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Research article

Stationary distribution of an SVIB Cholera model with reaction-diffusion and Brownian motion

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Abstract: In this study, we aimed to determine the stationary distribution of a Susceptible-Vaccine-Infected-Bacteria (SVIB) Cholera model that incorporates environmental noise and reaction-diffusion. First, we demonstrated the model's invariant set. Subsequently, a Lyapunov function was constructed to prove the existence and uniqueness of the solutions, and the model's finite-time stability was demonstrated. Furthermore, we derived the stationary distribution of the stochastic cholera model with reaction-diffusion. Finally, the theorem's results were verified through numerical simulation. Notably, the noise intensity could impact the model's stationary distribution. When the number of infected individuals and cholera bacteria decreases with reduced noise intensity, the system is characterized by a normal distribution. Therefore, appropriate measures should be taken to reduce the interference of external factors when a disease outbreak occurs.

Keywords: stationary distribution; SVIB Cholera model; brownian motion; reaction-diffusion
Mathematics Subject Classification: 35Q92, 37H05, 92B05

1. Introduction

Since 2021, cholera cases have increased worldwide. Globally, cholera cases reported to WHO in 2022 more than doubled those reported in 2021. In 2023, there were over 700,000 cholera cases worldwide. In 2024, global cholera cases decreased by 21% compared with the year, while deaths from cholera increased by 5%. As of 31 December 2024, there have been 49,554 cholera infections in Sudan, with 1316 fatalities. The WHO data on December 23, 2024, showed that Yemen reported approximately 250,000 suspected cholera cases, accounting for 35% of the global cholera cases, with 861 related deaths. From January 1 to May 16, 2025, the Democratic Republic of the Congo reported 24,395 suspected cholera cases and 513 deaths. From January 1 to May 9, 2025, Angola reported 17,967 cholera cases, including 937 confirmed cases and 576 deaths. Because of the high infection and mortality rates of cholera, it is of practical significance to study the pathogenesis and dynamic behavior of avian influenza models, respectively.

Apart from helping us understand the epidemic's dynamic behavior, mathematical models also provide effective information for controlling outbreaks. By reviewing the literature, we observed that there were ordinary differential equation models and partial differential equation models, respectively. First, we listed the research for the ordinary differential equation model. Musundi [1] established cholera models across and within hosts and explored the model's dynamic behavior. Yang et al. [2] proposed a time-delay stochastic cholera model, including media coverage and those driven by Lévy noise, and revealed how Lévy jumps, time-delays, and media coverage affect the model's asymptotic characteristics. Additionally, Ghosh et al. [3] developed a cholera model based on machine learning data of the epidemic and predicted short-term cholera cases in a certain area. In order to simulate the impacts of various control strategies, Edward and Nyerere [4] considered a cholera model with four different control strategies, namely public health education campaigns, vaccination, health, and treatment. Furthermore, Mustapha et al. [5] introduced personal hygiene as a control strategy into the cholera model with vaccines to evaluate its impact on the vaccines. Madubueze et al. [6] incorporated health education information, as well as treatment and vaccination details into the model, demonstrating the impact of different measures on diseases. Additionally, Mohammad et al. [7] designed a two-strain cholera model to evaluate the impact of basic control measures, treatment, and dosages of large-scale vaccination on the cholera outbreak in the population. Many researchers have considered other models for ordinary differential equations [8–13].

Spatial heterogeneity is crucial in epidemiological models and natural ecosystems. Therefore, researchers introduced the reaction-diffusion equation to capture the complex influence of heterogeneity and human mobility on the dynamics of cholera [14–18]. Hu et al. [14] established a nonlocal delayed reaction-spread cholera model, and discussed the effects of delay and diffusion. Moreover, Wu et al. [15] proposed a degenerative reaction-spread cholera model. This model combined various human mobility and behavioral changes in a spatially heterogeneous environment to study the dynamic behavior of cholera. Bai and Han [16] built a partially degraded reactive spread cholera model and investigated the effects of spread rate, seasonality, and heterogeneity on the basic reproduction number. Furthermore, Wang et al. [17] evaluated a reaction-advection-diffusion model that involved human hosts and Vibrio cholera aquatic hosts. Based on this, they established the global dynamics of the system. [18] and discussed the effects of nonlocal transmission, spread rate, and incubation period on the transmission speed and wave pattern in a reaction-diffusion cholera model with highly contagious Vibrio cholera and spatiotemporal delay. Many researchers have assessed other partial differential equation models [19–22].

Some researchers have evaluated the stationary distribution of cholera models under ordinary differential equations [23–28]. However, researchers have not assessed the stationary distribution of stochastic cholera models with reaction-diffusion due to the cholera models' strong complexity under partial differential equations during calculation and analysis. Thus, our objectives of this study were as follows: (i) Since vaccinations reduce the infection and mortality rates of diseases, incorporating their effects into epidemic models can help evaluate and design optimal intervention policies. Therefore, to better consider the cholera model's dynamic behavior, we introduced vaccination into the cholera model with reaction-diffusion; (ii) because epidemic models are inevitably affected by environmental noise, it is necessary to introduce random fluctuations in cholera models. In epidemic models, diseases are significantly influenced by the transmission rate. Hence, we introduced a random term through the transmission rate to establish a stochastic cholera model.

Our innovations were as follows: (1) We first established an SVIB cholera model with stochastic noise and spatial diffusion; (2) we also proved the well-posedness of the cholera model's solution and demonstrated that the model was stable within a finite time; (3) we provided the sufficient conditions for the stationary distribution of cholera model's solution and proved the existence and uniqueness of the stationary distribution of cholera model's solution. The remainder of this paper has been organized as follows. In Section 2 introduced the model's formulation. In Section 3, we proved the well-posedness of the solution and the finite-time stable. In Section 4, we derived the sufficient conditions for a stationary distribution. In Section 5 verified the theorem's results through numerical simulations. Conclusions are presented in Section 6.

2. Model formulation

Based on the motivation (i) in the introduction, we first built the following model:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \triangle S + \Lambda(x) - \beta_1(x)SI - \beta_2(x)SB - \mu_1 S - r_V(x)S, \\ \frac{\partial V}{\partial t} = d_V \triangle V + r_V(x)S - \beta_3(x)b_1 VI - \beta_4(x)b_2 VB - \mu_1(x)V, \\ \frac{\partial I}{\partial t} = d_I \triangle I + \beta_1(x)SI + \beta_2(x)SB + \beta_3(x)b_1 VI + \beta_4(x)b_2 VB - \mu_1(x)I - d(x)I, \\ \frac{\partial B}{\partial t} = d_B \triangle B + k(x)I - \mu_2(x)B, \\ S(\cdot,0) = S_0(\cdot), V(\cdot,0) = V_0(\cdot), I(\cdot,0) = I_0(\cdot), B(\cdot,0) = B_0(\cdot), \\ x \in \mathbb{O}, t > 0, \end{cases} \tag{2.1}$$

with boundary condition

$$\frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial B}{\partial \nu} = 0,\ x \in \partial\mathbb{O},\ t > 0.$$

Where S, V, and I denote the number of suspected humans, vaccinated humans, and infected humans, respectively. B represents the concentration of Cholera bacteria in the environment. $S := S(x,t), V := V(x,t), I := I(x,t), B := B(x,t), S(\cdot,0) := S(x,0), V(\cdot,0) := V(x,0), I(\cdot,0) := I(x,0), B(\cdot,0) := B(x,0)$. d_S, d_V, d_I, and d_B are the diffusion coefficients of S, V, I, and B. The other parameters are in Table 1.

Table 1. Parameters and their description.

Parameter	Description of parameter
$\Lambda(x)$	the recruitment rate of humans
$\beta_1(x)$	the transmission rate from infected to suspected
$\beta_2(x)$	the transmission rate from environment to human
$\beta_3(x)$	the transmission rate from infected to vaccinated
$\beta_4(x)$	the transmission rate from environment to vaccinated
$\mu_1(x)$	the natural death rate of humans
$r_V(x)$	the vaccination rate
$d(x)$	the recovery rate
$k(x)$	the rate of human contribution to bacteria
$\mu_2(x)$	the natural death rate of bacteria

Based on the main motivation (ii) in the introduction, stochastic noise is introduced into model (2.1) through the transmission rate, that is $\beta_i = \beta_i + \sigma_i\eta_i(t)$, $dW_i(t) = \eta_i(t)dt$ $(i = 1,2,3,4)$, and η_i $(i = 1,2,3,4)$, denoting the noise intensity, and $W_i(t),(i = 1,2,3,4)$ representing the standard Brownian motion. $W_i(t),(i = 1,2)$ defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$, $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ is a complete probability space. $\|\cdot\|$ denotes the norm.

$$\begin{cases} dS = \{d_S\triangle S + \Lambda(x) - \beta_1(x)SI - \beta_2(x)SB - \mu_1 S - r_V(x)S\}dt - \sigma_1 SIdW_1(t) - \sigma_2 SBdW_2(t), \\ dV = \{d_V\triangle V + r_V(x)S - \beta_3(x)b_1VI - \beta_4(x)b_2VB - \mu_1(x)V\}dt - \sigma_3 b_1 VIdW_3(t) - \sigma_4 b_2 VBdW_4(t), \\ dI = \{d_I\triangle I + \beta_1(x)SI + \beta_2(x)SB + \beta_3(x)b_1VI + \beta_4(x)b_2VB - \mu_1(x)I - d(x)I\}dt + \sigma_1 SIdW_1(t) \\ \qquad + \sigma_2 SBdW_2(t) + \sigma_3 b_1 VIdW_3(t) + \sigma_4 b_2 VBdW_4(t), \\ dB = \{d_B\triangle B + k(x)I - \mu_2(x)B\}dt, \\ S(\cdot,0) = S_0(\cdot), V(\cdot,0) = V_0(\cdot), I(\cdot,0) = I_0(\cdot), B(\cdot,0) = B_0(\cdot), \\ x \in \mathbb{O}, t > 0, \end{cases} \tag{2.2}$$

with boundary condition

$$\frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial B}{\partial \nu} = 0,\ x \in \partial\mathbb{O},\ t > 0.$$

Where $\frac{\partial S}{\partial \nu} = \frac{\partial V}{\partial \nu} = \frac{\partial I}{\partial \nu} = \frac{\partial B}{\partial \nu}$ denote the directional derivative, with the ν being the outer normal direction on $\partial\mathbb{O}$. Next, we explored the properties of system (2.2). First, we prove the boundedness of solutions, the existence and uniqueness of positive solutions, and the existence of an attractive set for system (2.2).

3. Well-posedness of the solution

Lemma 3.1. *For system* (2.2)*, there exists an invariant set*
$\mathbb{D}_1 := \{(S,V,I) \in Y_+, \int_{\mathbb{O}}(S + V + I)dx < \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\}$ *and* $\mathbb{D}_2 := \{B \in X_+, \int_{\mathbb{O}} Bdx < \frac{\bar{k}\bar{\Lambda}|\mathbb{O}|^2}{\underline{\mu}_1\underline{\mu}_2}\}$,
where $\underline{\mu}_1$ *and* $\underline{\mu}_2$ *represent the minimum of* $\mu_1(\cdot)$ *and* $\mu_2(\cdot)$*, respectively.* $\bar{\Lambda}$ *and* $\bar{k}$ *denote the maximum of* $\Lambda(\cdot)$ *and* $k(\cdot)$*, respectively.* $|\mathbb{O}|$ *indicates the modulus of* $\mathbb{O}$.

Proof. By virtue of system (2.2) and formula of integration by parts, we have

$$\begin{aligned} \frac{\partial}{\partial t}\int_{\mathbb{O}}(S + V + I)dx &= \int_{\mathbb{O}}\{d_S\triangle S + d_V\triangle V + d_I\triangle I + \Lambda(x) - \mu_1(\cdot)S - \mu_1(\cdot)V - \mu_1(\cdot)I - d(\cdot)I\}dx \\ &= \int_{\mathbb{O}}\{\Lambda(x) - \mu_1(\cdot)S - \mu_1(\cdot)V - (\mu_1(\cdot) + d(\cdot))I\}dx \\ &\leq \bar{\Lambda}|\mathbb{O}| - \underline{\mu}_1\int_{\mathbb{O}}\{S + V + I\}dx, \end{aligned} \tag{3.1}$$

Integrating both sides of Eq (3.1), we have

$$\int_{\mathbb{O}}(S + V + I)dx \leq \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + (S_0(\cdot) + V_0(\cdot) + I_0(\cdot) - \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1})e^{-\underline{\mu}_1 t}.$$

Hence, as $t \to \infty$, $\int_{\mathbb{O}}(S + V + I)dx \leq \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}$.

For the fourth equation of system (2.2), according to formula of integration by parts, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{O}} Bdx &= \int_{\mathbb{O}}\{d_B\triangle B + k(\cdot)I - \mu_2(\cdot)B\}dx\\ &\leq \int_{\mathbb{O}}\{k(\cdot)I - \mu_2(\cdot)B\}dx\\ &\leq \int_{\mathbb{O}}\{k(\cdot)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} - \mu_2(\cdot)B\}dx.\end{aligned} \tag{3.2}$$

Therefore, we have $\int_{\mathbb{O}} B(x,t)dx \leq \frac{\bar{k}\bar{\Lambda}|\mathbb{O}|^2}{\underline{\mu}_1\underline{\mu}_2}$ as $t \to \infty$. The Lemma is proved.

Remark 1. Lemma 3.1 indicated that system (2.2) has an invariant set.

Furthermore, we prove the uniqueness of the positive solution of system (2.2). For the needs of the following calculation, we first present the following theorem.

Theorem 3.1. *(Itô's formula) [29, Theorem 6.2] Let $y(t)$ be on Itô process on $t > 0$ with the stochastic differential*

$$dy(t) = f(t)dt + g(t)dW_t,$$

where $f \in \mathcal{L}^1(R_+; R)$ and $g \in \mathcal{L}^1(R_+; R)$. Let $V \in C^{2,1}(R \times R_+ : R)$. Then $V(y(t), t)$ is again an Itô process on $t > 0$ with the stochastic differential given by

$$dV(y(t),t) = [V_t(y(t),t) + V_y(y(t),t)f(t) + \frac{1}{2}V_{yy}(y(t),t)g^2(t)]dt + V_t(y(t),t)g(t)dW_t \ \ a.s.$$

Theorem 3.2. *If initial value $(S_0(x), V_0(x), I_0(x), B_0(x))$ is positive, then system (2.2) has a unique positive solution $(S(x,t), V(x,t), I(x,t), B(x,t))$ for $t > 0$ on $\mathbb{O}$.*

Proof. Since the coefficients of system (2.2) satisfy the local Lipschitz condition, there is a unique local solution on $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $l_0 > 1$ be sufficiently large for

$$\frac{1}{l_0} \leq \min_{0<t<\tau_e} ||\mathcal{N}|| \leq \max_{0<t<\tau_e} ||\mathcal{N}|| \leq l_0,$$

where $\mathcal{N} = (S, V, I, B)$. For each integer $l > l_0$, define the stopping time

$$\tau_l = inf\{t \in [0, \tau_e] : min(S, V, I, B) \leq \frac{1}{l} \ \ or \ \ max(S, V, I, B) \geq l\}.$$

Let $inf\emptyset = \infty$ ($\emptyset$ represents the empty set). τ_l is increasing as $l \to \infty$. Let $\tau_\infty = \lim_{l\to\infty}\tau_l$, then $\tau_\infty < \tau_e$ a.s. In the following, we need to show $\tau_\infty = \infty$ a.s.

Define the Lyapunov function as follows:

$$\mathbb{L}_1(S, V, I, B) = (S - 1 - \log S) + (V - 1 - \log V) + (I - 1 - \log I) + (P - 1 - \log P).$$

Therefore, according to Itô's formula, we have

$$
\begin{aligned}
&d\mathbb{L}_1(S,V,I,B)\\
&=\{(1-\frac{1}{S})(d_S\triangle S+\Lambda(x)-\beta_1(x)SI-\beta_2(x)SB-\mu_1S-r_V(x)S)+\frac{1}{2}(\sigma_1^2I^2+\sigma_2^2B^2)\}dt\\
&-(1-\frac{1}{S})\sigma_1SIdW_1(t)-(1-\frac{1}{S})\sigma_2SBdW_2(t)+\{(1-\frac{1}{V})(d_V\triangle V+r_V(x)S-\beta_3(x)b_1VI\\
&-\beta_4(x)b_2VB-\mu_1(x)V)+\frac{1}{2}(\sigma_3^2b_1^2I^2+\sigma_4^2b_2^2B^2)\}dt-(1-\frac{1}{V})\sigma_3b_1VIdW_3(t)\\
&-(1-\frac{1}{V})\sigma_4b_2VBdW_4(t)+\{(1-\frac{1}{I})(d_I\triangle I+\beta_1(x)SI+\beta_2(x)SB+\beta_3(x)b_1VI+\beta_4(x)b_2VB\\
&-\mu_1(x)I-d(x)I)+\frac{1}{2}\frac{1}{I^2}(\sigma_1^2S^2I^2+\sigma_2^2S^2B^2+\sigma_3^2b_1^2V^2I^2+\sigma_4^2b_2^2V^2B^2)\}dt+(1-\frac{1}{I})(\sigma_1SIdW_1(t)\\
&+\sigma_2SBdW_2(t)+\sigma_3b_1VIdW_3(t)+\sigma_4b_2VBdW_4(t))+\{(1-\frac{1}{B})(d_B\triangle B+k(x)I-\mu_2(x)B)\}.
\end{aligned}
\tag{3.3}
$$

Moreover, integrating both sides for system (3.3) from 0 to $\tau_l\wedge T$ and taking the expectations

$$
\begin{aligned}
&E\mathbb{L}_1(S(\cdot,\tau_l\wedge T),V(\cdot,\tau_l\wedge T),I(\cdot,\tau_l\wedge T),B(\cdot,\tau_l\wedge T))-\mathcal{L}_1(S_0(\cdot)+V_0(\cdot)+I_0(\cdot)+B_0(\cdot))\\
&=E\int_0^{\tau_l\wedge T}\{(1-\frac{1}{S})(d_S\triangle S+\Lambda(x)-\beta_1(x)SI-\beta_2(x)SB-\mu_1S-r_V(x)S)+\frac{1}{2}(\sigma_1^2I^2+\sigma_2^2B^2)\\
&+(1-\frac{1}{V})(d_V\triangle V+r_V(x)S-\beta_3(x)b_1VI-\beta_4(x)b_2VB-\mu_1(x)V)+\frac{1}{2}(\sigma_3^2b_1^2I^2+\sigma_4^2b_2^2B^2)\\
&+(1-\frac{1}{I})(d_I\Delta I+\beta_1(x)SI+\beta_2(x)SB+\beta_3(x)b_1VI+\beta_4(x)b_2VB-\mu_1(x)I-d(x)I)\\
&+\frac{1}{2}\frac{1}{I^2}(\sigma_1^2S^2I^2+\sigma_2^2S^2B^2+\sigma_3^2b_1^2V^2I^2+\sigma_4^2b_2^2V^2B^2)+(1-\frac{1}{B})(d_B\triangle B+k(x)I-\mu_2(x)B)\}dt\\
&\le E\int_0^{\tau_l\wedge T}\{4\lambda_0+(1-\frac{1}{S})(\Lambda(x)-\beta_1(x)SI-\beta_2(x)SB-\mu_1S-r_V(x)S)+\frac{1}{2}(\sigma_1^2I^2+\sigma_2^2B^2)\\
&+(1-\frac{1}{V})(r_V(x)S-\beta_3(x)b_1VI-\beta_4(x)b_2VB-\mu_1(x)V)+\frac{1}{2}(\sigma_3^2b_1^2I^2+\sigma_4^2b_2^2B^2)\\
&+(1-\frac{1}{I})(\beta_1(x)SI+\beta_2(x)SB+\beta_3(x)b_1VI+\beta_4(x)b_2VB-\mu_1(x)I-d(x)I)\\
&+\frac{1}{2}\frac{1}{I^2}(\sigma_1^2S^2I^2+\sigma_2^2S^2B^2+\sigma_3^2b_1^2V^2I^2+\sigma_4^2b_2^2V^2B^2)+(1-\frac{1}{B})(k(x)I-\mu_2(x)B)\}dt,
\end{aligned}
\tag{3.4}
$$

where $\lambda_0=max\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, $\lambda_i=\inf_{u\in\mathcal{H}}\|\nabla u_i\|^2/\|u_i\|^2$. By virtue of Lemma 3.1, we have

$$
\begin{aligned}
&(1-\frac{1}{S})(\Lambda(x)-\beta_1(x)SI-\beta_2(x)SB-\mu_1S-r_V(x)S)+\frac{1}{2}(\sigma_1^2I^2+\sigma_2^2B^2)+\\
&(1-\frac{1}{V})(r_V(x)S-\beta_3(x)b_1VI-\beta_4(x)b_2VB-\mu_1(x)V)+\frac{1}{2}(\sigma_3^2b_1^2I^2+\sigma_4^2b_2^2B^2)+\\
&(1-\frac{1}{I})(\beta_1(x)SI+\beta_2(x)SB+\beta_3(x)b_1VI+\beta_4(x)b_2VB-\mu_1(x)I-d(x)I)+\\
&\frac{1}{2}\frac{1}{I^2}(\sigma_1^2S^2I^2+\sigma_2^2S^2B^2+\sigma_3^2b_1^2V^2I^2+\sigma_4^2b_2^2V^2B^2)++(1-\frac{1}{B})(k(x)I-\mu_2(x)B)\le M.
\end{aligned}
$$

Hence, Eq (3.4) be equivalent to

$$\begin{aligned}&E\mathbb{L}_1(S(\cdot,\tau_l\wedge T),V(\cdot,\tau_l\wedge T),I(\cdot,\tau_l\wedge T),B(\cdot,\tau_l\wedge T))\\&\leq \mathbb{L}_1(S_0(\cdot)+V_0(\cdot)+I_0(\cdot)+B_0(\cdot))+(4\lambda_0+M)E(\tau_l\wedge T).\end{aligned}$$

Let $\Omega_k=\{\tau_l\leq T\}$, we know that $P(\Omega_l)\geq\epsilon$. Note that for any $\varpi\in\Omega_l$, there is at least one of $S(\tau_l,\varpi)$, $V(\tau_l,\varpi)$, $I(\tau_l,\varpi)$, $P(\tau_l,\varpi)$ equaling either l or $\frac{1}{l}$. Therefore,

$$\mathbb{L}_1(S(\cdot,\tau_l\wedge T),V(\cdot,\tau_l\wedge T),I(\cdot,\tau_l\wedge T),B(\cdot,\tau_l\wedge T))\geq(l-1-\log l)\wedge(\frac{1}{l}-1-\log\frac{1}{l}).$$

Furthermore,

$$\begin{aligned}&\mathbb{L}_1(S_0(\cdot)+V_0(\cdot)+I_0(\cdot)+B_0(\cdot))+(4\lambda_0+M)E(\tau_l\wedge T)\\&\geq E\{1_{\Omega_k}\mathcal{L}(S(\cdot,\tau_l\wedge T),V(\cdot,\tau_l\wedge T),I(\cdot,\tau_l\wedge T),B(\cdot,\tau_l\wedge T))\}\\&\geq\epsilon\{(k-1-\log k)\wedge(\frac{1}{k}-1-\log\frac{1}{k})\},\end{aligned}$$

where 1_{Ω_k} denotes the indicator function. As $l\to\infty$, we have

$$\infty>\mathbb{L}_1(S_0(\cdot)+V_0(\cdot)+I_0(\cdot)+B_0(\cdot))+(4\lambda_0+M)T=\infty.$$

This is a contradiction. Hence, we have $\tau_\infty=\infty$.

Remark 2. By virtue of Theorem 3.2, we prove the existence and uniqueness of the solution for system (2.2).

Before proving the stationary distribution of the system, we first show that the system is stablity in finite time.

Theorem 3.3. *There exists positive number T, K_1, K_2(where $K_2>K_1$) such that system* (2.2) *is finite-time stability, if $S^2(\cdot,0)+V^2(\cdot,0)+I^2(\cdot,0)+B^2(\cdot,0)\leq K_1$, then, for any $t\in[0,T]$, we have*

$$E(S^2+V^2+I^2+B^2)\leq K_2.$$

Proof. Define

$$\mathcal{L}_2=\|S\|^2+\|V\|^2+\|I\|^2+\|B\|^2.$$

By virtue of Itô's formula, we can obtain

$$\begin{aligned}&d(\|S\|^2+\|V(\|^2+\|I\|^2+\|B\|^2)\\&=\{2\langle S,d_S\triangle S+\Pi(x)-\beta_1(x)SI-\beta_2(x)SB-\mu_1S-r_V(x)S\rangle+2\langle V,d_V\triangle V+r_V(x)S\\&\quad-\beta_3(x)b_1VI-\beta_4(x)b_2VB-\mu_1(x)V\rangle+2\langle I,d_I\triangle I+\beta_1(x)SI+\beta_2(x)SB+\beta_3(x)b_1VI\\&\quad+\beta_4(x)b_2VB-\mu_1(x)I-d(x)I\rangle+2\langle B,d_B\triangle B+k(x)I-\mu_2(x)B\rangle+2\sigma_1^2\|S\|^2\|I\|^2\\&\quad+2\sigma_2^2\|S\|^2\|B\|^2+2\sigma_3^2b_1^2\|V\|^2\|I\|^2+2\sigma_4^2b_2^2\|V\|^2\|B\|^2\}dt+2\langle S,-\sigma_1SIdW_1(t)-\sigma_2SBdW_2(t)\rangle\\&\quad+2\langle V,-\sigma_3b_1VIdW_3(t)-\sigma_4b_2VBdW_4(t)\rangle+2\langle I,\sigma_1SIdW_1(t)+\sigma_2SBdW_2(t)+\sigma_3b_1VIdW_3(t)\\&\quad+\sigma_4b_2VBdW_4(t)\rangle.\end{aligned}\tag{3.5}$$

Moreover, integrating both sides of system (3.5) and taking expectation for $t \in [0, T]$ leads to

$$
\begin{aligned}
& E[\|S(\cdot,T)\|^2 + \|V(\cdot,T)\|^2 + \|I(\cdot,T)\|^2 + \|B(\cdot,T)\|^2] - (\|S_0(\cdot)\|^2 + \|V_0(\cdot)\|^2 + \|I_0(\cdot)\|^2 + \|B_0(\cdot)\|^2) \\
& \leq E\int_0^T \{-2\langle \nabla S(\cdot,t), d_S \nabla S(\cdot,t)\rangle - 2\langle \nabla V(\cdot,t), d_V \nabla V(\cdot,t)\rangle - 2\langle \nabla I(\cdot,t), d_I \nabla I(\cdot,t)\rangle \\
& - 2\langle \nabla B(\cdot,t), d_B \nabla B(\cdot,t)\rangle + \overline{\Pi}^2 + \|S\|^2 - 2\underline{\mu}_1\|S\|^2 - 2\underline{r}_v\|S\|^2 + \overline{r}_v\|S\|^2 + \overline{r}_v\|V\|^2 \\
& - 2\underline{\mu}_1\|V\|^2 + 2(\beta_1 + \beta_3 b_1)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|I\|^2 + 2\beta_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|S\|^2 + 2(\beta_2 + \beta_4 b_2)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|B\|^2 \\
& \quad + 2\beta_4 b_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|V\|^2 - 2\underline{\mu}_1\|I\|^2 - 2\underline{d}\|I\|^2 + \overline{k}\|I\|^2 + \overline{k}\|B\|^2 - 2\underline{\mu}_2\|B\|^2 + 2\sigma_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|S\|^2 \\
& \quad + 2\sigma_2^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|B\|^2 + 2\sigma_3^2 b_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|I\|^2 + 2\sigma_4^2 b_2^2\frac{\overline{k}\bar{\Lambda}|\mathbb{O}|^2}{\underline{\mu}_1\underline{\mu}_2}\|V\|^2\}dt \\
& \quad \leq E\int_0^T \{\overline{\Pi}^2 + \|S\|^2 + \overline{r}_v\|S\|^2 + \overline{r}_v\|V\|^2 + \overline{k}\|I\|^2 + \overline{k}\|B\|^2 + 2\sigma_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|S\|^2 \\
& \quad + 2\sigma_2^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|B\|^2 + 2\sigma_3^2 b_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|I\|^2 + 2\sigma_4^2 b_2^2\frac{\overline{k}\bar{\Lambda}|\mathbb{O}|^2}{\underline{\mu}_1\underline{\mu}_2}\|V\|^2 + 2(\beta_1 + \beta_3 b_1)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|I\|^2 \\
& \quad + 2\beta_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|S\|^2 + 2(\beta_2 + \beta_4 b_2)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|B\|^2 + 2\beta_4 b_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}\|V\|^2\}dt.
\end{aligned}
$$

Next, we can obtain

$$
\begin{aligned}
& E[\|S(\cdot,T)\|^2 + \|V(\cdot,T)\|^2 + \|I(\cdot,T)\|^2 + \|B(\cdot,T)\|^2] \\
& \leq \|S_0(\cdot)\|^2 + \|V_0(\cdot)\|^2 + \|I_0(\cdot)\|^2 + \|B_0(\cdot)\|^2 + \overline{\Pi}^2 T + E\int_0^T \{(1 + 2\sigma_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\beta_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{r}_v)\|S\|^2 \\
& + (\overline{r}_v + 2\beta_4 b_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\sigma_4^2 b_2^2\frac{\overline{k}\bar{\Lambda}|\mathbb{O}|^2}{\underline{\mu}_1\underline{\mu}_2})\|V\|^2 + (2(\beta_1 + \beta_3 b_1)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\sigma_3^2 b_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{k} - 2\underline{\mu}_1)\|I\|^2 \\
& + (2(\beta_2 + \beta_4 b_2)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\sigma_2^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{k})\|B\|^2\}dt.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& E[\|S(\cdot,T)\|^2 + \|V(\cdot,T)\|^2 + \|I(\cdot,T)\|^2 + \|B(\cdot,T)\|^2] \\
& \leq A_1 + A_2 E\int_0^T \{\|S\|^2 + \|V\|^2 + \|I\|^2 + \|B\|^2\}dt,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 =& \|S_0(\cdot)\|^2 + \|V_0(\cdot)\|^2 + \|I_0(\cdot)\|^2 + \|B_0(\cdot)\|^2 + \overline{\Pi}^2 T, \\
A_2 =& max\{1 + 2\sigma_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{r}_v + 2\beta_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{r}_v, \overline{r}_v + 2\beta_4 b_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\sigma_4^2 b_2^2\frac{\overline{k}\bar{\Lambda}|\mathbb{O}|^2}{\underline{\mu}_1\underline{\mu}_2}, \\
& 2(\beta_1 + \beta_3 b_1)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\sigma_3^2 b_1^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{k}, 2(\beta_2 + \beta_4 b_2)\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + 2\sigma_2^2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + \overline{k}\}.
\end{aligned}
$$

By the Gronwall inequality

$$E[\|S(\cdot,T)\|^2+\|V(\cdot,T)\|^2+\|I(\cdot,T)\|^2+\|B(\cdot,T)\|^2]\leq A_1e^{A_2T}:=K_2.$$

Remark 3. According to Theorem 3.3, we demonstrate that system (2.2) can reach a stable state within a finite time.

4. Stationary distribution

In this section, we prove that the system admits a stationary distribution.

Definition 4.1. [30–32] A stationary distribution for $\mathcal{N}$, system (2.2) is defined as a probability measure $\lambda\in P(\mathcal{H})$ satisfying

$$\lambda(f)=\lambda(P_tf), t>0,$$

here

$$\lambda(f):=\int_{\mathcal{H}}f(\zeta)\lambda(d\zeta), P_tf(\zeta):=Ef(\mathcal{N}(x,t,\zeta)), f\in C_b(\mathcal{H}).$$

For $\lambda_1,\lambda_2\in P(\mathcal{H})$, define a metric on $P(\mathcal{H})$ by

$$d(\lambda_1,\lambda_2)=\sup_{f\in\mathcal{A}}|\int_{\mathcal{H}}f(\zeta)\lambda_1(d\zeta)-f(\varrho)\lambda_2(d\varsigma)|,$$

where

$$\mathcal{A}:=\{f:\mathcal{H}\to R,|f(\zeta)-f(\varrho)|\leq|\zeta-\varrho|_{\mathcal{H}},\ \zeta,\ \varrho\in\mathcal{H}\ and\ |f(\cdot)|\leq 1\}.$$

$P(\mathcal{H})$ is complete under the metric $d(\cdot,\cdot)$.

$$\mathcal{H}=\{\phi\mid\phi\in L^2(\Omega),\frac{\partial\phi}{\partial x}\in L^2(\Omega),\ \ where\ \ \frac{\partial\phi}{\partial x}\ are\ generalized\ partial\ derivatives\}.$$

Moreover, we have the following theorem

Theorem 4.1. *For any bounded subset B of $\mathcal{H}$, $\alpha\geq 1$, if*

$$(i)\lim_{t\to\infty}\sup_{\psi,\varphi\in B}E\|\mathcal{N}(\cdot,t,\psi)-\mathcal{N}(\cdot,t,\varphi)\|^{\alpha}_{\mathcal{H}}=0;$$

$$(ii)\ there\ is\ a\ constant\ \chi>0,\ \alpha>2,\ such\ that\ \chi+Q>0,$$

then, system (2.2) *has a unique stationary distribution.*

Proof. First, we define the difference of two mild solutions of system (2.2) with distinct initial data $\zeta,\varrho\in\mathbb{O}$

$$\vartheta(\cdot,t)=\begin{pmatrix}\vartheta_1(\cdot,t,\zeta,\varrho)\\\vartheta_2(\cdot,t,\zeta,\varrho)\\\vartheta_3(\cdot,t,\zeta,\varrho)\\\vartheta_4(\cdot,t,\zeta,\varrho)\end{pmatrix}=\begin{pmatrix}S(\cdot,t,\zeta)-S(\cdot,t,\varrho)\\V(\cdot,t,\zeta)-V(\cdot,t,\varrho)\\I(\cdot,t,\zeta)-I(\cdot,t,\varrho)\\B(\cdot,t,\zeta)-B(\cdot,t,\varrho)\end{pmatrix}.\tag{4.1}$$

According to the $It\hat{o}'s$ formula, we have

$$
\begin{aligned}
& d(e^{\chi t}\|\vartheta(\cdot,t,\zeta,\varrho)\|^{\alpha}) \\
=& \chi e^{\chi t}\|\vartheta(\cdot,t,\zeta,\varrho)\|^{\alpha}dt + e^{\chi t}\{\alpha\|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\langle\vartheta_1(\cdot,t,\zeta,\varrho), d_S\triangle\vartheta_1(\cdot,t,\zeta,\varrho) - \mu_1\vartheta_1(\cdot,t,\zeta,\varrho) \\
& - \beta_1(\cdot)(S(\cdot,t,\zeta)I(\cdot,t,\zeta) - S(\cdot,t,\varrho)I(\cdot,t,\varrho)) - \beta_2(\cdot)(S(\cdot,t,\zeta)B(\cdot,t,\zeta) - S(\cdot,t,\varrho)B(\cdot,t,\varrho)) \\
& - r_V(\cdot)\vartheta_1(\cdot,t,\zeta,\varrho)\rangle + \alpha\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\langle\vartheta_2(\cdot,t,\zeta,\varrho), d_V\triangle\vartheta_2(\cdot,t,\zeta,\varrho) + r_V(\cdot)\vartheta_1(\cdot,t,\zeta,\varrho) \\
& - \beta_3(\cdot)b_1(V(\cdot,t,\zeta)I(\cdot,t,\zeta) - V(\cdot,t,\varrho)I(\cdot,t,\varrho)) - \beta_4(\cdot)b_2(V(\cdot,t,\zeta)B(\cdot,t,\zeta) - V(\cdot,t,\varrho)B(\cdot,t,\varrho)) \\
& - \mu_1(\cdot)\vartheta_2(\cdot,t,\zeta,\varrho)\rangle + \alpha\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\langle\vartheta_3(\cdot,t,\zeta,\varrho), d_I\triangle\vartheta_3(\cdot,t,\zeta,\varrho) - \mu_1(\cdot)\vartheta_3(\cdot,t,\zeta,\varrho) \\
& - d(\cdot)\vartheta_3(\cdot,t,\zeta,\varrho) + \beta_1(\cdot)(S(\cdot,t,\zeta)I(\cdot,t,\zeta) - S(\cdot,t,\varrho)I(\cdot,t,\varrho)) + \beta_2(\cdot)(S(\cdot,t,\zeta)B(\cdot,t,\zeta) - S(\cdot,t,\varrho)B(\cdot,t,\varrho)) \\
& + \beta_3(\cdot)b_1(V(\cdot,t,\zeta)I(\cdot,t,\zeta) - V(\cdot,t,\varrho)I(\cdot,t,\varrho)) + \beta_4(\cdot)b_2(V(\cdot,t,\zeta)B(\cdot,t,\zeta) - V(\cdot,t,\varrho)B(\cdot,t,\varrho))\rangle \\
& \alpha\|\vartheta_4(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\langle\vartheta_4(\cdot,t,\zeta,\varrho), d_B\triangle B + k(x)I - \mu_2(x)B\rangle \\
& + \frac{1}{2}\alpha(\alpha-1)\sigma_1^2\|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-2}(S^2(\cdot,t,\zeta)I^2(\cdot,t,\zeta) - S^2(\cdot,t,\varrho)I^2(\cdot,t,\varrho)) + \frac{1}{2}\alpha(\alpha-1)\sigma_2^2\times \\
& \|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-2}(S^2(\cdot,t,\zeta)B^2(\cdot,t,\zeta) - S^2(\cdot,t,\varrho)B^2(\cdot,t,\varrho)) + \frac{1}{2}\alpha(\alpha-1)\sigma_3^2b_1^2\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-2} \\
& (V^2(\cdot,t,\zeta)I^2(\cdot,t,\zeta) - V^2(\cdot,t,\varrho)I^2(\cdot,t,\varrho)) + \frac{1}{2}\alpha(\alpha-1)\sigma_4^2b_2^2\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-2}(V^2(\cdot,t,\zeta)B^2(\cdot,t,\zeta) \\
& - V^2(\cdot,t,\varrho)B^2(\cdot,t,\varrho)) + \frac{1}{2}\alpha(\alpha-1)\sigma_1^2\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}(S^2(\cdot,t,\zeta)I^2(\cdot,t,\zeta) - S^2(\cdot,t,\varrho)I^2(\cdot,t,\varrho)) \\
& + \frac{1}{2}\alpha(\alpha-1)\sigma_2^2\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}(S^2(\cdot,t,\zeta)B^2(\cdot,t,\zeta) - S^2(\cdot,t,\varrho)B^2(\cdot,t,\varrho)) + \frac{1}{2}\alpha(\alpha-1)\sigma_3^2b_1^2\times \\
& \|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}(V^2(\cdot,t,\zeta)I^2(\cdot,t,\zeta) - V^2(\cdot,t,\varrho)I^2(\cdot,t,\varrho)) + \frac{1}{2}\alpha(\alpha-1)\sigma_4^2b_2^2\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2} \\
& (V^2(\cdot,t,\zeta)B^2(\cdot,t,\zeta) - V^2(\cdot,t,\varrho)B^2(\cdot,t,\varrho))\}dt - \alpha\sigma_1\|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-1}(S(\cdot,t,\zeta)I(\cdot,t,\zeta) \\
& - S(\cdot,t,\varrho)I(\cdot,t,\varrho))dW_1(t) - \alpha\sigma_2\|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-1}(S(\cdot,t,\zeta)B(\cdot,t,\zeta) - S(\cdot,t,\varrho)B(\cdot,t,\varrho))dW_2(t) \\
& - \sigma_3b_1\alpha\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-1}(V(\cdot,t,\zeta)I(\cdot,t,\zeta) - V(\cdot,t,\varrho)I(\cdot,t,\varrho))dW_3(t) - \sigma_4b_2\alpha\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-1} \\
& \times(V(\cdot,t,\zeta)B(\cdot,t,\zeta) - V(\cdot,t,\varrho)B(\cdot,t,\varrho))dW_4(t) + \alpha\sigma_1\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}(S(\cdot,t,\zeta)I(\cdot,t,\zeta) - \\
& S(\cdot,t,\varrho)I(\cdot,t,\varrho))dW_1(t) + \alpha\sigma_2\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}(S(\cdot,t,\zeta)B(\cdot,t,\zeta) - S(\cdot,t,\varrho)B(\cdot,t,\varrho))dW_2(t) \\
& + \sigma_3b_1\alpha\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}(V(\cdot,t,\zeta)I(\cdot,t,\zeta) - V(\cdot,t,\varrho)I(\cdot,t,\varrho))dW_3(t) + \sigma_4b_2\alpha\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1} \\
& \times(V(\cdot,t,\zeta)B(\cdot,t,\zeta) - V(\cdot,t,\varrho)B(\cdot,t,\varrho))dW_4(t).
\end{aligned}
$$

To proceed, the following equation must be considered:

$$
\begin{aligned}
& S(\cdot,t,\zeta)I(\cdot,t,\zeta) - S(\cdot,t,\varrho)I(\cdot,t,\varrho) \\
& = S(\cdot,t,\zeta)I(\cdot,t,\zeta) - S(\cdot,t,\zeta)I(\cdot,t,\varrho) + S(\cdot,t,\zeta)I(\cdot,t,\varrho) - S(\cdot,t,\varrho)I(\cdot,t,\varrho) \\
& = S(\cdot,t,\zeta)\vartheta_3(\cdot,t,\zeta,\varrho) + I(\cdot,t,\varrho)\vartheta_1(\cdot,t,\zeta,\varrho) \\
& \le \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\vartheta_1(\cdot,t,\zeta,\varrho) + \vartheta_3(\cdot,t,\zeta,\varrho)).
\end{aligned}
$$

Using the same method, we can obtain

$$S(\cdot,t,\zeta)B(\cdot,t,\zeta)-S(\cdot,t,\varrho)B(\cdot,t,\varrho)\le\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\vartheta_4(\cdot,t,\zeta,\varrho)+\vartheta_1(\cdot,t,\zeta,\varrho)),$$
$$V(\cdot,t,\zeta)I(\cdot,t,\zeta)-V(\cdot,t,\varrho)I(\cdot,t,\varrho)\le\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\vartheta_3(\cdot,t,\zeta,\varrho)+\vartheta_2(\cdot,t,\zeta,\varrho)),$$
$$V(\cdot,t,\zeta)B(\cdot,t,\zeta)-V(\cdot,t,\varrho)B(\cdot,t,\varrho)\le\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\vartheta_4(\cdot,t,\zeta,\varrho)+\vartheta_2(\cdot,t,\zeta,\varrho)).$$

For $\langle\vartheta_1(\cdot,t,\zeta,\varrho),d_S\triangle\vartheta_1(\cdot,t,\zeta,\varrho)\rangle=-d_S\lambda_0\vartheta_1^2(\cdot,t,\zeta,\varrho)$. Integrating on both sides of $d(e^{\chi t}\|\vartheta(\cdot,t,\zeta,\varrho)\|^{\alpha})$ and taking expectations

$$\begin{aligned}
&E(e^{\chi t}\|\vartheta(\cdot,t,\zeta,\varrho)\|^{\alpha})\\
&\le\|\vartheta(\cdot,0,\zeta,\varrho)\|^{\alpha}+E\int_0^T\chi e^{\chi t}\|\vartheta(\cdot,t,\zeta,\varrho)\|^{\alpha}dt+E\int_0^T\alpha e^{\chi t}\{(-d_S\lambda_0-mu_1-r_V)\|\vartheta_1(\cdot,t,\zeta,\varrho\|^{\alpha})\\
&+(-d_V\lambda_0-\mu_1)\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha}+r_V\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-1}\|\vartheta_1(\cdot,t,\zeta,\varrho)\|+(-d_I\lambda_0-\mu_1-d)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha}\\
&+\beta_1\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\|\vartheta_1(\cdot,t,\zeta,\varrho)\|+\|\vartheta_3(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}\\
&+\beta_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\|\vartheta_4(\cdot,t,\zeta,\varrho)\|+\|\vartheta_1(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}\\
&+\beta_3\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\|\vartheta_3(\cdot,t,\zeta,\varrho)\|+\|\vartheta_2(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}\\
&+\beta_4\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\|\vartheta_4(\cdot,t,\zeta,\varrho)\|+\|\vartheta_2(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-1}\\
&-d_B\lambda_0\|\vartheta_4(\cdot,t,\zeta,\varrho)\|^{\alpha}+k\|\vartheta_4(\cdot,t,\zeta,\varrho)\|^{\alpha-1}\|\vartheta_3(\cdot,t,\zeta,\varrho)\|-\mu_2\|\vartheta_4(\cdot,t,\zeta,\varrho)\|^{\alpha}\\
&+(\alpha-1)\sigma_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_1(\cdot,t,\zeta,\varrho)\|+\|\vartheta_3(\cdot,t,\zeta,\varrho)\|)\|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\\
&+(\alpha-1)\sigma_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_1(\cdot,t,\zeta,\varrho)\|+\|\vartheta_4(\cdot,t,\zeta,\varrho)\|)\|\vartheta_1(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\\
&+(\alpha-1)\sigma_3^2b_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_3(\cdot,t,\zeta,\varrho)\|+\|\vartheta_2(\cdot,t,\zeta,\varrho)\|)\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\\
&+(\alpha-1)\sigma_4^2b_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_4(\cdot,t,\zeta,\varrho)\|+\|\vartheta_2(\cdot,t,\zeta,\varrho)\|)\|\vartheta_2(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\\
&+(\alpha-1)\sigma_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_1(\cdot,t,\zeta,\varrho)\|+\|\vartheta_3(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\\
&+(\alpha-1)\sigma_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_1(\cdot,t,\zeta,\varrho)\|+\|\vartheta_4(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}\\
&+(\alpha-1)\sigma_3^2b_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\|\vartheta_3(\cdot,t,\zeta,\varrho)\|+\|\vartheta_2(\cdot,t,\zeta,\varrho)\|)\|\vartheta_3(\cdot,t,\zeta,\varrho)\|^{\alpha-2}
\end{aligned}$$

$$+(\alpha-1)\sigma_4^2 b_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(||\vartheta_4(\cdot,t,\zeta,\varrho)||+||\vartheta_2(\cdot,t,\zeta,\varrho)||)||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha-2}\}dt. \tag{4.2}$$

For Eq (4.2), applying the Young inequality and Basic inequality, we have

$$
\begin{aligned}
&E(e^{\chi t}||\vartheta(\cdot,t,\zeta,\varrho)||^{\alpha})\\
&\leq ||\vartheta(\cdot,0,\zeta,\varrho)||^{\alpha}+E\int_0^T \chi e^{\chi t}||\vartheta(\cdot,t,\zeta,\varrho)||^{\alpha}dt+E\int_0^T \alpha e^{\chi t}\{(-d_S\lambda_0-\mu_1-r_V)||\vartheta_1(\cdot,t,\zeta,\varrho||^{\alpha})\\
&+(-d_V\lambda_0-\mu_1)||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+r_V(\frac{\alpha-1}{\alpha}||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha})+(-d_I\lambda_0-\mu_1-d)||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}\\
&+\beta_1\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\frac{1}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{\alpha-1}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha})\\
&+\beta_2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\frac{1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2(\alpha-1)}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha})\\
&+\beta_3\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{\alpha-1}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha})\\
&+\beta_4\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\frac{1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2(\alpha-1)}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha})\\
&+(-d_B\lambda_0-\mu_2)||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+k\frac{\alpha-1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}\\
&+(\alpha-1)\sigma_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{2\alpha-3}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{2\alpha-3}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_3^2 b_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{2\alpha-3}{\alpha}||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_4^2 b_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{2\alpha-3}{\alpha}||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{2\alpha-3}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{1}{\alpha}||\vartheta_1(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2\alpha-4}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_3^2 b_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{1}{\alpha}||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2\alpha-3}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\\
&+(\alpha-1)\sigma_4^2 b_2^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\frac{1}{\alpha}||\vartheta_2(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{1}{\alpha}||\vartheta_4(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2\alpha-4}{\alpha}||\vartheta_3(\cdot,t,\zeta,\varrho)||^{\alpha}+\frac{2}{\alpha})\}dt.
\end{aligned}
$$

Due to $\alpha > 2$, we can get the following result:

$$
\begin{aligned}
&E(e^{\chi t}||\vartheta(\cdot,t,\zeta,\varrho)||^{\alpha})\\
&\leq ||\vartheta(\cdot,0,\zeta,\varrho)||^{\alpha}+2\alpha(\alpha-1)\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}\frac{e^{\chi T-1}}{\chi}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2)+E\int_0^T(\chi+Q)e^{\chi t}||\vartheta(\cdot,t,\zeta,\varrho)||^{\alpha}dt,
\end{aligned}
$$

where

$$\begin{aligned}\chi + Q = \max\{&\chi + \alpha\{3(\alpha-1)\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\sigma_1^2+\sigma_2^2) + \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\beta_1+\beta_2+\beta_4) - d_S\lambda_0 - \mu_1\},\\
&\chi + \alpha\{3(\alpha-1)b_1^2\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\sigma_3^2+\sigma_4^2) + \beta_3\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1} + r_V - d_V\lambda_0 - \mu_1\},\\
&\chi + \alpha\{(\alpha-1)\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(3\sigma_1^2+2\sigma_2^2+3\sigma_3^2b_1^2+2\sigma_4^2b_2^2) + 1\\
&+ 2\frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\beta_1+\beta_2+\beta_3+\beta_4) - d_I\lambda_0 - \mu_1 - d\},\\
&\chi + \alpha\{2(\alpha-1)\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}(\sigma_2^2+\sigma_4^2b_2^2) + \frac{\bar{\Lambda}|\mathbb{O}|}{\underline{\mu}_1}(\beta_2+\beta_4) + k - d_B\lambda_0 - \mu_2\}\}.\end{aligned}$$

By virtue of the Gronwall inequality, we obtain

$$e^{\chi t}\|\vartheta(\cdot,t,\zeta,\varrho)\|^\alpha \le \|\vartheta(\cdot,0,\zeta,\varrho)\|^\alpha + 2\alpha(\alpha-1)\frac{\bar{\Lambda}^3|\mathbb{O}|^3}{\underline{\mu}_1^3}\frac{e^{\chi T-1}}{\chi}(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2)e^{-(\chi+Q)t},$$

moreover

$$\lim_{t\to\infty} E\|\vartheta(\cdot,t,\zeta,\varrho)\|^\alpha = 0.$$

Remark 4. Theorem 4.1 declares the existence and uniqueness of the stationary distribution of the solution for system (2.2).

5. Numerical simulations

In this section, the theoretical results are validated by means of numerical simulations. By virtue of the Milstein's method [33], the discrete form of system (2.2) is given and the algorithm process is shown.

$$\begin{aligned}S_{(i,j+1)} = S_{(i,j)} &+ [d_S\frac{S_{(i+1,j)} - 2S_{(i,j)} + S_{(i-1,j)}}{(\triangle x)^2} + \Lambda - \beta_1 S_{(i,j)}I_{(i,j)} - \beta_2 S_{(i,j)}B_{(i,j)}\\
&- \mu_1 S_{(i,j)} - r_V S_{(i,j)}]\triangle t - \sigma_1 S_{(i,j)}I_{(i,j)}\varsigma_j\sqrt{\triangle t} - \frac{\sigma_1^2}{2}S_{(i,j)}^2 I_{(i,j)}^2(\varsigma_j^2-1)\triangle t\\
&- \sigma_2 S_{(i,j)}B_{(i,j)}\varsigma_j\sqrt{\triangle t} - \frac{\sigma_2^2}{2}S_{(i,j)}^2 B_{(i,j)}^2(\varsigma_j^2-1)\triangle t,\\
V_{(i,j+1)} = V_{(i,j)} &+ [d_V\frac{V_{(i+1,j)} - 2V_{(i,j)} + V_{(i-1,j)}}{(\triangle x)^2} + r_V S_{(i,j)} - \beta_3 b_1 V_{(i,j)I_{(i,j)}} - \beta_4 b_2 V_{(i,j)}B_{(i,j)}\\
&- \mu_1 V_{(i,j)}]\triangle t - \sigma_3 b_1 V_{(i,j)}I_{(i,j)}\varsigma_j\sqrt{\triangle t} - \frac{\sigma_3^2}{2}b_1^2 V_{(i,j)}^2 I_{(i,j)}^2(\varsigma_j^2-1)\triangle t\\
&- \sigma_4 b_2 V_{(i,j)}B_{(i,j)}\varsigma_j\sqrt{\triangle t} - \frac{\sigma_4^2}{2}b_2^2 V_{(i,j)}^2 B_{(i,j)}^2(\varsigma_j^2-1)\triangle t,\end{aligned}$$

$$I_{(i,j+1)} = I_{(i,j)} + [d_I \frac{I_{(i+1,j)} - 2I_{(i,j)} + I_{(i-1,j)}}{(\triangle x)^2} + \beta_1 S_{(i,j)} I_{(i,j)} + \beta_2 S_{(i,j)} B_{(i,j)} + \beta_3 b_1 V_{(i,j)I_{(i,j)}}$$
$$+ \beta_4 b_2 V_{(i,j)} B_{(i,j)} - (\mu_1 + d) I_{(i,j)}]\triangle t + \sigma_1 S_{(i,j)} I_{(i,j)} \varsigma_j \sqrt{\triangle t} + \frac{\sigma_1^2}{2} S_{(i,j)}^2 I_{(i,j)}^2 (\varsigma_j^2 - 1)\triangle t$$
$$+ \sigma_2 S_{(i,j)} B_{(i,j)} \varsigma_j \sqrt{\triangle t} + \frac{\sigma_2^2}{2} S_{(i,j)}^2 B_{(i,j)}^2 (\varsigma_j^2 - 1)\triangle t + \sigma_3 b_1 V_{(i,j)} I_{(i,j)} \varsigma_j \sqrt{\triangle t}$$
$$+ \frac{\sigma_3^2}{2} b_1^2 V_{(i,j)}^2 I_{(i,j)}^2 (\varsigma_j^2 - 1)\triangle t + \sigma_4 b_2 V_{(i,j)} B_{(i,j)} \varsigma_j \sqrt{\triangle t} - \frac{\sigma_4^2}{2} b_2^2 V_{(i,j)}^2 B_{(i,j)}^2 (\varsigma_j^2 - 1)\triangle t,$$
$$B_{(i,j+1)} = B_{(i,j)} + [d_B \frac{B_{(i+1,j)} - 2B_{(i,j)} + B_{(i-1,j)}}{(\triangle x)^2} + kI_{(i,j)} - \mu_2 B_{(i,j)}]\triangle t.$$

5.1. The influence of noise intensity

For Figures 1 and 2, let $\Lambda = 10$, $\gamma_V = 0.45$, $\mu_1 = 0.35$, $d = 0.35$, $k = 0.35$, $\mu_2 = 0.33$, $d_S = 0.35$, $d_V = 0.4$, $d_I = 0.25$, $d_B = 0.5$, $\beta_1 = 0.08$, $\beta_2 = 0.07$, $\beta_3 = 0.02$, $\beta_4 = 0.03$, $b_1 = 0.3$, $b_2 = 0.4$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.04$, $\triangle x = 0.4$, $\triangle t = 0.1$. Figure 1 represents the path diagrams of S, V, I, and B in time and space. Figure 2 is the corresponding histogram for viewing the stationary distributions of S, V, I, and B more intuitively.

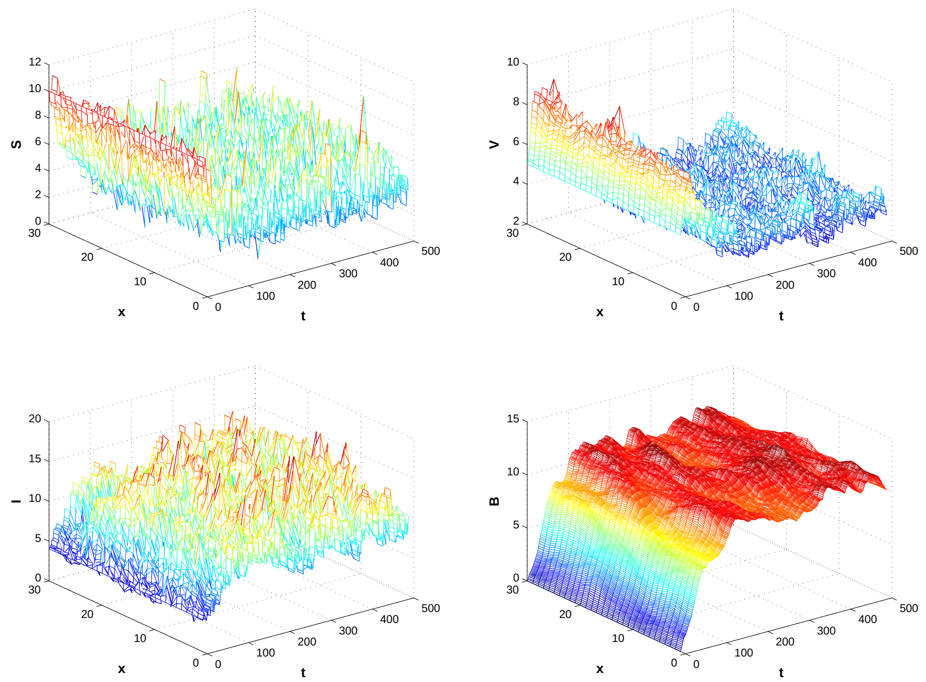

Figure 1. The evolution path of S, V, I, B for system (2.2) when the noise intensity is 0.04.

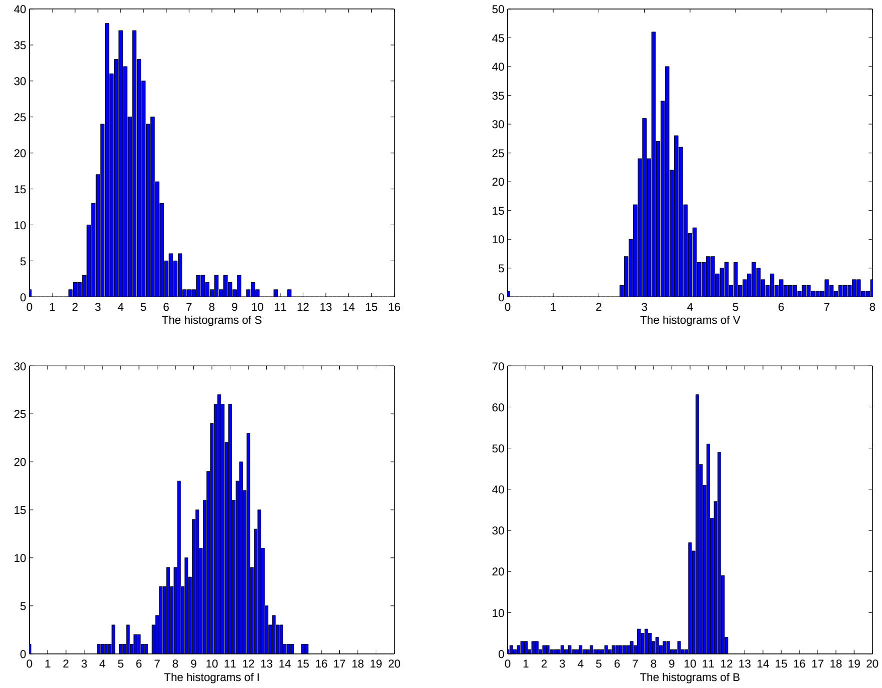

Figure 2. The histograms of S, V, I, B for system (2.2) when the noise intensity is 0.04.

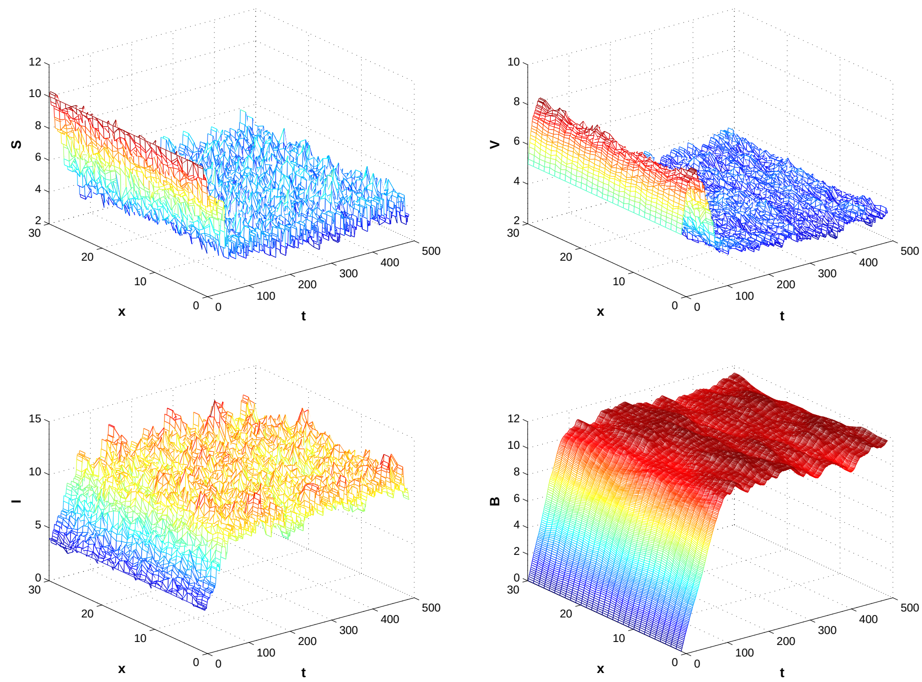

Figure 3. The evolution path of S, V, I, B for system (2.2) when the noise intensity is 0.02.

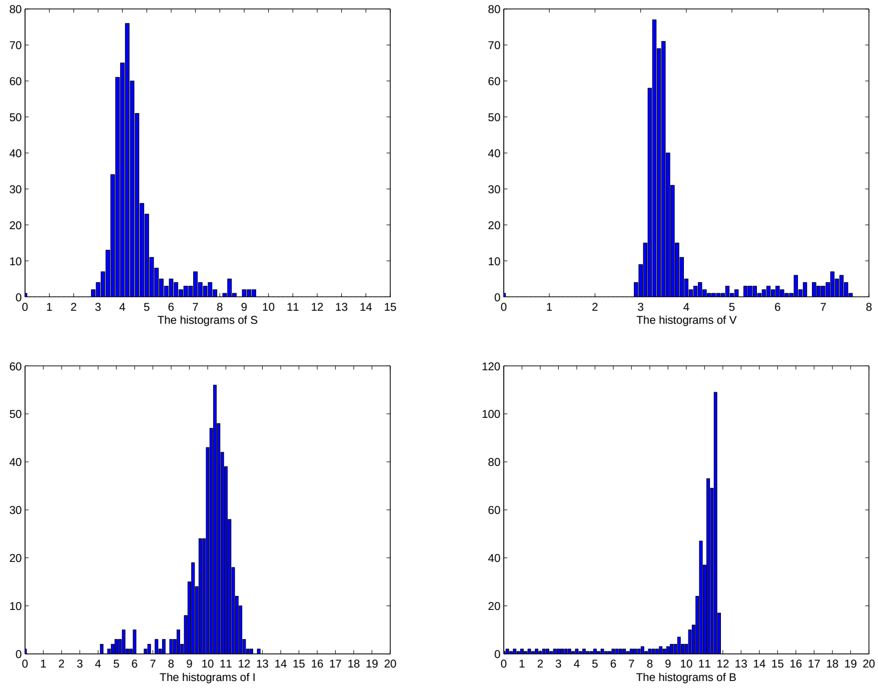

Figure 4. The histograms of S, V, I, B for system (2.2) when the noise intensity is 0.02.

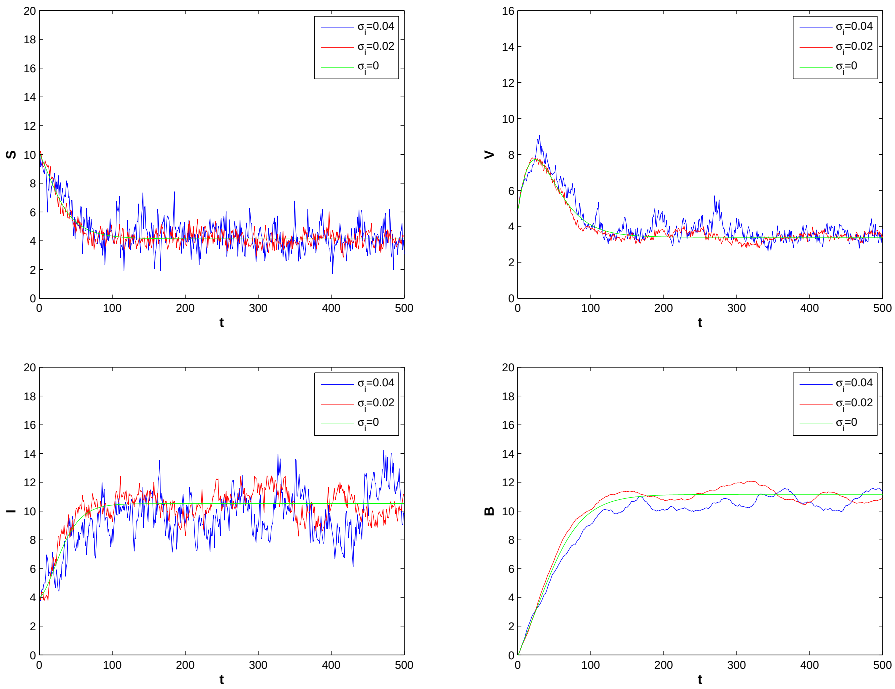

Figure 5. The evolution path of S, V, I, B for system (2.2) with different noise intensity.

To observe the influence of noise intensity on the stationary distribution of the solution, we adjust the noise intensity. In Figures 3 and 4, other parameters remain unchanged, let $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.02$, Figure 3 shows the path diagrams of S, V, I, and B in time and space. Figure 4 is the histogram corresponding to Figure 3. As the noise intensity increases, the amplitude of the image fluctuation increases (Figures 1 and 3). Furthermore, a lower noise intensity yields a stationary distribution for the solution, as evidenced in Figures 2 and 4.

Additionally, to observe the changes of the solutions more intuitively, Figure 5 depicts a two-dimensional variation graph. Hence, it can be observed that the image with noise fluctuates on both sides of the image without noise. Furthermore, a greater noise intensity is related to enhanced fluctuation.

5.2. *The influence of the transmission rate*

For Figures 6 and 7, let $\Lambda = 10$, $\gamma_V = 0.45$, $\mu_1 = 0.35$, $d = 0.35$, $k = 0.35$, $\mu_2 = 0.33$, $d_S = 0.35$, $d_V = 0.4$, $d_I = 0.25$, $d_B = 0.5$, $\beta_1 = 0.25$, $\beta_2 = 0.21$, $\beta_3 = 0.09$, $\beta_4 = 0.12$, $b_1 = 0.3$, $b_2 = 0.4$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.02$. Figure 6 shows the path diagrams of S, V, I, and B in time and space, while Figure 7 is the corresponding histogram.

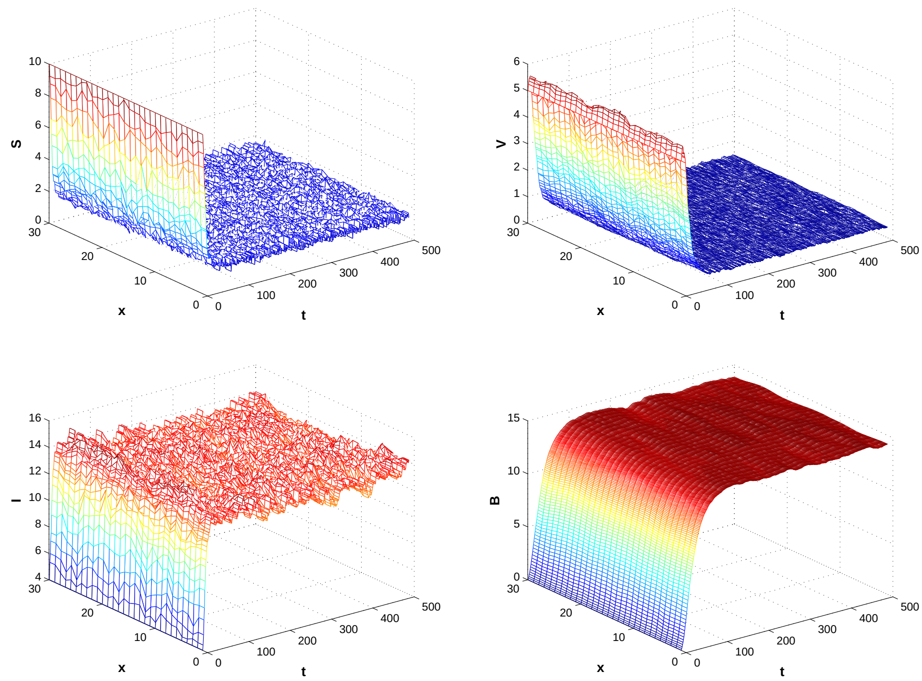

Figure 6. The evolution path of S, V, I, B for system (2.2) with $\beta_1 = 0.25$, $\beta_2 = 0.21$, $\beta_3 = 0.09, \beta_4 = 0.12$.

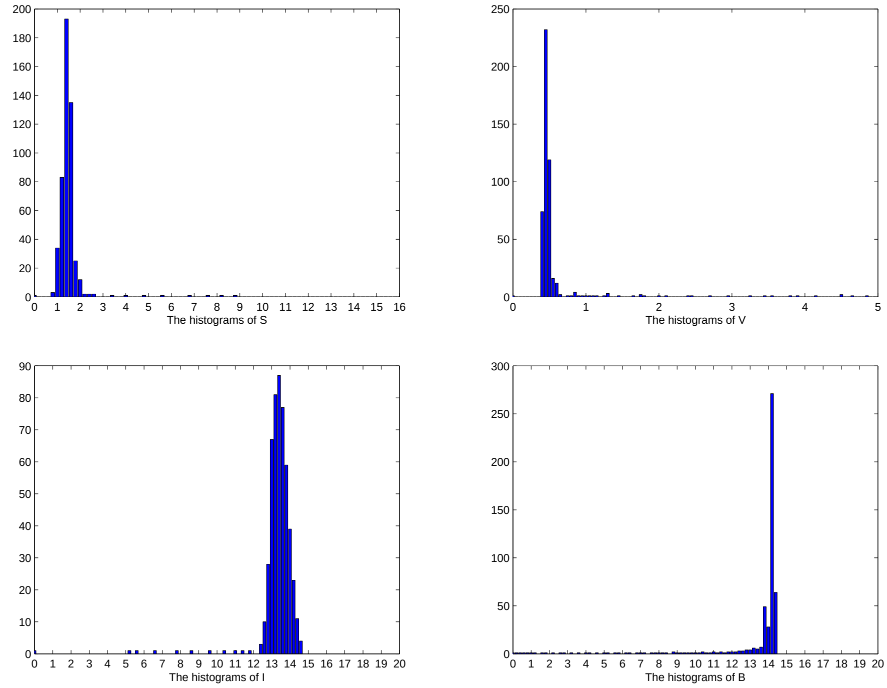

Figure 7. The histograms of S, V, I, B for system (2.2) with $\beta_1 = 0.25$, $\beta_2 = 0.21$, $\beta_3 = 0.09$, $\beta_4 = 0.12$.

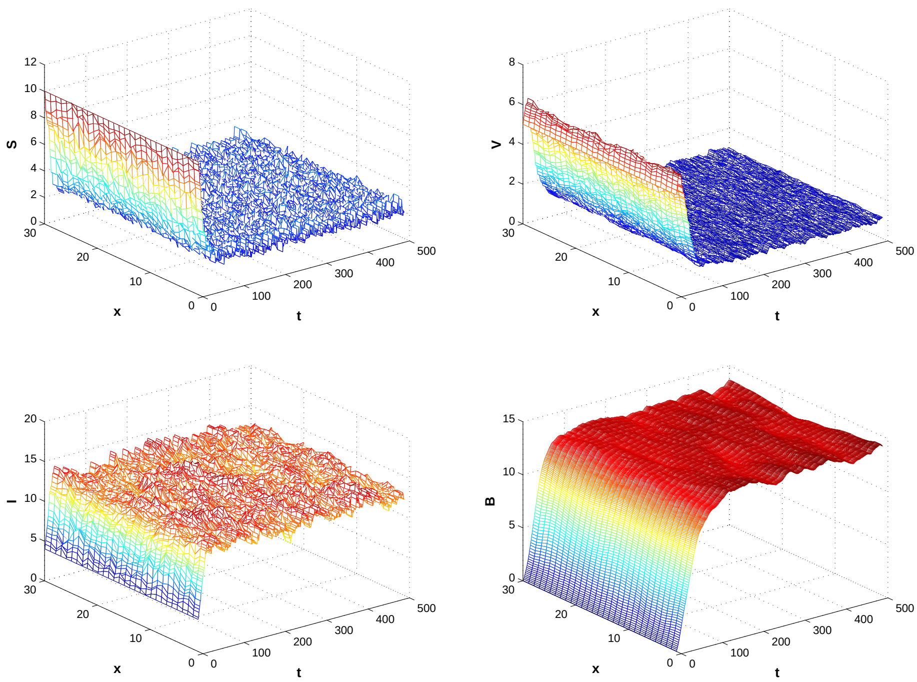

Figure 8. The evolution path of S, V, I, B for system (2.2) with $\beta_1 = 0.15$, $\beta_2 = 0.12$, $\beta_3 = 0.06$, $\beta_4 = 0.09$.

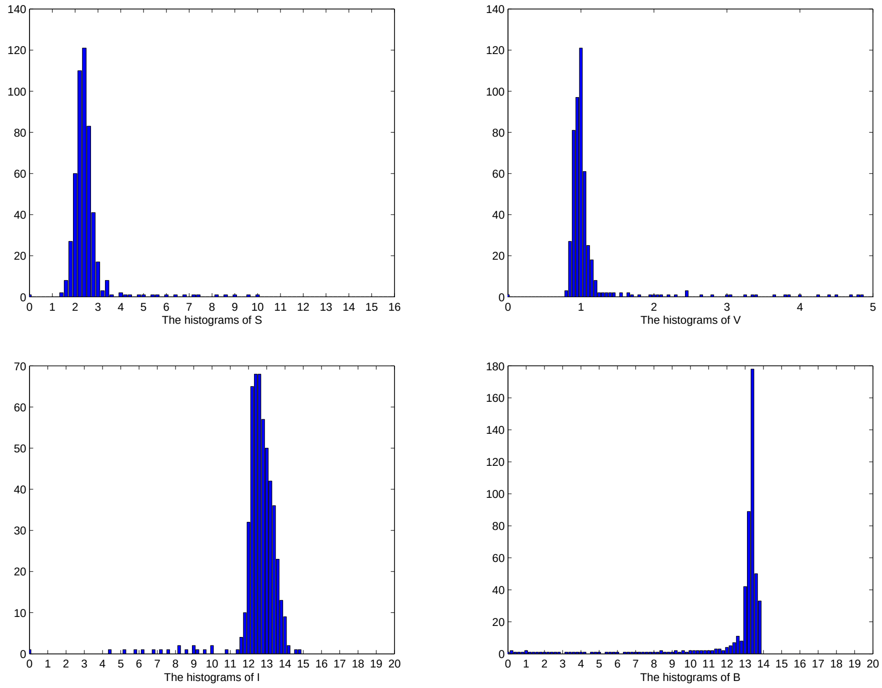

Figure 9. The histograms of S, V, I, B for system (2.2) with $\beta_1 = 0.15, \beta_2 = 0.12, \beta_3 = 0.06,$ $\beta_4 = 0.09$.

To observe the influence of the transmission rate on the stationary distribution of the model's solution, we adjust the dimensions of the transmission rate. In Figures 8 and 9, let $\Lambda = 10$, $\gamma_V = 0.45$, $\mu_1 = 0.35$, $d = 0.35$, $k = 0.35$, $\mu_2 = 0.33$, $d_S = 0.35$, $d_V = 0.4$, $d_I = 0.25$, $d_B = 0.5$, $\beta_1 = 0.15$, $\beta_2 = 0.12$, $\beta_3 = 0.06$, $\beta_4 = 0.09$, $b_1 = 0.3$, $b_2 = 0.4$, and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.02$. Figure 8 shows the path diagrams of S, V, I, and B in time and space, while Figure 9 is the corresponding histogram. Figures 3, 6, and 8 demonstrate how an enhanced transmission rate can weaken the image fluctuation. From Figures 4, 7, and 9, we can see that as the transmission rate increases, the center points of S and V migrate to the left, and the center points of I and B shift to the right. Therefore, we can conclude that the transmission rate can reduce randomness and change the stationary distribution of the model's solutions.

6. Conclusions

Cholera is an acute intestinal infectious disease caused by Vibrio cholera. Due to the rapid spread of cholera and its high incidence and mortality rates during the epidemic period, it is a grave threat. Therefore, an early, rapid, and correct diagnosis is crucial for the treatment and prevention of cholera. In this study, we determined the stationary distribution of an SVIB cholera model that incorporated stochastic noise and reaction-diffusion. Apart from the solution's well-posedness, we also proved the model's finite-time stability. Furthermore, we provided sufficient conditions for the stationary distribution of the solution of the model and proved the uniqueness of the stationary distribution of the solution. Finally, the theorem's results were verified through numerical simulation. For numerical results, we simulated the influences of noise intensity and propagation rate on the system. It could be observed that the noise intensity could impact the model's stationary distribution. When reducing noise intensity, the numbers of infected individuals and cholera bacteria also decreased, and the system tended to show a normal distribution. To conclude, the transmission rate reduces randomness and changes the stationary distribution of solutions. Therefore, we suggest that appropriate measures should be taken to reduce the interference of external factors in a disease outbreak. In our next phase of research, we aim to introduce non-local diffusion, Léyy noise, and Ornstein-Uhlenbeck processes into the model to explore the dynamic behavior of the disease.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. B. Musundi, An immuno-epidemiological model linking between-host and within-host dynamics of cholera, *Math. Biosci. Eng.*, **20** (2023), 16012–16030. https://doi.org/10.3934/mbe.2023714
2. W. Yang, L. Gan, S. Liao, Stochastic analysis of cholera model with Lévy jumps, *Int. J. Biomath.*, 2024, 2450067. https://doi.org/10.1142/S1793524524500670
3. A. Ghosh, P. Das, T. Chakraborty, P. Das, D. Ghosh, Developing cholera outbreak forecasting through qualitative dynamics: Insights into Malawi case study, *J. Theor. Biol.*, **605** (2025), 112097. https://doi.org/10.1016/j.jtbi.2025.112097
4. S. Edward, N. Nyerere, A mathematical model for the dynamics of cholera with control measures, *Appl. Comput. Math.*, **4** (2015), 53–63. https://doi.org/10.11648/j.acm.20150402.14
5. U. T. Mustapha, Y. A. Maigoro, A. Yusuf, S. Qureshi, Mathematical modeling for the transmission dynamics of cholera with an optimal control strategy, *Bull. Biomath.*, **2** (2024), 1–20. https://doi.org/10.59292/bulletinbiomath.2024001
6. C. E. Madubueze, S. C. Madubueze, S. Ajama, Bifurcation and stability analysis of the dynamics of cholera model with controls, *Int. J. Math. Comput. Sci.*, **9** (2015), 645–651.
7. M. A. Safi, D. Y. Melesse, A. B. Gumel, Dynamics analysis of a multi-strain cholera model with an imperfect vaccine, *Bull. Math. Biol.*, **75** (2013), 1104–1137. https://doi.org/10.1007/s11538-013-9845-2

8. S. Acharya, B. Mondal, R. K. Upadhyay, P. Das, Exploring noise-induced dynamics and optimal control strategy of iSIR cholera transmission model, *Nonlinear Dyn.*, **112** (2024), 3951–3975. https://doi.org/10.1007/s11071-023-09221-z
9. X. Cheng, Y. Wang, G. Huang, Global dynamics of a multiscale immuno-cholera transmission model with bacterial hyperinfectivity on complex networks, *Math. Method. Appl. Sci.*, **48** (2025), 5920–5945. https://doi.org/10.1002/mma.10646
10. S. Issam, K. Bouchaib, A. Labzai, G. Hicham, B. Mohamed, Mathematical modeling and optimal control strategy for a discrete-time cholera model, *Commun. Math. Biol. Neurosci*, **2023** (2023), 135. https://doi.org/10.28919/cmbn/8285
11. J. Wang, Mathematical models for cholera dynamics–A review, *Microorganisms*, **10** (2022), 2358. https://doi.org/10.3390/microorganisms10122358
12. J. Luo, J. Wang, H. Wang, Seasonal forcing and exponential threshold incidence in cholera dynamics, *Discrete Cont. Dyn. B*, **22** (2017), 2261–2290. https://doi.org/10.3934/dcdsb.2017095
13. Y. He, B. Bi, Threshold dynamics and density function of a stochastic cholera transmission model, *AIMS Math.*, **9** (2024), 21918–21939. https://doi.org/10.3934/math.20241065
14. Z. Hu, J. Li, L. Hu, L. Nie, Global dynamics of a nonlocal delayed reaction-diffusion Cholera model with phage-bacteria interaction, *Nonlinear Dyn.*, **113** (2025), 17257–17288. https://doi.org/10.1007/s11071-025-10938-2
15. W. Wu, Q. Zhang, H. Wang, S. Liu, Cholera dynamics driven by human behavior change via a degenerate reaction-diffusion model, *Z. Angew. Math. Phys.*, **76** (2025), 114. https://doi.org/10.1007/s00033-025-02495-w
16. Z. Bai, L. Han, A partially degenerate reaction-diffusion cholera model with temporal and spatial heterogeneity, *Appl. Anal.*, **102** (2023), 3167–3184. https://doi.org/10.1080/00036811.2022.2057302
17. X. Wang, R. Wu, X. Zhao, A reaction-advection-diffusion model of cholera epidemics with seasonality and human behavior change, *J. Math. Biol.*, **84** (2022), 34. https://doi.org/10.1007/s00285-022-01733-3
18. C. Song, R. Xu, Wave propagation of a reaction-diffusion cholera model with hyperinfectious vibrios and spatio-temporal delay, *J. Math. Phys.*, **65** (2024), 022706. https://doi.org/10.1063/5.0156896
19. X. Wang, D. Posny, J. Wang, A reaction-convection-diffusion model for cholera spatial dynamics, *Discrete Cont. Dyn. B*, **21** (2016), 2785–2809. https://doi.org/10.3934/dcdsb.2016073
20. S. Wang, L. Nie, Global stability and asymptotic profiles of a partially degenerate reaction diffusion Cholera model with asymptomatic individuals, *Adv. Nonlinear Anal.*, **13** (2024), 20240059. https://doi.org/10.1515/anona-2024-0059
21. J. Wang, W. Wu, T. Kuniya, Analysis of a degenerated reaction-diffusion cholera model with spatial heterogeneity and stabilized total humans, *Math. Comput. Simul.*, **198** (2022), 151–171. https://doi.org/10.1016/j.matcom.2022.02.026
22. J. Yang, P. Jia, J. Wang, Z. Jin, Rich dynamics of a bidirectionally linked immuno-epidemiological model for cholera, *J. Math. Biol.*, **87** (2023), 71. https://doi.org/10.1007/s00285-023-02009-0

23. B. hou, D. Jiang, B. Han, T. Hayat, Ergodic stationary distribution and practical application of a hybrid stochastic cholera transmission model with waning vaccine-induced immunity under nonlinear regime switching, *Math. Method. Appl. Sci.*, **45** (2022), 423–455. https://doi.org/10.1002/mma.7785
24. M. Song, W. Zuo, D. Jiang, T. Hayat, Stationary distribution and ergodicity of a stochastic cholera model with multiple pathways of transmission, *J. Franklin I.*, **357** (2020), 10773–10798. https://doi.org/10.1016/j.jfranklin.2020.04.061
25. X. Zhang, H. Peng, Stationary distribution of a stochastic cholera epidemic model with vaccination under regime switching, *Appl. Math. Lett.*, **102** (2020), 106095. https://doi.org/10.1016/j.aml.2019.106095
26. X. Zhang, The stationary distribution of a stochastic heroin epidemic model with distributed delay, *Int. J. Biomath.*, 2024, 2450077. https://doi.org/10.1142/S1793524524500773
27. W. Zuo, B. Liao, Ju. Ge, N. Zhao, D. Jiang , Ergodicity of a stationary distribution for a stochastic cholera model with a general functional response and higher-order perturbation, *Adv. Cont. Discr. Mod.*, **2024** (2024), 57. https://doi.org/10.1186/s13662-024-03822-7
28. Q. Liu, D. Jiang, Stationary distribution of a stochastic cholera model with imperfect vaccination, *Physica A*, **550** (2020), 124031. https://doi.org/10.1016/j.physa.2019.124031
29. X. Mao, *Stochastic differential equations and their applications*, Elsevier, 2007.
30. K. Liu, Stationary distributions of second order stochastic evolution equations with memory in Hilbert spaces, *Stoch. Proc. Appl.*, **130** (2020), 366–393. https://doi.org/10.1016/j.spa.2019.03.015
31. K. Chang, Q. Zhang, H. Yuan, Stationary distribution and control strategy of a stochastic dengue model with spatial diffusion, *J. Appl. Anal. Comput.*, **12** (2022), 153–178. https://doi.org/10.11948/20210094
32. Z. Zhang, G. Liang, K. Chang, Stationary distribution of a reaction-diffusionhepatitis B virus infection model driven by the Ornstein-Uhlenbeck process, *Plos one*, **18** (2023), e0292073. https://doi.org/10.1371/journal.pone.0292073
33. D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302