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Research article

Soliton solutions and dynamics of the (3+1)-dimensional Kadomtsev-Petviashvili equation via ϕ^6-model expansion method

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Abstract: In this work, we explored the three-dimensional Kadomtsev-Petviashvili equation by the ϕ^6-model expansion method to generate precise solutions, such as periodic waves and singular structures, kink-type solitons, and dark solitons. The periodic solutions are the repeating patterns of a wave, and the singular solutions are the profiles that have unlimited amplitude at a certain point. The kink-type solitons depict sudden transitions, and the dark solitons represent local amplitude depressions in optical systems. Two- and three-dimensional plots were used to show the dynamic behavior and interactions of these solutions. The ϕ^6-model expansion technique is highly effective to study nonlinear systems, and can potentially be applied to fluid mechanics, plasma physics, and optical communication. The obtained results contribute to the understanding of the soliton propagation in the multi-dimensional systems.

Keywords: (3+1)-dimensional Kadomtsev-Petviashvili (KP) equation; ϕ^6-model expansion method; exact periodic; singular soliton; dark soliton
Mathematics Subject Classification: 35Q55, 35C08, 37K10, 35A20

1. Introduction

Nonlinear partial differential equations (NLPDEs) are essential to describe nonlinear physical procedures in various areas. These equations are widely related in several disciplines, including science and society. NLPDEs are a useful tool for modeling and analyzing complicated procedures, providing information on real-world events. Researchers in various fields have used this concept. NLPDEs have become a popular multidisciplinary research field in recent decades because of their ability to accurately describe physical characteristics and processes in fields such as nano technology, digital communication, engineering, and economics. NLPDEs can reflect significant depressions and dynamic activity in a variety of fields, including physics, mechanics, chemistry, biology, geophysics, hydrodynamics, oceanography, biochemistry, medical sciences, and finance [1].

Soliton solutions for NLPDEs are notable for their distinct qualities. Scott Russell, a British physicist, detected the first single waves in early 1834 [2]. High devotion, good privacy, the waveform never changes, and the speed is consistent. In current times, nonlinear evolution equations (NLEEs) provide an excellent example of how NLPDEs generate higher-order spatial data. NLEEs provide sufficient conditions for researchers to simulate parameters in model equations. Modern engineering and mathematical physics aim to provide detailed responses to NLEEs using diverse methodologies. Scientists in this field are constantly looking for innovative methods to apply new methodologies and solutions [3].

A group of mathematicians and engineering teams have developed different methods including the Hirota technique [4], the simple equation technique [5], the $\left(\frac{1}{\phi(\zeta)}, \frac{\phi'(\zeta)}{\phi(\zeta)}\right)$ method [6], modified auxiliary equation technique [7], homogeneous balance technique [8], Mittag-Leffler function-based method [9], function transformation technique [10], modified double sub-equation technique [11], new extended direct algebraic method [12], binary bell polynomials technique [13], Lie symmetric investigation [14], Sardar-sub equation method [15], adomian decomposition technique [16], the technique of Painleve analysis [17], and there may be adequate others. Naher et al. [18], introduced the G'/G-expansion method as a necessary process to study closed soliton solutions of NLPDEs. The above method is simpler to use and enables a group of NLPDEs to produce many new and interesting closed soliton solutions that have extra parameters. Many scholars have employed this method to study the exact solutions of various NLPDEs. A number of experts have lately presented and studied some other innovative soliton solutions.

Two scientists, Boris Borisovich Kadomtsev and Vladimir Iosifovich Petviashvili, introduced the Kadomtsev-Petviashvili (KP) equation in 1970 [19]. This is a natural extension of the Kortewege-de Vries (KdV) equation, and its integrability has been demonstrated using Painleve analysis. The (3 + 1)-dimensional KP equation was initially developed to study the dynamics of long-wavelength, small-amplitude solitary waves; shallow water waves with weakly nonlinear restoring forces and frequency dispersion; ferromagnetic media waves; matter-wave pulses in Bose-Einstein condensates; super fluids; plasma physics; hydrodynamics; solid-state physics; fiber optics; water engineering; and oceanography. Surface waves and internal waves in channels or straits are shallow-water waves, where the surface tension and viscosity are smaller. The (3+1)-dimensional KP equation has been utilized to simulate more dispersion effects in nonlinear studies.

The (3+1)-dimensional Kadomtsev-Petviashvili equation:

$$(U_t + 6UU_x + U_{xxx})_x - 3U_{yy} - 3U_{zz} = 0, \tag{1.1}$$

where U_t governs temporal evolution.
UU_x causes wave steepening.
U_{xxx} balances nonlinearity to form solitons.
$U_{yy},\ U_{zz}$ accounts for variations in yy and zz.

The study of traveling wave solutions is fundamental to many domains, including quantum field theory and relativistic quantum mechanics, which describe complicated physical phenomena using nonlinear wave equations. Analytical and numerical strategies for obtaining precise results include the inverse scattering transform [20], the tanh-coth method [22], and the exp-function method [23].

The search for exact solutions to NLPDEs has sparked the development of numerous computational methods. These can be broadly classified as symbolic or non-symbolic ways. Non-symbolic approaches, such as the direct mapping method [24], provide a straightforward and efficient computing path by directly translating known solutions from a canonical equation to the target NLPDE, eliminating the need for heavy algebra. On the symbolic side, the Hirota bilinear approach is a cornerstone [25], known for discovering N-soliton solutions and exploring their interactions. We use a symbolic method. The ϕ^6-model expansion method, unlike Hirota's focus on multi-solitons, seeks a broader range of solutions inside a single framework. It employs a powerful auxiliary equation with a large solution space that produces a variety of results, including topological kinks, bell-shaped solitons, and periodic solutions. While more computationally intensive than direct procedures, this method's ability to generate a wide range of novel solutions makes it the best choice for our goal of exhaustively exploring the soliton landscape of the (3+1)-D KP equations.

In this study, we use the ϕ^6 model expansion method to solve the (3+1)-dimensional KP problem for accurate soliton solutions. We aim to expand the use of the ϕ^6 model expansion method to higher-dimensional nonlinear systems, which are vital for describing wave propagation in plasmas, fluid dynamics, and nonlinear optics. The motivation is the need for more wide and flexible approaches to address the complexity of (3+1)-dimensional problems, which often challenge existing analytical techniques. The proposed method has several advantages; the method offers a systematic and adaptable approach to creating a wide range of accurate soliton solutions, such as Jacobi elliptic, trigonometric, and hyperbolic function forms, revealing complex wave dynamics. Its value comes in its potential to produce more general and different solutions than traditional methods. However, like with other analytical techniques, the method may have limitations when working with highly complicated or non-integrable models, where numerical alternatives may be required. Its usefulness is shown through the successful derivation of exact solutions for the KP equation with implications in real-world applications for ion acoustic waves in plasma and shallow-water waves. The novelty of this research comes in the combination of the ϕ^6 model expansion into a higher-dimensional system, connecting a research gap where the use of conventional methods either fails or leads to partial solutions. Earlier research tended to concentrate on (1+1)- or (2+1)-dimensional conditions with the (3+1)-dimensional KP equation being less explored.

Our research provides value by providing a systematic way for getting exact solutions for such high-dimensional systems, thereby improving our understanding of nonlinear wave behavior. In future research, the approach can be extended to other high-dimensional nonlinear equations, such as the Jimbo-Miwa or Boiti-Leon-Manna-Pempinelli equations, to test its capability. Furthermore, numerical computations can be used to compare analytical answers to real-world observations, particularly in ocean engineering and plasma physics. Our findings of this study may also encourage the development of hybrid analytical-numerical methods that bridge the gap between computational accuracy and exactness. Overall, this research not only advances theoretical understanding but also opens up new avenues for engineering and nonlinear scientific applications.

2. Methodology

In this section, we describe the detailed methodology of the ϕ^6 model expansion technique [21]. The non-linear partial differential equation (PDE) is:

$$W(w, w_x, w_t, w_{xx}, w_{tt}, ...) = 0, \tag{2.1}$$

where $W(x, t)$ is a function holding partial derivatives.
The given technique preserves the following steps:
Step 1: Various transformations can used to convert PDE into odinary differential equation (ODE), but in this work, we will focus on traveling wave transformation:

$$U(W, W', W'', W''', \cdots), \tag{2.2}$$

where U is polynomial and W is a function with higher derivatives.
Step 2: Suppose that the formal solution of (2.2) is as follows:

$$W(\xi) = \sum_{g=0}^{2v} \alpha_g \phi^g(\xi), \tag{2.3}$$

where α_g, $g = 0, 1, 2...2k$ are constants and $\phi(\xi)$ satisfy the following auxiliary nonlinear ODEs:

$$\begin{aligned} \phi'(\xi)^2 &= p_0 + p_2\phi(\xi)^2 + p_4\phi(\xi)^4 + p_6\phi(\xi)^6, \\ \phi''(\xi) &= p_2\phi(\xi) + 2p_4\phi(\xi)^3 + 3p_6\phi(\xi)^5, \end{aligned} \tag{2.4}$$

where p_ℓ is a real constant for $\ell = 0, 2, 4, 6$.
Step 3: Balancing the terms of (2.2) to obtained the value of v in (2.3).
Step 4: The (2.4) preserve the following solution:

$$\phi(\xi) = \frac{\mho(\xi)}{\sqrt{\delta\mho(\xi)^2 + \mu}}, \tag{2.5}$$

where $\mho(\xi)$ and $\delta\mho(\xi)^2 + \mu > 0$ satisfy the Jacobian elliptic equation (JEE):

$$\mho'^2 = t_0 + t_2\mho^2(\xi) + t_4\mho^4(\xi), \tag{2.6}$$

where t_ℓ for $\ell = 0, 2, 4$ are constants.
Now, the values of δ and μ are defined in this way:

$$\delta = \frac{p_4 (t_2 - p_2)}{(t_2 - p_2)^2 - 2t_2 (t_2 - p_2) + 3t_0t_4}, \tag{2.7}$$

$$\mu = \frac{3p_4t_0}{(t_2 - p_2)^2 - 2t_2 (t_2 - p_2) + 3t_0t_4}, \tag{2.8}$$

under constraint conditions:

$$p_4^2 (t_2 - p_2) [9t_0t_4 - (t_2 - p_2) (p_2 + 2t_2)] + 3p_6 \left[3t_0t_4 - \left(t_2^2 - p_2^2\right)\right]^2 = 0. \tag{2.9}$$

Step 5: The (2.6) has the JEE demonstrated in the table below:

Sr. No	t_0	t_2	t_4	$P(\xi)$
1	1	$-(1+n^2)$	n^2	$sn(\xi)$ or $cd(\xi)$
2	$1-n^2$	$2n^2-1$	$-n^2$	$cn(\xi)$
3	n^2-1	$2-n^2$	-1	$dn(\xi)$
4	n^2	$-(1+n^2)$	1	$ns(\xi)$ or $dc(\xi)$
5	$-n^2$	$2n^2-1$	$1-n^2$	$nc(\xi)$
6	-1	$2-n^3$	$-(1-n^2)$	$nd(\xi)$
7	1	$2-n^2$	$1-n^2$	$sc(\xi)$
8	1	$2n^2-1$	$-n^2(1-n^2)$	$sd(\xi)$
9	$1-n^2$	$2-n^2$	1	$cs(\xi)$
10	$-n^2(1-n^2)$	$2n^2-1$	1	$ds(\xi)$
11	$\frac{1-n^2}{4}$	$\frac{1+n^2}{2}$	$\frac{1-n^2}{4}$	$nc(\xi) \pm sc(\xi)$ or $\frac{cn(\xi)}{1\pm sn(\xi)}$
12	$\frac{-(1-n^2)^2}{4}$	$\frac{1+n^2}{2}$	$\frac{-1}{4}$	$ncn(\xi)\pm dn(\xi)$
13	$\frac{1}{4}$	$\frac{1-2n^2}{2}$	$\frac{1}{4}$	$\frac{sn(\xi)}{1\pm cn(\xi)}$
14	$\frac{1}{4}$	$\frac{1+n^2}{2}$	$\frac{(1-n^2)^2}{4}$	$\frac{sn(\xi)}{cn(\xi)\pm dn(\xi)}$

Some limitations are defined to get exact solutions:

Function	$n \to 1$	$n \to 0$	**Function**	$n \to 1$	$n \to 0$
$sn(\xi, n)$	$\tanh(\xi)$	$\sin(\xi)$	$ns(\xi, n)$	$coth(\xi)$	$csc(\xi)$
$cd(\xi)$	1	$cos(\xi)$	$dc(\xi)$	1	$sec(\xi)$
$cn(\xi)$	$sech(\xi)$	$cos(\xi)$	$nc(\xi)$	$cosh(\xi)$	$sec(\xi)$
$dn(\xi)$	$sech(\xi)$	1	$nd(\xi)$	$cosh(\xi)$	1
$sc(\xi)$	$\sinh(\xi)$	$tan(\xi)$	$cs(\xi)$	$csch(\xi)$	$cot(\xi)$
$sd(\xi)$	$\sinh(\xi)$	$\sin(\xi)$	$ds(\xi)$	$csch(\xi)$	$csc(\xi)$

Step 6: By putting (2.4) and (2.5) into (2.3), we can obtain Jacobi elliptic function solutions of (2.1).

3. Execution of the expansion method

This method describes the detailed discussion of PDE conversion into ODE and illustrates detailed soliton solutions of a given equation, which is obtained using the defined analytical method.

By using the following transformation in (1.1):

$$U(x, y, z, t) = U(\xi), \quad \xi = x + y + z - \hbar t. \tag{3.1}$$

After two successive integrations, the following ODE is obtained:

$$U''(\xi) - \hbar U(\xi) + 3U(\xi)^2 - 6U(\xi) = 0. \tag{3.2}$$

By using the balancing principle on $[U'', U^3]$, we have $v = 2$, which reduces (2.3) into:

$$\mathrm{U}(\xi) = \alpha_0 + \alpha_1\theta(\xi) + \alpha_2\theta(\xi)^2 + \alpha_3\theta(\xi)^3 + \alpha_4\theta(\xi)^4, \tag{3.3}$$

where $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,$ and α_4 are constants and $\alpha_4 \neq 0$. By doing appropriate substitution through (3.1) and (3.2), we obtain the following algebraic equations:

$$\begin{aligned}
&\theta^0 : -\alpha_0\hbar + 3\alpha_0^2 - 6\alpha_0 + 2\alpha_2 p_0 = 0,\\
&\theta^1 : \alpha_1(-\hbar) + 6\alpha_0\alpha_1 - 6\alpha_1 + \alpha_1 p_2 + 6\alpha_3 p_0 = 0,\\
&\theta^2 : -\alpha_2\hbar + 3\alpha_1^2 + 6\alpha_0\alpha_2 - 6\alpha_2 + 4\alpha_2 p_2 + 12\alpha_4 p_0 = 0,\\
&\theta^3 : -\alpha_3\hbar + 6\alpha_2\alpha_1 + 6\alpha_0\alpha_3 - 6\alpha_3 + 2\alpha_1 p_4 + 9\alpha_3 p_2 = 0,\\
&\theta^4 : -\alpha_4\hbar + 3\alpha_2^2 + 6\alpha_1\alpha_3 + 6\alpha_0\alpha_4 - 6\alpha_4 + 6\alpha_2 p_4 + 16\alpha_4 p_2 = 0,\\
&\theta^5 : 6\alpha_4\alpha_1 + 6\alpha_2\alpha_3 + 3\alpha_1 p_6 + 12\alpha_3 p_4 = 0,\\
&\theta^6 : 3\alpha_3^2 + 6\alpha_2\alpha_4 + 8\alpha_2 p_6 + 20\alpha_4 p_4 = 0,\\
&\theta^7 : 6\alpha_4\alpha_3 + 15\alpha_3 p_6 = 0,\\
&\theta^8 : 3\alpha_4^2 + 24\alpha_4 p_6 = 0.
\end{aligned} \tag{3.4}$$

To compute parameters from above algebraic equations, we utilize Mathematica 13.0.

$$\alpha_0 = 0,\ \ \alpha_3 = 0,\ \ \alpha_4 = -\frac{\alpha_2^3}{\alpha_1^2},\ \ p_2 = -\frac{\alpha_1^2}{\alpha_2},\ \ p_4 = -3\alpha_2,\ \ p_6 = \frac{\alpha_2^3}{8\alpha_1^2},\ \ p_0 = 0,\ \ \hbar = \frac{-\alpha_1^2 - 6\alpha_2}{\alpha_2} \tag{3.5}$$

The obtained analytic solutions of (1.1) are:
Case 1: If $t_0 = 1;\ t_2 = -\left(1 + n^2\right);\ t_4 = n^2;$ **then** $\mho(\xi)$=sn(ξ) or cd(ξ), $0 < n < 1$ obtain the JEF solutions:

$$U_1 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2 + \mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2 + \mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2 + \mu}}, \tag{3.6}$$

where δ and μ are given as

$$\begin{aligned}
\delta &= \frac{p_4\left(-n^2 - p_2 - 1\right)}{(-n^2 - p_2 - 1)^2 - 2(-n^2 - 1)(-n^2 - p_2 - 1) + 3n^2},\\
\mu &= \frac{3p_4}{(-n^2 - p_2 - 1)^2 - 2(-n^2 - 1)(-n^2 - p_2 - 1) + 3n^2},
\end{aligned} \tag{3.7}$$

satisfy the following condition:

$$p_4^2\left(-n^2 - p_2 - 1\right)\left(9n^2 - \left(-n^2 - p_2 - 1\right)\left(2\left(-n^2 - 1\right) + p_2\right)\right) + 3p_6\left(3n^2 - \left(-n^2 - 1\right)^2 + p_2^2\right)^2 = 0,$$

where p_2, p_4 and p_6 are given above.
When $n \to 1$.
As $\mho(\xi) = \text{sn}(\xi) = \tanh(\xi)$, then we obtain

$$U_{1,1} = \frac{1}{3}\tanh(\xi)\left(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2(D)}{\alpha_1^4 - \alpha_2^2}}} + \frac{\left(\alpha_1^4 - \alpha_2^2\right)\tanh(\xi)}{\alpha_2\,(D)} - \frac{\left(\alpha_1^4 - \alpha_2^2\right)^2\tanh^3(\xi)}{3\alpha_1^2\alpha_2\,(-D)^2}\right), \tag{3.8}$$

where

$$D = \alpha_2\left(2\tanh^2(\xi) - 3\right) - \alpha_1^2\tanh^2(\xi).$$

When $\mho(\xi) = \mathrm{cd}(\xi) = 1$, then we obtain the following result:

$$U_{1,2} = \frac{1}{3}\left(-\frac{\left(\alpha_1^4 - \alpha_2^2\right)^2}{3\alpha_1^2\alpha_2\left(\alpha_1^2 + \alpha_2\right)^2} + \frac{\alpha_1^4 - \alpha_2^2}{\left(-\alpha_1^2 - \alpha_2\right)\alpha_2} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\left(-\alpha_1^2-\alpha_2\right)\alpha_2^2}{\alpha_1^4-\alpha_2^2}}}\right), \tag{3.9}$$

satisfy the condition:

$$p_4^2(-p_2 - 2)(9 - (-p_2 - 2)(p_2 - 4)) + 3p_6\left(p_2^2 - 1\right)^2 = 0.$$

When $n \to 0$

As $\mho(\xi) = \mathrm{sn}(\xi) = \sin(\xi)$, we then obtain:

$$\begin{aligned} U_{1,3} = \frac{1}{3}\sin(\xi)\Bigg(&\frac{\left(\alpha_1^4 - \alpha_2^2\right)\sin(\xi)}{\alpha_2\left(\alpha_2\left(\sin^2(\xi) - 3\right) - \alpha_1^2\sin^2(\xi)\right)} - \frac{\left(\alpha_1^4 - \alpha_2^2\right)^2\sin^3(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_1^2\sin^2(\xi) - \alpha_2\left(\sin^2(\xi) - 3\right)\right)^2} \\ &+ \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2\left(\sin^2(\xi)-3\right)-\alpha_1^2\sin^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}}\Bigg), \end{aligned} \tag{3.10}$$

or $\mho(\xi) = \mathrm{cd}(\xi) = \cos(\xi)$, then we obtain:

$$\begin{aligned} U_{1,4} = \frac{1}{3}\cos(\xi)\Bigg(&\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2\left(\cos^2(\xi)-3\right)-\alpha_1^2\cos^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}} + \frac{\left(\alpha_1^4 - \alpha_2^2\right)\cos(\xi)}{\alpha_2\left(\alpha_2\left(\cos^2(\xi) - 3\right) - \alpha_1^2\cos^2(\xi)\right)} \\ &- \frac{\left(\alpha_1^4 - \alpha_2^2\right)^2\cos^3(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_1^2\cos^2(\xi) - \alpha_2\left(\cos^2(\xi) - 3\right)\right)^2}\Bigg), \end{aligned} \tag{3.11}$$

satisfy the condition:

$$p_4^2(-p_2 - 1)(-(-p_2 - 1)(p_2 - 2)) + 3p_6\left(p_2^2 - 1\right)^2 = 0.$$

Case 2: If $t_0 = 1 - \mu^2$; $t_2 = 2\mu^2 - 1$; $t_4 = -\mu^2$**,** $0 < \mu < 1$**, then** $\mho(\xi)$**=cn**(ξ) and the JEF solution:

$$U_2 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2 + \mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2 + \mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2 + \mu}}, \tag{3.12}$$

where δ and μ are of the form:

$$\delta = \frac{p_4\left(2n^2 - p_2 - 1\right)}{(2n^2 - p_2 - 1)^2 - 2(2n^2 - 1)(2n^2 - p_2 - 1) - 3(1 - n^2)n^2},$$

$$\mu=\frac{3\left(1-n^2\right)p_4}{(2n^2-p_2-1)^2-2\,(2n^2-1)\,(2n^2-p_2-1)-3\,(1-n^2)\,n^2},$$

satisfy the condition:

$$\rho_4^2\left(2n^2-\rho_2-1\right)\left(-\left(2n^2-\rho_2-1\right)\left(2\left(2n^2-1\right)+n_2\right)\right.$$
$$\left.-9\left(1-n^2\right)n^2\right)+3\rho_6\left(-3\left(1-n^2\right)n^2-\left(2n^2-1\right)^2+\rho_2^2\right)^2=0.$$

When $n\rightarrow 1$
As $\mho(\xi)=\mathrm{sn}(\xi)=\mathrm{sech}(\xi)$, then we obtain

$$U_{2,1}=\alpha_0+\alpha_2\left(\frac{3\left(\alpha_0^2(-\delta)\kappa+4\kappa\left(c^2+\gamma^2+\kappa^2\right)+\sigma\right)}{2\alpha_0\alpha_2\delta\kappa}\right), \tag{3.13}$$

satisfy the condition:

$$\rho_4^2\,(1-\rho_2)\,(-\,(1-\rho_2)\,(\rho_2+2))+3\rho_6\left(\rho_2^2-1\right)^2=0.$$

Now, when $n\rightarrow 0$.
As $\mho(\xi)=\mathrm{sn}(\xi)=\cos(\xi)$, then we obtain

$$U_{2,2}=\alpha_0+\alpha_2\left(\frac{3\left(\alpha_0^2(-\delta)\kappa+4\kappa\left(c^2+\gamma^2+\kappa^2\right)+\sigma\right)}{2\alpha_0\alpha_2\delta\kappa}\right), \tag{3.14}$$

under the condition

$$\rho_4^2\,(-\rho_2-1)\,[-\,(-\rho_2-1)\,(\rho_2-2)]+3\rho_6\left(\rho_2^2-1\right)^2=0.$$

Case 3: If $\mathrm{t}_0=n^2-1$, $\mathrm{t}_2=2-n^2$, $\mathrm{t}_4=-1$**,** $0<n<1$**, then** $\mho(\xi)$**=dn**(ξ), we obtain the JEF solution:

$$U_3=-\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2}+\frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu}+\frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.15}$$

where δ and μ are as:

$$\delta=\frac{p_4\left(-n^2-p_2+2\right)}{(-n^2-p_2+2)^2-2\,(2-n^2)\,(-n^2-p_2+2)-3\,(n^2-1)},$$
$$\mu=\frac{3\left(n^2-1\right)p_4}{(-n^2-p_2+2)^2-2\,(2-n^2)\,(-n^2-p_2+2)-3\,(n^2-1)}.$$

Under the condition:

$$\rho_4^2\left(-n^2-\rho_2+2\right)\left(-\left(-n^2-\rho_2+2\right)\left(2\left(2-n^2\right)+\rho_2\right)-9\left(n^2-1\right)\right)+3\rho_6\left(-\left(2-n^2\right)^2-3\left(n^2-1\right)+\rho_2^2\right)^2=0.$$

If $n \to 1$.
As $\mho(\xi) = \text{dn}(\xi) = \text{sech}(\xi)$. Then, we obtain:

$$U_{3,1} = \alpha_0 + \alpha_2 \left(\frac{3\left(\alpha_0^2(-\delta)\kappa + 4\kappa\left(c^2 + \gamma^2 + \kappa^2\right) + \sigma\right)}{2\alpha_0\alpha_2\delta\kappa} \right), \tag{3.16}$$

satisfy the condition:

$$\rho_4^2\left(1-\rho_2\right)\left(-\left(1-\rho_2\right)\left(\rho_2+2\right)\right) + 3\rho_6\left(\rho_2^2-1\right)^2 = 0.$$

When $n \to 0$.
As $\mho(\xi) = \text{sn}(\xi) = 1$, then we obtain

$$U_{3,2} = \alpha_0 + \alpha_2 \left(\frac{3\left(\alpha_0^2(-\delta)\kappa + 4\kappa\left(c^2 + \gamma^2 + \kappa^2\right) + \sigma\right)}{2\alpha_0\alpha_2\delta\kappa} \right), \tag{3.17}$$

satisfy the condition:

$$\rho_4^2\left(2-\rho_2\right)\left(9-\left(2-\rho_2\right)\left(\rho_2+4\right)\right) + 3\rho_6\left(\rho_2^2-1\right)^2 = 0.$$

Case 4: If $\text{t}_0 = n^2$, $\text{t}_2 = -\left(n^2+1\right)$, $\text{t}_4 = 1$**,** $0 < n < 1$**, then** $\mho(\xi)$**=ns**(ξ) **or dc**(ξ) the JEF solution given as:

$$U_4 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.18}$$

where δ and μ are as:

$$\delta = \frac{p_4\left(-n^2-p_2-1\right)}{\left(-n^2-p_2-1\right)^2 - 2\left(-n^2-1\right)\left(-n^2-p_2-1\right) + 3n^2},$$
$$\mu = \frac{3n^2p_4}{\left(-n^2-p_2-1\right)^2 - 2\left(-n^2-1\right)\left(-n^2-p_2-1\right) + 3n^2},$$

satisfy the condition:

$$p_4^2\left(-n^2-p_2-1\right)\left(9n^2-\left(-n^2-p_2-1\right)\left(2\left(-n^2-1\right)+p_2\right)\right) + 3p_6\left(3n^2-\left(-n^2-1\right)^2+p_2^2\right)^2 = 0.$$

If $n \to 1$.
As $\mho(\xi) = \text{ns}(\xi) = \coth(\xi)$, then we get:

$$U_{4,1} = \frac{1}{3}\coth(\xi)\Bigg(\frac{\left(\alpha_1^4-\alpha_2^2\right)\coth(\xi)}{\alpha_2\left(\left(2\alpha_2-\alpha_1^2\right)\coth^2(\xi)-3\alpha_2\right)} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\left(\alpha_1^2-2\alpha_2\right)\coth^2(\xi)+3\alpha_2\right)}{\alpha_2^2-\alpha_1^4}}} - \frac{\left(\alpha_1^4-\alpha_2^2\right)^2\coth^3(\xi)}{3\alpha_1^2\alpha_2\left(\left(2\alpha_2-\alpha_1^2\right)\coth^2(\xi)-3\alpha_2\right)^2} \Bigg). \tag{3.19}$$

Or $\mho(\xi) = \mathrm{dc}(\xi) = 1$, so we get:

$$U_{4,2} = \frac{1}{3}\left(-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2}{3\alpha_1^2\left(-\alpha_1^2-\alpha_2\right)^2\alpha_2}+\frac{\alpha_1^4-\alpha_2^2}{\left(-\alpha_1^2-\alpha_2\right)\alpha_2}+\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_1^2+\alpha_2\right)}{\alpha_2^2-\alpha_1^4}}}\right), \tag{3.20}$$

satisfy the condition:

$$p_4^2\left(-p_2-2\right)\left(9-\left(-p_2-2\right)\left(p_2-4\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

When $n \to 0$.
As $\mho(\xi) = \mathrm{ns}(\xi) = \csc(\xi)$, we obtain

$$U_{4,3} = \frac{1}{3}\csc(\xi)\left(-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2\sin(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_2-\alpha_1^2\right)^2}+\frac{\left(\alpha_1^4-\alpha_2^2\right)\sin(\xi)}{\alpha_2\left(\alpha_2-\alpha_1^2\right)}+\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\left(\alpha_1^2-\alpha_2\right)\alpha_2^2\csc^2(\xi)}{\alpha_2^2-\alpha_1^4}}}\right), \tag{3.21}$$

or $\mho(\xi) = \mathrm{dc}(\xi) = \sec(\xi)$ so

$$U_{4,4} = \frac{1}{3}\sec(\xi)\left(-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2\cos(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_2-\alpha_1^2\right)^2}+\frac{\left(\alpha_1^4-\alpha_2^2\right)\cos(\xi)}{\alpha_2\left(\alpha_2-\alpha_1^2\right)}+\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\left(\alpha_1^2-\alpha_2\right)\alpha_2^2\sec^2(\xi)}{\alpha_2^2-\alpha_1^4}}}\right), \tag{3.22}$$

under constraint conditions:

$$p_4^2\left(-p_2-1\right)\left(-\left(-p_2-1\right)\left(p_2-2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

Case 5: If $\mathrm{t}_0 = -n^2$, $\mathrm{t}_2 = 2n^2-1$, $\mathrm{t}_4 = 1-n^2$**,** $0 < n < 1$**, then** $\eth(\xi)$**=nc**(ξ) obtain the JEF solution:

$$U_5 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2}+\frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu}+\frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.23}$$

where δ and μ are:

$$\delta = \frac{p_4\left(2n^2-p_2-1\right)}{\left(2n^2-p_2-1\right)^2-2\left(2n^2-1\right)\left(2n^2-p_2-1\right)-3\left(1-n^2\right)n^2},$$
$$\mu = -\frac{3n^2p_4}{\left(2n^2-p_2-1\right)^2-2\left(2n^2-1\right)\left(2n^2-p_2-1\right)-3\left(1-n^2\right)n^2},$$

satisfy the condition:

$$p_4^2\left(2n^2-p_2-1\right)\left(-\left(2n^2-p_2-1\right)\left(2\left(2n^2-1\right)+p_2\right)-9\left(1-n^2\right)n^2\right)+3p_6$$
$$\times\left(-3\left(1-n^2\right)n^2-\left(2n^2-1\right)^2+p_2^2\right)^2=0.$$

If $n \to 1$.
As $\mho(\xi) = \mathrm{nc}(\xi) = \cosh(\xi)$, we get:

$$U_{5,1} = \frac{1}{3}\cosh(\xi)\left(\frac{\left(\alpha_2^2 - \alpha_1^4\right)\cosh(\xi)}{\alpha_2\left(\left(\alpha_1^2 + \alpha_2\right)\cosh^2(\xi) - 3\alpha_2\right)} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\left(\alpha_1^2+\alpha_2\right)\cosh^2(\xi)-3\alpha_2\right)}{\alpha_2^2-\alpha_1^4}}} - \frac{\left(\alpha_1^4 - \alpha_2^2\right)^2\cosh^3(\xi)}{3\alpha_1^2\alpha_2\left(\left(\alpha_1^2 + \alpha_2\right)\cosh^2(\xi) - 3\alpha_2\right)^2}\right), \tag{3.24}$$

satisfy the condition:

$$p_4^2\left(1 - p_2\right)\left(-\left(1 - p_2\right)\left(p_2 + 2\right)\right) + 3p_6\left(p_2^2 - 1\right)^2 = 0.$$

When $n \to 0$
As $\mho(\xi) = \mathrm{nc}(\xi) = \sec(\xi)$, we get:

$$U_{5,2} = \frac{1}{3}\sec(\xi)\left(-\frac{\left(\alpha_1^4 - \alpha_2^2\right)^2\cos(\xi)}{3\alpha_1^2\left(\alpha_1^2 - \alpha_2\right)^2\alpha_2} + \frac{\left(\alpha_2^2 - \alpha_1^4\right)\cos(\xi)}{\left(\alpha_1^2 - \alpha_2\right)\alpha_2} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\left(\alpha_1^2-\alpha_2\right)\alpha_2^2\sec^2(\xi)}{\alpha_2^2-\alpha_1^4}}}\right), \tag{3.25}$$

under the computed constraint condition:

$$p_4^2\left(-p_2 - 1\right)\left(-\left(-p_2 - 1\right)\left(p_2 - 2\right)\right) + 3p_6\left(p_2^2 - 1\right)^2 = 0.$$

Case 6: If $t_0 = -1$, $t_2 = 2 - n^2$, $t_4 = -\left(1 - n^2\right)$**,** $0 < n < 1$**, then** $\mho(\xi)$**=nc**(ξ) obtain JEF solution:

$$U_6 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2 + \mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2 + \mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2 + \mu}}, \tag{3.26}$$

where δ and μ are:

$$\delta = \frac{p_4\left(-n^2 - p_2 + 2\right)}{\left(-n^2 - p_2 + 2\right)^2 - 2\left(2 - n^2\right)\left(-n^2 - p_2 + 2\right) - 3\left(n^2 - 1\right)},$$
$$\mu = -\frac{3p_4}{\left(-n^2 - p_2 + 2\right)^2 - 2\left(2 - n^2\right)\left(-n^2 - p_2 + 2\right) - 3\left(n^2 - 1\right)}.$$

under the condition:

$$p_4^2\left(-n^2 - p_2 + 2\right)\left(-\left(-n^2 - p_2 + 2\right)\left(2\left(2 - n^2\right) + p_2\right) - 9\left(n^2 - 1\right)\right) + 3p_6\left(-\left(2 - n^2\right)^2 - 3\left(n^2 - 1\right) + p_2^2\right)^2 = 0,$$

when $n \to 1$.
As $\mho(\xi) = \mathrm{nc}(\xi) = \cosh(\xi)$, we obtain:

$$U_{6,1} = \frac{1}{3}\cosh(\xi)\left(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2\left(3-\cosh^2(\xi)\right)-\alpha_1^2\cosh^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}} + \frac{\left(\alpha_1^4 - \alpha_2^2\right)\cosh(\xi)}{\alpha_2\left(\alpha_2\left(3 - \cosh^2(\xi)\right) - \alpha_1^2\cosh^2(\xi)\right)} - \frac{\left(\alpha_1^4 - \alpha_2^2\right)^2\cosh^3(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_1^2\cosh^2(\xi) - \alpha_2\left(3 - \cosh^2(\xi)\right)\right)^2}\right). \tag{3.27}$$

under condition interpret as:

$$p_4^2(1-p_2)(-(1-p_2)(p_2+2))+3p_6\left(p_2^2-1\right)^2=0.$$

If $n \to 0$
As $\mho(\xi) = \mathrm{nc}(\xi) = 1$, then we get:

$$U_{6,2}=\frac{1}{3}\left(-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2}{3\alpha_1^2\left(\alpha_1^2-\alpha_2\right)^2\alpha_2}+\frac{\alpha_1^4-\alpha_2^2}{\alpha_2\left(\alpha_2-\alpha_1^2\right)}+\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2-\alpha_1^2\right)}{\alpha_1^4-\alpha_2^2}}}\right), \tag{3.28}$$

under interpret condition:

$$p_4^2(2-p_2)(9-(2-p_2)(p_2+4))+3p_6\left(p_2^2-1\right)^2=0.$$

Case 7: If $t_0 = 1, t_2 = 2-n^2, t_4 = 1-n^2$**,** $0 < n < 1$**, then** $\mho(\xi)$**=sc**(ξ) obtain JEF solution:

$$U_7=-\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2}+\frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu}+\frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.29}$$

where δ and μ are:

$$\delta=\frac{p_4\left(-n^2-p_2+2\right)}{(-n^2-p_2+2)^2-2(2-n^2)(-n^2-p_2+2)+3(1-n^2)},$$
$$\mu=\frac{3p_4}{(-n^2-p_2+2)^2-2(2-n^2)(-n^2-p_2+2)+3(1-n^2)},$$

satisfy the condition:

$$p_4^2\left(-n^2-p_2+2\right)\left(9\left(1-n^2\right)-\left(-n^2-p_2+2\right)\left(2\left(2-n^2\right)+p_2\right)\right)+3p_6\left(-\left(2-n^2\right)^2+3\left(1-n^2\right)+p_2^2\right)^2=0.$$

When $n \to 1$.
As $\mho(\xi) = \mathrm{sc}(\xi) = \sinh(\xi)$, we get:

$$\begin{aligned}U_{7,1}=\frac{1}{3}\sinh(\xi)\Bigg(&\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2\left(-\sinh^2(\xi)-3\right)-\alpha_1^2\sinh^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}}+\frac{\left(\alpha_1^4-\alpha_2^2\right)\sinh(\xi)}{\alpha_2\left(\alpha_2\left(-\sinh^2(\xi)-3\right)-\alpha_1^2\sinh^2(\xi)\right)}\\&-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2\sinh^3(\xi)}{3\alpha_1^2\alpha_2\left(\left(\alpha_1^2+\alpha_2\right)\sinh^2(\xi)+3\alpha_2\right)^2}\Bigg),\end{aligned} \tag{3.30}$$

satisfy the condition:

$$p_4^2(1-p_2)(-(1-p_2)(p_2+2))+3p_6\left(p_2^2-1\right)^2=0.$$

If $n \to 0$.
As $\mho(\xi) = \mathrm{sc}(\xi) = \tan(\xi)$, then

$$U_{7,2} = \frac{1}{3}\tan(\xi)\left(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2\left(-2\tan^2(\xi)-3\right)-\alpha_1^2\tan^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}} + \frac{\left(\alpha_1^4-\alpha_2^2\right)\tan(\xi)}{\alpha_2\left(\alpha_2\left(-2\tan^2(\xi)-3\right)-\alpha_1^2\tan^2(\xi)\right)} - \frac{\left(\alpha_1^4-\alpha_2^2\right)^2\tan^3(\xi)}{3\alpha_1^2\alpha_2\left(\left(\alpha_1^2+2\alpha_2\right)\tan^2(\xi)+3\alpha_2\right)^2}\right), \tag{3.31}$$

under the condition:

$$p_4^2\left(2-p_2\right)\left(9-\left(2-p_2\right)\left(p_2+4\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

Case 8: If $t_0 = 1$, $t_2 = 2n^2 - 1$, $t_4 = -n^2\left(1-n^2\right)$**,** $0 < n < 1$ **then** $\mho(\xi)$**=sd**(ξ) obtain JEF solution:

$$U_8 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.32}$$

where δ and μ are:

$$\begin{aligned}\delta &= \frac{p_4\left(2n^2-p_2-1\right)}{\left(2n^2-p_2-1\right)^2-2\left(2n^2-1\right)\left(2n^2-p_2-1\right)-3\left(1-n^2\right)n^2},\\ \mu &= \frac{3p_4}{\left(2n^2-p_2-1\right)^2-2\left(2n^2-1\right)\left(2n^2-p_2-1\right)-3\left(1-n^2\right)n^2},\end{aligned} \tag{3.33}$$

satisfy the condition:

$$\begin{aligned}&p_4^2\left(2n^2-p_2-1\right)\left(-\left(2n^2-p_2-1\right)\left(2\left(2n^2-1\right)+p_2\right)-9\left(1-n^2\right)n^2\right)+\\ &3p_6\left(-3\left(1-n^2\right)n^2-\left(2n^2-1\right)^2+p_2^2\right)^2=0.\end{aligned}$$

If $n \to 1$.
As $\mho(\xi) = \mathrm{sd}(\xi) = \sinh(\xi)$ then:

$$U_{8,1} = \frac{1}{3}\sinh(\xi)\left(\frac{\left(\alpha_2^2-\alpha_1^4\right)\sinh(\xi)}{\alpha_2\left(\left(\alpha_1^2+\alpha_2\right)\sinh^2(\xi)+3\alpha_2\right)} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\left(\alpha_1^2+\alpha_2\right)\sinh^2(\xi)+3\alpha_2\right)}{\alpha_2^2-\alpha_1^4}}} - \frac{\left(\alpha_1^4-\alpha_2^2\right)^2\sinh^3(\xi)}{3\alpha_1^2\alpha_2\left(\left(\alpha_1^2+\alpha_2\right)\sinh^2(\xi)+3\alpha_2\right)^2}\right), \tag{3.34}$$

satisfy the condition:

$$p_4^2\left(1-p_2\right)\left(-\left(1-p_2\right)\left(p_2+2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

If $n \to 0$.
As $\mho(\xi) = \mathrm{sd}(\xi) = \sin(\xi)$, we get

$$\begin{aligned} \mathrm{U}_{8,2} = \frac{1}{3}\sin(\xi)\Bigg(& \frac{\left(\alpha_2^2-\alpha_1^4\right)\sin(\xi)}{\alpha_2\left(\left(\alpha_1^2-\alpha_2\right)\sin^2(\xi)+3\alpha_2\right)} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\left(\alpha_1^2-\alpha_2\right)\sin^2(\xi)+3\alpha_2\right)}{\alpha_2^2-\alpha_1^4}}} \\ & - \frac{\left(\alpha_1^4-\alpha_2^2\right)^2\sin^3(\xi)}{3\alpha_1^2\alpha_2\left(\left(\alpha_1^2-\alpha_2\right)\sin^2(\xi)+3\alpha_2\right)^2}\Bigg), \end{aligned} \tag{3.35}$$

under interpreted constraint condition:

$$p_4^2\left(-p_2-1\right)\left(-\left(-p_2-1\right)\left(p_2-2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

Case 9: If $\mathrm{t}=1-n^2$, $\mathrm{t}_2=2-n^2$, $\mathrm{t}_4=1$**,** $0<n<1$ **then** $\mho(\xi)$**=cs**(ξ) obtain solution:

$$U_9 = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.36}$$

where δ and μ are as:

$$\begin{aligned} \delta &= \frac{p_4\left(-n^2-p_2+2\right)}{\left(-n^2-p_2+2\right)^2-2\left(2-n^2\right)\left(-n^2-p_2+2\right)+3\left(1-n^2\right)}, \\ \mu &= \frac{3\left(1-n^2\right)p_4}{\left(-n^2-p_2+2\right)^2-2\left(2-n^2\right)\left(-n^2-p_2+2\right)+3\left(1-n^2\right)}, \end{aligned}$$

satisfy the condition:

$$p_4^2\left(-n^2-p_2+2\right)\left(9\left(1-n^2\right)-\left(-n^2-p_2+2\right)\left(2\left(2-n^2\right)+p_2\right)\right)+3p_6\left(-\left(2-n^2\right)^2+3\left(1-n^2\right)+p_2^2\right)^2=0.$$

If $n \to 1$.
As $\mho(\xi) = \mathrm{cs}(\xi) = \mathrm{csch}(\xi)$, then we obtain:

$$\begin{aligned} U_{9,1} = \frac{1}{3}\mathrm{csch}(\xi)\Bigg(& \frac{\left(\alpha_1^4-\alpha_2^2\right)\mathrm{csch}(\xi)}{\alpha_2\left(-\alpha_2\mathrm{csch}^2(\xi)-\alpha_1^2\mathrm{csch}^2(\xi)\right)} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(-\alpha_2\mathrm{csch}^2(\xi)-\alpha_1^2\mathrm{csch}^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}} \\ & - \frac{\left(\alpha_1^4-\alpha_2^2\right)^2\mathrm{csch}^3(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_1^2\mathrm{csch}^2(\xi)+\alpha_2\mathrm{csch}^2(\xi)\right)^2}\Bigg), \end{aligned} \tag{3.37}$$

satisfy the condition:

$$p_4^2\left(1-p_2\right)\left(-\left(1-p_2\right)\left(p_2+2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

If $n \to 0$.
As $\mho(\xi) = \text{cs}(\xi) = \cot(\xi)$, then we get:

$$U_{9,2} = \frac{1}{3}\cot(\xi)\Bigg(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_2\left(-2\cot^2(\xi)-3\right)-\alpha_1^2\cot^2(\xi)\right)}{\alpha_1^4-\alpha_2^2}}} + \frac{\left(\alpha_1^4-\alpha_2^2\right)\cot(\xi)}{\alpha_2\left(\alpha_2\left(-2\cot^2(\xi)-3\right)-\alpha_1^2\cot^2(\xi)\right)} - \frac{\left(\alpha_1^4-\alpha_2^2\right)^2\cot^3(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_1^2\cot^2(\xi)+\alpha_2\left(2\cot^2(\xi)+3\right)\right)^2}\Bigg), \tag{3.38}$$

satisfy the condition:

$$p_4^2\left(2-p_2\right)\left(9-\left(2-p_2\right)\left(p_2+4\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

Case 10: If $t_0 = -n^2\left(1-n^2\right)$, $t_2 = 2n^2-1$, $t_4 = 1$**,** $0 < n < 1$**, then** $\mho(\xi)$**=ds**(ξ) obtain JEF solution:

$$U_{10} = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.39}$$

where δ and μ are,

$$\delta = \frac{p_4\left(2n^2-p_2-1\right)}{\left(2n^2-p_2-1\right)^2-2\left(2n^2-1\right)\left(2n^2-p_2-1\right)-3\left(1-n^2\right)n^2},$$
$$\mu = -\frac{3n^2\left(1-n^2\right)p_4}{\left(2n^2-p_2-1\right)^2-2\left(2n^2-1\right)\left(2n^2-p_2-1\right)-3\left(1-n^2\right)n^2},$$

under the condition:

$$p_4^2\left(2n^2-p_2-1\right)\left(-\left(2n^2-p_2-1\right)\left(2\left(2n^2-1\right)+p_2\right)-9\left(1-n^2\right)n^2\right)+ 3p_6\left(-3\left(1-n^2\right)n^2-\left(2n^2-1\right)^2+p_2^2\right)^2=0.$$

If $n \to 1$.
As $\mho(\xi) = \text{ds}(\xi) = \text{csch}(\xi)$, then we get:

$$U_{10,1} = \frac{1}{3}\text{csch}(\xi)\left(-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2\sinh(\xi)}{3\alpha_1^2\alpha_2\left(\alpha_1^2+\alpha_2\right)^2} + \frac{\left(\alpha_2^2-\alpha_1^4\right)\sinh(\xi)}{\alpha_2\left(\alpha_1^2+\alpha_2\right)} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(\alpha_1^2+\alpha_2\right)\text{csch}^2(\xi)}{\alpha_2^2-\alpha_1^4}}}\right), \tag{3.40}$$

satisfy the condition:

$$p_4^2\left(1-p_2\right)\left(-\left(1-p_2\right)\left(p_2+2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

If $n \to 0$.
As $\eth(\xi) = \mathrm{ds}(\xi) = \csc(\xi)$, we get:

$$\mathtt{U}_{10,2} = \frac{1}{3}\csc(\xi)\left(-\frac{\left(\alpha_1^4-\alpha_2^2\right)^2\sin(\xi)}{3\alpha_1^2\left(\alpha_1^2-\alpha_2\right)^2\alpha_2} + \frac{\left(\alpha_2^2-\alpha_1^4\right)\sin(\xi)}{\left(\alpha_1^2-\alpha_2\right)\alpha_2} + \frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\left(\alpha_1^2-\alpha_2\right)\alpha_2^2\csc^2(\xi)}{\alpha_2^2-\alpha_1^4}}}\right), \tag{3.41}$$

under constraint condition:

$$p_4^2\left(-p_2-1\right)\left(-\left(-p_2-1\right)\left(p_2-2\right)\right) + 3p_6\left(p_2^2-1\right)^2 = 0.$$

Case 11: If $\mathrm{t}_0 = \frac{1}{4}\left(1-n^2\right), \mathrm{t}_2 = \frac{1}{2}\left(n^2+1\right), \mathrm{t}_4 = \frac{1}{4}\left(1-n^2\right)$**,** $0 < n < 1$**, then** $\eth(\xi)$**=nc**$(\xi)\pm$**sc**(ξ)**or** $\frac{\mathbf{cn}(\xi)}{1\pm\mathbf{sn}(\xi)}$ obtain JEF solution:

$$U_{11} = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2} + \frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu} + \frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.42}$$

where δ and μ are as:

$$\delta = \frac{p_4\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)}{\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)^2-\left(n^2+1\right)\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)+\frac{3}{16}\left(1-n^2\right)^2},$$

$$\mu = \frac{3\left(1-n^2\right)p_4}{4\left(\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)^2-\left(n^2+1\right)\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)+\frac{3}{16}\left(1-n^2\right)^2\right)},$$

satisfy the condition:

$$p_4^2\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)\left(\frac{9}{16}\left(1-n^2\right)^2-\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)\left(n^2+p_2+1\right)\right)+$$
$$3p_6\left(\frac{3}{16}\left(1-n^2\right)^2-\frac{1}{4}\left(n^2+1\right)^2+p_2^2\right)^2 = 0.$$

If $n \to 1$.
As $\mho(\xi) = \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi) = \cosh(\xi) \pm \sinh(\xi)$, then we get:

$$\begin{aligned}\mathtt{U}_{11,1} =& \frac{1}{6}(\cosh(\xi)\pm\sinh(\xi))\Bigg(\frac{\alpha_1}{\sqrt{2}\sqrt{-\frac{\alpha_2^2\left(2\alpha_1^2+2\alpha_2\right)(\cosh(\xi)\pm\sinh(\xi))^2}{48\alpha_1^4-48\alpha_2^2}}} \\ &+ \frac{(16\alpha_2^2-16\alpha_1^4)(\cosh(\xi)\pm\sinh(\xi))}{\alpha_2\left(8\alpha_1^2(\cosh(\xi)\pm\sinh(\xi))^2+8\alpha_2(\cosh(\xi)\pm\sinh(\xi))^2\right)}\Bigg) \\ &\times\left(-\frac{(16\alpha_2^2-16\alpha_1^4)^2(\cosh(\xi)\pm\sinh(\xi))^3}{24\alpha_1^2\alpha_2\left(4\alpha_1^2(\cosh(\xi)\pm\sinh(\xi))^2+4\alpha_2(\cosh(\xi)\pm\sinh(\xi))^2\right)^2}\right)\end{aligned} \tag{3.43}$$

and

$$\mho_{11,2}=\frac{\operatorname{sech}(\xi)\left(\frac{\alpha_1}{\sqrt{2}\sqrt{-\frac{\alpha_2^2\left(2\alpha_1^2+2\alpha_2\right)\operatorname{sech}^2(\xi)}{\left(48\alpha_1^4-48\alpha_2^2\right)(1\pm\tanh(\xi))^2}}}+\frac{\left(16\alpha_2^2-16\alpha_1^4\right)\operatorname{sech}(\xi)}{\alpha_2(1\pm\tanh(\xi))\left(\frac{8\alpha_1^2\operatorname{sech}^2(\xi)}{(1\pm\tanh(\xi))^2}+\frac{8\alpha_2\operatorname{sech}^2(\xi)}{(1\pm\tanh(\xi))^2}\right)}-\frac{\left(16\alpha_2^2-16\alpha_1^4\right)^2\operatorname{sech}^3(\xi)}{24\alpha_1^2\alpha_2(1\pm\tanh(\xi))^3\left(\frac{4\alpha_1^2\operatorname{sech}^2(\xi)}{(1\pm\tanh(\xi))^2}+\frac{4\alpha_2\operatorname{sech}^2(\xi)}{(1\pm\tanh(\xi))^2}\right)^2}\right)}{6(1\pm\tanh(\xi))} \tag{3.44}$$

satisfy the condition:

$$p_4^2\,(1-p_2)\,(-\,(1-p_2)\,(p_2+2))+3p_6\left(p_2^2-1\right)^2=0.$$

If $n\rightarrow 0$.
As $\mho(\xi)=\operatorname{nc}(\xi)\pm\operatorname{sc}(\xi)=\sec(\xi)\pm\tan(\xi)$, then we get:

$$\begin{aligned}\mho_{11,3}=\frac{1}{6}(\sec(\xi)\pm\tan(\xi))\Bigg(&\frac{\alpha_1}{\sqrt{\frac{\alpha_2^2\left(-2(2\alpha_1^2+\alpha_2)(\sec(\xi)\pm\tan(\xi))^2-3\alpha_2\right)}{48\alpha_1^4-3\alpha_2^2}}}\\&+\frac{(\alpha_2^2-16\alpha_1^4)(\sec(\xi)\pm\tan(\xi))}{\alpha_2\left(8\alpha_1^2(\sec(\xi)\pm\tan(\xi))^2+\alpha_2(4(\sec(\xi)\pm\tan(\xi))^2+6)\right)}\\&-\frac{(\alpha_2^2-16\alpha_1^4)^2(\sec(\xi)\pm\tan(\xi))^3}{24\alpha_1^2\alpha_2\left(4\alpha_1^2(\sec(\xi)\pm\tan(\xi))^2+\alpha_2(2(\sec(\xi)\pm\tan(\xi))^2+3)\right)^2}\Bigg)\end{aligned} \tag{3.45}$$

or $\mho(\xi)=\frac{\operatorname{cn}(\xi)}{1\pm\operatorname{sn}(\xi)}=\frac{\cos(\xi)}{1\pm\sin(\xi)}$, then we get:

$$\mho_{11,4}=\frac{\cos(\xi)\left(\frac{\alpha_1}{\sqrt{\frac{\alpha_2^2\left(-\frac{2\left(2\alpha_1^2+\alpha_2\right)\cos^2(\xi)}{(1\pm\sin(\xi))^2}-3\alpha_2\right)}{48\alpha_1^4-3\alpha_2^2}}}+\frac{\left(\alpha_2^2-16\alpha_1^4\right)\cos(\xi)}{\alpha_2(1\pm\sin(\xi))\left(\frac{8\alpha_1^2\cos^2(\xi)}{(1\pm\sin(\xi))^2}+\alpha_2\left(\frac{4\cos^2(\xi)}{(1\pm\sin(\xi))^2}+6\right)\right)}-\frac{\left(\alpha_2^2-16\alpha_1^4\right)^2\cos^3(\xi)}{24\alpha_1^2\alpha_2(1\pm\sin(\xi))^3\left(\frac{4\alpha_1^2\cos^2(\xi)}{(1\pm\sin(\xi))^2}+\alpha_2\left(\frac{2\cos^2(\xi)}{(1\pm\sin(\xi))^2}+3\right)\right)^2}\right)}{6(1\pm\sin(\xi))} \tag{3.46}$$

under conditions that are interpreted as:

$$p_4^2\left(\frac{1}{2}-p_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-p_2\right)(p_2+1)\right)+3p_6\left(p_2^2-\frac{1}{16}\right)^2=0.$$

Case 12: If $t_0=-\frac{1}{4}\left(1-n^2\right)^2$, $t_2=\frac{1}{2}\left(n^2+1\right)$, $t_4=-\frac{1}{4}$**,** $0<n<1$ **then** $\mho(\xi)$**=**n**cn**$(\xi)\pm$ **dn**(ξ) obtain JEF solution:

$$U_{12}=-\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2}+\frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu}+\frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.47}$$

where δ and μ are:

$$\delta=\frac{p_4\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)}{\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)^2-(n^2+1)\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)+\frac{3}{16}\left(1-n^2\right)^2},$$

$$\mu=-\frac{3\left(1-n^2\right)^2 p_4}{4\left(\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)^2-\left(n^2+1\right)\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)+\frac{3}{16}\left(1-n^2\right)^2\right)},$$

satisfy the condition:

$$p_4^2\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)\left(\frac{9}{16}\left(1-n^2\right)^2-\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)\left(n^2+p_2+1\right)\right)+$$
$$3p_6\left(\frac{3}{16}\left(1-n^2\right)^2-\frac{1}{4}\left(n^2+1\right)^2+p_2^2\right)^2=0.$$

If $n\to 1$.
As $\mho(\xi)=\mu\mathrm{cn}(\xi)\pm\mathrm{dn}(\xi)=\mu\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi)$, then we get:

$$U_{12,1}=-\frac{\left(\alpha_1^2-\alpha_2\right)\left(3\sqrt{3}\alpha_1^3\sqrt{-\frac{\alpha_2^2(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))^2}{\alpha_1^2-\alpha_2}}+\left(4\alpha_1^2-\alpha_2\right)\alpha_2(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))\right)}{9\alpha_1^2\alpha_2^2(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))}, \tag{3.48}$$

satisfy the condition:

$$p_4^2\left(1-p_2\right)\left(-\left(1-p_2\right)\left(p_2+2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

If $n\to 0$.
As $\mho(\xi)=\mu\mathrm{cn}(\xi)\pm\mathrm{dn}(\xi)=\mu\cos(\xi)\pm 1$, then we have:

$$\begin{aligned}U_{12,2}=\frac{1}{6}(0\pm1)\Bigg(&\frac{\left(16\alpha_1^4-\alpha_2^2\right)(0\pm1)}{6\alpha_2^2-4\alpha_2\left(2\alpha_1^2+\alpha_2\right)(0\pm1)^2}+\frac{\alpha_1}{\sqrt{\frac{\alpha_2^2\left(3\alpha_2-2\left(2\alpha_1^2+\alpha_2\right)(0\pm1)^2\right)}{48\alpha_1^4-3\alpha_2^2}}}\\&-\frac{\left(\alpha_2^2-16\alpha_1^4\right)^2(0\pm1)^3}{24\alpha_1^2\alpha_2\left(3\alpha_2-2\left(2\alpha_1^2+\alpha_2\right)(0\pm1)^2\right)^2}\Bigg),\end{aligned} \tag{3.49}$$

under computed condition:

$$p_4^2\left(\frac{1}{2}-p_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-p_2\right)\left(p_2+1\right)\right)+3p_6\left(p_2^2-\frac{1}{16}\right)^2=0.$$

Case 13: If $t_0=\frac{1}{4}$, $t_2=\frac{1}{2}\left(1-2n^2\right)$, $t_4=\frac{1}{4}$**,** $0<n<1$ **then** $\mho(\xi)=\frac{\mathbf{sn}(\xi)}{1\pm\mathbf{cn}(\xi)}$ obtain JEF solution:

$$U_{13}=-\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2}+\frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu}+\frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.50}$$

where δ and μ are:

$$\delta=\frac{p_4\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)}{\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)^2-\left(1-2n^2\right)\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)+\frac{3}{16}},$$

$$\mu = \frac{3p_4}{4\left(\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)^2-\left(1-2n^2\right)\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)+\frac{3}{16}\right)},$$

satisfy the condition:

$$p_4^2\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}\left(1-2n^2\right)-p_2\right)\left(-2n^2+p_2+1\right)\right)+3p_6\left(-\frac{1}{4}\left(1-2n^2\right)^2+p_2^2+\frac{3}{16}\right)^2=0.$$

If $n \to 1$.
As $\mho(\xi) = \frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)} = \frac{\tanh(\xi)}{1\pm\mathrm{sech}(\xi)}$, then we obtain

$$U_{13,1} = \frac{\tanh(\xi)\left(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(3\alpha_2(1\pm\mathrm{sech}(\xi))^2-2\left(\alpha_2-2\alpha_1^2\right)\tanh^2(\xi)\right)}{\left(\alpha_2^2-16\alpha_1^4\right)(1\pm\mathrm{sech}(\xi))^2}}}-\frac{\left(16\alpha_1^4-\alpha_2^2\right)\tanh(\xi)(1\pm\mathrm{sech}(\xi))\left(\left(64\alpha_1^4-24\alpha_2\alpha_1^2-\alpha_2^2\right)\tanh^2(\xi)+36\alpha_2\alpha_1^2(1\pm\mathrm{sech}(\xi))^2\right)}{24\alpha_1^2\alpha_2\left(3\alpha_2(1\pm\mathrm{sech}(\xi))^2-2\left(\alpha_2-2\alpha_1^2\right)\tanh^2(\xi)\right)^2}\right)}{6(1\pm\mathrm{sech}(\xi))} \tag{3.51}$$

satisfy the condition:

$$p_4^2\left(-p_2-\frac{1}{2}\right)\left(\frac{9}{16}-\left(-p_2-\frac{1}{2}\right)(p_2-1)\right)+3p_6\left(p_2^2-\frac{1}{16}\right)^2=0.$$

If $n \to 0$.
As $\mho(\xi) = \frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)} = \frac{\sin(\xi)}{1\pm\cos(\xi)}$, then we obtain

$$U_{13,2} = \frac{\sin(\xi)\left(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(2\left(2\alpha_1^2+\alpha_2\right)\sin^2(\xi)+3\alpha_2(1\pm\cos(\xi))^2\right)}{\left(\alpha_2^2-16\alpha_1^4\right)(1\pm\cos(\xi))^2}}}+\frac{\left(\alpha_2^2-16\alpha_1^4\right)\sin(\xi)(1\pm\cos(\xi))\left(64\alpha_1^4\sin^2(\xi)-\alpha_2^2\sin^2(\xi)+12\alpha_2\alpha_1^2\left(2\sin^2(\xi)+3(1\pm\cos(\xi))^2\right)\right)}{24\alpha_2\left(4\alpha_1^3\sin^2(\xi)+\alpha_2\alpha_1\left(2\sin^2(\xi)+3(1\pm\cos(\xi))^2\right)\right)^2}\right)}{6(1\pm\cos(\xi))} \tag{3.52}$$

satisfy the condition:

$$p_4^2\left(\frac{1}{2}-p_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-p_2\right)(p_2+1)\right)+3p_6\left(p_2^2-\frac{1}{16}\right)^2=0.$$

Case 14: If $t_0 = \frac{1}{4}$, $t_2 = \frac{1}{2}\left(n^2+1\right)$, $t_4 = \frac{1}{4}\left(1-n^2\right)^2$, $0 < n < 1$, **then** $\mho(\xi) = \frac{\mathbf{sn}(\xi)}{\mathbf{cn}(\xi)\pm\mathbf{dn}(\xi)}$ obtain JEF solution:

$$U_{14} = -\frac{\alpha_2^3\mho(\xi)^4}{\alpha_1^2\left(\delta\mho(\xi)^2+\mu\right)^2}+\frac{\alpha_2\mho(\xi)^2}{\delta\mho(\xi)^2+\mu}+\frac{\alpha_1\mho(\xi)}{\sqrt{\delta\mho(\xi)^2+\mu}}, \tag{3.53}$$

where δ and μ are defined as:

$$\delta = \frac{p_4\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)}{\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)^2-\left(n^2+1\right)\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)+\frac{3}{16}\left(1-n^2\right)^2},$$

$$\mu = \frac{3p_4}{4\left(\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)^2-\left(n^2+1\right)\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)+\frac{3}{16}\left(1-n^2\right)^2\right)},$$

satisfy the condition:

$$p_4^2\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)\left(\frac{9}{16}\left(1-n^2\right)^2-\left(\frac{1}{2}\left(n^2+1\right)-p_2\right)\left(n^2+p_2+1\right)\right)+$$
$$3p_6\left(\frac{3}{16}\left(1-n^2\right)^2-\frac{1}{4}\left(n^2+1\right)^2+p_2^2\right)^2=0.$$

If $n\rightarrow 1$.
As $\mho(\xi)=\frac{\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)\pm\mathrm{dn}(\xi)}=\frac{\tanh(\xi)}{\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi)}$, then we obtain:

$$7\mho_{14,1}=\frac{\tanh(\xi)}{6\left(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi)\right)}\left(\frac{4\sqrt{3}\,\alpha_1}{\sqrt{\frac{\alpha_2^2\left(4(\alpha_1^2+\alpha_2)\tanh^2(\xi)+3\alpha_2(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))^2\right)}{(\alpha_2^2-\alpha_1^4)(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))^2}}}\right.$$
$$\left.-\frac{8(\alpha_1^4-\alpha_2^2)\tanh(\xi)(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))\left(4(4\alpha_1^2-\alpha_2)(\alpha_1^2+\alpha_2)\tanh^2(\xi)+9\alpha_2\alpha_1^2(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))^2\right)}{3\alpha_1^2\alpha_2\left(4(\alpha_1^2+\alpha_2)\tanh^2(\xi)+3\alpha_2(\mathrm{sech}(\xi)\pm\mathrm{sech}(\xi))^2\right)^2}\right) \tag{3.54}$$

satisfy the condition:

$$p_4^2\left(1-p_2\right)\left(-\left(1-p_2\right)\left(p_2+2\right)\right)+3p_6\left(p_2^2-1\right)^2=0.$$

If $n\rightarrow 0$.
As $\mho(\xi)=\frac{\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)\pm\mathrm{dn}(\xi)}=\frac{\sin(\xi)}{\cos(\xi)\pm 1}$ then

$$\mho_{14,2}=\frac{\sin(\xi)\left(\frac{\sqrt{3}\alpha_1}{\sqrt{\frac{\alpha_2^2\left(2\left(2\alpha_1^2+\alpha_2\right)\sin^2(\xi)+3\alpha_2(\cos(\xi)\pm1)^2\right)}{\left(\alpha_2^2-16\alpha_1^4\right)(\cos(\xi)\pm1)^2}}}+\frac{\left(\alpha_2^2-16\alpha_1^4\right)\sin(\xi)(\cos(\xi)\pm1)\left(64\alpha_1^4\sin^2(\xi)-\alpha_2^2\sin^2(\xi)+12\alpha_2\alpha_1^2\left(2\sin^2(\xi)+3(\cos(\xi)\pm1)^2\right)\right)}{24\alpha_2\left(4\alpha_1^3\sin^2(\xi)+\alpha_2\alpha_1\left(2\sin^2(\xi)+3(\cos(\xi)\pm1)^2\right)\right)^2}\right)}{6(\cos(\xi)\pm1)} \tag{3.55}$$

satisfy the condition:

$$p_4^2\left(\frac{1}{2}-p_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-p_2\right)\left(p_2+1\right)\right)+3p_6\left(p_2^2-\frac{1}{16}\right)^2=0.$$

4. Results

The figures show a diverse range of exact solutions such as dark solitons, periodic wave, singular structure, and kink-type solutions of the (3+1)-dimensional Kadomtsev–Petviashvili (KP) equation obtained by the ϕ^6 model expansion method. The ϕ^6-model expansion method is efficient and applicable to a variety of nonlinear partial differential equations, particularly those that can be translated into polynomial form, enabling traveling wave solutions to reduce to ordinary differential equations. However, the method can face difficulties with highly complex, coupled, or non-polynomial NLPDEs, where the resulting algebraic systems are difficult to solve or the method's conditions fail.

In Figure 1, the dark soliton shape of $U_{1,1}$ and its corresponding intensity profile reveal a dip in its centre can be observed, which is useful in the framework of nonlinear optics. The Figure 1 shows a localized wave profile which, in the course of time, disperses gradually and indicates the spreading nature of the solution. This is an indication that the system facilitates dispersive wave propagation. Conversely, Figure 2 demonstrates periodic oscillatory behavior at constant amplitude, which relates to stationary traveling wave modes. Figure 3 shows the exponentially growing solution, which physically illustrates the instability within the system. In contrast, Figure 4 displays solutions with sharp peaks, indicating singular or blow-up behavior at some spatial points, pointing out strongly localized instabilities. Figures 2, 6, and 7 illustrate periodic soliton structures for the $U_{2,2}$, $U_{7,2}$, and $U_{11,4}$, respectively, showing comprehensible wave trains in shallow water or plasmas. As shown in Figure 3, the usual soliton solution for $U_{3,1}$ is stable, self-preserved in propagation, and is vital for ultra-high-speed communication at a large distance. In Figure 4, a singular soliton at the periodic boundary of $\mathbf{U}_{4,1}$ is characterized with dramatic gradient features, and demonstrates "shock waves" known in fluid and plasma dynamics. The cogency of the model in describing locally in-depth waves is further verified by an additional dark soliton in Figure 5 for solution $U_{7,1}$. Finally, in Figure 8, we show a kink-type soliton of $U_{13,1}$, which has a step-like conversion between the two asymptotic shapes and is of significance to the explanation of field walls and topological defects. These wide-ranging classes of solutions not only validate the flexibility of the ϕ^6-model expansion method but also demonstrate its potential applicability in nonlinear science, such as wave propagation, optical communications, and complex media modeling. These soliton-type solutions are of physical importance because similar patterns are observed in shallow water dynamics, nonlinear optics, and plasma physics, where stable, constrained wave packets are produced by striking a balance between dispersion and nonlinearity.

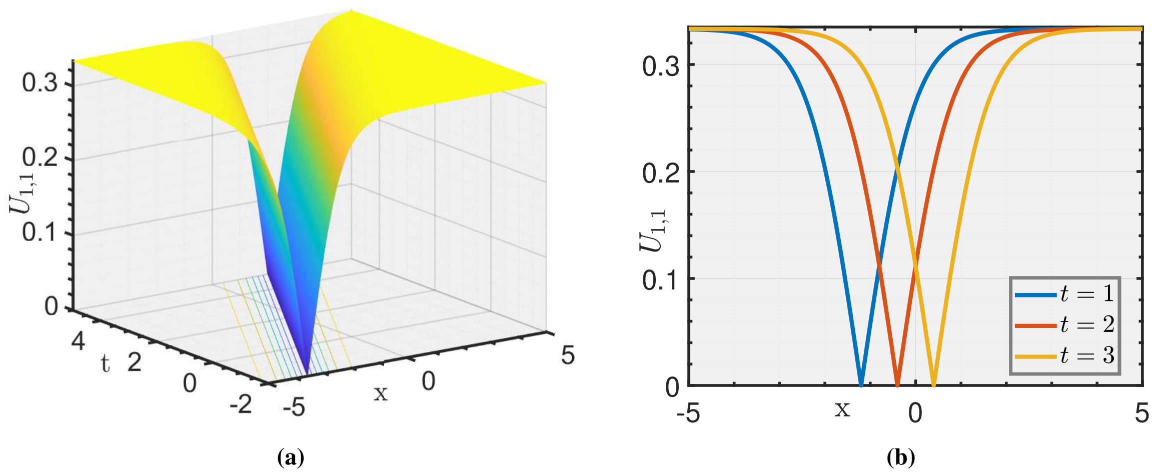
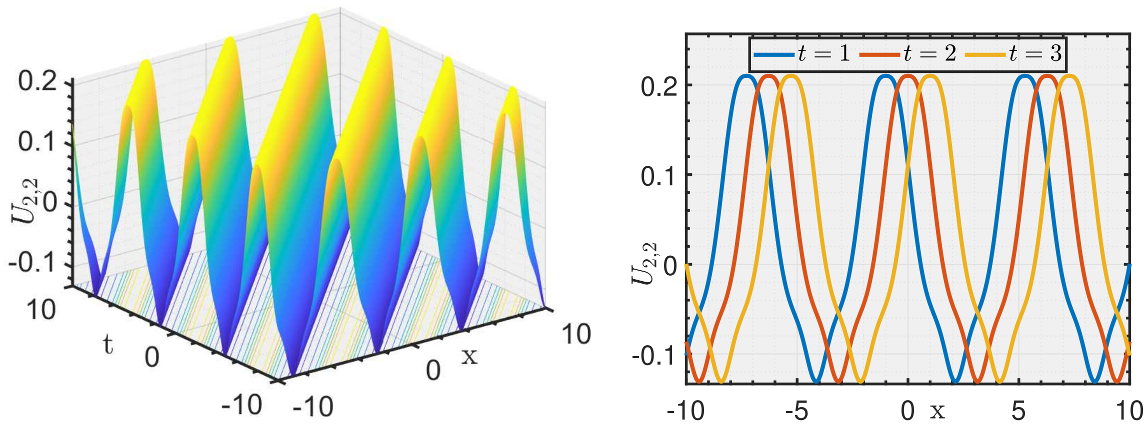
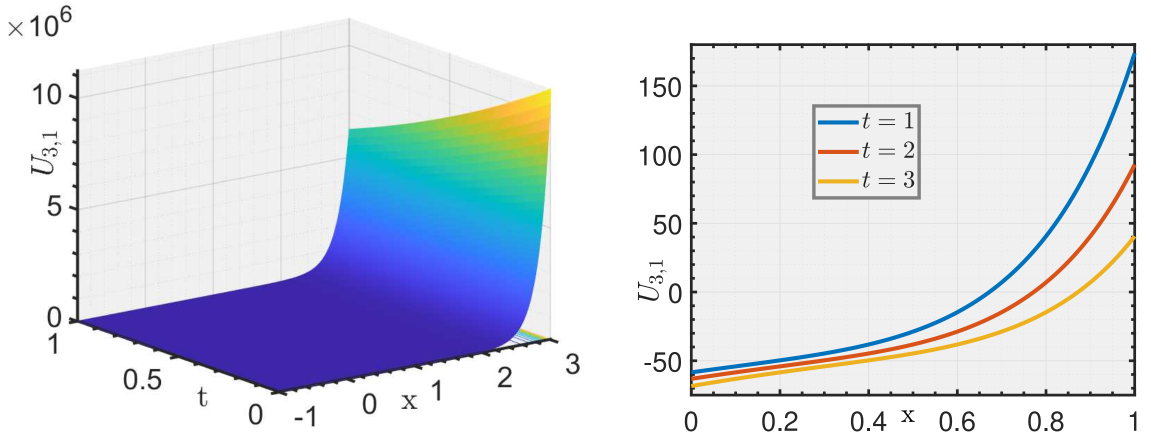


Figure 1. Visualization of $U_{1,1}$ for $\alpha_1 = 1$, $\alpha_2 = 1$, $y = 1$, $z = 1$, *and* $m = 0.8$. The 3D surface (1a) exhibits localized pulse at x=0, which is spreading with time showing dispersive behavior. This widening as t increases is confirmed by the 2D plot (1b).



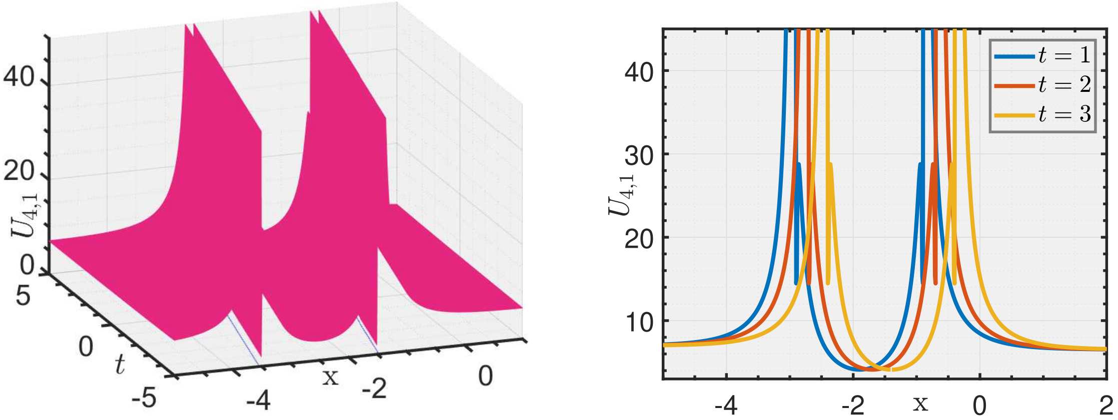
(a) (b)

Figure 2. Visualization of $U_{2,2}$ for $\alpha_1 = 1,\ \alpha_2 = 5,\ y = 1,\ z = 1,$ *and* $m = 1$. In the 3D plot (2a), there are periodic space and time oscillations. The wave propagation is constant in amplitude with the 2D slices (2b) being stable.



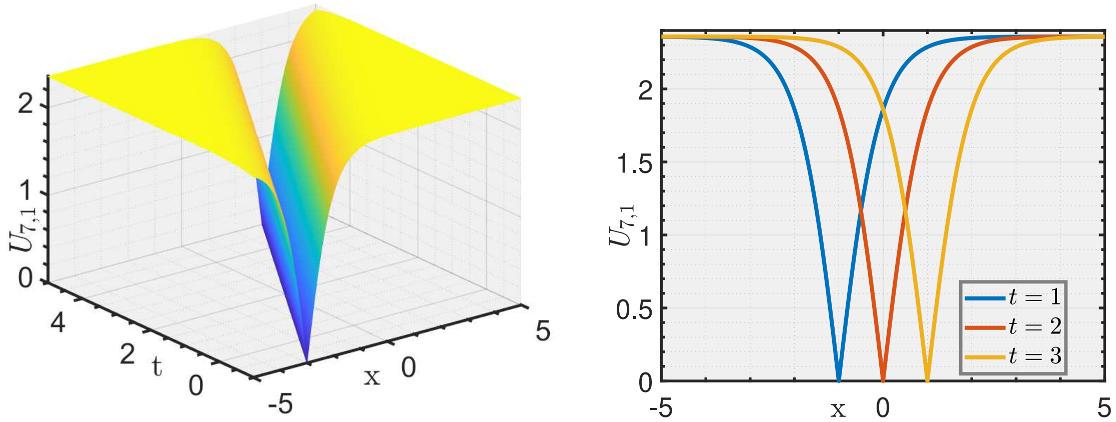
(a) (b)

Figure 3. Visualization of $U_{3,1}$ for $\alpha_1 = 0.1,\ \alpha_2 = 3,\ y = 0.5,\ z = 0.5,$ *and* $m = 0.1$. The 3D plot (3a) indicates a quick exponential increase of the solution as a function of x and t. The steep upward rise in time in the 2D curves (3b) confirms an unstable growing mode.



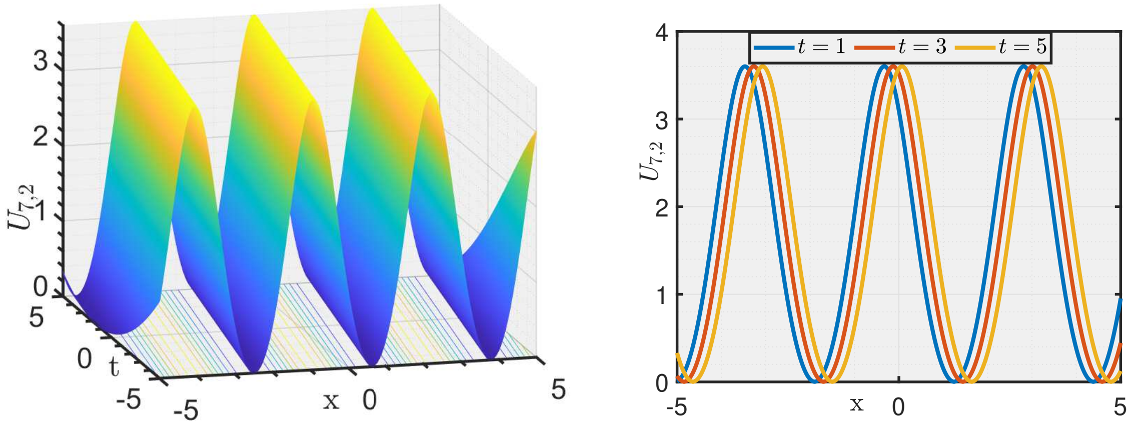
(a) **(b)**

Figure 4. Visualization of $U_{4,1}$ for $\alpha_1 = 1,\ \alpha_2 = 5,\ y = 1,\ z = 1,$ *and* $m = 0.1$. The 3D surface (4a) has many sharp peaks, which signifies singular-type behavior in the solution. In the 2D slices (4b), the amplitudes are large around certain points, implying that there are strongly localized instabilities.



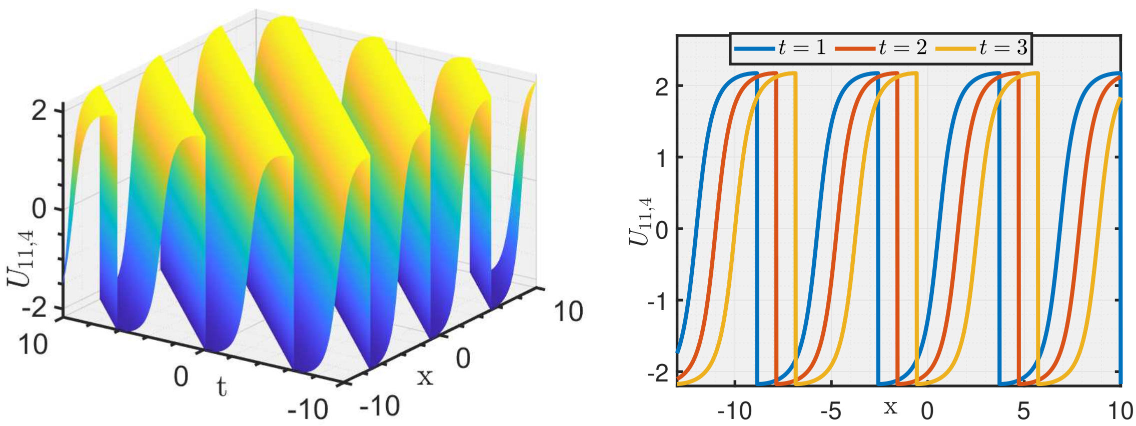
(a) **(b)**

Figure 5. Visualization of $U_{7,1}$ for $\alpha_1 = 1,\ \alpha_2 = 1,\ y = 1,\ z = 1,$ *and* $m = 0.8$. A dark and stable bell-shaped solitary wave is observed on the 3D surface (5a). The resulting 2D curves (5b) reveal its consistent and coherent time dependence, showing a strong and non-dispersive solution.



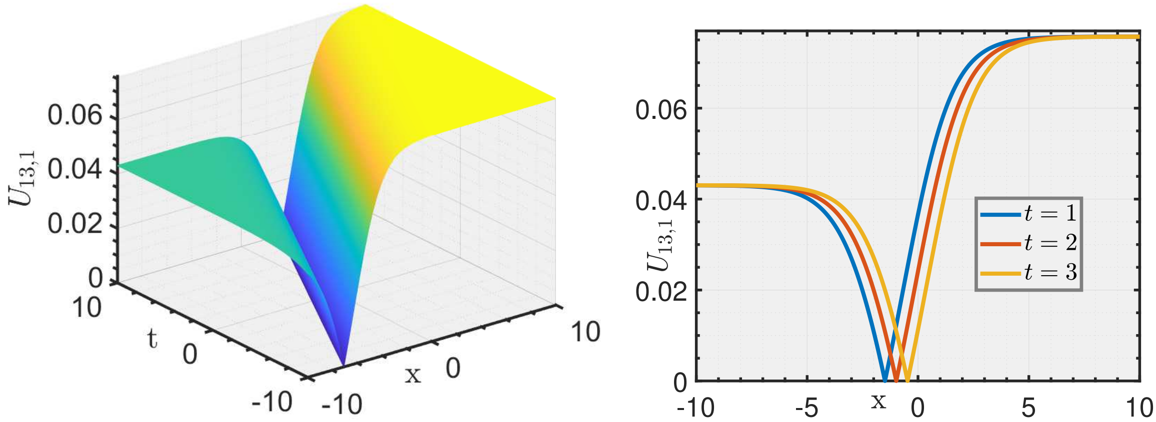
(a) (b)

Figure 6. Visualization of $U_{7,2}$ for $\alpha_1 = 1,\ \alpha_2 = 5,\ y = 1,\ z = 1,$ *and* $m = 0.1$. The surface (6a) is a 3D array, which is periodic with sharp and coherent peaks. The regular repeating of this waveform over the domain is verified in the 2D slices (6b), making this a periodic traveling wave solution.



(a) (b)

Figure 7. Visualization of $U_{11,4}$ for $\alpha_1 = 1,\ \alpha_2 = 1,\ y = 1,\ z = 1,$ *and* $m = 0.8$. The periodic wave pattern indicated by the 3D surface (7a) has crests and troughs that alternate. The 2D plots (7b) verify the existence of phase shifts with time, but the wave shape is maintained.



(a) **(b)**

Figure 8. Visualization of $U_{13,1}$ for $p = 1,\ q = 1,\ v = 1,\ r = 13,\ \beta_2 = 1,$ *and* $\beta_4 = 3$. The kink-type soliton structure is shown in the 3D surface (8a) with a sharp transition near the origin. The persistence of the kink over time is validated by the 2D profiles (8b) and indicates a stable localized wave.

5. Conclusions

In this work, we have applied the ϕ^6-model expansion technique successfully to integrate the (3+1)-dimensional Kadomtsev-Petviashvili (KP) equation and so derived a number of significant classes of exact solutions. Our method gave periodic solitons showing defined wave behavior, singular solitons with infinite amplitude at singular points, and dark solitons with amplitude depressions, which are extremely significant in optics systems. The ϕ^6-model expansion method is shown to be efficient through its ability to manage the high-order nonlinearity of the KP equation in more than one dimension. The graphical solutions in both 3D and 2D are insightful for the dynamics and interactions of these soliton solutions. These findings play an important role in the theoretical explanation of nonlinear wave propagation in multidimensional space and have possible applications to fluid dynamics, plasma physics, and optical communications. However, even though the method is successful, it has its limitations: It mainly aims to find analytic solutions under some presumptions, which can limit its application to more complicated or non-integrable systems. Additionally, the strength of soliton solutions with respect to different kinds of perturbations was not rigorously investigated in this work. It is possible that future research will entail the extension of the ϕ^6-model expansion technique to other high-dimensional nonlinear systems, the stability and strength of the obtained soliton solutions to different physical parameters, and the experimental verification of the theoretical results. Moreover, the application of the method to variable coefficients or fractional-order nonlinear equations can give further insight into more realistic models. This research repeats that the ϕ^6-model expansion method is a valuable resource for studying nonlinear complicated phenomena and can drive further innovations in mathematical physics and engineering.

Author contributions

S. Trabelsi and M. Balti: Formal analysis, investigation, supervision, project administration, funding acquisition; A. U. Khan: Conceptualization, formal analysis, investigation, writing—original draft, writing—review and editing, preparation, visualization; A. Khan: Conceptualization, software, methodology, validation, formal analysis, formatting, investigation, writing—original draft, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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