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Research article

DDT Theorem over ideal in quadratic field

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Abstract: Let K be a quadratic field and $\mathfrak{a}$ be a fixed integral ideal of O_K. In this paper, we investigate the distribution of ideals that divide $\mathfrak{a}$ using the Selberg-Delange method. This is a natural variation of a result studied by Deshouillers, Dress, and Tenenbaum (often referred to as the DDT Theorem), and we find that this distribution converges to the arcsine distribution.

Keywords: Selberg-Delange method; the distribution; quadratic field; the arcsine distribution
Mathematics Subject Classification: 11M06, 111N99, 11R04

1. Introduction

For each positive integer n, let $\tau(n)$ denote the number of divisors of n. We define the random variable D_n to take the value $(\log d)/\log n$, where d runs through the set of divisors of n, with each divisor having a uniform probability of $1/\tau(n)$. The distribution function F_n of D_n is given by the following:

$$F_n(t) = P(D_n \le t) = \frac{1}{\tau(n)} \sum_{d|n, d \le n^t} 1 \quad (0 \le t \le 1).$$

It is evident that the sequence $\{F_n\}_{n=1}^{\infty}$ does not converge pointwise on $[0, 1]$, since

$$F_p(t) = \begin{cases} 1/2, & 0 \le t < 1, \\ 1, & t = 1, \end{cases} \qquad F_{p^2}(t) = \begin{cases} 1/3, & 0 \le t < 1/2, \\ 2/3, & 1/2 \le t < 1, \\ 1, & t = 1. \end{cases}$$

However Deshouillers, Dress, and Tenenbaum [11] proved that its Cesàro means is uniformly convergent on $[0, 1]$. Remarkably, this limit corresponds to the distribution function of a probability law well-known to specialists the arcsine law with a density of $1/(\pi\sqrt{u(1-u)})$. More precisely,

$$\frac{1}{x}\sum_{n\le x} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right) \tag{1.1}$$

holds uniformly for $x \ge 2$ and $0 \le t \le 1$, and the error term in (1.1) is optimal. The main tool is the Selberg-Delange method, which was developed by Selberg [10] and Delange [2, 3]. This method has found application in various arithmetic problems, as demonstrated in works such as [4, 5, 7, 8].

Recently, Cui and Wu [1] considered the generalization of (1.1) for short intervals. More specifically, they proved the following theorem.

Let $\varepsilon > 0$ be an arbitrarily small positive constant. Then,

$$\frac{1}{y}\sum_{x<n\leqslant x+y} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O_\varepsilon\left(\frac{1}{\sqrt{\log x}}\right)$$

holds uniformly for $0 \leqslant t \leqslant 1$, $x \geqslant 2$, and $x^{\frac{62}{77}+\varepsilon} \leqslant y \leqslant x$, where the implied constant only depends on ε.

In this paper, we consider a similar problem within the context of quadratic fields. Let O_K denote the ring of integers of the quadratic field K, $\mathfrak{a}$ represent an integral ideal of O_K, and $N(\mathfrak{a})$ represent the norm of $\mathfrak{a}$. We introduce the following arithmetic function:

$$\tau_K(\mathfrak{a}) := \sum_{\mathfrak{b}\mid\mathfrak{a}} 1, \tag{1.2}$$

which is obviously multiplicative.

Let $\Omega_\mathfrak{a} = \{\mathfrak{b} \in O_K : \mathfrak{b} \mid \mathfrak{a}\}$, $\mathfrak{S}_\mathfrak{a}$ be the set of all subsets of $\Omega_\mathfrak{a}$, and let $\mu_\mathfrak{a}$ be the following uniform probability:

$$\mu_\mathfrak{a} = \frac{1}{\tau_K(\mathfrak{a})}.$$

It can be readily verified that $(\Omega_\mathfrak{a}, \mathfrak{S}_\mathfrak{a}, \mu_\mathfrak{a})$ forms a probability space. Now, let's consider the following random variable $\mathfrak{D}_\mathfrak{a}$:

$$\begin{aligned} \mathfrak{D}_\mathfrak{a} : \Omega_\mathfrak{a} &\to \mathbb{R}, \\ \mathfrak{b} &\mapsto \frac{\log N(\mathfrak{b})}{\log N(\mathfrak{a})}. \end{aligned}$$

The distribution function of $\mathfrak{D}_\mathfrak{a}$ is given by the following:

$$F_{K,\mathfrak{a}}(t) = P(\mathfrak{D}_\mathfrak{a} \le t) = \frac{1}{\tau_K(\mathfrak{a})}\sum_{\substack{\mathfrak{b}\mid\mathfrak{a}\\ \frac{\log N(\mathfrak{b})}{\log N(\mathfrak{a})}\le t}} 1 = \frac{1}{\tau_K(\mathfrak{a})}\sum_{\substack{\mathfrak{b}\mid\mathfrak{a}\\ N(\mathfrak{b})\le N(\mathfrak{a})^t}} 1.$$

It is evident that the sequence $\{F_{K,\mathfrak{a}}\}$ does not converge pointwise on $[0, 1]$. However, we shall see that

$$S(x, y; t) = \frac{1}{L(1,\chi')\,y}\sum_{x<N(\mathfrak{a})\le x+y} P(\mathfrak{D}_\mathfrak{a} \le t)$$

is uniformly convergent on $[0, 1]$, where $L(1,\chi')$ is a Dirichlet L-function with respect to the real primitive character modulo $|d(K)|$, and obtains the following result.

Theorem 1.1. *Let $\tau_K(\mathfrak{a})$ be defined in (1.2), and let $\varepsilon > 0$ be an arbitrarily small positive constant. We have*

$$\frac{1}{L(1,\chi')\,y}\sum_{x<N(\mathfrak{a})\le x+y} F_{K,\mathfrak{a}}(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\Big((\log x)^{\frac{\sqrt{2}}{2}-1}\Big),$$

uniformly for $0 \le t \le 1$, $x \ge 2$, and $x^{19/24+\varepsilon} \le y \le x$, where the implied constant only depends on ε.

Based on the results from the short interval, we can derive the following conclusion.

Theorem 1.2. *For $\mathfrak{a} \in O_K$, we have the following:*

$$\frac{1}{L(1,\chi')\,x}\sum_{N(\mathfrak{a})\le x} F_{K,\mathfrak{a}}(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\Big((\log x)^{\frac{\sqrt{2}}{2}-1}\Big),$$

which uniformly holds $x \ge 2$ and $0 \le t \le 1$.

2. Preliminaries

Let us fix some notation:
- $\Gamma(s)$ is the Gamma function;
- $\zeta(s)$ is the Riemann ζ-function;
- $\log_k$ stands for the k-fold logarithm;
- $L(s,\chi)$ is the Dirichlet L-function of χ;
- ε is an arbitrarily small positive constant;
- $r \in \mathbb{N}, \alpha > 0, \delta \geqslant 0, A \geqslant 0, M > 0$ (constants);
- $\boldsymbol{z} := (z_1, \dots, z_r) \in \mathbb{C}^r$ and $\boldsymbol{w} := (w_1, \dots, w_r) \in \mathbb{C}^r$;
- $\boldsymbol{\kappa} := (\kappa_1, \dots, \kappa_r) \in (\mathbb{R}^{+*})^r$ with $1 \leqslant \kappa_1 < \cdots < \kappa_r \leqslant 2\kappa_1$;
- $\boldsymbol{\chi} := (\chi_1, \dots, \chi_r)$ with χ_i non principal Dirichlet characters;
- $\boldsymbol{B} := (B_1, \dots, B_r) \in (\mathbb{R}^{+*})^r$ and $\boldsymbol{C} := (C_1, \dots, C_r) \in (\mathbb{R}^{+*})^r$;
- The notation $|\boldsymbol{z}| \leqslant \boldsymbol{B}$ means that $|z_i| \leqslant B_i$ for $1 \leqslant i \leqslant r$.

Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and its corresponding Dirichlet series is given by the following:

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}.$$

A Dirichlet series $\mathcal{F}(s)$ is said to be of the type $\mathcal{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{\chi}, \boldsymbol{B}, \boldsymbol{C}, \alpha, \delta, A, M)$ if the following conditions are verified:

(a) For any $\varepsilon > 0$ we have the following:

$$|f(n)| \ll_{\varepsilon} M n^{\varepsilon} \quad (n \geqslant 1),$$

where the implied constant depends only on ε;

(b) We have

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \leqslant M\,(\sigma - 1/\kappa_1)^{-\alpha} \quad (\sigma > 1/\kappa_1)\,;$$

(c) The Dirichlet series $\mathcal{F}(s)$ has the following expression:

$$\mathcal{F}(s) = \boldsymbol{\zeta}(\boldsymbol{\kappa} s)^{\boldsymbol{z}} \boldsymbol{L}(\boldsymbol{\kappa} s; \boldsymbol{\chi})^{\boldsymbol{w}} \mathcal{G}(s),$$

where

$$\boldsymbol{\zeta}(\boldsymbol{\kappa} s)^{\boldsymbol{z}} := \prod_{1 \leqslant i \leqslant r} \zeta(\kappa_i s)^{z_i},$$
$$\boldsymbol{L}(\boldsymbol{\kappa} s; \boldsymbol{\chi})^{\boldsymbol{w}} := \prod_{1 \leqslant i \leqslant r} L(\kappa_i s, \chi_i)^{w_i},$$

and the Dirichlet series $\mathcal{G}(s)$ is a holomorphic function in (some open set containing) $\sigma \geqslant (2\kappa_1)^{-1}$. In this region, $\mathcal{G}(s)$ satisfies the bound

$$|\mathcal{G}(s)| \leqslant M(|\tau| + 1)^{\max\{\delta(1-\kappa_1\sigma),0\}} \log^A(|\tau| + 3),$$

uniformly for $|\boldsymbol{z}| \leqslant \boldsymbol{B}$ and $|\boldsymbol{w}| \leqslant \boldsymbol{C}$. In the sequel, we implicitly define the real numbers σ and τ by the relation $s = \sigma + i\tau$ and choose the principal value of the complex logarithm.

Usually, we remember that $N(\sigma, T)$ is the number of zeros of $\zeta(s)$ in the region $\Re s \geqslant \sigma$ and $|\Im m s| \leqslant T$. It is well known that there are two constants ψ and η such that

$$N(\sigma, T) \ll T^{\psi(1-\sigma)}(\log T)^{\eta}$$

for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $T \geqslant 2$. In [6] Huxley showed that $\psi = \frac{12}{5}$ and $\eta = 9$ are admissible.

The following result is Corollary 1.2 of [12], which constitutes one of the key tools.

Lemma 2.1. *Suppose that the Dirchlet $\mathcal{F}(s)$ is of the type $\mathcal{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{\chi}, \boldsymbol{B}, \boldsymbol{C}, \alpha, \delta, A, M)$; for any $\varepsilon > 0$, we have*

$$\sum_{x < n \leqslant x + x^{1-1/\kappa_1} y} f(n) = y'(\log x)^{z-1} \left\{ \lambda_0(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{\chi}) + O\left(\frac{M}{\log x}\right) \right\},$$

uniformly for

$$x \geqslant 2, \quad x^{(1-1/(\psi+\delta))/\kappa_1 + \varepsilon} \leqslant y \leqslant x^{1/\kappa_1}, \quad |\boldsymbol{z}| \leqslant \boldsymbol{B}, \quad |\boldsymbol{w}| \leqslant \boldsymbol{C},$$

where

$$y' := \kappa_1 \left(\left(x + x^{1-1/\kappa_1} y \right)^{1/\kappa_1} - x^{1/\kappa_1} \right),$$
$$\lambda_0(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{\chi}) := \frac{\mathcal{G}(1/\kappa_1)}{\kappa_1^{z_1} \Gamma(z_1)} \prod_{2 \leqslant i \leqslant r} \zeta(\kappa_i/\kappa_1)^{z_i} \prod_{1 \leqslant i \leqslant r} L(\kappa_i/\kappa_1, \chi_i)^{w_i},$$

and the implied constant in the O-term depends only on $A, \boldsymbol{B}, \boldsymbol{C}, \alpha, \delta$ and ε.

Lemma 2.2. *For any $\varepsilon > 0$, we have*

$$\sum_{x < N(\mathfrak{a}) \leq x + y} \frac{1}{\tau_K(\mathfrak{c}\mathfrak{a})} = \frac{hy}{\sqrt{(\log x)}} \left\{ g(\mathfrak{c}) + O_\varepsilon\left(\frac{(3/4)^{\omega(\mathfrak{c})}}{\log x} \right) \right\},$$

uniformly for $N(\mathfrak{c}) \geq 1, x \geq 2$ and $x^{7/12+\varepsilon} \leq y \leq x$, where $\omega(\mathfrak{c})$ denotes the number of distinct prime ideal divisors of $\mathfrak{c}$ and

$$h := \delta \prod_{p | d(K)} p^{1/2} \sqrt{(p-1)} \log(1 - 1/p)^{-1} \prod_{\substack{p \nmid d(K) \\ p \geq 3, \left(\frac{d}{p}\right) = 1}} p^{3/2} \sqrt{(p-1)} \log^2(1 - 1/p) \prod_{\substack{p \nmid d(K) \\ p \geq 3, \left(\frac{d}{p}\right) = -1}} p^{3/2} \sqrt{(p-1)} \log(1 - 1/p^2)^{-1}.$$

$$\delta := \begin{cases} 2^{3/2}\log^2 2, \quad d \equiv 1(\text{mod} 8), \\ 2^{3/2}\log\left(4/3\right)^{-1}, \quad d \equiv 5(\text{mod} 8), \\ 1, \quad d \equiv 2,3(\text{mod} 8). \end{cases}$$

$$g(\mathfrak{c}) := \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+\nu+1} \right) \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+1} \right)^{-1}.$$

Proof. Use the following formula:

$$\tau_K(\mathfrak{c}\mathfrak{a}) = \prod_{\mathfrak{p}} (v_{\mathfrak{p}}(\mathfrak{c}) + v_{\mathfrak{p}}(\mathfrak{a}) + 1),$$

where $v_{\mathfrak{p}}(\mathfrak{a})$ denotes the $\mathfrak{p}$-adic valuation of $\mathfrak{a}$. We write for $\Re s > 1$,

$$\mathcal{F}_{\mathfrak{c}}(n) := \sum_{n=1} \frac{\sum_{N(\mathfrak{a})=n} \tau_K(\mathfrak{c}\mathfrak{a})^{-1}}{n^s} = \sum_{\mathfrak{a}} \tau_K(\mathfrak{c}\mathfrak{a})^{-1} N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+v_{\mathfrak{p}}(\mathfrak{c})+1}$$
$$= \prod_{\mathfrak{p}\nmid\mathfrak{c}} \sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1} = \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1} \right) \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \right)^{-1}.$$

Case 1. $d \equiv 1(\text{mod} 8)$.

$$\mathcal{F}_{\mathfrak{c}}(n) = \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1} \right) \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \right)^{-1}$$
$$= \prod_{p|d(K)} \sum_{j=0}^{\infty} \frac{p^{-js}}{j+1} \prod_{\substack{p\nmid d(K) \\ \left(\frac{d}{p}\right)=1}} \left(\sum_{j=0}^{\infty} \frac{p^{-js}}{j+1} \right)^2 \prod_{\substack{p\nmid d(K) \\ \left(\frac{d}{p}\right)=-1}} \sum_{j=0}^{\infty} \frac{p^{-2js}}{j+1} \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1} \right) \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \right)^{-1}.$$

Let χ' be the real primitive Dirichlet character modulo $|d(K)|$ and $L(s, \chi')$ be the Dirichlet L-function corresponding to χ'. According to the [9, Theorem 8, p. 194], for $\Re s > 1$, we have the following:

$$L(s, \chi') = \prod_{\substack{p\nmid d(K) \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\substack{p\nmid d(K) \\ p\geq 3, \left(\frac{d}{p}\right)=-1}} \left(1 + \frac{1}{p^s}\right)^{-1}.$$

Thus, we can write the following:

$$\mathcal{F}_{\mathfrak{c}}(n) = \zeta(s)^{\frac{1}{2}} \zeta(2s)^{-\frac{1}{6}} L(s, \chi')^{\frac{1}{2}} L(2s, \chi')^{-\frac{1}{6}} \mathcal{G}_{\mathfrak{c}}(s),$$

where $\mathcal{G}_{\mathfrak{c}}(s) = \mathcal{G}_1(s)\mathcal{G}_2(s)\mathcal{G}_3(s)\mathcal{G}_{4,\mathfrak{c}}(s)$ and

$$\mathcal{G}_1(s) = \prod_{p|d(K)} \sum_{j=0}^{\infty} \frac{p^{-js}}{j+1}\left(1-\frac{1}{p^s}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{6}},$$

$$\mathcal{G}_2(s) = \prod_{\substack{p\nmid d(K)\\ \left(\frac{d}{p}\right)=1}} \left(\sum_{j=0}^{\infty} \frac{p^{-js}}{j+1}\right)^2\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{3}},$$

$$\mathcal{G}_3(s) = \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=-1}} \left(\sum_{j=0}^{\infty} \frac{p^{-2js}}{j+1}\right)\left(1-\frac{1}{p^{2s}}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{4s}}\right)^{-\frac{1}{6}},$$

$$\mathcal{G}_{4,\mathfrak{c}}(s) = \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1}.$$

$\mathcal{G}_{\mathfrak{c}}(s)$ is a Dirichlet series that absolutely converges for $\Re s > \frac{1}{3}$. For $\Re s \geq \frac{1}{2}$, we easily see that

$$\left|\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1}\right| = \left|\frac{\log\left(1-N(\mathfrak{p})^{-s}\right)}{N(\mathfrak{p})^{-s}}\right| \geq \frac{\log\left(1+N(\mathfrak{p})^{-\sigma}\right)}{N(\mathfrak{p})^{-\sigma}} \geq \frac{1}{1+N(\mathfrak{p})^{-1/2}}.$$

This implies

$$|\mathcal{G}_{\mathfrak{c}}(s)| \ll \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left\{\frac{1}{1+\nu} + O\left(\frac{1}{\sqrt{N(\mathfrak{p})}}\right)\right\} \leq C\left(\frac{3}{4}\right)^{\omega(\mathfrak{c})},$$

for $\Re s \geq \frac{1}{2}$, where $C > 0$ is an absolute constant.

Consequently, $\mathcal{F}_{\mathfrak{c}}(s)$ is a Dirichlet series of the type $\mathcal{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{B}, \boldsymbol{C}, 1, 0, 0, M)$, where $\boldsymbol{\kappa} = (1, 2)$, $\boldsymbol{z} = \left(\frac{1}{2}, -\frac{1}{6}\right)$, $\boldsymbol{\omega} = \left(\frac{1}{2}, -\frac{1}{6}\right)$, $\boldsymbol{B} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\boldsymbol{C} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\boldsymbol{\chi} = (\chi', \chi')$ and $M = C\left(\frac{3}{4}\right)^{\omega(\mathfrak{c})}$. The proof for this case can be concluded by applying Lemma 2.1.

Case 2. $d \equiv 5(\mathrm{mod}\,8)$.

$$\begin{aligned}\mathcal{F}_{\mathfrak{c}}(n) =& \prod_{\mathfrak{p}} \sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1} \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1} \\ =& 2^{2s}\log\left(1-\frac{1}{2^{2s}}\right)^{-1} \prod_{p|d(K)} \sum_{j=0}^{\infty} \frac{p^{-js}}{j+1} \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=1}} \left(\sum_{j=0}^{\infty} \frac{p^{-js}}{j+1}\right)^2 \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=-1}} \sum_{j=0}^{\infty} \frac{p^{-2js}}{j+1} \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1}.\end{aligned}$$

Let χ' be the real primitive Dirichlet character of modulo $|d(K)|$ and $L(s, \chi')$ be the Dirichlet L-function corresponding to χ'. For $\Re s > 1$, we have the following:

$$L(s, \chi') = \left(1+\frac{1}{2^s}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=1}} \left(1-\frac{1}{p^s}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=-1}} \left(1+\frac{1}{p^s}\right)^{-1}.$$

Thus, we can write the following:

$$\mathcal{F}_{\mathfrak{c}}(n) = \zeta(s)^{\frac{1}{2}}\zeta(2s)^{-\frac{1}{6}}L(s,\chi')^{\frac{1}{2}}L(2s,\chi')^{-\frac{1}{6}}\mathcal{G}_{\mathfrak{c}}(s),$$

where $\mathcal{G}_{\mathfrak{c}}(s) = \mathcal{G}_1(s)\mathcal{G}_2(s)\mathcal{G}_3(s)\mathcal{G}_{4,\mathfrak{c}}(s)$, and

$$\mathcal{G}_1(s) = 2^{2s}\log\left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{2^{2s}}\right)^{\frac{1}{2}}\left(1-\frac{1}{2^{4s}}\right)^{-\frac{1}{6}}\prod_{p|d(K)}\sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\left(1-\frac{1}{p^s}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{6}},$$

$$\mathcal{G}_2(s) = \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=1}}\left(\sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\right)^2\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{3}},$$

$$\mathcal{G}_3(s) = \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=-1}}\left(\sum_{j=0}^{\infty}\frac{p^{-2js}}{j+1}\right)\left(1-\frac{1}{p^{2s}}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{4s}}\right)^{-\frac{1}{6}},$$

$$\mathcal{G}_{4,\mathfrak{c}}(s) = \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}}\left(\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1},$$

$\mathcal{G}_{\mathfrak{c}}(s)$ is a Dirichlet series that absolutely converges for $\Re s > \frac{1}{3}$, and

$$|\mathcal{G}_{\mathfrak{c}}(s)| \ll \prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}}\left\{\frac{1}{1+\nu}+O\left(\frac{1}{\sqrt{N(\mathfrak{p})}}\right)\right\} \leq C\left(\frac{3}{4}\right)^{\omega(\mathfrak{c})},$$

for $\Re s \geq \frac{1}{2}$, where $C > 0$ is an absolute constant.

Consequently, $\mathcal{F}_{\mathfrak{c}}(s)$ is a Dirichlet series of the type $\mathcal{P}(\boldsymbol{\kappa}, z, \boldsymbol{\omega}, \boldsymbol{B}, \boldsymbol{C}, 1, 0, 0, M)$, where $\boldsymbol{\kappa} = (1, 2)$, $\boldsymbol{z} = \left(\frac{1}{2}, -\frac{1}{6}\right)$, $\boldsymbol{\omega} = \left(\frac{1}{2}, -\frac{1}{6}\right)$, $\boldsymbol{B} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\boldsymbol{C} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\boldsymbol{\chi} = (\chi', \chi')$ and $M = C\left(\frac{3}{4}\right)^{\omega(\mathfrak{c})}$. The proof for this case can be concluded by applying Lemma 2.1.

Case 3. $d \equiv 2, 3 (\mathrm{mod} 4)$.

$$\begin{aligned}\mathcal{F}_{\mathfrak{c}}(n) &= \prod_{\mathfrak{p}}\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+1}\prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}}\left(\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1}\\ &= \prod_{p|d(K)}\sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=1}}\left(\sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\right)^2\prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=-1}}\sum_{j=0}^{\infty}\frac{p^{-2js}}{j+1}\prod_{\mathfrak{p}^{\nu}\|\mathfrak{c}}\left(\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty}\frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1}.\end{aligned}$$

Let χ' be the real primitive Dirichlet character of modulo $|d(K)|$ and $L(s, \chi')$ be the Dirichlet L-function corresponding to χ'. For $\Re s > 1$, we have the following:

$$L(s,\chi') = \prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=1}}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{\substack{p\nmid d(K)\\ p\geq 3,\left(\frac{d}{p}\right)=-1}}\left(1+\frac{1}{p^s}\right)^{-1}.$$

Thus, we can write the following:

$$\mathcal{F}_{\mathfrak{c}}(n) = \zeta(s)^{\frac{1}{2}}\zeta(2s)^{-\frac{1}{6}}L(s,\chi')^{\frac{1}{2}}L(2s,\chi')^{-\frac{1}{6}}\mathcal{G}_{\mathfrak{c}}(s),$$

where $\mathcal{G}_{\mathfrak{c}}(s) = \mathcal{G}_1(s)\mathcal{G}_2(s)\mathcal{G}_3(s)\mathcal{G}_{4,\mathfrak{c}}(s)$, and

$$\mathcal{G}_1(s) = \prod_{p|d(K)} \sum_{j=0}^{\infty} \frac{p^{-js}}{j+1}\left(1-\frac{1}{p^s}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{6}},$$

$$\mathcal{G}_2(s) = \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=1}} \left(\sum_{j=0}^{\infty} \frac{p^{-js}}{j+1}\right)^2\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{3}},$$

$$\mathcal{G}_3(s) = \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=-1}} \left(\sum_{j=0}^{\infty} \frac{p^{-2js}}{j+1}\right)\left(1-\frac{1}{p^{2s}}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{4s}}\right)^{-\frac{1}{6}},$$

$$\mathcal{G}_{4,\mathfrak{c}}(s) = \prod_{\mathfrak{p}^{\nu}||\mathfrak{c}} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-js}}{j+1}\right)^{-1},$$

$\mathcal{G}_{\mathfrak{c}}(s)$ is a Dirichlet series that absolutely converges for $\Re s > \frac{1}{3}$, and

$$|\mathcal{G}_{\mathfrak{c}}(s)| \ll \prod_{\mathfrak{p}^{\nu}||\mathfrak{c}} \left\{\frac{1}{1+\nu} + O\left(\frac{1}{\sqrt{N(\mathfrak{p})}}\right)\right\} \leq C\left(\frac{3}{4}\right)^{\omega(\mathfrak{c})},$$

for $\Re s \geq \frac{1}{2}$, where $C > 0$ is an absolute constant.

Consequently, $\mathcal{F}_{\mathfrak{c}}(s)$ is a Dirichlet series of the type $\mathcal{P}(\boldsymbol{\kappa}, \boldsymbol{z}, \boldsymbol{\omega}, \boldsymbol{B}, \boldsymbol{C}, 1, 0, 0, M)$, where $\boldsymbol{\kappa} = (1, 2)$, $\boldsymbol{z} = \left(\frac{1}{2}, -\frac{1}{6}\right)$, $\boldsymbol{\omega} = \left(\frac{1}{2}, -\frac{1}{6}\right)$, $\boldsymbol{B} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\boldsymbol{C} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\boldsymbol{\chi} = (\chi', \chi')$. The proof for this case can be concluded by applying Lemma 2.1. □

Lemma 2.3. *Let* $g(\mathfrak{a})$ *be defined as in Lemma 2.2. Then,*

$$\sum_{N(\mathfrak{a})\leq x} g(\mathfrak{a}) = \frac{L(1,\chi')x}{h\sqrt{(\pi \log x)}}\left\{1 + O_{\varepsilon}\left(\frac{1}{\log x}\right)\right\},$$

uniformly for $x \geq 2$*, where* h *is defined as in Lemma 2.2.*

Proof. Note that g is the multiplicative function defined by the following:

$$g(\mathfrak{p}^{\nu}) := \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+1}\right)^{-1}.$$

We write for $\Re s > 1$,

$$\mathcal{F}(n) := \sum_{n=0} \frac{\sum_{N(\mathfrak{a})=n} g(\mathfrak{a})}{n^s} = \sum_{\mathfrak{a}} g(\mathfrak{a})^{-1}N(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \sum_{\nu=0}^{\infty} N(\mathfrak{p})^{-\nu s}\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+1}\right)^{-1}.$$

Case 1. $d \equiv 1(\mathrm{mod}\,8)$.

$$\begin{aligned}\mathcal{F}(n) &= \prod_{\mathfrak{p}} \sum_{v=0}^{\infty} N(\mathfrak{p})^{-vs} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+1}\right)^{-1} \\ &= \prod_{p|d(K)} \sum_{v=0}^{\infty} p^{-vs} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+1}\right)^{-1} \\ &\quad \prod_{\substack{p\nmid d(K)\\ \left(\frac{d}{p}\right)=1}} \left(\sum_{v=0}^{\infty} p^{-vs} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+1}\right)^{-1}\right)^{2} \\ &\quad \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=-1}} \sum_{v=0}^{\infty} p^{-2vs} \left(\sum_{j=0}^{\infty} \frac{p^{-2j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{p^{-2j}}{j+1}\right)^{-1}.\end{aligned}$$

Let χ' be the real primitive Dirichlet character of modulo $|d(K)|$ and $L(s,\chi')$ be the Dirichlet L-function corresponding to χ'. For $\Re s > 1$, we have

$$L(s,\chi') = \left(1 - \frac{1}{2^s}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=1}} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=-1}} \left(1 + \frac{1}{p^s}\right)^{-1},$$

and

$$\mathcal{F}(n) = \zeta(s)^{\frac{1}{2}} L(s,\chi')^{\frac{1}{2}} \mathcal{G}(s),$$

where $\mathcal{G}(s)$ absolutely converges for $\Re s > 1/2$. The proof for this case can be concluded by applying the Selberg-Delange theorem [11, Theorem 5.2, p. 281] with $N = 0$.

Case 2. $d \equiv 5(\mathrm{mod}\,8)$.

$$\begin{aligned}\mathcal{F}(n) &= \prod_{\mathfrak{p}} \sum_{v=0}^{\infty} N(\mathfrak{p})^{-vs} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+1}\right)^{-1} \\ &= \prod_{p|d(K)} \sum_{v=0}^{\infty} p^{-vs} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+1}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=1}} \left(\sum_{v=0}^{\infty} p^{-vs} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+1}\right)^{-1}\right)^{2} \\ &\quad \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=-1}} \sum_{v=0}^{\infty} p^{-2vs} \left(\sum_{j=0}^{\infty} \frac{p^{-2j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{p^{-2j}}{j+1}\right)^{-1} \times \sum_{v=0}^{\infty} 2^{-2vs} \left(\sum_{j=0}^{\infty} \frac{2^{-2j}}{j+v+1}\right)\left(\sum_{j=0}^{\infty} \frac{2^{-2j}}{j+1}\right)^{-1}.\end{aligned}$$

Let χ' be the real primitive Dirichlet character modulo $|d(K)|$ and $L(s,\chi')$ be the Dirichlet L-function corresponding to χ'. For $\Re s > 1$, we have

$$L(s,\chi') = \left(1 + \frac{1}{2^s}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=1}} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\substack{p\nmid d(K)\\ p\geq 3, \left(\frac{d}{p}\right)=-1}} \left(1 + \frac{1}{p^s}\right)^{-1},$$

and

$$\mathcal{F}(n) = \zeta(s)^{\frac{1}{2}} L(s,\chi')^{\frac{1}{2}} \mathcal{G}(s),$$

where $\mathcal{G}(s)$ absolutely converges for $\Re s > 1/2$. The proof for this case can be concluded by applying the Selberg-Delange theorem [11, Theorem 5.2 p.281] with $N = 0$.

Case 3. $d \equiv 2, 3 (\bmod 4)$.

$$\begin{aligned}\mathcal{F}(n) &= \prod_{\mathfrak{p}} \sum_{v=0}^{\infty} N(\mathfrak{p})^{-vs} \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+v+1} \right) \left(\sum_{j=0}^{\infty} \frac{N(\mathfrak{p})^{-j}}{j+1} \right)^{-1} \\ &= \prod_{p \mid d(K)} \sum_{v=0}^{\infty} p^{-vs} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+v+1} \right) \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+1} \right)^{-1} \\ &\quad \prod_{\substack{p \nmid d(K) \\ p \geq 3, \left(\frac{d}{p}\right) = 1}} \left(\sum_{v=0}^{\infty} p^{-vs} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+v+1} \right) \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j+1} \right)^{-1} \right)^2 \\ &\quad \prod_{\substack{p \nmid d(K) \\ p \geq 3, \left(\frac{d}{p}\right) = -1}} \sum_{v=0}^{\infty} p^{-2vs} \left(\sum_{j=0}^{\infty} \frac{p^{-2j}}{j+v+1} \right) \left(\sum_{j=0}^{\infty} \frac{p^{-2j}}{j+1} \right)^{-1}.\end{aligned}$$

Let χ' be the real primitive Dirichlet character modulo $|d(K)|$ and $L(s,\chi')$ be the Dirichlet L-function corresponding to χ'. For $\Re s > 1$, we have

$$L(s,\chi') = \prod_{\substack{p \nmid d(K) \\ p \geq 3, \left(\frac{d}{p}\right) = 1}} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{\substack{p \nmid d(K) \\ p \geq 3, \left(\frac{d}{p}\right) = -1}} \left(1 + \frac{1}{p^s}\right)^{-1},$$

and

$$\mathcal{F}(n) = \zeta(s)^{\frac{1}{2}} L(s,\chi')^{\frac{1}{2}} \mathcal{G}(s),$$

where $\mathcal{G}(s)$ absolutely converges for $\Re s > 1/2$. The proof for this case can be concluded by applying the Selberg-Delange theorem [11, Theorem 5.2, p. 281] with $N = 0$. □

3. Proof of Theorems 1.1 and 1.2

Now, we are ready to prove Theorem 1.1.

$$\begin{aligned}F_{K,\mathfrak{a}}(t) = P(\mathfrak{D}_{\mathfrak{a}} \leq t) &= \frac{1}{\tau_K(\mathfrak{a})} \sum_{\substack{\mathfrak{b} \mid \mathfrak{a} \\ N(\mathfrak{b}) \leq N(\mathfrak{a})^t}} 1 = \frac{1}{\tau_K(\mathfrak{a})} \sum_{\substack{\mathfrak{c} \mid \mathfrak{a} \\ N(\mathfrak{c}) \geq N(\mathfrak{a})^{1-t}}} 1 = P(\mathfrak{D}_{\mathfrak{a}} \geq 1 - t) \\ &= 1 - P(\mathfrak{D}_{\mathfrak{a}} < 1 - t) = 1 - F_{K,\mathfrak{a}}(1-t) + O\left(\tau_K(\mathfrak{a})^{-1/2}\right).\end{aligned}$$

Summing over $x < N(\mathfrak{a}) \leq x + y$, we handle the O-term using Lemma 2.1 in a similar manner to the proof of Lemma 2.2. We find that

$$S(x,y;t) + S(x,y;1-t) = 1 + O\left((\log x)^{\frac{\sqrt{2}}{2}-1}\right) \quad (0 \leq t \leq 1),$$

where

$$S(x,y;t) := \frac{1}{L(1,\chi')y} \sum_{x<N(\mathfrak{a})\le x+y} F_{K,\mathfrak{a}}(t).$$

On the other hand, we have the following identity:

$$\frac{2}{\pi}\arcsin\sqrt{t} + \frac{2}{\pi}\arcsin\sqrt{1-t} = 1 \quad (0 \le t \le 1).$$

Therefore it is sufficient to prove Theorem 1.1 for $0 \le t \le \frac{1}{2}$.

For $0 \le t \le \frac{1}{2}$, we can write

$$S(x,y;t) = \frac{1}{L(1,\chi')y} \sum_{x<N(\mathfrak{a})\le x+y} \frac{1}{\tau_K(\mathfrak{a})} \sum_{\mathfrak{b}|\mathfrak{a}, N(\mathfrak{b})\le N(\mathfrak{a})^t} 1 =: S_1(x,y;t) - S_2(x,y;t), \tag{3.1}$$

where

$$S_1(x,y;t) := \frac{1}{L(1,\chi')y} \sum_{x<N(\mathfrak{a})\le x+y} \frac{1}{\tau_K(\mathfrak{a})} \sum_{\mathfrak{b}|\mathfrak{a}, N(\mathfrak{b})\le (x+y)^t} 1,$$
$$S_2(x,y;t) := \frac{1}{L(1,\chi')y} \sum_{x<N(\mathfrak{a})\le x+y} \frac{1}{\tau_K(\mathfrak{a})} \sum_{\mathfrak{b}|\mathfrak{a}, N(\mathfrak{a})^t< N(\mathfrak{b})\le (x+y)^t} 1.$$

First, we evaluate $S_1(x,y;t)$. By changing the order of summations, we have the following:

$$S_1(x,y;t) = \frac{1}{L(1,\chi')y} \sum_{N(\mathfrak{b})\le (x+y)^t} \sum_{x/N(\mathfrak{b})<N(\mathfrak{c})\le (x+y)/N(\mathfrak{b})} \frac{1}{\tau_K(\mathfrak{bc})}.$$

For $N(\mathfrak{b}) \le (x+y)^t \le (2x)^{1/2}$ and $y \ge x^{19/24+\varepsilon}$, it is easy to verify that

$$(y/N(\mathfrak{b})) \ge (x/N(\mathfrak{b}))^{7/12+\varepsilon}.$$

Thus we can apply Lemma 2.2 with $(x/N(\mathfrak{b}), y/N(\mathfrak{b}))$ in place of (x,y) to write

$$S_1(x,y;t) = \frac{h\sqrt{\pi}}{L(1,\chi')} \sum_{N(\mathfrak{b})\le (x+y)^t} \frac{1}{N(\mathfrak{b})\sqrt{\log(x/N(\mathfrak{b}))}} \left\{ g(\mathfrak{b}) + O_\varepsilon\left(\frac{(3/4)^{\omega(\mathfrak{b})}}{\log x}\right)\right\},$$

uniformly for $0 \le t \le \frac{1}{2}$, $x \ge 2$ and $x \ge y \ge x^{19/24+\varepsilon}$. Bounding $(3/4)^{\omega(d)}$ by 1, in the summation over d, we see that the contribution of the error terms is $\ll 1/\sqrt{\log x}$. The main term is handled by a partial integration. Let us write the following:

$$\mathcal{G}(t) := \sum_{N(\mathfrak{a})\le e^t} g(\mathfrak{a}) = \frac{\sqrt{\pi}e^t}{4h\sqrt{t}} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad (t > 1).$$

We have the following:

$$\begin{aligned}\frac{h\sqrt{\pi}}{L(1,\chi')}\sum_{N(\mathfrak{b})\le x^u}\frac{g(\mathfrak{b})}{N(\mathfrak{b})\sqrt{\log x/N(\mathfrak{b})}} &= \frac{h\sqrt{\pi}}{L(1,\chi')}\int_{0-}^{u\log x}\frac{e^{-t}}{\sqrt{\log x-t}}\mathrm{d}\mathcal{G}(t)\\ &= \frac{h\sqrt{\pi}}{L(1,\chi')}\int_{0}^{u\log x}\mathcal{G}(t)\frac{e^{-t}}{\sqrt{\log x-t}}\left\{1-\frac{1}{2(\log x-t)}\right\}\mathrm{d}t+O\left(\frac{1}{\sqrt{\log x}}\right)\\ &= \frac{1}{\pi}\int_{0}^{u\log x}\frac{1+O(1/(t+1))}{\sqrt{t(\log x-t)}}\mathrm{d}t+O\left(\frac{1}{\sqrt{\log x}}\right)\\ &= \frac{1}{\pi}\int_0^u\frac{dv}{\sqrt{v(1-v)}}+O\left(\frac{1}{\sqrt{\log x}}\right)=\frac{2}{\pi}\arcsin\sqrt{u}+O\left(\frac{1}{\sqrt{\log x}}\right),\end{aligned}$$

which implies that

$$\frac{h\sqrt{\pi}}{L(1,\chi')}\sum_{N(\mathfrak{b})\le(x+y)^t}\frac{g(\mathfrak{b})}{N(\mathfrak{b})\sqrt{\log(x/N(\mathfrak{b}))}}=\frac{2}{\pi}\arcsin\sqrt{t}+O\left(\frac{1}{\sqrt{\log x}}\right),$$

since

$$\sum_{x^t<N(\mathfrak{b})\le(x+y)^t}\frac{g(\mathfrak{b})}{N(\mathfrak{b})\sqrt{\log(x/d)}}\ll\frac{1}{\sqrt{\log x}}\sum_{x^t<N(\mathfrak{b})\le(x+y)^t}\frac{1}{N(\mathfrak{b})}\ll\frac{1}{\sqrt{\log x}}.$$

By combining these estimates, we obtain

$$S_1(x,y;t)=\frac{2}{\pi}\arcsin\sqrt{t}+O\left(\frac{1}{\sqrt{\log x}}\right),\tag{3.2}$$

uniformly for $0\le t\le\frac{1}{2}$, $x\ge 2$ and $x\ge y\ge x^{19/24+\varepsilon}$. Note that,

$$\begin{aligned}S_2(x,y;t)&\le\frac{h}{L(1,\chi')y}\sum_{x^t<N(\mathfrak{b})\le(x+y)^t}\sum_{x/d<N(\mathfrak{c})\le(x+y)/N(\mathfrak{b})}\frac{1}{\tau_K(\mathfrak{c})}\\ &\ll\frac{1}{\sqrt{\log x}}\sum_{x^t<N(\mathfrak{b})\le(x+y)^t}\frac{1}{N(\mathfrak{b})}\ll\frac{1}{\sqrt{\log x}}.\end{aligned}\tag{3.3}$$

By inserting (3.2) and (3.3) into (3.1), we find that

$$S(x,y;t)=\frac{2}{\pi}\arcsin\sqrt{t}+O_\varepsilon\left(\frac{1}{\sqrt{\log x}}\right),$$

uniformly for $0\le t\le\frac{1}{2}$, $x\ge 2$ and $x\ge y\ge x^{19/24+\varepsilon}$.

Finally, we prove that Theorem 1.2 follows from Theorem 1.1 with $y=x$. Since $0\le F_{\mathfrak{a}}(t)\le 1$, we have the following:

$$\sum_{N(\mathfrak{a})\le x}F_{K,\mathfrak{a}}(t)=\sum_{\sqrt{x}<N(\mathfrak{a})\le x}F_{K,\mathfrak{a}}(t)+O(\sqrt{x})=\sum_{0\le k\le(\log x)/(2\log 2)}\sum_{x/2^{k+1}<N(\mathfrak{a})\le x/2^k}F_{K,\mathfrak{a}}(t)+O(\sqrt{x}).$$

Applying Lemma 2.2 with $y = x$ to the inner sum, we deduce the following

$$\sum_{N(\mathfrak{a})\le x} F_{K,\mathfrak{a}}(t) = \sum_{k=0}^{[(\log x)/(2\log 2)]} \left\{ L(1,\chi')\frac{x}{2^{k+1}}\frac{2}{\pi}\arcsin\sqrt{t} + O\left(\frac{x/2^{k+1}}{(\log(x/2^{k+1}))^{1-\frac{\sqrt{2}}{2}}}\right)\right\} + O(\sqrt{x})$$
$$= L(1,\chi')\frac{2x}{\pi}\arcsin\sqrt{t} + O\left(x(\log x)^{\frac{\sqrt{2}}{2}-1}\right).$$

This completes the proof of Theorem 1.2.

Author contributions

Zhishan Yang: Conceptualization, methodology; Zongqi Yu: Writing-original draft, writing-review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no competing interest.

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