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Research article

A new scheme for simple asymmetric bivariate copulas and applications

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Abstract: Bivariate copulas play a central role in modeling the dependence structure between two random variables and serve as a fundamental tool in various applied fields. In this article, we develop a new theoretical framework aimed at constructing simple asymmetric bivariate copulas of the form $C(u, v) = uv\,[\phi(v) + u(1 - \phi(v))]$, $(u, v) \in [0, 1]^2$. This framework relies on a tuning univariate function to achieve the desired asymmetry. We study this pioneering scheme, emphasizing its theoretical foundations, and illustrating it with several examples. More precisely, we establish important properties of the proposed copulas and derive analytical expressions for concordance measures such as Spearman's rho, Kendall's tau, Gini's gamma, and Blomqvist's beta. In addition, we investigate the estimation procedure for the dependence parameter using the maximum likelihood approach. Finally, we conduct a simulation study to evaluate the performance of the proposed estimator. A real climatological dataset from the city of Abu Dhabi is used to demonstrate the applicability of the proposed copulas, with very convincing results.

Keywords: asymmetric copulas; dependence models; concordance measures; maximum likelihood method
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1. Introduction

Copulas are key multivariate functions in statistical modeling and risk analysis. They are designed to capture complex dependence structures between random variables by isolating marginal distributions from the underlying dependence structure. This was first pointed out by Sklar in [1], and has been studied over time. If we restrict our attention to the bivariate case, most of the existing work on the subject focuses on symmetric copulas. In this case, the two random variables involved can be exchanged without affecting their dependence structure; see [2–4].

Asymmetric copulas are thus a class of copulas designed to model dependence structures between random variables that exhibit non-symmetric relationships. Unlike symmetric copulas, asymmetric copulas adapt to situations where the strength of the dependence varies according to the specific "direction" of the relationship. These copulas can capture skewed or tail dependencies, making them indispensable in fields such as financial risk management, actuarial science, environmental science, hydrology, and water resources modeling, where such features are common. The related theory can also be found in [2–4]. For more detailed insights into the applications of asymmetric copulas, see [5–7].

Several methods have been proposed for constructing asymmetric copulas. Khoudraji's device [8] introduced asymmetry by multiplying copulas and incorporating shape parameters to control tail behavior, while Liebscher [9] extended this idea through an iterative construction that combines multiple base copulas with nonlinear marginal distortions. Alfonsi and Brigo [10] proposed a novel method for constructing asymmetric copulas by employing periodic functions. Durante [11] constructed asymmetric copulas by taking products of copulas evaluated at powered arguments, with the powers controlling the asymmetry between margins. Wu [12] made a significant contribution by presenting a practical and interpretable framework for constructing asymmetric copulas. The study introduces a novel approach that incorporates a convex combination of asymmetric copulas, enabling the modeling of different tail dependencies along distinct directions, which is particularly useful in reliability analysis. Overall, these approaches offer powerful tools for modeling asymmetric dependence, but their practical use is limited by complex parameter estimation, model selection challenges, interpretability issues, and computational demands.

There is still a need to create asymmetric bivariate copulas to better model the complex dependence structure and tail behavior observed in different data sets, which existing models may not adequately address. There is also a need to consider the simplicity of such dependence models: a model that is too complex may be difficult to handle from a computational point of view. With this in mind, this article proposes the development of a new theoretical scheme for the construction of simple asymmetric bivariate copulas. The need for simplicity in asymmetric copulas is based on practical and interpretable considerations. Moreover, our scheme is very flexible; it involves a certain function "$\phi(x)$" which can be chosen to be of different nature, provided that certain technical assumptions are met. The corresponding copula densities have a wide variety of shapes, making them applicable to many practical situations. We support this claim with various examples, backed up by numerical work.

The rest of the article is organized as follows: Section 2 discusses basic concepts related to copulas. Section 3 presents the main theoretical results. Section 4 gives some examples of asymmetric copulas. Section 5 examines the properties of selected asymmetric copulas and evaluates the performance of the estimated dependence parameter through a simulation study. Section 6 presents an analysis of a real climatological dataset from the city of Abu Dhabi, followed by concluding remarks and recommendations for future research in Section 7.

2. Materials and methods

In this section, we revisit the definition of a bivariate copula (asymmetric or not) and review the concepts of left-tail increasing, right-tail decreasing, and stochastically decreasing random variables.

Definition 2.1. *We consider the bivariate absolutely continuous context. Let $D(u, v)$, $(u, v) \in [0, 1]^2$,* *be a differentiable function on* $(0,1)^2$*. Then,* $D(u,v)$ *is called a copula if it meets the following two requirements:*

(I) $D(u,1) = u$, $D(1,v) = v$, $D(u,0) = D(0,u) = 0$.

(II) $\dfrac{\partial^2}{\partial u \partial v} D(u,v) \geq 0$.

We refer to [4] for more information on these technical requirements.

Having established the definition of a bivariate copula, we can now introduce the concepts of left-tail increasing, right-tail decreasing, and stochastically decreasing random variables.

Definition 2.2. *Let* (X,Y) *be a continuous random vector and* D *be the corresponding copula.*

(i) We say that Y is *left-tail increasing* in X, denoted LTI$(Y|X)$, if $D(u,v)/u$ is a nondecreasing function in u for any $v \in [0,1]$.

(ii) We say that Y is *right-tail decreasing* in X, denoted RTD$(Y|X)$, if $[1 - u - v + D(u,v)]/(1-u)$ is a nonincreasing function in u for any $v \in [0,1]$.

(iii) We say that Y is stochastically *decreasing* in X, denoted SD$(Y|X)$, if $\partial D(u,v)/\partial u$ is a nondecreasing function in u for any $v \in [0,1]$.

Detailed explanations of left-tail increasing, right-tail decreasing, and stochastically decreasing random variables can be found in [2–4].

Based on Theorems 5.2.4 and 5.2.12 from [4], we can deduce that if Y is stochastically *decreasing* in X, then

- Y is left-tail increasing in X.
- Y is right-tail decreasing in X.
- The copula D exhibits negative quadrant dependence (NQD), i.e., $D(u,v) \leq \Pi(u,v)$ for any $(u,v) \in [0,1]^2$, where $\Pi(u,v) = uv$ is the independent copula.

Building on these definitions and concepts, in the next section we introduce a new class of asymmetric copulas. This innovative approach allows for modeling dependence structures, capturing asymmetric relationships, and providing more flexibility in applications.

3. Asymmetric copula scheme

In this section, we construct a new class of asymmetric copula. The considered asymmetric copula scheme is described in the following proposition.

Proposition 3.1. *Let* $\phi(v)$*,* $v \in [0,1]$*, be a differentiable function over* $(0,1)$ *satisfying the following four requirements:*

(a) for any $v \in [0,1]$, $\phi(v) \leq 1$ and $\phi(1) = 1$,

(b) $\lim_{v \to 0} v\phi(v) = 0$,

(c) for any $v \in (0,1)$, $\phi(v) + v\phi'(v) \geq 0$,

(d) for any $v \in (0,1)$, $v\phi'(v) \leq 1$.

Then, the following bivariate function is a copula:

$$C(u,v) = uv\left[u + (1-u)\phi(v)\right], \quad (u,v) \in [0,1]^2. \tag{3.1}$$

Proof. The proof is based on the validation of **(I)** and **(II)** in Definition 2.1. First, we look at **(I)**, starting with $u = 1$ or $v = 1$. For any $u \in [0,1]$, from the equality $\phi(1) = 1$ in **(a)**, since $u + (1-u)\phi(1) = u + (1-u) = 1$, we have

$$C(u,1) = u \times 1 \times [u + (1-u)\phi(1)] = u.$$

For any $v \in [0,1]$, it is immediately true that

$$C(1,v) = 1 \times v \times [1 + (1-1)\phi(v)] = v.$$

Let us now examine the cases $u = 0$ or $v = 0$. For any $u \in [0,1]$, owing to **(b)**, we obtain

$$C(u,0) = \lim_{v\to 0} uv\left[u + (1-u)\phi(v)\right] = u^2 \lim_{v\to 0} v + u(1-u)\lim_{v\to 0} v\phi(v) = 0.$$

More directly, for any $v \in [0,1]$, we get

$$C(0,v) = 0 \times v \times [0 + (1-0)\phi(v)] = 0.$$

Thus, **(I)** is confirmed.

Let us now concentrate on **(II)**. For any $(u,v) \in (0,1)^2$, by applying standard differentiation rules and operating arranging factorizations, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial u \partial v} C(u,v) &= -2uv\phi'(v) + v\phi'(v) - 2u\phi(v) + \phi(v) + 2u \\ &= (1-u)[\phi(v) + v\phi'(v)] + u[1 - \phi(v)] + u[1 - v\phi'(v)].\end{aligned}$$

It is clear that $u \geq 0$ and $1 - u \geq 0$. Furthermore, from **(c)**, we have $\phi(v) + v\phi'(v) \geq 0$; from the inequality $\phi(v) \leq 1$ in **(a)**, we have $1 - \phi(v) \geq 0$; and from **(d)**, we have $1 - v\phi'(v) \geq 0$. We conclude that $\partial^2 C(u,v)/(\partial u \partial v) \geq 0$, so **(II)** holds. As a result, $C(u,v)$ is validated, and the proof is complete. □

The copula in Eq (3.1) can be expressed in the following way:

$$C(u,v) = uv\left[\phi(v) + u(1 - \phi(v))\right], \quad (u,v) \in [0,1]^2.$$

Note that for a continuous random vector (X,Y) with the copula C defined in Eq (3.1), SD$(Y|X)$ holds since $\partial^2 C(u,v)/\partial^2 u = 2v(1 - \phi(v)) \geq 0$. It follows that LTI$(Y|X)$ and RTD$(Y|X)$ are also satisfied. Furthermore, the copula C shows NQD. Consequently, as noted in [2], Spearman's rho and Kendall's tau are related by $\rho \leq \tau \leq 0$. For a short proof, see [13].

4. Examples

In this section, we consider the case where a cumulative distribution function (CDF) is chosen as a candidate to apply Proposition 3.1. We will also derive some examples of asymmetric copulas according to the following lemma.

Lemma 4.1. *If* $\phi(v)$, $v \in [0,1]$, *is a differentiable CDF of a distribution with support* $[0,1]$, *then* **(a)**, **(b)**, *and* **(c)** *in Proposition 3.1 are immediately satisfied.*

Proof. By the definition of a CDF of a distribution with support $[0,1]$, we have $\phi(0) = 0$ (or $\lim_{v\to 0}\phi(v) = 0$) and $\phi(1) = 1$ (or $\lim_{v\to 1}\phi(v) = 1$), and $\phi(v)$ is increasing over $(0,1)$, i.e., $\phi'(v) \geq 0$, which also implies that $\phi(v) \geq 0$.

As a result, for any $v \in [0,1]$, since $\phi(v)$ is increasing over $(0,1)$ and $\phi(1) = 1$, we have $\phi(v) \leq \phi(1)$ and $\phi(1) = 1$, implying **(a)**. Since $\phi(0) = 0$ (or $\lim_{v\to 0}\phi(v) = 0$), we get $\lim_{v\to 0} v\phi(v) = 0$, which corresponds to **(b)**. Since, for any $v \in (0,1)$, we have $\phi(v) \geq 0$ and $\phi'(v) \geq 0$, it is immediate that $\phi(v) + v\phi'(v) \geq 0$, which is **(c)**. □

Based on Lemma 4.1, Proposition 3.1 can be applied to any CDF of a distribution with support $[0,1]$ that satisfies the assumption **(d)**, which can be the one that drives the potential values of the parameters involved. This is illustrated in the proposition below.

Proposition 4.2. *Twelve examples of copulas are described below:*

Copula 1. For $\alpha \in [0,1]$, the following copula is valid:

$$C_{1,\alpha}(u,v) = uv[u + (1-u)v^\alpha], \quad (u,v) \in [0,1]^2.$$

Copula 2. For $\alpha \geq 1$, the following copula is valid:

$$C_{2,\alpha}(u,v) = uv[1 - (1-u)(1-v)^\alpha], \quad (u,v) \in [0,1]^2.$$

Copula 3. For $\alpha \in [1,2]$, the following copula is valid:

$$C_{3,\alpha}(u,v) = uv\left\{u + (1-u)[v(2-v)]^\alpha\right\}, \quad (u,v) \in [0,1]^2.$$

Copula 4. For $\alpha \in [0,4/\pi]$, the following copula is valid:

$$C_{4,\alpha}(u,v) = uv\left\{u + (1-u)\left[\sin\left(\frac{\pi}{2}v\right)\right]^\alpha\right\}, \quad (u,v) \in [0,1]^2.$$

Copula 5. For $\alpha \in [0,1]$, the following copula is valid:

$$C_{5,\alpha}(u,v) = uv\left[u + (1-u)\frac{v^\alpha - 1}{\alpha\ln(v)}\right], \quad (u,v) \in [0,1]^2.$$

Copula 6. For $\alpha \in [0,1]$, the following copula is valid:

$$C_{6,\alpha}(u,v) = uv\left[1 - \frac{\alpha}{1+\alpha}(1-u)\ln(v)v^\alpha\right], \quad (u,v) \in [0,1]^2.$$

Copula 7. For $\alpha \in [0,1]$, the following copula is valid:

$$C_{7,\alpha}(u,v) = uv\left[u + (1-u)ve^{\alpha(1-v)}\right], \quad (u,v) \in [0,1]^2.$$

Copula 8. For $\alpha > 0$, the following copula is valid:

$$C_{8,\alpha}(u,v) = uv\left[u + (1-u)\frac{\ln(1+\alpha v)}{\ln(1+\alpha)}\right], \quad (u,v) \in [0,1]^2.$$

Copula 9. For $\alpha > 0$, the following copula is valid:

$$C_{9,\alpha}(u,v) = uv\left[u + (1-u)\frac{\arctan(\alpha v)}{\arctan(\alpha)}\right], \quad (u,v) \in [0,1]^2.$$

Copula 10. For $\alpha > 0$, the following copula is valid:

$$C_{10,\alpha}(u,v) = uv\left[u + (1-u)\frac{\operatorname{erf}(\alpha v)}{\operatorname{erf}(\alpha)}\right], \quad (u,v) \in [0,1]^2,$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, \quad x \geq 0.$$

Copula 11. For $\alpha > 0$, the following copula is valid:

$$C_{11,\alpha}(u,v) = uv\left[u + (1-u)\frac{1-e^{-\alpha v}}{1-e^{-\alpha}}\right], \quad (u,v) \in [0,1]^2.$$

Copula 12. For $\alpha > 0$, the following copula is valid:

$$C_{12,\alpha}(u,v) = uv\left[u + (1+\alpha)(1-u)\frac{v}{\alpha+v}\right], \quad (u,v) \in [0,1]^2.$$

Proof. **Copula 1.** For $\alpha \in [0,1]$, let us consider the CDF of the power distribution defined as

$$\phi(v) = v^\alpha, \quad v \in [0,1].$$

Then, based on Lemma 4.1, in order to apply Proposition 3.1, it is sufficient to demonstrate **(d)**. For any $v \in (0,1)$, we have

$$v\phi'(v) = v\alpha v^{\alpha-1} = \alpha v^\alpha.$$

Therefore, for $\alpha \in [0,1]$, we have $v\phi'(v) = \alpha v^\alpha \leq \alpha \leq 1$. Hence, **(d)** is fulfilled.

Copula 2. For $\alpha \geq 1$, let us consider the CDF of the type II power distribution defined as

$$\phi(v) = 1-(1-v)^\alpha, \quad v \in [0,1].$$

Then, taking into account Lemma 4.1 and Proposition 3.1, it is sufficient to prove **(d)**. For any $v \in (0, 1)$, we have

$$\psi(v) = v\phi'(v) = \alpha v(1 - v)^{\alpha-1}.$$

Let us now study this function. We have $\psi'(v) = \alpha(1 - \alpha v)(1 - v)^{\alpha-2}$. Therefore, for $\alpha \geq 1$, we have $\psi'(v) = 0$ for $v = 1/\alpha \in [0, 1]$, and we can check that it is argmaximum. Hence, we have $\sup_{v\in[0,1]} \psi(v) = \psi(1/\alpha) = (1 - 1/\alpha)^{\alpha-1}$. As a result, for $\alpha \geq 1$, we have $\psi(v) \leq (1 - 1/\alpha)^{\alpha-1} \leq 1$. Condition **(d)** is verified.

Copula 3. For $\alpha \in [1, 2]$, let us consider the CDF of the Topp-Leone distribution defined as

$$\phi(v) = [v(2 - v)]^{\alpha}, \quad v \in [0, 1].$$

Then, based on Lemma 4.1, in order to apply Proposition 3.1, it is sufficient to demonstrate **(d)**. For any $v \in (0, 1)$, we have

$$v\phi'(v) = 2\alpha v(1 - v)[v(2 - v)]^{\alpha-1}.$$

See [14, 15]. For $\alpha = 1$, it is clear that $[v(2 - v)]^{\alpha-1} = 1$, and, for $\alpha > 1$, we can notice that $G(v) = [v(2-v)]^{\alpha-1}$ corresponds to the CDF of the Topp-Leone distribution with parameter $\alpha - 1$. Hence, we have $G(v) \leq 1$. Furthermore, we have $\sup_{v\in[0,1]} v(1 - v) = 1/4$ (with a argmaximum at $v = 1/2 \in [0, 1]$). Therefore, for $\alpha \in [1, 2]$, we have

$$v\phi'(v) = 2\alpha v(1 - v)[v(2 - v)]^{\alpha-1} \leq 2\alpha v(1 - v) \leq \frac{\alpha}{2} \leq 1.$$

Condition **(d)** is thus satisfied.

Copula 4. For $\alpha \in [0, 4/\pi]$, let us consider the CDF of the exponentiated sine distribution defined as

$$\phi(v) = \left[\sin\left(\frac{\pi}{2}v\right)\right]^{\alpha}, \quad v \in [0, 1].$$

This distribution was extensively studied in [16]. Then, with Lemma 4.1 and Proposition 3.1 in mind, it is sufficient to prove **(d)**. For any $v \in (0, 1)$, we have

$$\psi(v) = v\phi'(v) = \alpha\frac{\pi}{2}v\cos\left(\frac{\pi}{2}v\right)\left[\sin\left(\frac{\pi}{2}v\right)\right]^{\alpha-1}.$$

Now, let us distinguish the cases $\alpha \in [0, 1]$ and $\alpha \in (1, 4/\pi]$.

For $\alpha \in [0, 1]$, applying the well-known trigonometric inequality $\sin(x) \geq x\cos(x)$ for $x \in [0, \pi/2]$ and using $\phi(v) \leq 1$, we have

$$\psi(v) \leq \alpha\sin\left(\frac{\pi}{2}v\right)\left[\sin\left(\frac{\pi}{2}v\right)\right]^{\alpha-1} = \alpha\phi(v) \leq \alpha \leq 1.$$

On the other hand, for $\alpha \in (1, 4/\pi]$, it is clear that $\{\sin[(\pi/2)v]\}^{\alpha-1} \leq 1$. Furthermore, the Jordan inequality says that $\sin(x) \geq (2/\pi)x$ for $x \in [0, \pi/2]$, and we have $\sup_{v\in[0,1]} v\sqrt{1 - v^2} = 1/2$ (with an argmaximum at $v = 1/\sqrt{2} \in [0, 1]$). Therefore, for $\alpha \in (1, 4/\pi]$, we obtain

$$v\phi'(v) = \alpha\frac{\pi}{2}v\cos\left(\frac{\pi}{2}v\right)\left[\sin\left(\frac{\pi}{2}v\right)\right]^{\alpha-1}$$

$$\leq \alpha\frac{\pi}{2}v\cos\left(\frac{\pi}{2}v\right) = \alpha\frac{\pi}{2}v\sqrt{1-\left[\sin\left(\frac{\pi}{2}v\right)\right]^2}$$
$$\leq \alpha\frac{\pi}{2}v\sqrt{1-v^2} \leq \alpha\frac{\pi}{4} \leq 1.$$

Hence, **(d)** is satisfied.

Copula 5. For $\alpha \in [0,1]$, let us consider the CDF of the ratio-power-logarithmic distribution defined as

$$\phi(v) = \frac{v^\alpha - 1}{\alpha\ln(v)}, \quad v \in [0,1].$$

It was established in [17]. Then, based on Lemma 4.1, in order to apply Proposition 3.1, we only need to prove **(d)**. For any $v \in (0,1)$, we have

$$v\phi'(v) = v\frac{\alpha v^\alpha\ln(v) + 1 - v^\alpha}{\alpha v[\ln(v)]^2} = \frac{\alpha v^\alpha\ln(v) + 1 - v^\alpha}{\alpha[\ln(v)]^2}.$$

The following logarithmic inequality is well-known: $\ln(1+x) \leq x$, for $x > -1$. Therefore, we have $\alpha\ln(v) = \ln[1+(v^\alpha - 1)] \leq v^\alpha - 1$, and $1 - v^\alpha \leq -\alpha\ln(v)$. Using this inequality and $\phi(v) \leq 1$ as for any CDFs, we obtain

$$v\phi'(v) \leq \frac{\alpha v^\alpha\ln(v) - \alpha\ln(v)}{\alpha[\ln(v)]^2} = \frac{v^\alpha - 1}{\ln(v)} = \alpha\phi(v) \leq \alpha \leq 1.$$

Condition **(d)** is satisfied.

Copula 6. For $\alpha \in [0,1]$, let us consider the CDF of the log-Lindley distribution defined as

$$\phi(v) = \left[1 - \frac{\alpha}{1+\alpha}\ln(v)\right]v^\alpha, \quad v \in [0,1].$$

It was established in [17]. Then, taking into account Lemma 4.1 and Proposition 3.1, it is sufficient to prove **(d)**. For any $v \in (0,1)$, we have

$$\psi(v) = v\phi'(v) = v\frac{\alpha^2}{1+\alpha}[1-\ln(v)]v^{\alpha-1} = \frac{\alpha^2}{1+\alpha}[1-\ln(v)]v^\alpha.$$

Let us now study this function. We have

$$\psi'(v) = \frac{\alpha^2}{1+\alpha}[\alpha - 1 - \alpha\ln(v)]v^{\alpha-1}.$$

Therefore, for $\alpha \in [0,1]$, we have $\psi'(v) = 0$ for $v = e^{(\alpha-1)/\alpha} \in [0,1]$, and we can check that it is argmaximum. Hence, we have $\sup_{v\in[0,1]}\psi(v) = \psi(e^{(\alpha-1)/\alpha}) = [\alpha/(1+\alpha)]e^{\alpha-1}$. As a result, for $\alpha \in [0,1]$, we have

$$\psi(v) \leq \frac{\alpha}{1+\alpha}e^{\alpha-1} \leq \frac{\alpha}{1+\alpha} \leq 1.$$

Hence, **(d)** is fulfilled.

Copula 7. For $\alpha \in [0,1]$, let us consider the CDF of the polynomial-exponential distribution defined as

$$\phi(v) = ve^{\alpha(1-v)}, \quad v \in [0,1].$$

Then, based on Lemma 4.1, in order to apply Proposition 3.1, we only need to prove **(d)**. For any $v \in (0,1)$, we have

$$\psi(v) = v\phi'(v) = v(1-\alpha v)e^{\alpha(1-v)}.$$

Therefore, for $\alpha \in [0,1]$, since $1-\alpha v \leq 1$ and $\phi(v) \leq 1$, we have

$$\psi(v) \leq ve^{\alpha(1-v)} = \phi(v) \leq 1.$$

Condition **(d)** is thus fulfilled.

Copula 8. For $\alpha > 0$, let us consider the CDF of the truncated logarithmic distribution defined as

$$\phi(v) = \frac{\ln(1+\alpha v)}{\ln(1+\alpha)}, \quad v \in [0,1].$$

Then, with Lemma 4.1 and Proposition 3.1 in mind, it is sufficient to prove **(d)**. For any $v \in (0,1)$, we have

$$\psi(v) = v\phi'(v) = \frac{\alpha v}{(1+\alpha v)\ln(1+\alpha)}.$$

The following logarithmic inequality is well-known: $\ln(1+x) \geq x/(1+x)$, for $x > -1$. This inequality, combined with the fact that the logarithmic function is increasing, implies that

$$\ln(1+\alpha) \geq \ln(1+\alpha v) \geq \frac{\alpha v}{1+\alpha v},$$

which gives $\psi(v) \leq 1$. Hence, **(d)** is fulfilled.

Copula 9. For $\alpha > 0$, let us consider the CDF of the truncated arctangent distribution defined as

$$\phi(v) = \frac{\arctan(\alpha v)}{\arctan(\alpha)}, \quad v \in [0,1].$$

Then, based on Lemma 4.1, in order to apply Proposition 3.1, it is sufficient to prove **(d)**. For any $v \in (0,1)$, we have

$$\psi(v) = v\phi'(v) = \frac{\alpha v}{(1+\alpha^2 v^2)\arctan(\alpha)}.$$

The following arctangent inequality is well-known: $\arctan(x) \geq x/(1+x^2)$, for $x \geq 0$. This inequality, combined with the fact that the arctangent function is increasing, implies that

$$\arctan(\alpha) \geq \arctan(\alpha v) \geq \frac{\alpha v}{1+\alpha^2 v^2},$$

which gives $\psi(v) \leq 1$. Condition **(d)** is thus satisfied.

Copula 10. For $\alpha > 0$, let us consider the CDF of the truncated error function distribution defined as

$$\phi(v) = \frac{\mathrm{erf}(\alpha v)}{\mathrm{erf}(\alpha)}, \quad v \in [0, 1].$$

Then, we aim to apply Lemma 4.1 and Proposition 3.1. To do this, it is sufficient to prove **(d)**. For any $v \in (0, 1)$, we have

$$\psi(v) = v\phi'(v) = \frac{2\alpha}{\sqrt{\pi}\,\mathrm{erf}(\alpha)} v e^{-\alpha^2 v^2}.$$

Since $\alpha > 0$ and e^{-t^2} is a decreasing positive function, we have

$$\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(\alpha v) = \int_0^{\alpha v} e^{-t^2} dt \geq e^{-\alpha^2 v^2} \int_0^{\alpha v} dt = \alpha v e^{-\alpha^2 v^2}.$$

This inequality, combined with the fact that the error function is increasing, implies that

$$\mathrm{erf}(\alpha) \geq \mathrm{erf}(\alpha v) \geq \frac{2\alpha}{\sqrt{\pi}} v e^{-\alpha^2 v^2}.$$

Therefore, we have $\psi(v) \leq 1$. Hence, **(d)** is fulfilled.

Copula 11. For $\alpha > 0$, let us consider the CDF of the truncated exponential distribution defined as

$$\phi(v) = \frac{1 - e^{-\alpha v}}{1 - e^{-\alpha}}, \quad v \in [0, 1].$$

Then, based on Lemma 4.1, in order to apply Proposition 3.1, it is sufficient to prove **(d)**. For any $v \in (0, 1)$, we have

$$\psi(v) = v\phi'(v) = \frac{\alpha}{1 - e^{-\alpha}} v e^{-\alpha v}.$$

The following exponential inequality is well-known: $e^x \geq 1 + x$, for $x \in \mathbb{R}$, and it implies that $e^{\alpha v} \geq 1 + \alpha v$, which can be rewritten as $1 - e^{-\alpha v} \geq \alpha v e^{-\alpha v}$. Therefore, using the fact that the exponential function is increasing, we get

$$1 - e^{-\alpha} \geq 1 - e^{-\alpha v} \geq \alpha v e^{-\alpha v},$$

which implies that $\psi(v) \leq 1$. Hence, **(d)** is fulfilled.

Copula 12. For $\alpha > 0$, let us consider the CDF of the ratio-polynomial distribution defined as

$$\phi(v) = (1 + \alpha)\frac{v}{v + \alpha}, \quad v \in [0, 1].$$

Then, thanks to Lemma 4.1, in order to apply Proposition 3.1, it is sufficient to prove **(d)**. For any $v \in (0, 1)$, since $\phi(v) \leq 1$, we have

$$\psi(v) = v\phi'(v) = \alpha(1 + \alpha)\frac{v}{(v + \alpha)^2} = \frac{\alpha}{v + \alpha}\phi(v) \leq \frac{\alpha}{v + \alpha} \leq 1.$$

Condition **(d)** is thus satisfied.

□

Having established the theoretical framework and derived various asymmetric copulas in Section 4, we now turn our attention to studying the properties of some specific examples of copulas given in Proposition 4.2.

5. Properties of some proposed copulas

5.1. Framework

In this section, we present the properties of copulas outlined in Proposition 4.2, focusing on Copula 1, Copula 2, Copula 11, and Copula 12. They have the advantage of being simple, with a wide range for the single parameter that is involved. For these copulas, we compute the conditional copula, the copula density, and several association measures. We also use maximum likelihood estimation to estimate the dependence parameter for each copula. We conclude this section by evaluating the performance of the estimated dependence parameter through a simulation study.

Figure 1 exhibits the shape of the copula $C_{1,\alpha}$ based on different values of α.

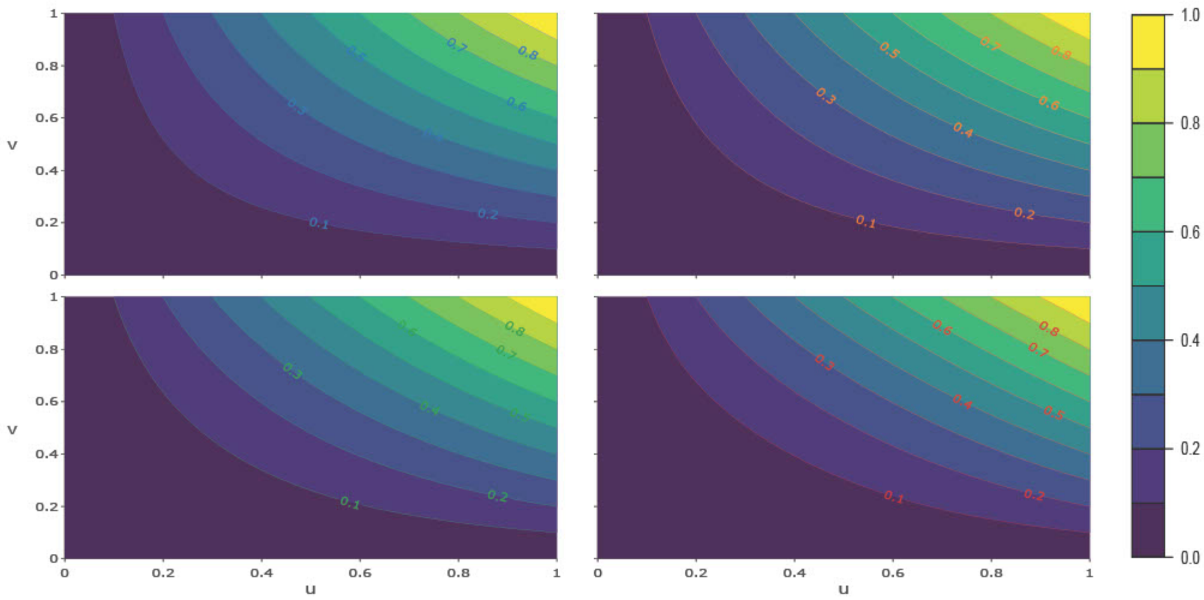

Figure 1. Intensity contours for $C_{1,\alpha}$, where $\alpha = 0.05$ (upper left), $\alpha = 0.25$ (upper right), $\alpha = 0.55$ (lower left), and $\alpha = 0.99$ (lower right).

Let us now consider a continuous random vector (U, V) with copula $\{C_{1,\alpha}, \alpha \in [0, 1]\}$. The conditional copula of V given $U = u$, corresponding to the Copula 1 is expressed as

$$\mathcal{A}_{1,u,\alpha}(v) = \frac{\partial C_{1,\alpha}(u, v)}{\partial u} = 2uv + (1 - 2u)v^{\alpha+1}, \quad (u, v) \in [0, 1]^2. \tag{5.1}$$

In a similar way, the conditional copula of V given $U = u$ for Copulas 2, 11, and 12 are, respectively, given by

$$\mathcal{A}_{2,u,\alpha}(v) = v\left[1 - (1 - 2u)(1 - v)^{\alpha}\right], \quad (u, v) \in [0, 1]^2,$$

$$\mathcal{A}_{11,u,\alpha}(v) = \frac{v\left[(1 - 2u)e^{-\alpha v} + 2ue^{-\alpha} - 1\right]}{e^{-\alpha} - 1}, \quad (u, v) \in [0, 1]^2,$$

and

$$\mathcal{A}_{12,u,\alpha}(v) = \frac{\left[(2u - 1)v - 2u\right]\alpha - v}{\alpha + v}, \quad (u, v) \in [0, 1]^2,$$

respectively.

The conditional copula of V given $U = u$, as given in Eq (5.1), can be used to generate data from the copula $C_{1,\alpha}$. This process can be implemented using the following algorithm:

1) Generate two independent values u and v_1 from the uniform distribution over $[0, 1]$.

2) Find v which satisfies $\mathcal{A}_{1,u,\alpha}(v) = 2uv + (1 - 2u)v^{\alpha+1} = v_1$.

3) The desired pair generated by the copula $C_{1,\alpha}$ is then (u, v).

Note that the previous algorithm can also be used to generate data from the copulas $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$ by using, in Step 2, the conditional copulas $\mathcal{A}_{2,u,\alpha}$, $\mathcal{A}_{11,u,\alpha}$, and $\mathcal{A}_{12,u,\alpha}$, respectively.

Figure 2 presents scatterplots of $n = 50$ data pairs generated using the asymmetric copula $C_{1,\alpha}$ for different values of the dependence parameter α. These scatterplots effectively show the effect of α on the relationship between the variables. As α increases, the dependence between the variables becomes more pronounced, moving from weak to moderate. The asymmetric copula $C_{1,\alpha}$ captures the nuances of this dependence, highlighting the varying strength of the relationship in different regions.

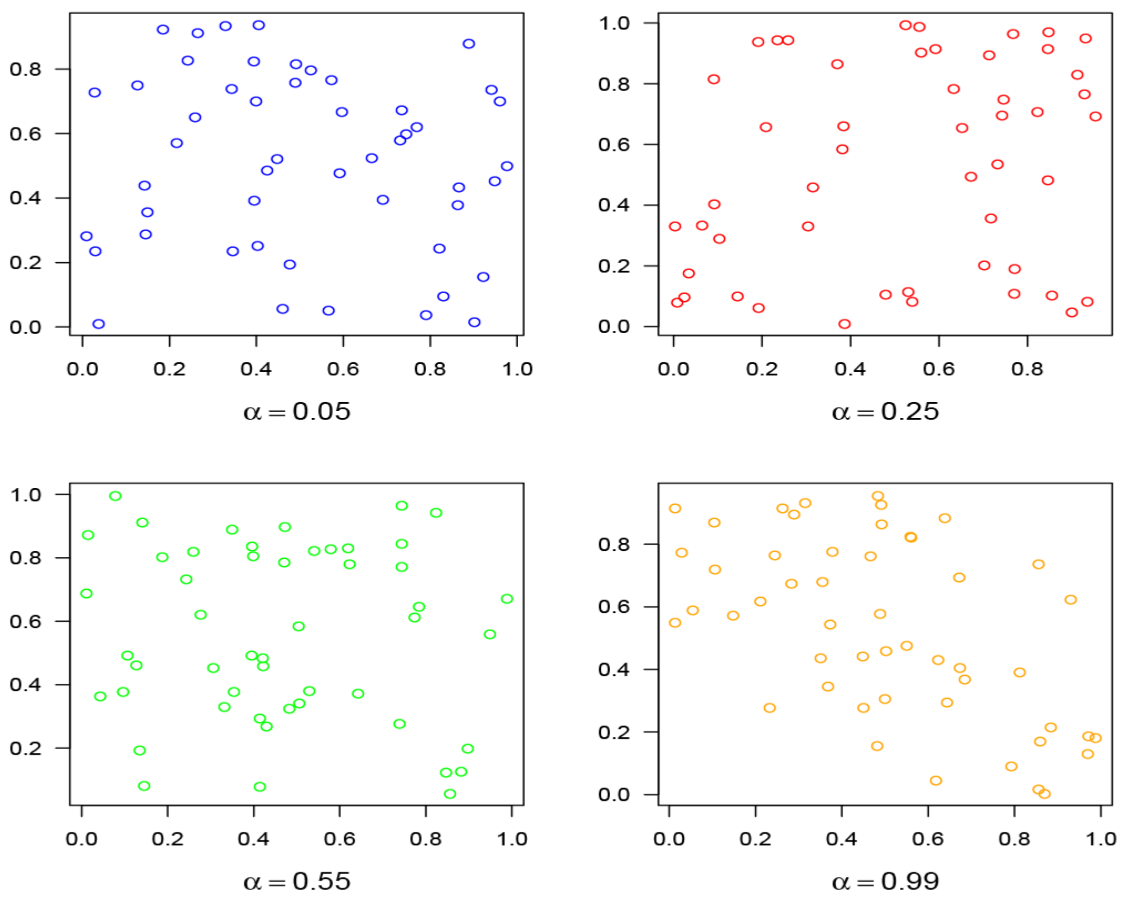

Figure 2. Scatterplots of $n = 50$ pairs of data based on $C_{1,\alpha}$ for different values of α.

The copula density of $C_{1,\alpha}$ is given by

$$c_{1,\alpha}(u,v) = \frac{\partial^2 C_{1,\alpha}(u,v)}{\partial u \partial v} = 2u + (1+\alpha)(1-2u)v^{\alpha}, \quad (u,v) \in [0,1]^2. \tag{5.2}$$

Similarly, the copula densities associated with $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$ are obtained as

$$c_{2,\alpha}(u,v) = (1-2u)(1-v)^{\alpha-1}\left[(\alpha+1)v - 1\right] + 1, \quad (u,v) \in [0,1]^2,$$

$$c_{11,\alpha}(u,v) = \frac{(1-2u)(1-\alpha v)e^{-\alpha v} + 2ue^{-\alpha} - 1}{e^{-\alpha} - 1}, \quad (u,v) \in [0,1]^2,$$

and

$$c_{12,\alpha}(u,v) = \frac{\left[(2-4u)v + 2u\right]\alpha^2 + \left[(1-2u)v^2 + 2v\right]\alpha + v^2}{(\alpha+v)^2}, \quad (u,v) \in [0,1]^2,$$

respectively.

Figure 3 displays the shape of the copula density $c_{1,\alpha}$ associated with the asymmetric copula $C_{1,\alpha}$ for different values of α. This figure illustrates how the dependence between variables changes as the parameter α increases from 0.05 to 0.99. At $\alpha = 0.05$, the density is almost uniform, indicating a very weak dependence. At $\alpha = 0.25$, there is a slight structuring, still indicating a very weak dependence. At $\alpha = 0.55$, the density becomes more pronounced. Finally, at $\alpha = 0.99$, the density is highly structured with sharp peaks.

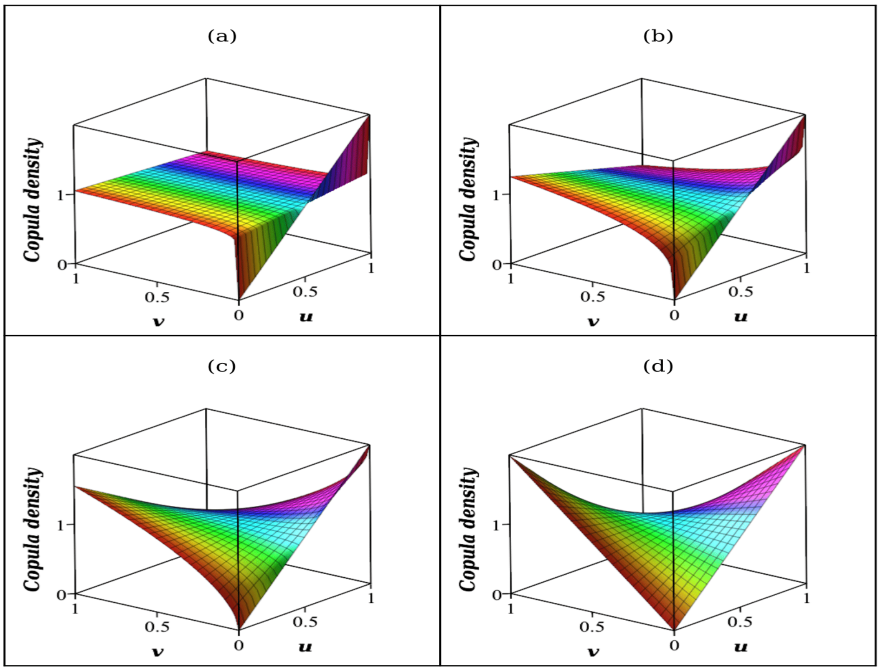

Figure 3. Copula density $c_{1,\alpha}$ for (a) $\alpha = 0.05$, (b) $\alpha = 0.25$, (c) $\alpha = 0.55$, and (d) $\alpha = 0.99$.

5.2. Concordance measures of the proposed copulas

We now turn to the most common measures of association, known as concordance measures. These include Spearman's rho, Kendall's tau, Gini's gamma, and Blomqvist's beta. Our aim is to derive explicit expressions for these measures corresponding to the proposed copulas $C_{1,\alpha}$, $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$. Let (U, V) be a continuous random vector with copula C_α. The population versions of Spearman's rho, Kendall's tau, Gini's gamma, and Blomqvist's beta can be expressed in terms of the copula C_α and corresponding density c_α as follows (see [4] for more details):

$$\rho_\alpha = 12 \int_0^1 \int_0^1 C_\alpha(u, v) du dv - 3,$$

$$\tau_\alpha = 4 \int_0^1 \int_0^1 C_\alpha(u, v) c_\alpha(u, v) du dv - 1,$$

$$\gamma_\alpha = 4 \left[\int_0^1 C_\alpha(u, 1-u) du - \int_0^1 (u - C_\alpha(u, u))\, du \right],$$

and

$$\beta_\alpha = 4 C_\alpha\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

Tables 1 and 2 summarize the expressions and the ranges of the association measures ρ_α, τ_α, γ_α, and β_α corresponding to the copulas $C_{1,\alpha}$, $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$.

Table 1. Association measures of the copulas $C_{1,\alpha}$, $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$.

Copula	ρ_α	τ_α	γ_α	β_α
$C_{1,\alpha}$	$-\frac{\alpha}{\alpha+2}$	$-\frac{2\alpha}{3\alpha+6}$	$-\frac{2\alpha(\alpha+7)}{3\alpha^2+21\alpha+36}$	$-\frac{1}{2} + \frac{1}{2^{\alpha+1}}$
$C_{2,\alpha}$	$-\frac{2}{\alpha^2+3\alpha+2}$	$-\frac{4}{3\alpha^2+9\alpha+6}$	$-\frac{16}{\alpha^3+9\alpha^2+26\alpha+24}$	$-\frac{1}{2^{\alpha+1}}$
$C_{11,\alpha}$	$\frac{\alpha^2+2\alpha-2e^\alpha+2}{\alpha^2(e^\alpha-1)}$	$\frac{2\alpha^2+4\alpha-4e^\alpha+4}{3\alpha^2(e^\alpha-1)}$	$\frac{2\alpha^4-24\alpha^2-96\alpha-(48\alpha-144)e^\alpha-144}{3\alpha^4(e^\alpha-1)}$	$-\frac{1}{2+2e^{\alpha/2}}$
$C_{12,\alpha}$	$-2(\alpha^3+\alpha^2)\ln\left(\frac{\alpha}{\alpha+1}\right)$- $(2\alpha^2+\alpha)$	$-\frac{4}{3}(\alpha^3+\alpha^2)\ln\left(\frac{\alpha}{\alpha+1}\right)$- $\frac{2}{3}(2\alpha^2+\alpha)$	$8\alpha^3(\alpha+1)^2\ln\left(\frac{\alpha}{\alpha+1}\right)+$ $(8\alpha^2+4\alpha)(\alpha^2+\alpha-\frac{1}{6})$	$-\frac{\alpha}{4\alpha+2}$

Table 2. Ranges of the association measures corresponding to the copulas $C_{i,\alpha}$, $i = 1, 2, 11,$ and 12.

Copula	ρ_α	τ_α	γ_α	β_α
$C_{i,\alpha}$	$[-0.3333, 0]$	$[-0.2222, 0]$	$[-0.2667, 0]$	$[-0.25, 0]$

Remarks

1) According to Table 1, one can observe that $\tau_\alpha = (2/3)\rho_\alpha$.

2) The copulas $C_{1,\alpha}$, $C_{2,\alpha}$ $C_{11,\alpha}$, and $C_{12,\alpha}$ can model weak negative dependence to moderate negative dependence as evidenced by $-1/3 \leq \rho_\alpha \leq 0$, $-2/9 \leq \tau_\alpha \leq 0$, $-4/15 \leq \gamma_\alpha \leq 0$, and $-1/4 \leq \beta_\alpha \leq 0$.

As noted earlier, the concordance measures Spearman's rho and Kendall's tau corresponding to the copula Eq (3.1); in particular, Copula 1, Copula 2, Copula 11, and Copula 12, satisfy $\rho_\alpha \leq \tau_\alpha \leq 0$. This relationship is illustrated visually in Figure 4.

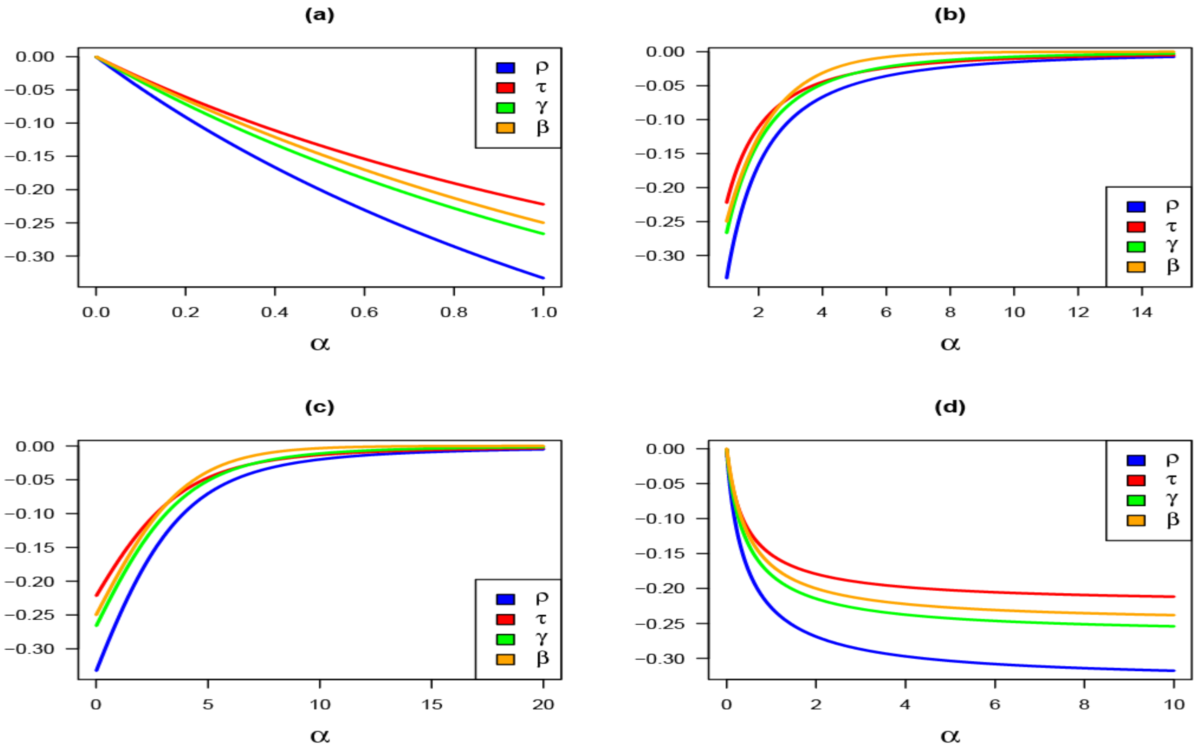

Figure 4. Graph of association measures for (a) $C_{1,\alpha}$, (b) $C_{2,\alpha}$, (b) $C_{11,\alpha}$, and (d) $C_{12,\alpha}$.

5.3. *Dependence parameter estimation of the proposed copulas*

To estimate the dependence parameter α of the proposed copulas $C_{1,\alpha}$, $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$, we adopt the maximum likelihood (ML) parameter estimation procedure. To achieve this, let n be a positive integer and $(u_1, v_1), \ldots, (u_n, v_n)$ denote mutually independent copies of (U, V) from the copula $C_{1,\alpha}$. The ML estimation of the copula parameter is based on the copula density in Eq (5.2). The likelihood function for $\alpha \in [0, 1]$ can be expressed as

$$L_1(\alpha \mid u, v) = \prod_{i=1}^{n} \{c_{1,\alpha}(u_i, v_i)\} = \prod_{i=1}^{n} \{2u_i + (\alpha + 1)(1 - 2u_i)v_i^{\alpha}\},$$

and its corresponding log-likelihood is

$$\ell_1(\alpha \mid u, v) = \sum_{i=1}^{n} \ln\{2u_i + (1 + \alpha)(1 - 2u_i)v_i^{\alpha}\}, \tag{5.3}$$

with

$$\frac{\partial}{\partial \alpha}\ell_1(\alpha \mid u, v) = \sum_{i=1}^{n} \frac{(1-2u_i)\, v_i^{\alpha} + (1+\alpha)\,(1-2u_i) v_i^{\alpha} \ln(v_i)}{2u_i + (1+\alpha)(1-2u_i) v_i^{\alpha}}.$$

The latter lacks a closed-form solution for deriving $\hat{\alpha}_{\mathrm{ML}}$, the ML estimator of the dependence parameter α for the copula $C_{1,\alpha}$. Thus, the numerical maximization of Eq (5.3) is employed.

The ML estimator $\hat{\alpha}_{\mathrm{ML}}$ is asymptotically normal, meaning that

$$\sqrt{n}\,(\hat{\alpha}_{\mathrm{ML}} - \alpha) \xrightarrow{d} \mathcal{N}\left(0, I_1^{-1}(\alpha)\right) \quad \text{as} \quad n \to \infty,$$

where $I_1(\alpha)$ is the fisher information given by

$$I_1(\alpha) = -\int_0^1 \int_0^1 \frac{\partial^2}{\partial^2 \alpha}\ell_1(\alpha \mid u, v) c_{1,\alpha}(u, v)\, du dv, \tag{5.4}$$

and

$$\frac{\partial^2}{\partial^2 \alpha}\ell_1(\alpha \mid u, v) = \frac{(u-1/2)\left\{(u-1/2)v^{2\alpha} + [2 + (1+\alpha)\ln(v)]\, u v^{\alpha} \ln(v)\right\}}{[(1+\alpha)(u-1/2)v^{\alpha} - u]^2}.$$

Since there is no closed form of the integral in Eq (5.4), we will calculate it numerically.

The log-likelihood functions for the copulas $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$ are

$$\ell_2(\alpha \mid u, v) = \sum_{i=1}^{n} \ln\left\{(1-2u_i)(1-v_i)^{\alpha-1}\,[(\alpha+1)v_i - 1] + 1\right\}, \tag{5.5}$$

$$\ell_{11}(\alpha \mid u, v) = \sum_{i=1}^{n} \ln\left\{\frac{(1-2u_i)(1-\alpha v_i)e^{-\alpha v_i} + 2u_i e^{-\alpha} - 1}{e^{-\alpha} - 1}\right\}, \tag{5.6}$$

and

$$\ell_{12}(\alpha \mid u, v) = \sum_{i=1}^{n} \ln\left\{\frac{[(2-4u_i)v_i + 2u_i]\,\alpha^2 + \left[(1-2u_i)v_i^2 + 2v_i\right]\alpha + v_i^2}{(\alpha + v_i)^2}\right\}, \tag{5.7}$$

respectively.

Since, $\partial\ell_{2,\alpha}(\alpha \mid u, v)/\partial\alpha$, $\partial\ell_{11,\alpha}(\alpha \mid u, v)/\partial\alpha$, and $\partial\ell_{12,\alpha}(\alpha \mid u, v)/\partial\alpha$ do not have closed forms, a numerical maximization of Eqs (5.5)–(5.7) is used to find $\hat{\alpha}_{\mathrm{ML}}$, the ML estimator of α corresponding to the copulas $C_{2,\alpha}$ $C_{11,\alpha}$, and $C_{12,\alpha}$, respectively.

Similar to the Fisher information for Copula 1 given in Eq (5.4), the Fisher information for copulas $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$, denoted as $I_2(\alpha)$, $I_{11}(\alpha)$, and $I_{12}(\alpha)$, respectively, can be calculated numerically.

5.4. Simulation study

We now evaluate the performance of the ML estimator for the dependence parameter α of Copulas 1, 2, 11, and 12. In this evaluation, we calculate the bias and the mean squared error (MSE) of the ML estimator $\hat{\alpha}_{ML}$. Specifically, the bias is given by the empirical version of $\mathrm{Bias}(\hat{\alpha}_{ML}) = E\left[\hat{\alpha}_{ML}\right] - \alpha$, and the MSE is given by the empirical version of $\mathrm{MSE}(\hat{\alpha}_{ML}) = E\left[(\hat{\alpha}_{ML} - \alpha)^2\right]$. We also provide the 95% asymptotic confidence intervals for α. To do this, we generate n mutually independent copies $(u_1, v_1), \ldots, (u_n, v_n)$ of the random vector (U, V) associated with each of the specified copulas, and for different sample sizes $n \in \{100, 200, 300, 400, 500, 1000\}$. For each generated sample, we apply the ML estimation approach to estimate the parameter of interest using the *optim* function, a general purpose optimization tool in R 4.2.1. This process is repeated $k = 500$ times to ensure robust results.

Table 3. ML estimation for α corresponding to $C_{1,\alpha}$, Bias, and MSE of $\hat{\alpha}_{ML}$, based on sample sizes $n \in \{100, 200, 300, 400, 500, 1000\}$.

	n	$\hat{\alpha}_{ML}$	Bias($\hat{\alpha}_{ML}$)	MSE($\hat{\alpha}_{ML}$)	95% C.I(α)
$\alpha = 0.1$	100	0.1273	0.0273	0.0153	(0.0876, 0.1670)
	200	0.1145	0.0145	0.0109	(0.0948, 0.1342)
	300	0.1114	0.0114	0.0093	(0.0983, 0.1246)
	400	0.1063	0.0063	0.0079	(0.0965, 0.1161)
	500	0.1011	0.0011	0.0064	(0.0932, 0.1089)
	1000	0.0992	−0.0008	0.0028	(0.0953, 0.1032)
$\alpha = 0.3$	100	0.3274	0.0274	0.0590	(0.2810, 0.3737)
	200	0.3155	0.0155	0.0301	(0.2917, 0.3393)
	300	0.3127	0.0127	0.0220	(0.2967, 0.3287)
	400	0.3076	0.0076	0.0170	(0.2955, 0.3197)
	500	0.3001	−0.0010	0.0127	(0.2905, 0.3098)
	1000	0.2991	−0.0009	0.0035	(0.2942, 0.304)
$\alpha = 0.5$	100	0.5232	0.0232	0.0725	(0.4708, 0.5757)
	200	0.5214	0.0214	0.0435	(0.4943, 0.5486)
	300	0.5158	0.0158	0.0308	(0.4974, 0.5341)
	400	0.5121	0.0121	0.0219	(0.4982, 0.526)
	500	0.4948	−0.0052	0.0167	(0.4838, 0.5059)
	1000	0.4973	−0.0027	0.0042	(0.4917, 0.5029)
$\alpha = 0.7$	100	0.6632	−0.0368	0.0739	(0.6079, 0.7185)
	200	0.6864	−0.0136	0.0425	(0.6575, 0.7153)
	300	0.7117	0.0117	0.0284	(0.6920, 0.7314)
	400	0.7062	0.0062	0.0208	(0.6913, 0.7211)
	500	0.7031	0.0031	0.0197	(0.6911, 0.7150)
	1000	0.6993	−0.0007	0.0045	(0.6933, 0.7054)
$\alpha = 0.9$	100	0.8454	−0.0546	0.0394	(0.7875, 0.9033)
	200	0.8764	−0.0236	0.0221	(0.8470, 0.9057)
	300	0.8857	−0.0143	0.0144	(0.8660, 0.9055)
	400	0.8941	−0.0059	0.0106	(0.8792, 0.9089)
	500	0.8957	−0.0043	0.0099	(0.8838, 0.9076)
	1000	0.8982	−0.0018	0.0045	(0.8922, 0.9043)

The overall results of the simulation studies for Copulas 1, 2, 11, and 12 are summarized in Tables 3–6, respectively. These results indicate that the ML estimator performs well across different values of the dependence parameter for each specified copula. In particular, as the sample size increases, both the bias and the mean squared error of the estimator decrease, and the 95% confidence interval for α becomes narrower. This demonstrates the consistency and efficiency of the ML estimator. Further discussion on the asymptotic efficiency of the maximum likelihood estimator is provided in [18].

Table 4. ML estimation for α corresponding to $C_{2,\alpha}$, Bias, and MSE of $\hat{\alpha}_{ML}$, based on sample sizes $n \in \{100, 200, 300, 400, 500, 1000\}$.

	n	$\hat{\alpha}_{ML}$	Bias($\hat{\alpha}_{ML}$)	MSE($\hat{\alpha}_{ML}$)	95% C.I(α)
$\alpha = 1.1$	100	1.1189	0.0189	0.0158	(1.0879, 1.1499)
	200	1.1139	0.0139	0.0130	(1.0977, 1.1301)
	300	1.1085	0.0085	0.0118	(1.0977, 1.1194)
	400	1.1075	0.0075	0.0093	(1.0988, 1.1161)
	500	1.1051	0.0051	0.0086	(1.0981, 1.1121)
	1000	1.1040	0.0040	0.0065	(1.1001, 1.1079)
$\alpha = 1.5$	100	1.5199	0.0199	0.0774	(1.438, 1.6018)
	200	1.5186	0.0186	0.0652	(1.4764, 1.5608)
	300	1.5143	0.0143	0.0540	(1.4856, 1.5429)
	400	1.5079	0.0079	0.0439	1(1.486, 1.5298)
	500	1.5056	0.0056	0.0381	(1.4879, 1.5234)
	1000	1.5044	0.0044	0.0272	(1.4953, 1.5135)
$\alpha = 1.9$	100	1.9236	0.0236	0.0953	(1.784, 2.0631)
	200	1.9197	0.0197	0.0811	(1.8491, 1.9903)
	300	1.9171	0.0171	0.0742	(1.8698, 1.9644)
	400	1.8950	−0.0050	0.0551	(1.8596, 1.9304)
	500	1.8972	−0.0028	0.0511	(1.8687, 1.9257)
	1000	1.9019	0.0019	0.0022	(1.8866, 1.9171)
$\alpha = 2.3$	100	2.2826	−0.0174	0.0748	(2.0843, 2.4809)
	200	2.3133	0.0133	0.0719	(2.2118, 2.4149)
	300	2.3094	0.0094	0.0697	(2.2418, 2.3771)
	400	2.3070	0.0070	0.0646	(2.2562, 2.3579)
	500	2.3068	0.0068	0.0598	(2.266, 2.3476)
	1000	2.3061	0.0061	0.0290	(2.2852, 2.3269)
$\alpha = 4$	100	3.9805	−0.0195	0.0849	(3.4772, 4.4838)
	200	4.0162	0.0162	0.0804	(3.7609, 4.2716)
	300	3.9838	−0.0162	0.0789	(3.8157, 4.1520)
	400	4.0080	0.0080	0.0788	(3.8807, 4.1353)
	500	3.9958	−0.0042	0.0769	(3.8944, 4.0972)
	1000	4.0038	0.0038	0.0093	(3.9522, 4.0554)

Table 5. ML estimation for α corresponding to $C_{11,\alpha}$, Bias, and MSE of $\hat{\alpha}_{ML}$, based on sample sizes $n \in \{100, 200, 300, 400, 500, 1000\}$.

	n	$\hat{\alpha}_{ML}$	Bias($\hat{\alpha}_{ML}$)	MSE($\hat{\alpha}_{ML}$)	95% C.I(α)
$\alpha = 0.1$	100	0.1452	0.0452	0.0210	(0.0024, 0.2879)
	200	0.1266	0.0266	0.0184	(0.0561, 0.1971)
	300	0.0884	−0.0116	0.0079	(0.0422, 0.1346)
	400	0.0902	−0.0098	0.0076	(0.0555, 0.1250)
	500	0.1029	0.0029	0.0076	(0.0748, 0.1310)
	1000	0.0982	−0.0018	0.0022	(0.0839, 0.1124)
$\alpha = 1$	100	0.9797	−0.0203	0.0765	(0.7725, 1.1869)
	200	0.9808	−0.0192	0.0701	(0.8771, 1.0845)
	300	1.0122	0.0122	0.0677	(0.9424, 1.0821)
	400	1.0062	0.0062	0.0657	(0.9539, 1.0585)
	500	0.9956	−0.0044	0.0597	(0.9539, 1.0374)
	1000	1.0015	0.0015	0.0004	(0.9804, 1.0226)
$\alpha = 2$	100	2.0303	0.0303	0.0810	(1.7401, 2.3205)
	200	1.9892	−0.0108	0.0778	(1.8459, 2.1324)
	300	1.9907	−0.0093	0.0726	(1.8951, 2.0863)
	400	2.003	0.0030	0.0714	(1.9311, 2.0750)
	500	1.9977	−0.0023	0.0093	(1.9399, 2.0556)
	1000	2.0005	0.0005	0.0089	(1.9715, 2.0294)
$\alpha = 3$	100	2.9867	−0.0133	0.0838	(2.5913, 3.3821)
	200	2.9892	−0.0108	0.0792	(2.7913, 3.1872)
	300	3.0097	0.0097	0.0777	(2.8768, 3.1425)
	400	2.9943	−0.0057	0.0764	(2.8951, 3.0934)
	500	3.0036	0.0036	0.0726	(2.924, 3.0832)
	1000	2.9966	−0.0034	0.0093	(2.9567, 3.0366)
$\alpha = 4$	100	4.0181	0.0181	0.0856	(3.4673, 4.5688)
	200	3.9821	−0.0179	0.0827	(3.7097, 4.2545)
	300	4.0138	0.0138	0.0810	(3.8304, 4.1972)
	400	3.9889	−0.0111	0.0795	(3.8524, 4.1255)
	500	3.9924	−0.0076	0.0789	(3.883, 4.1017)
	1000	3.9989	−0.0011	0.0094	(3.9436, 4.0541)

Table 6. ML estimation for α corresponding to $C_{12,\alpha}$, Bias, and MSE of $\hat{\alpha}_{ML}$, based on sample sizes $n \in \{100, 200, 300, 400, 500, 1000\}$.

	n	$\hat{\alpha}_{ML}$	Bias($\hat{\alpha}_{ML}$)	MSE($\hat{\alpha}_{ML}$)	95% C.I(α)
$\alpha = 0.1$	100	0.1188	0.0188	0.0061	(0.0966, 0.1410)
	200	0.1100	0.0100	0.0032	(0.0965, 0.1235)
	300	0.1045	0.0045	0.0028	(0.0955, 0.1135)
	400	0.0983	−0.0017	0.0003	(0.0905, 0.1062)
	500	0.1010	0.0010	0.0003	(0.0946, 0.1074)
	1000	0.1003	0.0003	0.0001	(0.0970, 0.1035)
$\alpha = 0.3$	100	0.3209	0.0209	0.0496	(0.2902, 0.3516)
	200	0.3138	0.0138	0.0272	(0.2931, 0.3344)
	300	0.3103	0.0103	0.0208	(0.2946, 0.3260)
	400	0.3076	0.0076	0.0197	(0.2944, 0.3208)
	500	0.3066	0.0066	0.0185	(0.2957, 0.3176)
	1000	0.2994	−0.0006	0.0004	(0.2922, 0.3066)
$\alpha = 0.5$	100	0.4764	−0.0236	0.0841	(0.4340, 0.5188)
	200	0.4869	−0.0131	0.0677	(0.4601, 0.5136)
	300	0.5106	0.0106	0.0535	(0.4850, 0.5361)
	400	0.5073	0.0073	0.0461	(0.4870, 0.5277)
	500	0.5050	0.0050	0.0432	(0.4876, 0.5224)
	1000	0.5003	0.0003	0.0004	(0.4887, 0.5118)
$\alpha = 0.9$	100	0.8787	−0.0213	0.0551	(0.6939, 1.0636)
	200	0.8829	−0.0171	0.0506	(0.7890, 0.9767)
	300	0.9083	0.0083	0.0330	(0.8402, 0.9764)
	400	0.9063	0.0063	0.0327	(0.8554, 0.9572)
	500	0.9064	0.0064	0.0312	(0.8655, 0.9473)
	1000	0.9037	0.0037	0.0006	(0.8819, 0.9255)
$\alpha = 2$	100	2.0688	0.0688	0.0634	(1.4451, 2.6925)
	200	2.0374	0.0374	0.0605	(1.7322, 2.3426)
	300	2.0275	0.0275	0.0490	(1.8243, 2.2307)
	400	2.0106	0.0106	0.0481	(1.8599, 2.1613)
	500	2.0047	0.0047	0.0455	(1.8845, 2.1249)
	1000	2.0037	0.0037	0.0006	(1.9419, 2.0655)

6. Real data application

This section compares the fit of our proposed Copulas 1, 2, 11, and 12 with those modeling negative dependence, specifically Clayton, Frank, and Ali-Mikhail-Haq (C_{AMH}) copulas presented in [4], as well as more recent copulas introduced in [6, 19–21]. The analysis uses climatological data from the city of Abu Dhabi, covering the period from January 1, 2023, to December 31, 2023, with a sample size of

$n = 365$. This data set is accessible in [22]. The variables examined in this data set include wind speed (Wind Speed) (km/h) and air pressure (Air Pressure) (hPa).

The scatter plot in Figure 5 shows a negative dependence between wind speed and air pressure. This relationship is quantified by the empirical correlation coefficients, Spearman's rho at -0.22, and Kendall's tau at -0.15. These negative values indicate a moderate correlation between the two variables.

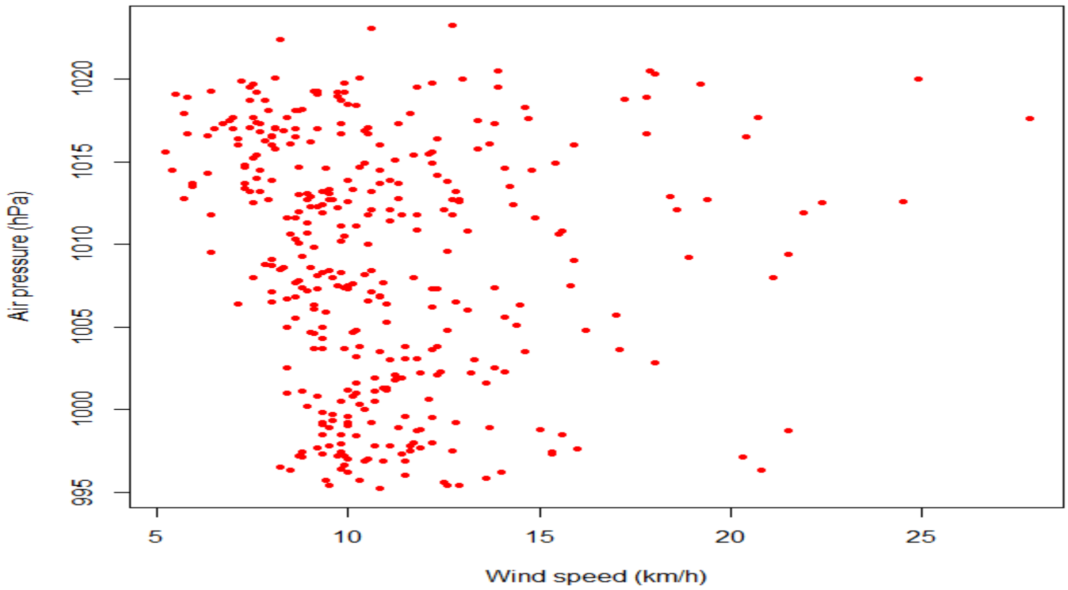

Figure 5. Scatter plot of wind speed versus air pressure.

The next step is to perform a goodness-of-fit (gof) test using the Cramér-von Mises (CvM) statistic to determine which copula best fits our dataset. This involves numerically maximizing Eqs (5.3), (5.5), (5.6), and (5.7) to obtain the maximum likelihood estimator $\hat{\alpha}_{ML}$ for α corresponding to the copulas $C_{1,\alpha}$, $C_{2,\alpha}$, $C_{11,\alpha}$, and $C_{12,\alpha}$, respectively. We then assess the gof tests for Copulas 1, 2, 11, and 12 using the CvM statistic, applying the bootstrap algorithm proposed by Genest et al. [23]. Finally, we compare the results with Clayton, Frank, and Ali-Mikhail-Haq copulas, as well as other copula families commonly used for modeling negative dependence introduced in [6, 19–21]. There are, for any $(u, v) \in [0, 1]^2$, $(\eta, \gamma, \theta, \alpha) \in (0, 1)^4$ and $\beta \in [-1, 1]$,

$$C_{\eta,\gamma}(u, v) = u^{1-\eta} v^{1-\gamma} \max(u^{\eta} + v^{\gamma} - 1, 0),$$

$$A_{\theta}(u, v) = u^{1-\theta} v^{1-\theta} \max(u^{\theta} + v^{\theta} - 1, 0),$$

$$B_{\alpha}(u, v) = \begin{cases} v - (1-u) + \dfrac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}(1-u)^{1+\alpha} v^{-\alpha}, & 0 < v \le \dfrac{\alpha}{1+\alpha}, 1 - \dfrac{(1+\alpha)v}{\alpha} < u < 1 \\ u - (1-v)\left[1 - (1-u)^{1+\alpha}\right], & u \in [0,1], \dfrac{\alpha}{1+\alpha} < v < 1, \\ 0 & otherwise, \end{cases}$$

and

$$D_\beta(u,v) = \frac{uv}{1-\beta(1-u)(1-v)e^{-(u+v)}}.$$

Table 7 shows significant differences in how well different copulas fit our data set. The gof tests revealed that Copula 11 has the strongest fit, with a p-value of 0.942, significantly outperforming the other copulas tested. In comparison, Copula 1 (p-value = 0.240), Copula 2 (p-value = 0.186), Copula 12 (p-value = 0.246), Frank Copula (p-value = 0.212), C_{AMH} (p-value = 0.230), and D_β (p-value = 0.308) showed relatively weaker fits. Among other copula families used for modeling negative dependence, the copula $C_{\eta,\gamma}$ (p-value = 0.796) demonstrates a good fit, followed by the copula A_θ (p-value = 0.540) and Clayton copula (p-value = 0.448). The copula B_α has a borderline fit with a p-value of 0.069. Overall, Copula 11 offers the best model to capture the dependence structure between wind speed and air pressure.

Table 7. Goodness-of-fit test for the selected copulas.

	Cramér–von Mises	
	Test statistic S_n	p-value
Copula 1	0.149	0.240
Copula 2	0.164	0.186
Copula 11	0.303	0.942
Copula 12	0.157	0.246
Clayton Copula	0.107	0.448
Frank Copula	0.148	0.212
C_{AMH}	0.168	0.230
$C_{\eta,\gamma}$	0.083	0.796
A_θ	0.100	0.540
B_α	0.262	0.069
D_β	0.129	0.308

Table 8 presents the ML estimator for the dependence parameter of Copula 11, along with the corresponding estimated concordance measures, which are calculated using the explicit formulas provided in Table 1.

Table 8. ML estimator and estimated concordance measures of $C_{11,\alpha}$.

	$\hat{\alpha}$	$\rho_{\hat{\alpha}}$	$\tau_{\hat{\alpha}}$	$\gamma_{\hat{\alpha}}$	$\beta_{\hat{\alpha}}$
Copula 11	1.326	−0.230	−0.154	−0.184	−0.170

Note that, the dependence parameter α of Copula 11 controls the strength of the negative dependence between wind speed and air pressure. A value of $\hat{\alpha} = 1.326$ corresponds to the moderate negative dependence observed in the data, as also reflected by the Spearman's rho ($\rho_{\hat{\alpha}} = -0.230$) and

Kendall's tau ($\tau_{\hat{\alpha}} = -0.154$). Therefore, α provides a quantitative measure of the dependence intensity, allowing the copula to accurately capture the observed relationship between these two climatological variables.

7. Conclusions

We have introduced a novel theoretical framework for the study of simple asymmetric bivariate copulas. Using this framework, we have derived several examples of this class of copulas and conducted an in-depth study of the properties of Copulas 1, 2, 11, and 12. We have also investigated the ML estimator for the dependence parameter associated with the aforementioned copulas. Our results provide important probabilistic tools for better understanding and modeling negative dependencies in asymmetric bivariate distributions. A general class of asymmetric copula describing both positive and negative dependence will be explored in a forthcoming article.

Author contributions

Rachid Bentoumi: Conceptualization, data curation, formal analysis, writing-original draft; Farid El Ktaibi: Conceptualization, data curation, formal analysis, writing-original draft; Christophe Chesneau: Conceptualization, data curation, formal analysis, writing-original draft. All authors have read and agreed to the submitted version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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