Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(10): 23547–23563.
DOI: 10.3934/math.20251046
Received: 04 July 2025
Revised: 19 September 2025
Accepted: 25 September 2025
Published: 16 October 2025

Research article

Novel insights into the precise boundedness and Lipschitz continuity of Hardy-type operators within central Morrey spaces

Muhammad Asim[1,*] **and Ghada AlNemer**[2]

[1] Department of NUSASH, National University of Technology (NUTECH), Islamabad 44000, Pakistan

[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, Riyadh 11671, Saudi Arabia

* **Correspondence:** Email: masim@math.qau.edu.pk, gnnemer@pnu.edu.sa.

Abstract: In this study, precise evaluations of the optimal constants associated with weighted weak-type Hardy inequalities are established. The analysis extends to the computation of the weak-type operator norm of the fractional Hardy operator within the framework of weighted central Morrey spaces. Furthermore, it is demonstrated that the commutators of the Hardy operator exhibit boundedness on these weighted central Morrey spaces when the symbol functions reside in the weighted Lipschitz space.

Keywords: Hardy-type operators; sharp constants; Lipschitz space; weights
Mathematics Subject Classification: 42B35, 26D10, 47B38, 47G10

1. Introduction

Averaging operators play a vital role in harmonic analysis, with the Hardy operator standing out as one of the most important among them. This operator has captured the interest of a wide range of researchers. In [1] the Hardy operator is defined as follows:

$$Hg(z) = \frac{1}{|z|^n} \int_{|y|<|z|} g(y)dy, \qquad z \in \mathbb{R}^n \setminus \{0\},$$

its adjoint operator is denoted as H^* and can be defined as:

$$H^*g(z) = \int_{|y|\geq|z|} \frac{g(y)}{|y|^n} dy, \qquad z \in \mathbb{R}^n \setminus \{0\}.$$

In the seminal contribution [2], Christ and Grafakos not only extended the classical Hardy inequality to its full n-dimensional counterpart but also furnished the precise determination of the best possible constants associated with these inequalities. The pursuit of Hardy-type inequalities, particularly the ascertainment of their sharp bounds, has historically constituted a central and challenging theme in the analytical landscape, with complete resolution achieved in only a restricted set of scenarios. The enduring vitality of this domain is attested by a substantial body of contemporary investigations, including but not limited to [3, 4]. Moreover, the determination of optimal constants for Hardy-type inequalities within the framework of product spaces of function classes has been addressed in a number of refined analyses, as evidenced by [5–10].

The study of the continuity properties of commutator operators has made significant contributions to the theory of partial differential equations (PDEs). Specifically, estimating commutators is closely linked to understanding the regularity properties of solutions to certain PDEs. These insights have motivated numerous authors to investigate the boundedness of commutator operators. In this context, Coifman, Rochberg, and Weiss [11] derived estimates for the commutators of the Calderón-Zygmund singular operator on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$, establishing that such estimates hold if and only if $b \in BMO(\mathbb{R}^n)$. Many results discussing the boundedness properties of such commutators are available in the literature. Song and Wang [12] examined the one-dimensional Hardy operator and its commutator on Lebesgue spaces. For a locally integrable function b, the commutator of the Hardy operator and its adjoint operator is defined as follows:

$$H_b f = b(Hf) - H(bf),$$

and

$$H_b^* f = b(H^* f) - H^*(bf).$$

Furthermore, Fu et al. [13] rigorously established the boundedness of these operators within the framework of classical Lebesgue spaces. In parallel, it has been substantiated that the commutators associated with Hardy operators possess significant utility in characterizing certain nuanced function spaces, as elucidated in [14, 15]. In a related vein, Gao and Wang [16] obtained weighted norm estimates for the commutator forms H_b and H_b^* acting on weighted Lebesgue spaces. This line of inquiry was further advanced in [17], wherein the authors extended such weighted continuity results to encompass the commutators of rough Hardy operators. Subsequently, two-weight norm inequalities pertaining to these commutator-type operators were meticulously derived in [18], thereby broadening the scope of weighted theory in this context.

This manuscript advances the trajectory of inquiry delineated in the aforementioned references by establishing weak-type operator norm estimates for the fractional Hardy operator within the framework of power-weighted central Morrey spaces. In pursuit of this objective, we shall first delineate the requisite definitions and notational conventions that underpin the analytical apparatus necessary for the derivation of our principal results. The discourse proceeds with a detailed investigation of the weak-type sharp boundedness properties of the Hardy operator in the setting of weighted central Morrey spaces, wherein particular attention is devoted to notable special cases. Furthermore, we rigorously derive a sufficiency criterion ensuring the boundedness of commutators generated by Hardy operators when the associated symbol functions are drawn from the weighted Lipschitz class, thereby extending the functional analytic scope of these operators within this weighted Morrey framework. The principal contributions of this work are concisely encapsulated in the following formulations:

Theorem 1.1. *Let* $1 < p < \infty, 1 \le q < \infty, 0 \le \alpha < m(p-1), m > -\rho$, $0 \le \beta \le \frac{\alpha}{p-1} < \frac{n+\rho}{q} + \beta < m$, $\max\{-1, \beta - m\} < \mu(m+\rho) < \beta - \frac{\alpha}{p-1}$, *and* $-\frac{1}{p} \le \lambda < -\frac{\alpha}{(p-1)(m+\alpha)}$. *If* $v(x) = |x|^\rho$ *and* $u(x) = |x|^\alpha$ *are power weight functions, and* $(m+\alpha)\lambda + \beta = (m+\rho)\mu$, *then*

$$\|H_\beta\|_{\dot{B}_u^{p,\lambda} \to \omega\dot{B}_v^{q,\mu}} = v_m^{\frac{\beta}{m}+\lambda-\mu} \left(\frac{m}{m+\rho}\right)^{-\mu} \left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda} \left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}.$$

Theorem 1.2. *Let* $1 < p < \infty, 1 \le q < \infty, 0 \le \alpha < m(p-1), m+\rho > 0$, $\frac{\alpha}{p-1} < \frac{m+\rho}{q} < m$, $-\frac{1}{q} \le \mu(m+\rho) < -\frac{\alpha}{p-1}$, *and* $-\frac{1}{p} \le \lambda < -\frac{\alpha}{(p-1)(m+\alpha)}$. *If* $u(x) = |x|^\alpha$ *and* $v(x) = |x|^\rho$ *are power weight functions and* $(m+\alpha)\lambda = (m+\rho)\mu$, *then*

$$\|H\|_{\dot{B}_u^{p,\lambda} \to \omega\dot{B}_v^{q,\mu}} = v_m^{\lambda-\mu} \left(\frac{m}{m+\rho}\right)^{-\mu} \left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda} \left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}.$$

Theorem 1.3. *Let* $\omega \in A_1$, $b \in Lip_{\beta,\omega}$ *where* $0 < \beta < 1$, $1 < q < \infty$, *and the parameters* λ *and* μ *satisfy* $-\frac{1}{q} < \lambda < \mu < 0$ *with the relation* $\mu = \lambda + \frac{\beta}{n}$. *Under these conditions, the commutator operators* H_b *and* H_b^* *are bounded from the homogeneous weighted central Morrey space* $\dot{M}^{q,\lambda}(\omega)$ *into* $\dot{M}^{q,\mu}(\omega^{1-q})$.

Now, we will provide several definitions and notations.

2. Preliminaries

The Morrey space $M^{q,\lambda}(\mathbb{R}^n)$ was introduced in [15] for a set of locally integrable functions f that satisfy the inequality:

$$\|f\|_{M^{q,\lambda}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{|B|^{1+\lambda q}} \int_B |f(x)|^q dx\right)^{1/q} < \infty,$$

where $1 \le q < \infty$, $-p^{-1} < \lambda < 0$, and $B(x,R)$ designates the Euclidean ball in $\mathbb{R}^n$ centered at the point x with radius R. Additionally, $|B|$ denotes the Lebesgue measure of the ball $B(x,R)$. Recently, in [19], the authors introduced the central Morrey space $\dot{M}^{q,\lambda}(\mathbb{R}^n)$, which is defined as:

$$\|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0} \left(|B(0,R)|^{-1-\lambda q} \int_{B(0,R)} |f(x)|^q dx\right)^{1/q} < \infty.$$

Function spaces with weighted norms also play a significant role in various branches of mathematical analysis. The theory of A_p weights began in 1972 with Muckenhoupt's work [20] and has been further developed by several mathematicians [21]. The weighted measure is denoted by $\omega(B)$, i.e., $\omega(B) = \int_B \omega(x)dx$, and p' is the conjugate index of p, which must satisfy $1/p + 1/p' = 1$. A weight ω is known to be of class A_p if it satisfies the following condition:

$$\left(\frac{1}{|B|} \int_B \omega(z)dz\right) \left(\frac{1}{|B|} \int_B \omega(z)^{-(p-1)^{-1}} dz\right)^{p-1} \le C.$$

For every Euclidean ball B, there exists a constant C, independent of B, such that the inequality holds whenever $\omega \in A_1$, that is, for all ω belonging to the Muckenhoupt class A_1 characterized by the condition

$$\left(\frac{1}{|B|} \int_B \omega(z)dz\right) \le C \operatorname*{ess\,inf}_{z\in B} \omega(z).$$

Furthermore, it is true that $A_p \subset A_q$ for all $p < q$. In [22], Komori utilized the weighted Morrey space and discussed estimates for classical operators.

Definition 2.1. [23] Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The weighted central Morrey space $\dot{M}^{q,\lambda}(\omega)$ can be characterized by the condition:

$$\|f\|_{\dot{M}^{q,\lambda}(\omega)} = \sup_{R>0}\left(\omega(B(0,R))^{-\lambda q-1}\int_{B(0,R)}|f(x)|^q\omega(x)dx\right)^{1/q} < \infty.$$

Definition 2.2. [24] f belongs to the weighted Lipschitz class if it satisfies the following condition:

$$\|f\|_{Lip^q_{\beta,\omega}} = \sup_{B\in\mathbb{R}^n}\frac{1}{\omega(B)^{\beta/n}}\left(\frac{1}{\omega(B)}\int_B|f(x)-f_B|^q\omega(x)^{1-q}dx\right)^{1/q} \leq C < \infty,$$

wherein the supremum is taken over the entire ensemble of Euclidean balls $B \subseteq \mathbb{R}^n$, the symbol $\int_B$ signifies the normalized mean oscillation of the function f upon the domain B, and the governing parameters are constrained by the inequalities $0 < \beta < 1$ and $1 \leq p \leq \infty$. It is a matter of elementary discernment that, in the particular instance when $\omega = 1$, the generalized Lipschitz-type space $Lip^q_{\beta,\omega}$ collapses into its classical counterpart Lip^q_β namely, the traditional Lipschitz space.

Definition 2.3. Let $1 \leq p < \infty$, $\lambda \in \mathbb{R}$, and w be a locally integrable and positive function on $\mathbb{R}^n$. Then, the weighted weak central Morrey space $\omega\dot{B}^{p,\lambda}_w(\mathbb{R}^n)$ is defined as

$$\omega\dot{B}^{p,\lambda}_w(\mathbb{R}^n) = \{f : \|f\|_{\omega\dot{B}^{p,\lambda}_\omega(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\omega\dot{B}^{p,\lambda}_w(\mathbb{R}^n)} = \sup_{R>0}\frac{1}{\omega(B(0,R))^{\frac{1}{p}+\lambda}}\|f\|_{\omega L^p_w(B(0,R))},$$

and

$$\|f\|_{\omega L^p_w(B(0,R)} = \sup_{t>0} t\omega(\{x \in B(0,R) : |f(x)| > t\})^{\frac{1}{p}}.$$

If $\omega = 1$, we recover the weak central Morrey spaces as introduced in [25,26], which are defined by

$$\dot{B}^{p,\lambda}_w(\mathbb{R}^n) = \{f : \|f\|_{\dot{B}^{p,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{B}^{p,\lambda}_w(\mathbb{R}^n)} = \sup_{R>0}\frac{1}{(B(0,R))^{\frac{1}{p}+\lambda}}\|f\|_{L^p_w(B(0,R))},$$

and

$$\|f\|_{L^p_w(B(0,R)} = \sup_{t>0} tw(\{x \in B(0,R) : |f(x)| > t\})^{\frac{1}{p}}.$$

Additionally, in the case $\lambda = -p^{-1}$, we obtain the weighted weak Lebesgue space $WL^p_w(\mathbb{R}^n) = L^{p,\infty}_w(\mathbb{R}^n)$.

3. Analytical Elucidation of the fundamental results

In the present section, we embark upon the rigorous demonstration of the principal results previously articulated in Section 1. To this end, it is imperative to first establish a collection of auxiliary lemmas that will serve as essential instruments in the ensuing analysis.

Lemma 3.1. *([27]) Let $\omega \in A_1$. Then there exist positive constants C_1, and C_2 and a real number δ with $0 < \delta < 1$ such that, for every measurable subset S of a ball B, the following inequality is fulfilled:*

$$C_1\frac{|S|}{|B|} \leq \frac{\omega(S)}{\omega(B)} \leq C_2\left(\frac{|S|}{|B|}\right)^{\delta}.$$

Lemma 3.2. *([28]) If $b \in Lip_{\beta,\omega}$ and $\omega \in A_1$, then there exist a constant C such that for all $i, k \in \mathbb{Z}$ with $i > k$,*

$$|b_{B_i} - b_{B_k}| \leq C(i-k)\|b\|_{Lip_{\beta,\omega}}\omega(B_i)^{\beta/n}\frac{\omega(B_k)}{|B_k|}.$$

Lemma 3.3. *([28]) Suppose ω belongs to A_1. Then, for any $1 \leq p < \infty$,*

$$\int_B \omega(z)^{1-p'}dz \leq C|B|^{p'}\omega(B)^{1-p'}.$$

Now, let us proceed to examine our results.

Proof of Theorem 1.1. Since

$$\begin{aligned}
|H_\beta f(x)| \leq& |B(0,|x|)|^{\frac{\beta}{m}-1}\int_{B(0,|x|)}|f(z)|dz \\
\leq& v_m^{\frac{\beta}{m}-1}|x|^{\beta-m}\left(\int_{B(0,|x|)}|f(z)|^p|z|^\alpha dz\right)^{\frac{1}{p}}\left(\int_{B(0,|x|)}|z|^{\frac{-\alpha p'}{p}}dz\right)^{\frac{1}{p'}} \\
=& v_m^{\frac{\beta}{m}+\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\|f\|_{\dot{B}_u^{p,\lambda}}|x|^{(m+\alpha)\lambda+\beta}.
\end{aligned} \tag{3.1}$$

Next, the conditions $\beta \leq \frac{\alpha}{p-1}$ and $\lambda < -\frac{\alpha}{(p-1)(m+\alpha)}$ ensure that $(m+\alpha)\lambda + \beta < 0$ and validate the equation:

$$(m+\alpha)\lambda + \beta = (m+\rho)\mu. \tag{3.2}$$

Let

$$A = v_m^{\frac{\beta}{m}+\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\|f\|_{\dot{B}_u^{p,\lambda}},$$

then from (3.1) and (3.2), we have

$$|H_\beta f(x)| \leq A|x|^{(m+\rho)\mu}.$$

Now, by the definition of weighted weak central Morrey space:

$$\begin{aligned}
\|H_\beta f\|_{\omega\dot{B}_v^{q,\mu}} =& \sup_{R>0}\sup_{t>0} tv(B(0,R))^{-\frac{1}{q}-\mu}v\left(\left\{x \in B(0,R) : |H_\beta f(x)| > t\right\}\right)^{\frac{1}{q}} \\
\leq& \sup_{R>0}\sup_{t>0} tv(B(0,R))^{-\frac{1}{q}-\mu}v\left(\left\{x \in B(0,R) : A|x|^{(n+\rho)\mu} > t\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $m+\rho>0$ and $\mu<0$, we can establish that

$$\|H_\beta f\|_{\omega\dot{B}_v^{q,\mu}} \le \sup_{R>0}\sup_{t>0} tv(B(0,R))^{-\frac{1}{q}-\mu} v\bigg(\bigg\{x\in B(0,R): |x|<(\frac{t}{A})^{\frac{1}{(m+\rho)\mu}}\bigg\}\bigg)^{\frac{1}{q}}. \tag{3.3}$$

To approximate the right-hand side of the inequality (3.3), one needs to compare R and $(t/A)^{1/(m+\rho)\mu}$. Therefore, if $R\le (t/A)^{1/(m+\rho)\mu}$, then, given that $\mu<0$, we obtain:

$$\begin{aligned}
&\|H_\beta f\|_{\omega\dot{B}_v^{q,\mu}}\\
&\le \sup_{t>0} t \sup_{0<R\le (t/A)^{1/(m+\rho)\mu}} v(B(0,R))^{-\frac{1}{q}-\mu} v\bigg(\bigg\{x\in B(0,R): |x|<(\frac{t}{A})^{\frac{1}{(m+\rho)\mu}}\bigg\}\bigg)^{\frac{1}{q}}\\
&= \sup_{t>0}\sup_{0<R\le (t/A)^{1/(m+\rho)\mu}} t\left(\frac{mv_m}{m+\rho}\right)^{-\mu} R^{-(m+\rho)\mu}\\
&\le v_m^{\frac{\beta}{m}+\lambda-\mu}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\|f\|_{\dot{B}_u^{p,\lambda}},
\end{aligned}$$

and if $R>(t/A)^{1/(m+\rho)\mu}$, then, for $-1/q\le\mu$, we have

$$\begin{aligned}
&\|H_\beta f\|_{\omega\dot{B}_v^{q,\mu}}\\
&\le \sup_{t>0} t \sup_{R>(t/A)^{1/(m+\rho)\mu}} v(B(0,R))^{-\frac{1}{q}-\mu} v\bigg(\bigg\{x\in B(0,R): |x|<(\frac{t}{A})^{\frac{1}{(m+\rho)\mu}}\bigg\}\bigg)^{\frac{1}{q}}\\
&= \sup_{t>0} t \sup_{R>(t/A)^{1/(m+\rho)\mu}} \left(\frac{mv_m}{m+\rho}\right)^{-\mu}\left(\frac{t}{A}\right)^{\frac{1}{q\mu}} R^{-(m+\rho)(\frac{1}{q}+\mu)}\\
&\le v_n^{\frac{\beta}{m}+\lambda-\mu}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\|f\|_{\dot{B}_u^{p,\lambda}}.
\end{aligned}$$

Hence, in either case, we obtain

$$\|H_\beta\|_{\dot{B}_u^{p,\lambda}\to\omega\dot{B}_v^{q,\mu}} \le v_m^{\frac{\beta}{m}+\lambda-\mu}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}. \tag{3.4}$$

To prove that the constant appearing in the inequality (3.4) is sharp, we will consider

$$f_0(x)=\begin{cases}|x|^{-\frac{\alpha}{p-1}} & |x|\le 1,\\ 0 & |x|>1,\end{cases}$$

and compute the norm:

$$\|f_0\|_{\dot{B}_u^{p,\lambda}} = \sup_{R>0} u(B(0,R))^{-\frac{1}{p}-\lambda}\left(\int_{B(0,R)} |x|^{-\frac{\alpha p}{p-1}+\alpha}\chi\{x:|x|\le 1\}(x)dx\right)^{\frac{1}{p}}.$$

The value of this norm depends on whether $R \le 1$ or $R > 1$. If $R \le 1$, then for $\lambda < -\frac{\alpha}{(p-1)(m+\alpha)}$, it is evident that

$$\begin{aligned}\|f_0\|_{\dot{B}_u^{p,\lambda}} &= \sup_{R\le 1} v_m^{-\lambda}\left(\frac{m}{m+\alpha}\right)^{-\frac{1}{p}-\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p}} R^{-\frac{\alpha}{p-1}-\lambda(m+\alpha)}\\ &= v_m^{-\lambda}\left(\frac{m}{m+\alpha}\right)^{-(\frac{1}{p}+\lambda)}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p}},\end{aligned}$$

and if $R > 1$, then for $-\frac{1}{p} \le \lambda$, it is not difficult to observe that

$$\begin{aligned}\|f_0\|_{\dot{B}_u^{p,\lambda}} &= \sup_{R>1} v_m^{-\lambda}\left(\frac{m}{m+\alpha}\right)^{-\frac{1}{p}-\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p}} R^{-(m+\alpha)(\frac{1}{p}+\lambda)}\\ &= v_m^{-\lambda}\left(\frac{m}{m+\alpha}\right)^{-\frac{1}{p}-\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p}}.\end{aligned}$$

$$H_\beta f_0(x) = C\begin{cases}|x|^{\beta-\frac{\alpha}{p-1}} & |x| \le 1,\\ |x|^{\beta-m} & |x| > 1,\end{cases}$$

where $C = mv_m^{\beta/m}/(m-\frac{\alpha}{p-1})$. Once again, using Definition 2.3, one can conclude:

$$\begin{aligned}\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}} &= \sup_{R>0}\sup_{t>0} tv(B(0,R))^{-\frac{1}{q}-\mu} v\left(\{x\in B(0,R) : |Hf_0(x)| > t\}\right)^{\frac{1}{q}}\\ &= C_1 \sup_{R>0}\sup_{t>0} tR^{-(m+\rho)(\frac{1}{q}+\mu)} v\left(\{x\in B(0,R) : |Hf_0(x)| > t\}\right)^{\frac{1}{q}},\end{aligned} \tag{3.5}$$

where $C_1 = [mv_m/(m+\rho)]^{-1/q-\mu}$. Since $H_\beta f_0(x)$ is piecewise defined, its norm will be calculated by the necessary division of the domain $B(0,R)$. Guided by this consideration, we proceed to categorize the remainder of the analysis into the ensuing distinct scenarios:

Case-1. When $R \le 1$:

In the present configuration, under the condition $\beta \le \alpha/(p-1)$, the inequality labeled as (3.5) yields the following estimate:

$$\begin{aligned}&\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}}\\ &= C_1 \sup_{t>0}\sup_{0<R\le 1} tR^{-(m+\rho)(\frac{1}{q}+\mu)} v\left(\{x\in B(0,R) : C|x|^{\beta-\frac{\alpha}{p-1}} > t\}\right)^{\frac{1}{q}}\\ &= C_1 \sup_{t>0}\sup_{0<R\le 1} tR^{-(m+\rho)(\frac{1}{q}+\mu)} v\left(\left\{x\in B(0,R) : |x| < (\frac{C}{t})^{\frac{1}{\frac{\alpha}{p-1}-\beta}}\right\}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.6}$$

Here, C and t are positive real numbers, so either $t < C$ or $t \ge C$.

Case-1(a): If $t < C$, then (3.6) gives us:

$$\begin{aligned}\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}} =& C_1\left(\frac{mv_m}{m+\rho}\right)^{\frac{1}{q}} \sup_{0<t<C}\sup_{0<R\le 1} tR^{-(m+\rho)\mu}\\ =& v_m^{\frac{\beta}{m}-\mu}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)\\ =& v_m^{\frac{\beta}{m}-\mu+\lambda}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\|f_0\|_{\dot{B}_u^{p,\lambda}}.\end{aligned}$$

Case-1(b): If $t \ge C$, then (3.6) suggests the following two subcases:

(i) When R and C/t satisfy $R < (C/t)^{1/(-\beta+\alpha/(p-1))}$, then from (3.6), we get

$$\begin{aligned}\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}} =& C_1\left(\frac{mv_m}{m+\rho}\right)^{\frac{1}{q}} \sup_{t\ge C} t \sup_{0<R<(C/t)^{1/(-\beta+\alpha/(p-1))}} R^{-\mu(m+\rho)}\\ =& \left(\frac{mv_m}{m+\rho}\right)^{-\mu} C^{-\frac{\mu(m+\rho)}{\frac{\alpha}{p-1}-\beta}} \sup_{t\ge C} t^{1+\frac{\mu(m+\rho)}{\frac{\alpha}{p-1}-\beta}}.\end{aligned} \tag{3.7}$$

Since, $(m+\rho)\mu < \beta - \frac{\alpha}{p-1}$, therefore

$$\begin{aligned}\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}} =& v_m^{\frac{\beta}{m}-\mu}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)\\ =& v_m^{\frac{\beta}{m}-\mu+\lambda}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}\|f_0\|_{\dot{B}_u^{p,\lambda}}.\end{aligned} \tag{3.8}$$

(ii) When R and C/t satisfy $0 < (C/t)^{1/(-\beta+\alpha/(p-1))} < R$, then from (3.6), we obtain

$$\begin{aligned}&\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}}\\ &= C_1\left(\frac{mv_m}{m+\rho}\right)^{\frac{1}{q}} \sup_{t\ge C} t\left(\frac{C}{t}\right)^{\frac{m+\rho}{q(\frac{\alpha}{p-1}-\beta)}} \sup_{(C/t)^{1/(-\beta+\alpha/(p-1))}<R} R^{-(\mu+\frac{1}{q})(m+\rho)}\\ &= \left(\frac{mv_m}{m+\rho}\right)^{-\mu} C^{-\frac{\mu(m+\rho)}{\frac{\alpha}{p-1}-\beta}} \sup_{t\ge C} t^{1+\frac{\mu(m+\rho)}{\frac{\alpha}{p-1}-\beta}},\end{aligned}$$

which is the same as (3.7). By an argument similar to Case-1(b), the part (i) above, we obtain (3.8).

Case-2. When $R > 1$:

In this case, we infer from (3.5) that

$$
\begin{aligned}
&\|H_\beta f_0\|_{\omega\dot{B}^{q,\mu}_v}\\
&= C_1 \sup_{R>1}\sup_{t>0} tR^{-(m+\rho)(\frac{1}{q}+\mu)}\\
&\times v\left(\{x\in B(0,1): |Hf_0(x)|>t\}\cup\{1\le |x|<R: |Hf_0(x)|>t\}\right)^{\frac{1}{q}}\\
&= C_1 \sup_{R>1}\sup_{t>0} tR^{-(m+\rho)(\frac{1}{q}+\mu)}\\
&\times v\left(\left\{x\in B(0,1): |x|<(\frac{C}{t})^{\frac{1}{\frac{\alpha}{p-1}-\beta}}\right\}\cup\left\{1\le |x|<R: |x|<(\frac{C}{t})^{\frac{1}{m-\beta}}\right\}\right)^{\frac{1}{q}}.
\end{aligned}
\tag{3.9}
$$

Again, we have to resolve the comparison between R and C/t by considering the relationship between t and C.

Case-2(a): If $t<C$, then (3.9) suggests the following two subcases:

(i) When R and C/t satisfy $1<R<C/t$, then from (3.9), we get

$$
\begin{aligned}
\|H_\beta f_0\|_{\omega\dot{B}^{q,\mu}_v} &= C_1 \sup_{0<t<C} t \sup_{1<R<C/t} R^{-(m+\rho)(\frac{1}{q}+\mu)} v(\{x:0<|x|<R\})^{\frac{1}{q}}\\
&= \left(\frac{mv_m}{m+\rho}\right)^{-\mu} \sup_{0<t<C} t \sup_{1<R<C/t} R^{-(m+\rho)\mu}\\
&= \left(\frac{mv_m}{m+\rho}\right)^{-\mu} C^{-(m+\rho)\mu} \sup_{0<t<C} t^{1+(m+\rho)\mu}.
\end{aligned}
\tag{3.10}
$$

By virtue of the condition $-1<(m+\rho)\mu<0$, we have

$$
\begin{aligned}
\|H_\beta f_0\|_{\omega\dot{B}^{q,\mu}_v} &= v_m^{\frac{\beta}{m}-\mu}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)\\
&= v_m^{\frac{\beta}{m}-\mu+\lambda}\left(\frac{m}{m+\rho}\right)^{-\mu}\left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}\left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}} \|f_0\|_{\dot{B}^{p,\lambda}_u}.
\end{aligned}
\tag{3.11}
$$

(ii) When R and C/t satisfy $1<(C/t)^{1/(m-\beta)}<R$, then from (3.9), we get

$$
\begin{aligned}
&\|H_\beta f_0\|_{\omega\dot{B}^{q,\mu}_v}\\
&= C_1 \sup_{t<C} t \sup_{R>(C/t)^{1/(m-\beta)}} R^{-(m+\rho)(\frac{1}{q}+\mu)} v\left(\left\{x:0<|x|<(\frac{C}{t})^{\frac{1}{m-\beta}}\right\}\right)^{\frac{1}{q}}\\
&= C_1\left(\frac{mv_m}{m+\rho}\right)^{\frac{1}{q}} \sup_{t<C} t\left(\frac{C}{t}\right)^{-(\frac{m+\rho}{m-\beta})(\frac{1}{q}+\mu)}\left(\frac{C}{t}\right)^{\frac{m+\rho}{q(m-\beta)}}\\
&= \left(\frac{mv_m}{m+\rho}\right)^{-\mu} C^{-\mu(\frac{m+\rho}{m-\beta})} \sup_{t<C} t^{1+\mu(\frac{m+\rho}{m-\beta})},
\end{aligned}
$$

which is the same as (3.10). Therefore, considering the condition $\beta-m<\mu(m+\rho)$, we obtain (3.11).

Case-2(b): If $t \geq C$, then (3.9) gives us:

$$\begin{aligned}
&\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}} \\
&= C_1 \sup_{t\geq C} t \sup_{R>1} R^{-(m+\rho)(\frac{1}{q}+\mu)} v\left(\left\{x : 0 < |x| < (\frac{C}{t})^{\frac{1}{\frac{\alpha}{p-1}-\beta}}\right\}\right)^{\frac{1}{q}} \\
&= C_1 \left(\frac{m v_m}{m+\rho}\right)^{\frac{1}{q}} \sup_{t\geq C} t \left(\frac{C}{t}\right)^{\frac{m+\rho}{q(\frac{\alpha}{p-1}-\beta)}} \\
&= \left(\frac{m v_m}{m+\rho}\right)^{-\mu} C^{\frac{m+\rho}{q(\frac{\alpha}{p-1}-\beta)}} \sup_{t\geq C} t^{1-\frac{m+\rho}{q(\frac{\alpha}{p-1}-\beta)}}.
\end{aligned}$$

The condition $\alpha/(p-1) - \beta < (m+\rho)/q$ implies that

$$\begin{aligned}
\|H_\beta f_0\|_{\omega\dot{B}_v^{q,\mu}} &= v_m^{\frac{\beta}{m}-\mu} \left(\frac{m}{m+\rho}\right)^{-\mu} \left(\frac{m}{m-\frac{\alpha}{p-1}}\right) \\
&= v_m^{\frac{\beta}{m}-\mu+\lambda} \left(\frac{m}{m+\rho}\right)^{-\mu} \left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda} \left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}} \|f_0\|_{\dot{B}_u^{p,\lambda}}.
\end{aligned}$$

Combining the results of each case, we obtain

$$\|H_\beta\|_{\dot{B}_u^{p,\lambda}\to\omega\dot{B}_v^{q,\mu}} \geq v_m^{\frac{\beta}{m}+\lambda-\mu} \left(\frac{m}{m+\rho}\right)^{-\mu} \left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}} \left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}. \tag{3.12}$$

Finally, combining (3.4) and (3.12), we arrive at

$$\|H_\beta\|_{\dot{B}_u^{p,\lambda}\to\omega\dot{B}_v^{q,\mu}} = v_m^{\frac{\beta}{m}+\lambda-\mu} \left(\frac{m}{m+\rho}\right)^{-\mu} \left(\frac{m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}} \left(\frac{m}{m+\alpha}\right)^{\frac{1}{p}+\lambda}.$$

If we let $\lambda = -1/p$ and $\mu = -1/q$, then we obtain the following corollary.

Corollary 3.4. *Let* $1 < p < \infty, 1 \leq q < \infty, 0 \leq \alpha < m(p-1), m+\rho > 0, 0 \leq \beta \leq \frac{\alpha}{p-1}$. *If* $u(x) = |x|^\alpha$ *and* $v(x) = |x|^\rho$ *are power weight functions and*

$$\frac{m+\alpha}{p} - \beta = \frac{m+\rho}{q},$$

then

$$\|H_\beta\|_{L_u^p\to\omega L_v^q} = v_m^{\frac{\beta}{m}-1} \left(\frac{m v_m}{m+\rho}\right)^{\frac{1}{q}} \left(\frac{m v_m}{m-\frac{\alpha}{p-1}}\right)^{\frac{1}{p'}}.$$

Proof of Theorem 1.2. The proof is very similar to the proof of Theorem 1.1. If $\alpha = \rho = 0$, then in this case $\mu = \lambda$ and $p = q$, so for these parameter values, we obtain one of the main results of [25] in the form of the following corollary.

Corollary 3.5. *For* $p \in [1, \infty)$, *and* $-\frac{1}{p} \leq \lambda < 0$, *we have:*

$$\|H\|_{\dot{B}^{p,\lambda}\to\omega\dot{B}^{p,\lambda}} = 1.$$

Proof of Theorem 1.3. Let $B_l = \{t \in \mathbb{R}^n : |t| \le 2^l\}$, $C_l = B_l \backslash B_{l-1}$. Without loss of generality, we assume $B(0,R) = B_{k_0}$ for $k_0 \in \mathbb{Z}$. Here, we need to establish the following two inequalities:

$$\left(\frac{1}{\omega(B_{k_0})^{1+q\mu}}\int\limits_{B_{k_0}} |H_b f(t)|^q \omega(t)^{1-q} dt\right)^{1/q} \le C\|b\|_{Lip_{\beta,\omega}}\|f\|_{\dot{M}^{q,\lambda}(\omega)}, \tag{3.13}$$

$$\left(\frac{1}{\omega(B_{k_0})^{1+q\mu}}\int\limits_{B_{k_0}} |H_b^* f(t)|^q \omega(t)^{1-q} dt\right)^{1/q} \le C\|b\|_{Lip_{\beta,\omega}}\|f\|_{\dot{M}^{q,\lambda}(\omega)}. \tag{3.14}$$

To construct (3.13), we begin by considering

$$\begin{aligned}
\int\limits_{B_{k_0}} |H_b f(t)|^q \omega(t)^{1-q} dt &= \int\limits_{B_{k_0}} \left|\frac{1}{|t|^n}\int\limits_{|z|<|t|} (b(t)-b(z))\, f(z)dz\right|^q \omega(t)^{1-q} dt \\
&\le \sum_{k=-\infty}^{k_0}\int\limits_{C_k} \left|\frac{1}{|t|^n}\int\limits_{B_k} (b(t)-b(z))\, f(z)dz\right|^q \omega(t)^{1-q} dt \\
&\le C\sum_{k=-\infty}^{k_0}\int\limits_{C_k} \left|\frac{1}{|t|^n}\sum_{i=-\infty}^{k}\int\limits_{C_i} \left|b(t)-b_{B_k}\right| |f(z)|dz\right|^q \omega(t)^{1-q} dt \\
&+ C\sum_{k=-\infty}^{k_0}\int\limits_{C_k} \left|\frac{1}{|t|^n}\sum_{i=-\infty}^{k}\int\limits_{C_i} \left|b(z)-b_{B_k}\right| |f(z)|dz\right|^q \omega(t)^{1-q} dt \\
&=: I_1 + I_2.
\end{aligned}$$

Taking into account the fact that $A_1 \subset A_q$, Lemma 3.3, and Hölder's inequality, we have

$$\begin{aligned}
\int\limits_{C_i} |f(z)|dz &\le \left(\int\limits_{B_i} |f(z)|^q \omega(z)dz\right)^{1/q}\left(\int\limits_{B_i} \omega^{-1/(q-1)}(z)dz\right)^{(q-1)/q} \\
&\le C\omega(B_i)^{\lambda}|B_i|\|f\|_{\dot{M}^{q,\lambda}(\omega)}.
\end{aligned} \tag{3.15}$$

Therefore, using Lemma 3.1 and inequality (3.15), we can estimate I_1 as:

$$\begin{aligned}
I_1 &\leq C \sum_{k=-\infty}^{k_0} 2^{-knq} \int_{C_k} \left|b(t) - b_{B_k}\right|^q \omega(t)^{1-q} dt \left|\sum_{i=-\infty}^{k} \int_{C_i} |f(t)| dt\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+q\beta/n} \left|\sum_{i=-\infty}^{k} 2^{n(i-k)} \omega(B_i)^{\lambda}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+q\mu} \left|\sum_{i=-\infty}^{k} 2^{n(i-k)(1+\lambda)}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu} \sum_{k=-\infty}^{k_0} 2^{n\delta(k-k_0)(1+\mu q)} \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}$$

Here, we have used the condition $\lambda + \beta/n = \mu$, along with the fact that $-1/q < \lambda < \mu < 0$.

Next, we approximate I_2 by decomposing it as follows:

$$\begin{aligned}
I_2 &\leq C \sum_{k=-\infty}^{k_0} 2^{-knq} \int_{C_k} \left|\sum_{i=-\infty}^{k} \int_{C_i} \left|b(z) - b_{B_i}\right| |f(z)| dz\right|^q \omega(t)^{1-q} dt \\
&\quad + C \sum_{k=-\infty}^{k_0} 2^{-knq} \int_{C_k} \left|\sum_{i=-\infty}^{k} \int_{C_i} \left|b_{B_k} - b_{B_i}\right| |f(z)| dz\right|^q \omega(t)^{1-q} dt \\
&=: I_{21} + I_{22}.
\end{aligned}$$

Now we will address each of I_{21} and I_{22} separately. First, we need to establish an inequality similar to (3.15). By applying Hölder's inequality, it is easy to observe that

$$\int_{C_i} |b(z) - b_{B_i}||f(z)| dz \leq C\omega(B_i)^{1+\mu} \|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)}. \tag{3.16}$$

With the aid of inequality (3.16) and Lemmas 3.1 and 3.3, I_{21} reduces to:

$$\begin{aligned}
I_{21} &\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} 2^{-knq} \int_{C_k} \omega(t)^{1-q} dt \left|\sum_{i=-\infty}^{k} \omega(B_i)^{1+\mu}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1-q} \left|\sum_{i=-\infty}^{k} \omega(B_i)^{1+\mu}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+q\mu} \left|\sum_{i=-\infty}^{k} 2^{n\delta(i-k)(1+\mu)}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu} \sum_{k=-\infty}^{k_0} 2^{n\delta(k-k_0)(1+q\mu)} \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}$$

The convergence of the above series is ensured by the fact that $-1/q < \mu < 0$.

Estimates for I_{22} are still in progress. To accomplish this, we again rely on Lemmas 3.1–3.3 and inequality (3.15), leading to:

$$\begin{aligned}
I_{22} &\leq C\|b\|^q_{Lip_{\beta,\omega}}\|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1-q+q\beta/n} \left|\sum_{i=-\infty}^{k}(k-i)\omega(B_i)^{1+\lambda}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}}\|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \sum_{k=-\infty}^{k_0} \omega(B_k)^{1+q\mu} \left|\sum_{i=-\infty}^{k}(k-i)2^{n\delta(i-k)(1+\lambda)}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}}\|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu} \sum_{k=-\infty}^{k_0} 2^{n\delta(k-k_0)(1+q\mu)} \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}}\|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}$$

By combining the estimates for I_1, I_{21}, and I_{22}, we arrive at the inequality (3.13).

Now, let us proceed to establish (3.14). For this purpose, we consider:

$$\begin{aligned}
\int_{B_{k_0}} |H^*_b f(t)|^q \omega(t)^{1-q} dt &= \int_{B_{k_0}} \left| \int_{|z|\geq|t|} \frac{(b(t)-b(z))}{|z|^n} f(z)dz \right|^q \omega(t)^{1-q} dz \\
&\leq C \int_{B_{k_0}} \left| \int_{2^{nk_0}\geq|z|\geq|t|} \frac{(b(t)-b(z))}{|z|^n} f(z)dz \right|^q \omega(t)^{1-q} dt \\
&+ C \int_{B_{k_0}} \left| \int_{|z|>2^{nk_0}} \frac{(b(t)-b(z))}{|z|^n} f(z)dz \right|^q \omega(t)^{1-q} dt \\
&=: J + J'.
\end{aligned}$$

The approximation of J can be executed similarly to that of (3.13). However, estimating J' requires more computational effort. An analysis analogous to H_b implies

$$\begin{aligned}
J &\leq C \int_{B_{k_0}} \left| \frac{1}{|t|^n} \int_{|z|<2^{nk_0}} |(b(t)-b(z))f(z)|dz \right|^q \omega(t)^{1-q} dt \\
&\leq C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|t|^n} \sum_{i=-\infty}^{k} \int_{C_i} |b(t)-b(z)f(z)|dz \right|^q \omega(t)^{1-q} dt \\
&\leq C\|b\|^q_{Lip_{\beta,\omega}}\|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}$$

To estimate the term J', we proceed to demonstrate that

$$
\begin{aligned}
J' &\leq C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(t) - b_{B_{k_0}}|}{|z|^n} |f(z)| dz \right|^q \omega(t)^{1-q} dt \\
&+ C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(z) - b_{B_{k_0}}|}{|z|^n} |f(z)| dz \right|^q \omega(t)^{1-q} dt \\
&=: J'_1 + J'_2.
\end{aligned}
$$

We employ inequality (3.15) and Lemma 3.1 for the analysis of J'_1, resulting in

$$
\begin{aligned}
J'_1 &\leq C \int_{B_{k_0}} |b(t) - b_{B_{k_0}}|^q \omega(t)^{1-q} dt \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|f(z|}{|z|^n} dz \right|^q \\
&\leq C \|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\beta/n} \left| \sum_{k=k_0}^{\infty} \omega(B_k)^{\lambda} \right|^q \\
&\leq C \|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu} \left| \sum_{k=k_0}^{\infty} 2^{n\delta(k-k_0)\lambda} \right|^q \\
&\leq C \|b\|^q_{Lip_{\beta,\omega}} \|f\|^q_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}
$$

To establish the boundedness of J'_2, we require the following decomposition:

$$
\begin{aligned}
J'_2 &\leq C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(z) - b_{B_k}|}{|z|^n} |f(z)| dz \right|^q \omega(t)^{1-q} dt \\
&+ C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b_{B_k} - b_{B_{k_0}}|}{|z|^n} |f(z)| dz \right|^q \omega(t)^{1-q} dt \\
&=: J'_{21} + J'_{22}.
\end{aligned}
$$

Let's start by computing J'_{21}. To accomplish this, we employ Lemmas 3.1 and 3.3 and inequality (3.16), yielding:

$$
\begin{aligned}
J'_{21} &\le C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} 2^{-kn} \int_{C_k} |b(z) - b_{B_k}||f(z)|dz \right|^q \omega(t)^{1-q} dt \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} \int_{B_{k_0}} \omega(t)^{1-q} dt \left| \sum_{k=k_0}^{\infty} 2^{-kn} \omega(B_k)^{1+\mu} \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} |B_{k_0}|^q \omega(B_{k_0})^{1-q} \left| \sum_{k=k_0}^{\infty} 2^{-kn} \omega(B_k)^{1+\mu} \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu} \left| \sum_{k=k_0}^{\infty} 2^{n(k-k_0)\mu} \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}
$$

Now, let's analyze the integral J'_{22}. To achieve this, we utilize Lemmas 3.1–3.3 and the inequality (3.15), yielding

$$
\begin{aligned}
J'_{22} &\le C \int_{B_{k_0}} \omega(t)^{1-q} dt \left| \sum_{k=k_0}^{\infty} 2^{-kn} |b_{B_k} - b_{B_{k_0}}| \int_{C_k} |f(z)|dz \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} |B_{k_0}|^q \omega(B_{k_0})^{1-q} \left| \sum_{k=k_0}^{\infty} (k-k_0) \omega(B_k)^{\lambda+\beta/n} \frac{\omega(B_{k_0})}{|B_{k_0}|} \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0}) \left| \sum_{k=k_0}^{\infty} (k-k_0) \omega(B_k)^{\mu} \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu} \left| \sum_{k=k_0}^{\infty} (k-k_0) 2^{n\delta(k-k_0)\mu} \right|^q \\
&\le C\|b\|_{Lip_{\beta,\omega}} \|f\|_{\dot{M}^{q,\lambda}(\omega)} \omega(B_{k_0})^{1+q\mu}.
\end{aligned}
$$

Here, we have utilized the fact that $\mu < 0$. By combining the estimates for J, J'_1, J'_{21}, and J'_{22}, we establish (3.14). Thus, the proof of Theorem 1.3 is complete.

4. Conclusions

In this article, we have discussed both weak and strong Hardy inequalities. We assumed that $w^{p(\cdot)} \in A_1$ to obtain the sharp constants for H and H_β on weak weighted central Morrey space. Additionally, we have explored Lipschitz estimates of Hardy operators on weighted central Morrey space. The results of this article will open new avenues for research in the fields of differential equations and quantum mechanics.

Author contributions

The authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R45), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no competing interests.

References

1. W. G. Faris, Weak Lebesgue spaces and quantum mechanical binding, *Duke Math. J.*, **43** (1976), 365–373. https://doi.org/10.1215/S0012-7094-76-04332-5
2. M. Christ, L. Grafakos, Best constants for two nonconvolution inequalities, *P. Am. Math. Soc.*, **123** (1995), 1687–1693. https://doi.org/10.2307/2161122
3. Z. W. Fu, Q. Y. Wu, S. Z. Lu, Sharp estimates of p-adic Hardy and Hardy-Littlewood-Pólya operators, *Acta Math. Sin.,* **29** (2013), 137–150. https://doi.org/10.1007/s10114-012-0149-3
4. S. Z. Lu, D. C. Yang, The central BMO space and Littlewood-Paley operators, *Approx. Theory Appl.*, **11** (1995), 72–94. https://doi.org/10.1007/BF02836580
5. S. Z. Lu, D. C. Yang, F. Y. Zhao, Sharp bounds for Hardy type operators on higher-dimensional product spaces, *J. Inequal. Appl.*, **2013** (2013). https://doi.org/10.1186/1029-242X-2013-148
6. R. H. Liu, J. Zhou, Sharp estimates for the p-adic Hardy type operator on higher-dimensional product spaces, *J. Inequal. Appl.*, **2017** (2017). https://doi.org/10.1186/s13660-017-1491-z
7. Q. J. He, X. Li, D. Y. Yan, Sharp bounds for Hardy type operators on higher-dimensional product spaces, *Front. Math. China,* **13** (2018), 1341–1353. https://doi.org/10.1007/s11464-018-0740-x
8. M. Q. Wei, D. Y. Yan, Sharp bounds for Hardy type operators on product spaces, *Acta Math. Sci.*, **38** (2018), 441–449. https://doi.org/10.1016/S0252-9602(18)30759-8
9. O. A. Omer, K. Saibi, M. Z. Abidin, M. Osman, Parametric Marcinkiewicz integral and its higher-order commutators on variable exponents Morrey-Herz spaces, *J. Funct. Space.*, 2022. https://doi.org/10.1155/2022/7209977
10. M. Z. Abidin, J. C. Chen, Global well-posedness of generalized magnetohydrodynamics equations in variable exponent Fourier-Besov-Morrey spaces, *Acta Math. Sin.*, **38** (2022), 2187–2198. https://doi.org/10.1007/s10114-022-0581-0

11. R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.*, **103** (1976), 611–635. https://doi.org/10.2307/1970954
12. S. Long, J. Wang, Commutators of Hardy operators, *J. Math. Anal. Appl.*, **274** (2002), 626–644.
13. Z. Fu, Z. Liu, S. Lu, H. Wang, Characterization for commutators of n-dimensional fractional Hardy operators, *Sci. China Ser. A*, **50** (2007), 1418–1426. https://doi.org/10.1007/s11425-007-0094-4
14. Z. Fu, S. Lu, A characterization of λ-central BMO spaces, *Front. Math. China*, **8** (2013), 229–238.
15. S. Shi, S. Lu, Characterization of the central Campanato space via the commutator operator of Hardy type, *J. Math. Anal. Appl.*, **429** (2015), 713–732. https://doi.org/10.1016/j.jmaa.2015.03.083
16. G. Gao, M. Wang, Weighted estimates for commutators of higher dimensional Hardy operators, *Appl. Math. J. Chin. Univ. Ser. A*, **27** (2012), 104–111.
17. Z. Ren, S. Tao, Weighted estimates for commutators of n-dimensional rough Hardy operators, *J. Funct. Space. Appl.*, 2013. https://doi.org/10.1155/2013/568202
18. W. Li, T. Zhang, L. Xue, Two-weight inequalities for the Hardy operator and commutators, *J. Math. Inequal.*, **9** (2015), 653–664. https://doi.org/10.7153/jmi-09-55
19. J. Álvarez, M. Guzmán-Partida, J. Lakey, Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measure, *Collect. Math.*, **51** (2000), 1–47.
20. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *T. Am. Math. Soc.*, **165** (1972), 207–226. https://doi.org/10.1090/S0002-9947-1972-0293384-6
21. J. García-Cuerva, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
22. Y. Komori, S. Shirai, Weighted Morrey spaces and a singular integral operator, *Math. Nachr.*, **282** (2009), 219–231. https://doi.org/10.1002/mana.200610733
23. X. Yu, H. Zhang, G. Zhao, Weighted boundedness of some integral operators on weighted λ-central Morrey space, *Appl. Math. J. Chin. Univ. Ser. B*, **31** (2016), 331–342. https://doi.org/10.1007/s11766-016-3348-5
24. J. García-Cuerva, Weighted H^p spaces, *Diss. Math.*, **162** (1979).
25. G. Gao, X. Hu, C. Zhong, Sharp weak estimates for Hardy-type operators, *Ann. Funct. Anal.*, **7** (2016), 421–433. https://doi.org/10.1215/20088752-3605447
26. Q. Y. Wu, Z. W. Fu, Hardy–Littlewood–Sobolev inequalities on p-adic central Morrey spaces, *J. Funct. Space.*, 2015. https://doi.org/10.1155/2015/419532
27. J. Journé, Calderón–Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón, *Lect. Notes Math.*, **994** (1983), 1–127. https://doi.org/10.1007/BFb0061463
28. Y. Lin, Z. Liu, M. Song, Lipschitz estimates for commutators of singular integral operators on weighted Herz spaces, *Jordan J. Math. Stat.*, **3** (2010), 53–64.

AIMS Press