{\alpha}\,{}_2F_1\left(r,\frac{\delta d}{\alpha};\frac{\delta d}{\alpha}+1;-Dz\right)\frac{\Gamma\left(\frac{\delta d}{\alpha}\right)\Gamma(1)}{\Gamma\left(\frac{\delta d}{\alpha}+1\right)}\\&=(1+Dz)^{-r}\,{}_2F_1\left(r,1;\frac{\delta d}{\alpha}+1;\frac{Dz}{Dz+1}\right)={}_2F_1\left(r,\frac{\delta d}{\alpha};\frac{\delta d}{\alpha}+1;-Dz\right).\end{aligned} \tag{2.9}$$

(ii) If $C\neq 0$, we have the following two subcases:

(a) If $D=0$, since

$$p(z)=\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\int_0^z t^{\frac{\delta d}{\alpha}-1}(1+Ct)^r\,\mathrm{d}t,$$

using similar computations as above, we get

$$p(z)=(1+Cz)^r\,{}_2F_1\left(-r,1;\frac{\delta d}{\alpha}+1;\frac{Cz}{Cz+1}\right)={}_2F_1\left(-r,\frac{\delta d}{\alpha};\frac{\delta d}{\alpha}+1;-Cz\right). \tag{2.10}$$

(b) If $D\neq 0$, we have

$$t^{\frac{\delta d}{\alpha}-1}\left(\frac{1+Ct}{1+Dt}\right)^r=t^{\frac{\delta d}{\alpha}-1}\left(\frac{C}{D}\right)^r\left(1-\frac{C-D}{C+CDt}\right)^r.$$

For the last factor for the right-hand-side of the above relation, we will use the binomial formula taking $\rho:=r\in(0,1]$ and $\zeta:=\dfrac{C-D}{C(1+Dt)}$. If we consider the case $r\in(0,1]\setminus\{1\}=(0,1)$, then $r\notin\mathbb{N}$ and the formula (2.8) holds only if

$$|\zeta|=\left|\frac{C-D}{C(1+Dt)}\right|<1,\ \text{for all } |t|<1.$$

We have $|t|<1$ because the integral of an analytic function doesn't depend on the path, so we could consider that we integrate on the segment connecting 0 to $z\in\mathcal{U}$, hence $t\in\mathcal{U}$. Also, it is necessary to assume $|C|\leq 1$, $|D|\leq 1$ with $C\neq D$ (the first two conditions for the analyticity of the function p, and the last one for not being a constant function).

In order to use the binomial power series formula (2.8) for $r\in(0,1)$ (excepting the case $r=1$), we should have

$$\left|\frac{C-D}{C(1+Dt)}\right|<1,\ \forall|t|<1\Leftrightarrow 0\neq\left|1-\frac{D}{C}\right|<|1+Dt|,\ \forall|t|<1. \tag{2.11}$$

Since $0<|C|\leq 1$ and $0<|D|\leq 1$, the left-hand-side of (2.11) is well-defined, and

$$\inf\{|1+Dt|:|t|<1\}=1-|D|\geq 0.$$

Consequently, the condition (2.11) is equivalent to

$$\left|1-\frac{D}{C}\right|\leq 1-|D|, \tag{2.12}$$

then the binomial power series formula (2.8) for $r\in(0,1)$ can be used if and only if we assume that the inequality (2.12) holds. With this additional assumption, using first the substitution $t=uz$, then taking in (1.5) $b:=\dfrac{\delta d}{\alpha}$, $a:=j$, $c:=b+1$, and using finally (1.7), it follows that

$$\begin{aligned}p(z)&=\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\int_0^z t^{\frac{\delta d}{\alpha}-1}\left(\frac{1+Ct}{1+Dt}\right)^r\mathrm{d}t=\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\int_0^z t^{\frac{\delta d}{\alpha}-1}\left(\frac{C}{D}\right)^r\left(1-\frac{C-D}{C(1+Dt)}\right)^r\mathrm{d}t\\&=\frac{\delta d}{\alpha}z^{-\frac{\delta d}{\alpha}}\left(\frac{C}{D}\right)^r\int_0^z t^{\frac{\delta d}{\alpha}-1}\left(\sum_{j\geq 0}\frac{(-r)_j}{j!}\left(\frac{C-D}{C(1+Dt)}\right)^j\right)\mathrm{d}t\end{aligned}$$

$$= \frac{\delta d}{\alpha} z^{-\frac{\delta d}{\alpha}} \left(\frac{C}{D}\right)^r \int_0^1 z^{\frac{\delta d}{\alpha}-1} u^{\frac{\delta d}{\alpha}-1} \left(\sum_{j\geq 0} \frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j (1+Dzu)^{-j} \right) z\mathrm{d}u$$

$$= \frac{\delta d}{\alpha} \left(\frac{C}{D}\right)^r \int_0^1 u^{\frac{\delta d}{\alpha}-1} \left(\sum_{j\geq 0} \frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j (1+Dzu)^{-j} \right) \mathrm{d}u$$

$$= \frac{\delta d}{\alpha} \left(\frac{C}{D}\right)^r \sum_{j\geq 0} \left[\frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j \int_0^1 u^{\frac{\delta d}{\alpha}-1} (1+Dzu)^{-j} \mathrm{d}u \right]$$

$$= \frac{\delta d}{\alpha} \left(\frac{C}{D}\right)^r \sum_{j\geq 0} \left[\frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j {}_2F_1\left(j, \frac{\delta d}{\alpha}; \frac{\delta d}{\alpha}+1; -Dz\right) \frac{\Gamma\left(\frac{\delta d}{\alpha}\right)\Gamma(1)}{\Gamma\left(\frac{\delta d}{\alpha}+1\right)} \right]$$

$$= \left(\frac{C}{D}\right)^r \sum_{j\geq 0} \left[\frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j {}_2F_1\left(j, \frac{\delta d}{\alpha}; \frac{\delta d}{\alpha}+1; -Dz\right) \right],$$

thus

$$p(z) = \left(\frac{C}{D}\right)^r \sum_{j\geq 0} \left[\frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j {}_2F_1\left(j, \frac{\delta d}{\alpha}; \frac{\delta d}{\alpha}+1; -Dz\right) \right]. \tag{2.13}$$

Concluding, from (2.9), (2.10), and (2.13) combined in this last case with the assumption (2.12), under the assumptions $|C| \leq 1$, $|D| \leq 1$ with $C \neq D$, and $r \in (0,1)$, we have

$$p(z) = \begin{cases} {}_2F_1\left(r, \frac{\delta d}{\alpha}; \frac{\delta d}{\alpha}+1; -Dz\right), & \text{if} \quad C = 0, \\ {}_2F_1\left(-r, \frac{\delta d}{\alpha}; \frac{\delta d}{\alpha}+1; -Cz\right), & \text{if} \quad C \neq 0,\ D = 0, \\ \left(\frac{C}{D}\right)^r \sum_{j\geq 0} \left[\frac{(-r)_j}{j!} \left(\frac{C-D}{C}\right)^j {}_2F_1\left(j, \frac{\delta d}{\alpha}; \frac{\delta d}{\alpha}+1; -Dz\right) \right], & \text{if} \quad DC \neq 0,\ \left|1-\frac{D}{C}\right| \leq 1-|D|. \end{cases}$$

For $r = 1$, like in [17], we similarly get that

$$p(z) = \begin{cases} \frac{C}{D} + \left(1-\frac{C}{D}\right)(1+Dz)^{-1} {}_2F_1\left(1; 1; \frac{\delta d}{\alpha}+1; \frac{Dz}{1+Dz}\right), & \text{if} \quad D \neq 0, \\ 1 + \frac{\delta d}{\delta d+\alpha} Cz, & \text{if} \quad D = 0. \end{cases}$$

Now we will prove that

$$\inf\{\operatorname{Re} p(z) : |z| < 1\} = p(-1). \tag{2.14}$$

Thus, if $0 < r \leq 1$ we have

$$\operatorname{Re}\left(\frac{1+Cz}{1+Dz}\right)^r \geq \left(\frac{1-C\sigma}{1-D\sigma}\right)^r,\ |z| < \sigma \leq 1.$$

Setting

$$\hbar(s,z) := \left(\frac{1+Csz}{1+Dsz}\right)^r,\ z \in \mathcal{U},\ (0 \leq s \leq 1),$$

and

$$dv(s) := \frac{\delta d}{\alpha} s^{\frac{\delta d}{\alpha}-1} ds,$$

where $dv(s)$ is a positive measure on $[0, 1]$, we get

$$p(z) = \int_0^1 \hbar(s, z) \mathrm{d}v(s),$$

therefore

$$\operatorname{Re} p(z) \geq \int_0^1 \left(\frac{1 - Cs\sigma}{1 - Ds\sigma}\right)^r \mathrm{d}v(s) = p(-\sigma), \ |z| < \sigma < 1.$$

Taking $\sigma \to 1^-$ we get that (2.14) holds, while the inequality (2.5) is the best possible since p is the best dominant of the subordination (2.4). □

Next, we will give some particular and special cases of the above theorem, followed by some examples.

If we choose $\alpha = \delta = 1$ in Theorem 1, we get the next corollary:

Corollary 1. *Let* $0 < r \leq 1$*, and* $|C| \leq 1$*,* $|D| \leq 1$ *with* $C \neq D$*. If*

$$z^p \mathcal{L}_{d,p}^{s+1}(n, m) f(z) \prec \left(\frac{1 + Cz}{1 + Dz}\right)^r,$$

then

$$\operatorname{Re}\left(z^p \mathcal{L}_{d,p}^{s+1}(n, m) f(z)\right) > p(-1), \ z \in \mathcal{U}, \tag{2.15}$$

where the function p is given in Theorem 1. Moreover, the inequality (2.15) *is the best possible.*

If we choose $s = -1$ and $n = m$ in Theorem 1, we get the below result:

Corollary 2. *Let* $0 < r \leq 1$ *and for the function* $f \in \Sigma_p$ *assume that*

$$z^p f(z) \neq 0, \ z \in \mathcal{U}.$$

Define the function Φ_1 *by*

$$\widetilde{\Phi}(z) := \left(1 + \frac{\alpha p}{d}\right)(z^p f(z))^\delta + \frac{\alpha z^{p+1}}{d} f'(z) \left(z^p f(z)\right)^{\delta-1}, \ z \in \mathcal{U},$$

where all the powers are considered at the main branch, i.e., log$1 = 0$*. If*

$$\widetilde{\Phi}(z) \prec \left(\frac{1 + Cz}{1 + Dz}\right)^r,$$

where $|C| \leq 1$*,* $|D| \leq 1$ *with* $C \neq D$*, then*

$$(z^p f(z))^\delta \prec p(z), \tag{2.16}$$

where the function p is given in Theorem 1, and it is the best dominant of (2.16)*. Moreover,*

$$\operatorname{Re}(z^p f(z))^\delta > p(-1), \ z \in \mathcal{U}, \tag{2.17}$$

and the inequality (2.17) *is the best possible.*

For $\delta = 1$ and $r = 1$, the Corollary 2 reduces to the next example:

Example 1. Let $0 < r \leq 1$ and for the function $f \in \Sigma_p$ assume that

$$z^p f(z) \neq 0,\ z \in \mathcal{U}.$$

Defining the function Φ_2 by

$$\Phi_2(z) := \left(1 + \frac{\alpha p}{d}\right) z^p f(z) + \frac{\alpha z^{p+1}}{d} f'(z),\ z \in \mathcal{U},$$

if

$$\Phi_2(z) \prec \frac{1 + Cz}{1 + Dz},$$

where $|C| \leq 1$, $|D| \leq 1$ with $C \neq D$, then

$$z^p f(z) \prec p(z), \tag{2.18}$$

where the function p is given in Theorem 1, and it is the best dominant of (2.18). Moreover,

$$\operatorname{Re}\left(z^p f(z)\right) > p(-1),\ z \in \mathcal{U}, \tag{2.19}$$

and the inequality (2.19) is the best possible.

For $C = 1$ and $D = -1$, the Example 1 leads to the next particular case:

Example 2. (i) If $0 < r \leq 1$, let the function $f \in \Sigma_p$ such that

$$z^p f(z) \neq 0,\ z \in \mathcal{U}.$$

Let the function Φ_2 defined by

$$\Phi_2(z) := \left(1 + \frac{\alpha p}{d}\right) z^p f(z) + \frac{\alpha z^{p+1}}{d} f'(z) \prec \frac{1 + z}{1 - z}.$$

Then, the subordination

$$z^p f(z) \prec \frac{1 + z}{1 - z}$$

implies

$$\operatorname{Re}\left(z^p f(z)\right) > \zeta_*,\ z \in \mathcal{U}, \tag{2.20}$$

where

$$\zeta_* = -1 + {}_2F_1\left(1, 1; \frac{d + \alpha}{\alpha}; \frac{1}{2}\right),$$

and the inequality (2.20) is the best possible.

(ii) For $d = \alpha = 1$ the above result leads us to the following one:

For $f \in \Sigma_p$ assume that

$$z^p f(z) \neq 0,\ z \in \mathcal{U}.$$

Then, the next implication holds:

$$(1 + p) z^p f(z) + z^{p+1} f'(z) \prec \frac{1 + z}{1 - z} \Rightarrow \operatorname{Re}\left(z^p f(z)\right) > -1 + 2 \ln 2,\ z \in \mathcal{U},$$

and the right-hand-side lower bound is the best possible.

The next result represents another subordination result involving the operator $\mathcal{L}_{d,p}^{s}$ defined by (1.3).

Theorem 2. *Let* $0 < r \leq 1$, $0 < \alpha < \dfrac{1}{1+p}$, *and* $|C| \leq 1$, $|D| \leq 1$ *with* $C \neq D$. *For* $f \in \Sigma_p$ *let's define the function* $\mathcal{F}_\alpha$ *by*

$$\mathcal{F}_\alpha(z) := \alpha d\mathcal{L}_{d,p}^{s}(n,m)f(z) + (1-\alpha-\alpha(d+p))\mathcal{L}_{d,p}^{s+1}(n,m)f(z),\ z \in \mathcal{U}. \tag{2.21}$$

If

$$z^p\mathcal{F}_\alpha(z) \prec (1-\alpha-\alpha p)\left(\frac{1+Cz}{1+Dz}\right)^r, \tag{2.22}$$

then the assumption (2.22) *implies*

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z) \prec p(z), \tag{2.23}$$

where

$$\widetilde{p}(z) = \begin{cases} {}_2F_1\left(r, \dfrac{\alpha}{1-\alpha-\alpha p}; \dfrac{1-\alpha p}{1-\alpha-\alpha p}; -Dz\right) = \\ \qquad (1+Dz)^{-r}\, {}_2F_1\left(r, 1; \dfrac{1-\alpha p}{1-\alpha-\alpha p}; \dfrac{Dz}{Dz+1}\right), \quad \textit{if} \quad C = 0, \\ {}_2F_1\left(-r, \dfrac{\alpha}{1-\alpha-\alpha p}; \dfrac{1-\alpha p}{1-\alpha-\alpha p}; -Cz\right) = \\ \qquad (1+Cz)^{r}\, {}_2F_1\left(-r, 1; \dfrac{1-\alpha p}{1-\alpha-\alpha p}; \dfrac{Cz}{Cz+1}\right), \quad \textit{if} \quad C \neq 0,\ D = 0, \\ \left(\dfrac{C}{D}\right)^r \displaystyle\sum_{j\geq 0} \dfrac{(-r)_j}{j!}\left(\dfrac{C-D}{C}\right)^j (1+Dz)^{-j}\, {}_2F_1\left(j, 1; \dfrac{1-\alpha p}{1-\alpha-\alpha p}; \dfrac{Dz}{1+Dz}\right), \quad \textit{if} \quad DC \neq 0, \\ \qquad \left|1 - \dfrac{D}{C}\right| \leq 1 - |D|, \end{cases}$$

if $r \in (0,1)$, *and*

$$\widetilde{p}(z) = \begin{cases} \dfrac{C}{D} + \left(1 - \dfrac{C}{D}\right)(1+Dz)^{-1}\, {}_2F_1\left(1; 1; \dfrac{1-\alpha p}{1-\alpha-\alpha p}; \dfrac{Dz}{1+Dz}\right), \quad \textit{if} \quad D \neq 0, \\ 1 + \dfrac{\alpha}{1-\alpha p}Cz, \quad \textit{if} \quad D = 0 \end{cases}$$

for $r = 1$. *Moreover, the* $\widetilde{p}$ *is the best dominant of* (2.23) *and the next inequality holds*

$$\operatorname{Re}\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right) > \widetilde{p}(-1),\ z \in \mathcal{U}, \tag{2.24}$$

while the inequality (2.24) *is the best possible.*

Proof. From the definition (2.21), using the relation (1.4) we get

$$\mathcal{F}_\alpha(z) = \alpha z\left(\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)' + (1-\alpha)\mathcal{L}_{d,p}^{s+1}(n,m)f(z). \tag{2.25}$$

Denoting

$$\phi(z) := z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z),\ z \in \mathcal{U}, \tag{2.26}$$

we have $\phi \in \mathcal{H}$. Differentiating (2.26), from the relation (2.25) we get

$$z^p \mathcal{F}_\alpha(z) = (1-\alpha-\alpha p)\left(\phi(z) + \frac{\alpha}{1-\alpha-\alpha p} z\phi'(z)\right), \ z \in \mathcal{U}, \tag{2.27}$$

then

$$\phi(z) + \frac{\alpha}{1-\alpha-\alpha p} z\phi'(z) \prec \left(\frac{1+Cz}{1+Dz}\right)^r.$$

From the assumption of the theorem, the conditions of Lemma 1 are satisfied, and following the techniques similarly that those of the proof of Theorem 1 our result follows immediately. □

If we choose $r = 1$ in Theorem 2 we get

Corollary 3. *For $f \in \Sigma_p$ let the function $\mathcal{F}_\alpha$ be defined by 2.21. If*

$$z^p \mathcal{F}_\alpha(z) \prec (1-\alpha-\alpha p)\frac{1+Cz}{1+Dz},$$

then

$$\operatorname{Re}\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z)\right) > \widetilde{p}(-1), \ z \in \mathcal{U},$$

where the function $\widetilde{p}$ was defined in the Theorem 2. The above inequality is the best possible.

Example 3. Taking $p = C = 1$, $D = -1$ and $\alpha = \dfrac{1}{3}$ in Corollary 3 we get the next implication: If $f \in \Sigma_p$, then

$$z\mathcal{F}_{\frac{1}{3}}(z) \prec \frac{1}{3} \cdot \frac{1+z}{1-z} \Rightarrow \operatorname{Re}\left(z\mathcal{L}_{d,p}^{s+1}(n,m) f(z)\right) > -1 + 2\ln 2, \ z \in \mathcal{U},$$

and the right hand side lower bound is the best possible.

The next theorem evaluate the connection between the arguments of $z^p\mathcal{F}_\alpha(z)$ and $z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)$, as follows:

Theorem 3. *If $f \in \Sigma_p$, $0 < \gamma \le 1$ and $0 < \alpha < \dfrac{1}{1+p}$, then*

$$\left|\arg\left(z^p \mathcal{F}_\alpha(z)\right)\right| < \frac{\pi}{2}\gamma, \ z \in \mathcal{U},$$

implies

$$\left|\arg\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m) f z)\right)\right| < \frac{\pi}{2}\gamma, \ z \in \mathcal{U}.$$

Proof. For $f \in \Sigma_p$ let define the function

$$g(z) := z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z), \ z \in \mathcal{U},$$

thus $g \in \mathcal{H}$. If we suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\left|\arg g(z)\right| < \frac{\pi}{2}\gamma, \ |z| < |z_0|,$$

and

$$\left|\arg g\left(z_0\right)\right| = \frac{\pi}{2}\gamma,$$

then, by Lemma 4, we have

$$\frac{z_0 g'\left(z_0\right)}{g\left(z_0\right)} = ik\gamma \quad \text{and} \quad \left(g\left(z_0\right)\right)^{\frac{1}{\gamma}} = \pm ia,\ a > 0,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right), \quad \text{when} \quad \arg p\left(z_0\right) = \frac{\pi}{2}\gamma,$$
$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right), \quad \text{when} \quad \arg p\left(z_0\right) = -\frac{\pi}{2}\gamma.$$

(i) If $\arg g\left(z_0\right) = \frac{\pi}{2}\gamma$, using (2.27) we get

$$\begin{aligned} z^p\mathcal{F}_\alpha(z_0) &= (1-\alpha-\alpha p)g\left(z_0\right)\left(1 + \frac{\alpha}{1-\alpha-\alpha p}\frac{z_0 g'\left(z_0\right)}{g\left(z_0\right)}\right) \\ &= (1-\alpha-\alpha p)a^\gamma e^{i\frac{\pi}{2}\gamma}\left(1 + \frac{\alpha}{1-\alpha-\alpha p}ik\gamma\right), \end{aligned}$$

which implies that

$$\arg\left(z^p\mathcal{F}_\alpha(z_0)\right) = \frac{\pi}{2}\gamma + \arg\left(1 + \frac{k\alpha\gamma i}{1-\alpha-\alpha p}\right) = \frac{\pi}{2}\gamma + \tan^{-1}\left(\frac{k\alpha\gamma}{1-\alpha-\alpha p}\right) \geq \frac{\pi}{2}\gamma,$$

with $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1$, which contradicts the assumption of the theorem.

(ii) Similarly, if $\arg g\left(z_0\right) = -\frac{\pi}{2}\gamma$ we obtain

$$\arg\left(z^p\mathcal{F}_\alpha(z_0)\right) \leq -\frac{\pi}{2}\gamma,$$

which also contradicts the assumption of the theorem.

Thus, the function g will satisfy the inequality $\left|\arg g(z)\right| < \frac{\pi}{2}\gamma$, $z \in \mathcal{U}$, that is

$$\left|\arg\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)\right| < \frac{\pi}{2}\gamma,\ z \in \mathcal{U}.$$

□

For $\theta > 0$, let define the integral operator $\mathrm{J}_{p,\theta} : \Sigma_p \to \Sigma_p$ by

$$\mathrm{J}_{p,\theta}(f)(z) = \frac{\theta}{z^{\theta+p}}\int_0^z t^{\theta+p-1}f(t)\mathrm{d}t,\ z \in \mathcal{U}^*. \tag{2.28}$$

The next sharp subordination result deals with the connection between the functions $z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)$ and $z^p\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z)$.

Theorem 4. *Let $0 < r \leq 1$, $\theta > 0$, and $|C| \leq 1$, $|D| \leq 1$ with $C \neq D$. If $f \in \Sigma_p$ and*

$$z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z) \prec \left(\frac{1+Cz}{1+Dz}\right)^r, \tag{2.29}$$

then

$$z^p \mathcal{L}_{d,p}^{s+1}(n,m) \mathrm{J}_{p,\theta}(f)(z) \prec \widehat{p}(z), \tag{2.30}$$

where

$$\widehat{p}(z) = \begin{cases} {}_2F_1\left(r,\theta;\theta+1;-Dz\right) = (1+Dz)^{-r}\,{}_2F_1\left(r,1;\theta+1;\dfrac{Dz}{Dz+1}\right), & \text{if} \quad C = 0, \\ {}_2F_1\left(-r,\theta;\theta+1;-Cz\right) = (1+Cz)^{r}\,{}_2F_1\left(-r,1;\theta+1;\dfrac{Cz}{Cz+1}\right), & \text{if} \quad C \neq 0,\ D = 0, \\ \left(\dfrac{C}{D}\right)^r \displaystyle\sum_{j\geq 0} \dfrac{(-r)_j}{j!}\left(\dfrac{C-D}{C}\right)^j (1+Dz)^{-j}\,{}_2F_1\left(j,1;1+\theta;\dfrac{Dz}{1+Dz}\right), & \text{if} \quad DC \neq 0, \\ & \left|1-\dfrac{D}{C}\right| \leq 1-|D|. \end{cases}$$

If $r \in (0,1)$, and

$$\widehat{p}(z) = \begin{cases} \dfrac{C}{D} + \left(1-\dfrac{C}{D}\right)(1+Dz)^{-1}\,{}_2F_1\left(1;1;\theta+1;\dfrac{Dz}{1+Dz}\right), & \text{if} \quad D \neq 0, \\ 1 + \dfrac{\theta}{\theta+1}Cz, \quad \text{if} \quad D = 0, \end{cases}$$

for $r = 1$, and $\widehat{p}$ is the best dominant of (2.30). *Moreover,*

$$\mathrm{Re}(z^p \mathcal{L}_{d,p}^{s+1}(n,m) \mathrm{J}_{p,\theta}(f)(z) > \widehat{p}(-1),\ z \in \mathcal{U}, \tag{2.31}$$

and the inequality (2.31) *is the best possible.*

Proof. For $f \in \Sigma_p$, if we let

$$\phi(z) := z^p \mathcal{L}_{d,p}^{s+1}(n,m) \mathrm{J}_{p,\theta}(f)(z),\ z \in \mathcal{U},$$

then $\phi \in \mathcal{H}$.

The definition formula (2.28) could be written as

$$z^{\theta+p}\,\mathrm{J}_{p,\theta}(f)(z) = \theta \int_0^z t^{\theta+p-1} f(t)\mathrm{d}t,$$

and applying the linear operator $\mathcal{L}_{d,p}^{s+1}(n,m)$ to the above equality, we get

$$z^{\theta+p}\,\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z) = \theta \int_0^z t^{\theta+p-1} \mathcal{L}_{d,p}^{s+1}(n,m) f(t)\mathrm{d}t,\ z \in \mathcal{U}^*.$$

Differentiating this equality and multiplying with " z " it follows

$$z^{\theta+p+1}\left(\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z)\right)' + (\theta+p) z^{\theta+p}\,\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z) = \theta z^{\theta+p} \mathcal{L}_{d,p}^{s+1}(n,m) f(t)\mathrm{d}t,\ z \in \mathcal{U}^*,$$

and dividing both sides by " z^{θ} " we have

$$z^{p+1}\left(\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z)\right)' = \theta z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(t)\mathrm{d}t - (\theta+p)z^{p}\,\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z),\ z\in\mathcal{U}, \tag{2.32}$$

mentioning that this last relation also holds for $z = 0$. The differentiation of the function ϕ combined with (2.32), leads to

$$\begin{aligned} z\phi'(z) &= pz^{p}\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z) + z^{p+1}\left(\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z)\right)' \\ &= pz^{p}\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z) + \theta z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(t)\mathrm{d}t - (\theta+p)z^{p}\,\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z) \\ &= \theta\Big(z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(t)\mathrm{d}t - z^{p}\,\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z)\Big),\ z\in\mathcal{U}, \end{aligned}$$

or

$$\begin{aligned} \frac{z\phi'(z)}{\theta} &= z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(t)\mathrm{d}t - z^{p}\,\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z) \\ &= z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(t)\mathrm{d}t - \phi(z),\ z\in\mathcal{U}. \end{aligned}$$

Using this last relation, from the assumption (2.29) we deduce

$$\phi(z) + \frac{z\phi'(z)}{\theta} = z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(z) \prec \left(\frac{1+Cz}{1+Dz}\right)^{r}.$$

Now, using a similar techniques with those used in the proof of the previous theorem, we obtain the subordination (2.30) and the inequality (2.31). □

For the particular case $r = 1$ in Theorem 4 we get:

Corollary 4. *Let* $0 < r \le 1$, $\theta > 0$, *and* $|C| \le 1$, $|D| \le 1$ *with* $C \ne D$. *If* $f \in \Sigma_p$ *and*

$$z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)f(z) \prec \frac{1+Cz}{1+Dz},$$

then

$$\operatorname{Re}\left(z^{p}\mathcal{L}_{d,p}^{s+1}(n,m)\mathrm{J}_{p,\theta}(f)(z)\right) > \beta_1,\ z\in\mathcal{U}, \tag{2.33}$$

where

$$\beta_1 = \begin{cases} \dfrac{C}{D} + \left(1-\dfrac{C}{D}\right)(1-D)^{-1}\ {}_2F_1\left(1,1;\theta+1;\dfrac{D}{D-1}\right), & \textit{if}\quad D\ne 0, \\ 1-\dfrac{\theta}{\theta+1}C, & \textit{if}\quad D = 0, \end{cases}$$

and the inequality (2.33) *is the best possible.*

Example 4. (i) If we put $p = C = 1$, $D = -1$, and $\theta = 2$ in Corollary 4, we get the next implication:
If $f \in \Sigma_p$, then

$$z\mathcal{L}_{d,1}^{s+1}(n,m)f(z) \prec \frac{1+z}{1-z} \Rightarrow \operatorname{Re}\left(z\mathcal{L}_{d,1}^{s+1}(n,m)\mathrm{J}_{1,2}(f)(z)\right) > -1 + 4(1-\ln 2),\ z\in\mathcal{U}.$$

(ii) Taking in the above corollary $p = C = 1$, $D = -1$, and $\theta = 1$, the following implication holds:
If $f \in \Sigma_p$, then

$$z\mathcal{L}_{d,1}^{s+1}(n,m)f(z) \prec \frac{1+z}{1-z} \Rightarrow \operatorname{Re}\left(z\mathcal{L}_{d,1}^{s+1}(n,m)\mathrm{J}_{1,1}(f)(z)\right) > -1 + 2\ln 2,\ z\in\mathcal{U}.$$

The following result shows that the best dominant q for the function $\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}$ if the linear combination defined by the function Φ given by (2.2) is subordinated to a linear between $q(z)$ and $zq'(z)$.

Theorem 5. *Let the function q b univalent in $\mathcal{U}$, such that*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\frac{\delta d}{\alpha}\right\},\ z\in\mathcal{U}.$$

Let $0<r\leq 1$ *and for* $f\in\Sigma_p$ *suppose that*

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\neq 0,\ z\in\mathcal{U}.$$

If the function Φ *defined by* (2.2) *fulfill the subordination*

$$\Phi(z)\prec q(z)+\frac{\alpha}{\delta d}zq'(z), \tag{2.34}$$

then

$$\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}\prec q(z), \tag{2.35}$$

and q is the best dominant of (2.35).

Proof. For $f\in\Sigma_p$, if ϕ is the function defined by (2.6), from the left-hand-side of (2.7) we have

$$\Phi(z)=\phi(z)+\frac{\alpha}{\delta d}z\phi'(z),\ z\in\mathcal{U}. \tag{2.36}$$

Combining (2.34) and (2.36), we deduce

$$\phi(z)+\frac{\alpha}{\delta d}z\phi'(z)\prec q(z)+\frac{\alpha}{\delta d}zq'(z), \tag{2.37}$$

and our result follows from (2.37) by using Lemma 2. □

Taking in Theorem 5 the function $q(z)=\left(\frac{1+Cz}{1+Dz}\right)^r$ with $|C|\leq 1$, $|D|\leq 1$ and $C\neq D$, we obtain the next particular case:

Corollary 5. *For* $0<r\leq 1$ *and* $|C|\leq 1$, $|D|\leq 1$ *with $C\neq D$, assume that*

$$\operatorname{Re}\left(\frac{1-Dz}{1+Dz}+\frac{(r-1)(C-D)z}{(1+Dz)(1+Cz)}\right)>\max\left\{0;-\frac{\delta d}{\alpha}\right\},\ z\in\mathcal{U}.$$

Let $f\in\Sigma_p$ *such that*

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\neq 0,\ z\in\mathcal{U}.$$

If the function Φ *defined by* (2.2) *satisfies*

$$\Phi(z)\prec\left(\frac{1+Cz}{1+Dz}\right)^r+\frac{\alpha}{\delta d}\left(\frac{1+Cz}{1+Dz}\right)^r\frac{r(C-D)z}{(1+Dz)(1+Cz)},$$

then

$$\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}\prec\left(\frac{1+Cz}{1+Dz}\right)^r, \tag{2.38}$$

and the function $\left(\frac{1+Cz}{1+Dz}\right)^r$ *is the best dominant of* (2.38).

For $q(z)=\dfrac{1+Cz}{1+Dz}$, Theorem 5 leads to the next corollary:

Corollary 6. *For $|C|\le 1$, $|D|\le 1$ with $C\neq D$, suppose that*

$$\operatorname{Re}\frac{1-Dz}{1+Dz}>\max\left\{0;-\frac{\delta d}{\alpha}\right\},\ z\in\mathcal{U},$$

and let $f\in\Sigma_p$ such that

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\neq 0,\ z\in\mathcal{U}.$$

If the function Φ defined by (2.2) *satisfies the subordination*

$$\Phi(z)\prec\frac{1+Cz}{1+Dz}+\frac{\alpha}{\delta d}\frac{(C-D)z}{(1+Dz)^2},$$

then

$$\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}\prec\frac{1+Cz}{1+Dz} \tag{2.39}$$

and the function $\dfrac{1+Cz}{1+Dz}$ is the best dominant of (2.39).

If we put $s=0$ and $n=m=d=1$ in Theorem 5, we get the below particular case:

Corollary 7. *Let the function q be univalent in $\mathcal{U}$ such that*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\frac{\delta}{\alpha}\right\},\ z\in\mathcal{U}.$$

For $f\in\Sigma_p$ assume that

$$z^pf(z)\neq 0,\ z\in\mathcal{U}.$$

If the function Φ defined by (2.36) *satisfies*

$$\Phi(z)\prec q(z)+\frac{\alpha}{\delta}zq'(z),$$

then

$$(z^pf(z))^{\delta}\prec q(z), \tag{2.40}$$

and q is the best dominant of (2.40).

Taking $C=1$ and $D=-1$ in Corollaries 5 and 6, we get the next examples, respectively:

Example 5. (i) Let $0<r\le 1$ such that

$$\operatorname{Re}\left(\frac{1+z}{1-z}+\frac{2(r-1)z}{1-z^2}\right)>\max\left\{0;-\frac{\delta d}{\alpha}\right\},\ z\in\mathcal{U},$$

and for $f\in\Sigma_p$ assume that

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\neq 0,\ z\in\mathcal{U}.$$

If the function Φ is defined by (2.2), then

$$\Phi(z) \prec \left(\frac{1+z}{1-z}\right)^r + \frac{\alpha}{\delta}\left(\frac{1+z}{1-z}\right)^r \frac{2rz}{1-z^2}$$

implies

$$\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^\delta \prec \left(\frac{1+z}{1-z}\right)^r, \tag{2.41}$$

and the function $\left(\frac{1+z}{1-z}\right)^r$ is the best dominant of (2.41).

(ii) For $f \in \Sigma_p$ suppose that

$$z^p \mathcal{L}_{d,p}^{s+1}(n,m)f(z) \neq 0, \ z \in \mathcal{U}.$$

If the function Φ defined by (2.2) satisfies

$$\Phi(z) \prec \frac{1+z}{1-z} + \frac{\alpha}{\delta}\frac{2z}{1-z^2},$$

then

$$\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^\delta \prec \frac{1+z}{1-z}, \tag{2.42}$$

and the function $\frac{1+z}{1-z}$ is the best dominant of (2.42).

Example 6. If we put in the Corollary 7 $q(z) = \left(\frac{1+Cz}{1+Dz}\right)^r$ with $p = C = \alpha = \delta = 1$ and $D = -1$, we get the below result.

For $f \in \Sigma_p$ suppose that

$$zf(z) \neq 0, \ z \in \mathcal{U}.$$

Then,

$$2zf(z) + z^2 f'(z) \prec \left(\frac{1+z}{1-z}\right)^r + \left(\frac{1+z}{1-z}\right)^r \frac{2rz}{1-z^2},$$

implies

$$zf(z) \prec \left(\frac{1+z}{1-z}\right)^r, \tag{2.43}$$

and the function $\left(\frac{1+z}{1-z}\right)^r$ is the best dominant of (2.43).

The next result represents a superordination theorem that is the dual of Theorem 5.

Theorem 6. *For the function $f \in \Sigma_p$ assume that*

$$z^p \mathcal{L}_{d,p}^{s+1}(n,m)f(z) \neq 0, \ z \in \mathcal{U}.$$

Suppose that

$$\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^\delta \in \mathcal{H} \cap Q,$$

and the function Φ *defined by* (2.2) *is univalent in* $\mathcal{U}$.

If the function q *is convex (univalent) in* $\mathcal{U}$, *then*

$$q(z) + \frac{\alpha}{\delta d} z q'(z) \prec \Phi(z)$$

implies

$$q(z) \prec \left(z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z)\right)^{\delta}, \tag{2.44}$$

and q *is the best subordinant of* (2.44).

Proof. If ϕ is the function defined by (2.6), from the left hand side of (2.7) and (2.36) we deduce

$$q(z) + \frac{\alpha}{\delta d} z q'(z) \prec \Phi(z) = \phi(z) + \frac{\alpha}{\delta d} z \phi'(z),$$

and our result follows immediately from Lemma 3. □

For the particular case $q(z) = \left(\frac{1+Cz}{1+Dz}\right)^r$, Theorem 6 becomes:

Corollary 8. *Let* $0 < r \leq 1$, *and for* $f \in \Sigma_p$ *suppose that*

$$z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z) \neq 0, \ z \in \mathcal{U}.$$

Suppose that

$$\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z)\right)^{\delta} \in \mathcal{H} \cap Q,$$

and the function Φ *defined by* (2.2) *is univalent in* $\mathcal{U}$.

If

$$\left(\frac{1+Cz}{1+Dz}\right)^r + \frac{\alpha}{\delta d}\left(\frac{1+Cz}{1+Dz}\right)^r \frac{r(C-D)z}{(1+Dz)(1+Cz)} \prec \Phi(z),$$

then

$$\left(\frac{1+Cz}{1+Dz}\right)^r \prec \left(z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z)\right)^{\delta}, \tag{2.45}$$

and the function $\left(\frac{1+Cz}{1+Dz}\right)^r$ *is the best subordinant of* (2.45).

Taking $q(z) = \frac{1+Cz}{1+Dz}$ in Theorem 6, we get:

Corollary 9. *For* $f \in \Sigma_p$ *assume that*

$$z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z) \neq 0, \ z \in \mathcal{U}.$$

Suppose that

$$\left(z^p \mathcal{L}_{d,p}^{s+1}(n,m) f(z)\right)^{\delta} \in \mathcal{H} \cap Q,$$

and the function Φ *defined by* (2.2) *is univalent in* $\mathcal{U}$.

If

$$\frac{1+Cz}{1+Dz}+\frac{\alpha}{\delta d}\frac{(C-D)z}{(1+Dz)^2}\prec\Phi(z),$$

then

$$\frac{1+Cz}{1+Dz}\prec\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta},\tag{2.46}$$

and the function $\frac{1+Cz}{1+Dz}$ *is the best dominant of* (2.46).

Combining the results of Theorems 5 and 6, we obtain the following sandwich-type theorem:

Theorem 7. *For $f\in\Sigma_p$ assume that*

$$z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\neq 0,\ z\in\mathcal{U}.$$

Suppose that

$$\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}\in\mathcal{H}[q(0),1]\cap Q,$$

and the function Φ *defined by* (2.2) *is univalent in* $\mathcal{U}$. *Let* q_1 *be a convex (univalent) function in* $\mathcal{U}$, *and assume that* q_2 *is univalent in* $\mathcal{U}$ *such that*

$$\operatorname{Re}\left(1+\frac{zq_2''(z)}{q_2'(z)}\right)>\max\left\{0;-\frac{\delta d}{\alpha}\right\},\ z\in\mathcal{U}.$$

If

$$q_1(z)+\frac{\alpha}{\delta d}zq_1'(z)\prec\Phi_j(z)\prec q_2(z)+\frac{\alpha}{\delta d}zq_2'(z),$$

then

$$q_1(z)\prec\left(z^p\mathcal{L}_{d,p}^{s+1}(n,m)f(z)\right)^{\delta}\prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and best dominant of the above double subordination.*

3. Conclusions

The results presented in this paper represent an interesting investigation of differential subordinations and superordinations connected with the convolution operator $\mathcal{L}_{d,p}^{s}(n,m)$ and defined on the class of meromorphic multivalent functions in $\mathcal{U}^*$.

The main aspects we would like to emphasize are the successful using of the general theory of the differential subordinations and superordinations together with the well-known Nunokawa's lemma, to obtain sharp subordination results for a generalized convolution operator. These techniques are not widely used, while some immediate consequences are given in the corollaries and examples we obtained for some special cases. We hope that this study and some possible similar ones could help the specialists in this field to solve other related aspects of this field of interest.

For some possible new studies, newly defined classes can be considered by using the methods of the theory of differential superordinations, which could connect the outcomes of this study with possible further results. We believe that our results will be useful for the specialists of the field of Geometric Function Theory for some new approaches in this area.

Author contributions

Ekram E. Ali: Conceptualization, writing–original draft and editing, formal analysis, supervision, funding acquisition; Teodor Bulboacă: Conceptualization, writing–original draft and editing, formal analysis, supervision; Rabha M. El-Ashwah: Conceptualization, writing–original draft and editing, formal analysis, supervision; Abeer H. Alblowy: Conceptualization, writing–original draft and editing, formal analysis; Fozaiyah A. Alhubairah: Conceptualization, writing–original draft and editing, formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research has been funded by the Scientific Research Deanship at the University of Ha'il Saudi Arabia through project number RG-25 006.

Conflict of interest

The authors declare no conflicts of interest.

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