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Research article

Riemann solitons on Egorov and Cahen-Wallach symmetric spaces

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Abstract: In this paper, we consider Egorov and Cahen-Wallach symmetric spaces and study the Riemann solitons on these spaces. We prove that Egorov and Cahen-Wallach symmetric spaces admit the Riemann solitons. Also, we classify the Riemann solitons on these spaces and show that the potential vector fields of the Riemann solitons are Killing, Ricci collineation, and Ricci bi-conformal vector fields.

Keywords: conformal vector fields; Walker manifolds; Riemannian metrics
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1. Introduction

On a pseudo-Riemannian manifold (M, g) with the Ricci tensor S, the Ricci flow was introduced by Hamilton [1] as follows:

$$\frac{\partial}{\partial t}g = -2S.$$

The Ricci soliton [2] is a special solution to the Ricci flow and is a generalization of the Einstein metric and is defined by

$$\mathcal{L}_Y g + S + \lambda g = 0,$$

for some constant λ and the vector field Y, where $\mathcal{L}_Y g$ denotes the Lie derivative of the metric g in the direction of Y. A Ricci soliton has applications in physics [3–7]. The Ricci solitons have been investigated in a pseudo-Riemannian setting [8, 9]. Many authors have generalized the Ricci flow and introduced new geometric flows. For instance, on a manifold (M, g) with the Riemann curvature tensor

R, the Riemann flow was introduced by Udrişte [10] as

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)),$$

where $G = \frac{1}{2}g \odot g$, and $\odot$ denotes the Kulkarni-Nomizu product. The Kulkarni-Nomizu product of two $(0, 2)$-tensors ω and θ is defined by

$$\begin{aligned}(\omega \odot \theta)(U_1, U_2, U_3, U_4) &= \omega(U_1, U_4)\theta(U_2, U_3) + \omega(U_2, U_3)\theta(U_1, U_4) \\ &\quad -\omega(U_1, U_3)\theta(U_2, U_4) - \omega(U_2, U_4)\theta(U_1, U_3),\end{aligned}$$

for all vector fields U_1, U_2, U_3 and U_4. A complete pseudo-Riemannian manifold (M^n, g) is called the Riemann soliton [11] and denoted by (M^n, g, μ, Y) if it satisfies

$$2R + \mu g \odot g + g \odot \mathcal{L}_Y g = 0, \tag{1.1}$$

for some constant μ and the vector field Y. If $\mu > 0$, or $\mu < 0$, or $\mu = 0$, then the Riemann soliton is called expanding, or shrinking, or steady. If $Y = \text{grad}h$ (for some smooth function h), then the Riemann soliton is said to be a gradient Riemann soliton, and the equation of the Riemann soliton turns to

$$2R + \mu g \odot g + 2g \odot \nabla^2 h = 0.$$

A lot of studies have been carried out on the Riemann solitons on different kinds of manifolds. For instance, Venkatesha et al. [12, 13] studied the Riemann solitons on contact geometry and almost Kenmotsu manifolds, Biswas et al. [14] investigated the Riemann solitons on a 3-dimensional almost co-Kahler manifold, and De et al. [15] investigated almost Riemann solitons on para-Sasakian manifolds and in a non-cosymplectic normal almost contact metric manifold [16]. In [17] explored some remarks on almost Riemann solitons with gradient or torse-forming vector field. In [18], the authors studied four classes of Riemann solitons on α-cosymplectic manifolds. Also, see [19–22].

On the other hand, Egorov spaces and ϵ-spaces are Lorentzian manifolds. Egorov spaces and ϵ-spaces have constant curvature, and we can write these manifolds as $N^{n+1} \times \mathbb{R}$. If the dimension of an isometry group of a Riemannian manifold M^n is at least $\frac{1}{2}n(n-1) + 1$ (this manifold is called a manifold with a large isometry group), then the manifold is either of constant curvature or a product of an $(n-1)$-dimensional manifold of constant curvature with a circle or a line. In 2003, Patrangenaru [23] proved that if the dimension of an isometry group of a Lorentzian manifold M^n is at least $\frac{1}{2}n(n-1)+2$, then the manifold has constant curvature. In [23], all Lorentzian manifolds with a large isometry group of dimension $n \geq 4$, $n \neq 7$ are classified. Let (M, g) be a pseudo-Riemannian manifold. A neutral metric g is called a Walker metric if there is a null distribution D with respect to g on M. Also, (M, g) is locally conformally flat if for each point $x \in M$, there exists a neighborhood U of x and a smooth function f defined on U such that $(U, e^{2f}g)$ is flat, meaning its curvature of $e^{2f}g$ vanishes in U. Egorov spaces and ϵ-spaces are Walker manifolds and are locally conformally flat while Egorov spaces are not homogeneous and ϵ-spaces are locally symmetric [24–26]. Also, Egorov spaces are geodesically complete. A pseudo-Riemannian manifold is called indecomposable if the holonomy group, acting at each point $p \in M$, stabilizes only nontrivial degenerate subspaces $V \subset T_pM$. Indecomposable Lorentzian symmetric spaces are either irreducible or the Cahen-Wallach symmetric spaces [27, 28]. On Egorov and Cahen-Wallach symmetric spaces, the Ricci solitons [29], algebraic properties of curvature operators [24, 25], and Killing magnetic trajectories [30] have been studied. Also, see [23, 26].

The exploration of geometric solitons, particularly the Riemann soliton, holds significant importance in the fields of differential geometry and physics, as previously noted. In general, verifying the existence of a Riemann soliton on a manifold poses significant challenges due to the involvement of non-linear differential equations, and in some cases, it may be impossible to ascertain.

Motivated by the above-mentioned works, we study the Riemann solitons on Egorov and Cahen-Wallach symmetric spaces. We demonstrate that the Egorov and Cahen-Wallach symmetric spaces support the existence of Riemann solitons and gradient Riemann solitons. Furthermore, we provide a classification of the Riemann solitons within these spaces and establish that the potential vector fields associated with the Riemann solitons are characterized as Killing, Ricci collineation, and Ricci bi-conformal vector fields.

The paper is organized as follows: In the following section, we recall some fundamental concepts on Egorov and Cahen-Wallach symmetric spaces, which will be used in the next sections. The non-vanishing Christoffel symbols of the Levi-Civita connection associated with the metric, Ricci tensor, and the Lie derivative of both the metric tensor and Ricci tensor along an arbitrary vector field are described in Egorov and Cahen-Wallach symmetric spaces. In Section 3, we study the Riemann solitons on Egorov spaces. We categorize all possible vector fields associated with Riemann solitons in Egorov spaces. Subsequently, we derive gradient Riemann solitons within these spaces. Additionally, we examine which of the potential vector fields related to Riemann solitons in Egorov spaces qualify as Killing fields, Ricci collineations, and Ricci bi-conformal vector fields. Similar to Section 3, in Section 4 we classify the Riemann solitons on Cahen-Wallach symmetric spaces.

2. Preliminaries

In this section, we recall some fundamental concepts on Egorov and Cahen-Wallach symmetric spaces.

2.1. *Egorov spaces*

In this subsection, we will discuss the Levi-Civita connection, the curvature tensor, and the Ricci tensor of Egorov spaces, focusing on their components relative to the basis of coordinate vector fields. This analysis equips us with the essential geometric tools necessary for our investigation. Subsequently, we will calculate the Lie derivative of both the metric and the Ricci tensor in relation to an arbitrary vector field Y.

An Egorov space [24, 25] is a Lorentzian manifold $(\mathbb{R}^{n+2}, g_f)$, where $f : \mathbb{R} \to (0, +\infty)$ is a positive function, and with respect to the coordinates $\{u, v, x_1, \ldots, x_n\}$ on $\mathbb{R}^{n+2}$, the metric g_f is respectively defined by

$$g_f(u, v, x_1, \ldots, x_n) = 2dudv + f(u) \sum_{i=1}^{n} (dx_i)^2. \tag{2.1}$$

Let ∇ be the Levi-Civita connection of $(\mathbb{R}^{n+2}, g_f)$, and let R be its curvature tensor, where

$$R(X_1, X_2)X_3 = \nabla_{[X_1,X_2]}X_3 - [\nabla_{X_1}, \nabla_{X_2}]X_3,$$

for all vector fields X_1, X_2, and X_3. The Ricci tensor [29] is defined by

$$S(X_1, X_2) = \text{trace}(X_3 \to R(X_1, X_3)X_2).$$

As proved in [24, 29], on $(\mathbb{R}^{n+2}, g_f)$, with respect to the basis $\{\partial_u = \frac{\partial}{\partial u}, \partial_v = \frac{\partial}{\partial v}, \partial_i = \frac{\partial}{\partial x_i}\}$ for $i = 1, 2, \ldots, n$, the non-zero components of the covariant derivative are

$$\nabla_{\partial_i}\partial_i = -\frac{1}{2}f'\partial_v, \qquad \nabla_{\partial_i}\partial_u = \frac{f'}{2f}\partial_i, \quad i = 1, 2, ..., n.$$

The only non-zero components of the Riemann curvature tensor [25] are determined by

$$R_{uiui} = \frac{1}{4f}(f'^2 - 2ff''), \qquad i = 1, 2, ..., n, \tag{2.2}$$

and the only non-zero component of the Ricci tensor is given by

$$S_{uu} = \frac{n}{4f^2}(f'^2 - 2ff'').$$

Suppose $Y = Y^u\partial_u + Y^v\partial_v + \sum_{i=1}^{n} Y^i\partial_i$ is an arbitrary vector field on $(\mathbb{R}^{n+2}, g_f)$, where $Y^u = Y^u(u, v, x_1, \ldots, x_n)$, $Y^v = Y^v(u, v, x_1, \ldots, x_n)$, and $Y^i = Y^i(u, v, x_1, \ldots, x_n)$, $i = 1, \ldots, n$ are smooth functions. By the direct computation, we obtain

$$\begin{aligned}
&(\mathcal{L}_Y g_f)(\partial_u, \partial_u) = 2\partial_u Y^v,\\
&(\mathcal{L}_Y g_f)(\partial_u, \partial_v) = \partial_u Y^u + \partial_v Y^v,\\
&(\mathcal{L}_Y g_f)(\partial_u, \partial_i) = \partial_i Y^v + f\partial_u Y^i, \quad 1 \le i \le n,\\
&(\mathcal{L}_Y g_f)(\partial_v, \partial_v) = 2\partial_v Y^u,\\
&(\mathcal{L}_Y g_f)(\partial_v, \partial_i) = \partial_i Y^u + f\partial_v Y^i, \quad 1 \le i \le n,\\
&(\mathcal{L}_Y g_f)(\partial_i, \partial_j) = \partial_i Y^j + \partial_j Y^i, \quad 1 \le i \ne j \le n,\\
&(\mathcal{L}_Y g_f)(\partial_i, \partial_i) = f'Y^u + 2f\partial_i Y^i, \quad 1 \le i \le n,
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&(\mathcal{L}_Y S)(\partial_u, \partial_u) = Y^u\left(\frac{n}{4f^2}(f'^2 - 2ff'')\right)' + 2(\partial_u Y^u)\left(\frac{n}{4f^2}(f'^2 - 2ff'')\right),\\
&(\mathcal{L}_Y S)(\partial_u, \partial_v) = \frac{n\partial_v Y^u}{4f^2}(f'^2 - 2ff''),\\
&(\mathcal{L}_Y S)(\partial_u, \partial_i) = \frac{n\partial_i Y^u}{4f^2}(f'^2 - 2ff''), \quad 1 \le i \le n,\\
&(\mathcal{L}_Y S)(\partial_v, \partial_v) = 0,\\
&(\mathcal{L}_Y S)(\partial_v, \partial_i) = 0, \quad 1 \le i \le n,\\
&(\mathcal{L}_Y S)(\partial_i, \partial_j) = 0, \quad 1 \le i \ne j \le n,\\
&(\mathcal{L}_Y S)(\partial_i, \partial_i) = 0, \quad 1 \le i \le n.
\end{aligned} \tag{2.4}$$

2.2. Cahen-Wallach symmetric spaces

This subsection explores the Levi-Civita connection, curvature tensor, and Ricci tensor of Cahen-Wallach symmetric spaces, focusing on their components in relation to coordinate vector fields. We will then compute the Lie derivative of the metric and Ricci tensor with respect to an arbitrary vector field Y.

ϵ-spaces [29] are Lorentzian manifolds $(\mathbb{R}^{n+2}, g_\epsilon)$, and with respect to the coordinates $\{u, v, x_1, \dots, x_n\}$ on $\mathbb{R}^{n+2}$, the metrics g_ϵ are defined by

$$g_\epsilon = \epsilon\left(\sum_{i=1}^{n} x_i^2\right)(du)^2 + dudv + \sum_{i=1}^{n}(dx_i)^2.$$

Cahen-Wallach symmetric spaces [27, 28] are Lorentzian manifolds $(\mathbb{R}^{n+2}, g_{cw})$, where the metric g_{cw} is defined by

$$g_{cw}(u, v, x_1 \dots, x_n) = \left(\sum_{i=1}^{n} k_i x_i^2\right)(du)^2 + dudv + \sum_{i=1}^{n}(dx_i)^2, \tag{2.5}$$

where k_i, $i = 1, \dots, n$ are non-zero constants. If $k_1 = k_2 = \cdots = k_n$, then Cahen-Wallach symmetric spaces are locally conformally flat and conversely. In this case, a Cahen-Wallach symmetric space becomes ϵ-space, and k_i $(i = 1, .., n)$ are non-zero constants. As proved in [29], on $(\mathbb{R}^{n+2}, g_{cw})$, the non-vanishing Christoffel symbols of the Levi-Civita connection are described by

$$\nabla_{\partial_u}\partial_u = -\sum_{i=1}^{n} k_i x_i \partial_i, \qquad \nabla_{\partial_i}\partial_u = k_i x_i \partial_v, \quad i = 1, 2, ..., n.$$

The only non-zero components of the Riemann curvature tensor are defined by

$$R_{uiui} = -k_i, \qquad i = 1, 2, ..., n, \tag{2.6}$$

and the only non-zero component of the Ricci tensor is given by

$$S_{uu} = -\sum_{i=1}^{n} k_i.$$

Let $Y = Y^u\partial_u + Y^v\partial_v + \sum_{i=1}^{n} Y^i\partial_i$ be an arbitrary vector field on $(\mathbb{R}^{n+2}, g_{cw})$, where $Y^u = Y^u(u, v, x_1, \dots, x_n)$, $Y^v = Y^v(u, v, x_1, \dots, x_n)$, and $Y^i = Y^i(u, v, x_1, \dots, x_n)$, $i = 1, \dots, n$ are smooth functions. We obtain

$$\begin{aligned}
(\mathcal{L}_Y g_{cw})(\partial_u, \partial_u) &= 2\sum_{i=1}^{n} k_i x_i Y^i + 2\left(\sum_{i=1}^{n} k_i x_i^2\right)\partial_u Y^u + 2\partial_u Y^v, \\
(\mathcal{L}_Y g_{cw})(\partial_u, \partial_v) &= \left(\sum_{i=1}^{n} k_i x_i^2\right)\partial_v Y^u + \partial_u Y^u + \partial_v Y^v, \\
(\mathcal{L}_Y g_{cw})(\partial_u, \partial_i) &= \left(\sum_{i=1}^{n} k_i x_i^2\right)\partial_i Y^u + \partial_i Y^v + \partial_u Y^i, \quad 1 \le i \le n, \\
(\mathcal{L}_Y g_{cw})(\partial_v, \partial_v) &= 2\partial_v Y^u,
\end{aligned}$$

$$(\mathcal{L}_Y g_{cw})(\partial_v, \partial_i) = \partial_i Y^u + \partial_v Y^i, \quad 1 \le i \le n,$$
$$(\mathcal{L}_Y g_{cw})(\partial_i, \partial_j) = \partial_i Y^j + \partial_j Y^i, \quad 1 \le i \ne j \le n,$$
$$(\mathcal{L}_Y g_{cw})(\partial_i, \partial_i) = 2\partial_i Y^i, \quad 1 \le i \le n,$$

and

$$(\mathcal{L}_Y S)(\partial_u, \partial_u) = -2\left(\sum_{i=1}^{n} k_i\right)\partial_u Y^u,$$
$$(\mathcal{L}_Y S)(\partial_u, \partial_v) = -\left(\sum_{i=1}^{n} k_i\right)\partial_v Y^u,$$
$$(\mathcal{L}_Y S)(\partial_u, \partial_i) = -\left(\sum_{i=1}^{n} k_i\right)\partial_i Y^u, \quad 1 \le i \le n,$$
$$(\mathcal{L}_Y S)(\partial_v, \partial_v) = 0,$$
$$(\mathcal{L}_Y S)(\partial_v, \partial_i) = 0, \quad 1 \le i \le n,$$
$$(\mathcal{L}_Y S)(\partial_i, \partial_j) = 0, \quad 1 \le i \ne j \le n,$$
$$(\mathcal{L}_Y S)(\partial_i, \partial_i) = 0, \quad 1 \le i \le n.$$

3. Riemann solitons on $(\mathbb{R}^{n+2}, g_f)$

In this section, we study the Riemann solitons on $(\mathbb{R}^{n+2}, g_f)$.

Theorem 3.1. *$(\mathbb{R}^{n+2}, g_f, \mu, Y)$ is a Riemann soliton if and only if μ and $Y = Y^u\partial_u + Y^v\partial_v + \sum_{i=1}^{n} Y^i\partial_i$ are admitted by*

$$\begin{cases} Y^u = b_1 u + b_2, \\ Y^v = -(\mu + b_1)v + \int \frac{1}{4f^2}(f'^2 - 2ff'')du + b_3 + \sum_{i=1}^{n}(\frac{1}{2}dx_i^2 + e_i x_i), \\ Y^i = -(dx_i + e_i)\int \frac{1}{f}du + ax_i + \sum_{\substack{j=1 \\ j\ne i}}^{n} c_{ij}x_j + \alpha_i, \quad 1 \le i \le n, \\ (b_1 u + b_2)\frac{f'}{2f} - d\int \frac{1}{f}du + a = -\frac{1}{2}\mu, \end{cases} \tag{3.1}$$

or

$$\begin{cases} Y^u = b_1 u + b_2 + \sum_{i=1}^{n} c_i x_i, \\ Y^v = -(\mu + b_1)v + b_3 + \sum_{i=1}^{n} e_i x_i, \\ Y^i = -\frac{c_i}{k}v - \frac{e_i}{k}u - \frac{1}{2}\mu x_i + \sum_{\substack{j=1 \\ j\ne i}}^{n} c_{ij}x_j + \alpha_i, \quad 1 \le i \le n, \end{cases} \tag{3.2}$$

or

$$\begin{cases} Y^u = \left(b_1 + \sum_{i=1}^{n} a_i x_i\right)u + b_2 + \sum_{i=1}^{n} \frac{c_l}{a_l}a_i x_i, \\ Y^v = -(\mu + b_1 + \sum_{i=1}^{n} a_i x_i)v + b_3 + \sum_{i=1}^{n}(\frac{1}{2}\beta a_l(c_l b_1 - b_2 a_l)x_i^2 + e_i x_i), \\ Y^i = -\frac{a_i}{a_l\beta(a_l u + c_l)}v - \frac{\beta a_l(c_l b_1 - b_2 a_l)x_i + e_i}{a_l u + c_l} - (\frac{1}{2}\mu + b_1)x_i - \frac{1}{2}a_i x_i^2 - \sum_{\substack{j=1 \\ j\ne i}}^{n} a_j x_i x_j \\ \qquad - \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{1}{2}a_i x_j^2 + \sum_{\substack{j=1 \\ j\ne i}}^{n} c_{ij}x_j + \alpha_i, \end{cases} \tag{3.3}$$

where $b_i, b_2, b_3, d, a, c_i, c_{ij}, e_i, \alpha_i, a_i$ are constants for $1 \le i, j \le n$.

Proof. From (1.1), we get

$$\begin{aligned} 2R(U_1,U_2,U_3,U_4) &= -2\mu\left[g(U_1,U_4)g(U_2,U_3)-g(U_1,U_3)g(U_2,U_4)\right] \\ &\quad -\left[g(U_1,U_4)\mathcal{L}_Y g(U_2,U_3)+g(U_2,U_3)\mathcal{L}_Y g(U_1,U_4)\right] \\ &\quad +\left[g(U_1,U_3)\mathcal{L}_Y g(U_2,U_4)+g(U_2,U_4)\mathcal{L}_Y g(U_1,U_3)\right] \end{aligned} \tag{3.4}$$

for any vector fields U_1, U_2, U_3, U_4. By using (3.4), $(\mathbb{R}^{n+2}, g_f, \mu, Y)$ becomes a Riemann soliton if and only if

$$\begin{aligned} 2R_{iuiv} &= 2\mu g_{ii}g_{uv} + g_{ii}(\mathcal{L}_Y g)(\partial_u,\partial_v) + g_{uv}(\mathcal{L}_Y g)(\partial_i,\partial_i), \quad 1\le i\le n,\\ 2R_{iuui} &= -g_{ii}(\mathcal{L}_Y g)(\partial_u,\partial_u), \quad 1\le i\le n,\\ 2R_{iuuv} &= g_{uv}(\mathcal{L}_Y g)(\partial_i,\partial_u), \quad 1\le i\le n,\\ 2R_{ivvu} &= g_{uv}(\mathcal{L}_Y g)(\partial_i,\partial_v), \quad 1\le i\le n,\\ 2R_{ivvi} &= -g_{ii}(\mathcal{L}_Y g)(\partial_v,\partial_v), \quad 1\le i\le n,\\ 2R_{uvuv} &= -2\mu g_{uv}g_{uv} - 2g_{uv}(\mathcal{L}_Y g)(\partial_u,\partial_v),\\ 2R_{iujv} &= g_{uv}(\mathcal{L}_Y g)(\partial_i,\partial_j), \quad 1\le i\ne j\le n. \end{aligned}$$

Applying (2.1) and (2.2) in the above equations, we respectively have

$$\begin{aligned} (\mathcal{L}_Y g)(\partial_u,\partial_u) &= \frac{1}{2f^2}(f'^2-2ff''),\\ (\mathcal{L}_Y g)(\partial_u,\partial_v) &= -\mu,\\ (\mathcal{L}_Y g)(\partial_u,\partial_i) &= 0, \quad 1\le i\le n,\\ (\mathcal{L}_Y g)(\partial_v,\partial_v) &= 0,\\ (\mathcal{L}_Y g)(\partial_v,\partial_i) &= 0, \quad 1\le i\le n,\\ (\mathcal{L}_Y g)(\partial_i,\partial_j) &= 0, \quad 1\le i\ne j\le n,\\ (\mathcal{L}_Y g)(\partial_i,\partial_i) &= -\mu f, \quad 1\le i\le n. \end{aligned} \tag{3.5}$$

Applying (2.3) in the above equations, one respectively gets

$$\partial_u Y^v = \frac{1}{4f^2}(f'^2-2ff''), \tag{3.6}$$

$$\partial_u Y^u + \partial_v Y^v = -\mu, \tag{3.7}$$

$$\partial_i Y^v + f\partial_u Y^i = 0, \quad 1\le i\le n, \tag{3.8}$$

$$\partial_v Y^u = 0, \tag{3.9}$$

$$\partial_i Y^u + f\partial_v Y^i = 0, \quad 1\le i\le n, \tag{3.10}$$

$$\partial_i Y^j + \partial_j Y^i = 0, \quad 1\le i\ne j\le n, \tag{3.11}$$

$$f'Y^u + 2f\partial_i Y^i = -\mu f, \quad 1\le i\le n. \tag{3.12}$$

Now, we solve the above system of partial differential equations. Equation (3.9) implies that

$$Y^u = F(u, x_1, \ldots, x_n) \tag{3.13}$$

for some smooth function F. Inserting (3.13) in (3.7), we conclude that

$$Y^v = -(\mu + \partial_u F)v + G(u, x_1, \ldots, x_n) \tag{3.14}$$

for some smooth function G. Plugging (3.14) in (3.6), it follows that

$$-\partial^2_{uu} F v + \partial_u G = \frac{1}{4f^2}(f'^2 - 2ff''). \tag{3.15}$$

Equation (3.15) is a polynomial with respect to v, then $\partial^2_{uu} F = 0$ and $\partial_u G = \frac{1}{4f^2}(f'^2 - 2ff'')$. Thus, we have

$$F = F_1(x_1, \ldots, x_n)u + F_2(x_1, \ldots, x_n), \quad G = \int \frac{1}{4f^2}(f'^2 - 2ff'')du + G_1(x_1, \ldots, x_n),$$

for some smooth functions F_1, F_2, and G_1. From (3.10), we deduce

$$Y^i = -\frac{1}{f}(\partial_i F_1 u + \partial_i F_2)v + H_i(u, x_1, \ldots, x_n), \quad i = 1, \ldots, n \tag{3.16}$$

for some smooth functions H_i, $i = 1, \ldots, n$. Substituting (3.14) and (3.16) in (3.8), we obtain

$$-\partial_i F_1 v + \partial_i G_1 + \left(-\partial_i F_1 + \frac{f'}{f}(\partial_i F_1 u + \partial_i F_2)\right)v + f\partial_u H_i = 0, \quad i = 1, \ldots, n.$$

Equation (3.15) is a polynomial with respect to v, then

$$-2\partial_i F_1 + \frac{f'}{f}(\partial_i F_1 u + \partial_i F_2) = 0, \quad \partial_i G_1 + f\partial_u H_i = 0, \quad i = 1, \ldots, n. \tag{3.17}$$

Hence,

$$H_i = -\partial_i G_1 \int \frac{1}{f}du + L_i(x_{1,}, \ldots, x_n), \quad i = 1, \ldots, n,$$

for some smooth functions L_i, $i = 1, \ldots, n$. Equation (3.11) yields

$$-2\frac{1}{f}(\partial^2_{ij} F_1 u + \partial^2_{ij} F_2)v - 2\partial^2_{ij} G_1 \int \frac{1}{f}du + \partial_i L_j + \partial_j L_i = 0, \quad 1 \le i \ne j \le n,$$

and consequently,

$$\partial^2_{ij} F_1 = \partial^2_{ij} F_2 = 0, \quad \partial^2_{ij} G_1 = 0, \quad \partial_i L_j + \partial_j L_i = 0, \quad 1 \le i \ne j \le n. \tag{3.18}$$

Equation (3.12) leads to

$$(F_1 u + F_2)\frac{f'}{2f} - \frac{1}{f}(\partial^2_{ii} F_1 u + \partial^2_{ii} F_2)v - \partial^2_{ii} G_1 \int \frac{1}{f}du + \partial_i L_i = -\frac{\mu}{2}, \quad i = 1, \ldots, n.$$

The last equation implies that

$$\partial^2_{ii} F_1 = \partial^2_{ii} F_2 = 0, \quad (F_1 u + F_2)\frac{f'}{2f} - \partial^2_{ii} G_1 \int \frac{1}{f}du + \partial_i L_i = -\frac{\mu}{2}, \quad i = 1, \ldots, n, \tag{3.19}$$

and

$$\partial_{ii}^2 G_1 = \partial_{11}^2 G_1 \quad \partial_i L_i = \partial_1 L_1, \quad i = 2, \ldots, n. \tag{3.20}$$

From (3.18) and (3.19), we find

$$F_1 = b_1 + \sum_{i=1}^{n} a_i x_i, \qquad F_2 = b_2 + \sum_{i=1}^{n} c_i x_i,$$

for some constants $a_1 \ldots, a_n, c_1 \ldots, c_n, b_1$, and b_2. If $a_i = 0$ for all $i = 1, \ldots, n$. Then $F_1 = b_1$, and (3.17) implies that $f' c_i = 0$, $i = 1, \ldots, n$. If $c_i = 0$ for all $i = 1, \ldots, n$, then $F_2 = b_2$ and

$$G_1 = b_3 + \sum_{i=1}^{n} (\frac{1}{2} d x_i^2 + e_i x_i), \quad L_i = a x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} x_j + \alpha_i$$

for some constants $b_3, d, a, \alpha_i, e_i, c_{ij}$, $i, j = 1, \ldots, n$ such that $c_{ij} + c_{ji} = 0$. In this case, we have (3.1).

Now, we assume that $a_i = 0$ for all $i = 1, \ldots, n$ and there exists $l (1 \leq l \leq n)$ such that $c_l \neq 0$. In this case, $f = k$ for some constant k, and thus we have

$$G_1 = b_3 + \sum_{i=1}^{n} e_i x_i, \quad L_i = -\frac{1}{2} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} x_j + \alpha_i$$

such that $c_{ij} + c_{ji} = 0$. Hence, we infer (3.2). Now, suppose that there is $l (1 \leq l \leq n)$ such that $a_l \neq 0$. Then (3.17) implies that $f = \beta (a_l u + c_l)^2$ for some constant β. In this case, $c_i = \frac{c_l}{a_l} a_i$, $i = 1, \ldots, n$. From (3.18) and (3.20), it follows that

$$F_2 = b_2 + \sum_{i=1}^{n} \frac{c_l}{a_l} a_i x_i, \quad G_1 = b_3 + \sum_{i=1}^{n} (\frac{1}{2} d x_i^2 + e_i x_i).$$

Using (3.19), we get

$$\left[\left(b_1 + \sum_{i=1}^{n} a_i x_i \right) u + b_2 + \sum_{i=1}^{n} \frac{c_l}{a_l} a_i x_i \right] a_l + \frac{d}{\beta a_l} = -(\frac{1}{2} \mu + \partial_i L_i)(a_l u + c_l),$$

which is a polynomial with respect to u, then

$$\frac{1}{2} \mu + \partial_i L_i = -\left(b_1 + \sum_{i=1}^{n} a_i x_i \right), \qquad d = \beta a_l (c_l b_1 - b_2 a_l).$$

Since $\partial_j L_i + \partial_i L_j = 0$ for $1 \leq i \neq j \leq n$, we arrive at

$$L_i = -(\frac{1}{2} \mu + b_1) x_i - \frac{1}{2} a_i x_i^2 - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j x_i x_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{2} a_i x_j^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} x_j + \alpha_i,$$

where $c_{ij} + c_{ji} = 0$. Hence, we have (3.3). This completes the proof of theorem.

Now we investigate the gradient Riemann soliton on $(\mathbb{R}^{n+2}, g_f)$.

Corollary 3.1. *A Riemann soliton* $(\mathbb{R}^{n+2}, g_f, \mu, Y)$ *is a gradient Riemann soliton with the potential function h, which satisfies*

$$h = -\frac{1}{2}\mu uv + b_2 v + \int\int \frac{1}{4f^2}(f'^2 - 2ff'')dudu + b_3 u + \sum_{i=1}^{n}(\frac{1}{2}ax_i^2 + \alpha_i x_i) + \gamma \tag{3.21}$$

or

$$h = -\frac{1}{2}\mu uv + b_2 v + b_3 u + \sum_{i=1}^{n}(-\frac{1}{4}\mu x_i^2 + \alpha_i x_i) + \gamma \tag{3.22}$$

for some constant γ.

Proof. From (2.1) and Theorem 3.1, the Riemann soliton $(\mathbb{R}^{n+2}, g_f, \mu, Y)$ is a gradient Riemann soliton with $Y = \nabla h = \partial_v h \partial_u + \partial_u h \partial_v + \sum_{i=1}^{n} \partial_i h \partial_i$ for some smooth function h, if and only if $\partial_v h = Y^u$, $\partial_u h = Y^v$, $\partial_i h = Y^i$, $i = 1, \ldots, n$.

Since h is a smooth function, for the case (3.1), from the equation $\partial_u Y^u = \partial_u \partial_v h = \partial_v \partial_u h = \partial_v Y^v$, we conclude that $b_1 = -\frac{1}{2}\mu$. Similarly, using equation $\partial_i Y^v = \partial_i \partial_u h = \partial_u \partial_i h = \partial_u Y^i$, we deduce $d = e_i = 0$ for $i = 1, \ldots, n$. Also, for $1 \le i \ne j \le n$ by applying $\partial_j Y^i = \partial_j \partial_i h = \partial_i \partial_j h = \partial_i Y^j$, we arrive at $c_{ij} = 0$. Then, we have

$$\begin{cases} \partial_v h = -\frac{1}{2}\mu u + b_2, \\ \partial_u h = -\frac{1}{2}\mu v + \int \frac{1}{4f^2}(f'^2 - 2ff'')du + b_3 \\ \partial_i h = ax_i + \alpha_i, \quad 1 \le i \le n, \\ (-\frac{1}{2}\mu u + b_2)\frac{f'}{2f} + a = -\frac{1}{2}\mu. \end{cases} \tag{3.23}$$

Integrating the first equation in (3.23), we get

$$h = -\frac{1}{2}\mu uv + b_2 v + h_1(u, x_1, \cdots, x_n) \tag{3.24}$$

for some function h_1. By deriving Eq (3.25) with respect to u and using the second equation in (3.23), it is concluded that

$$-\frac{1}{2}\mu v + \partial_u h_1(u, x_1, \cdots, x_n) = -\frac{1}{2}\mu v + \int \frac{1}{4f^2}(f'^2 - 2ff'')du + b_3. \tag{3.25}$$

Then

$$h_1 = \int\int \frac{1}{4f^2}(f'^2 - 2ff'')dudu + b_3 u + h_1(x_1, \cdots, x_n)$$

and

$$h = -\frac{1}{2}\mu uv + b_2 v + \int\int \frac{1}{4f^2}(f'^2 - 2ff'')dudu + b_3 u + h_2$$

for some function h_2. Putting the last equation in the third equation in (3.23) gives $\partial_i h_2 = ax_i + \alpha_i$ for $1 \le i \le n$. Hence, $h_2 = \sum_{i=1}^{n}(\frac{1}{2}ax_i^2 + \alpha_i x_i) + \gamma$ and we infer (3.21). For the case (3.2), from the equation $\partial_u Y^u = \partial_u \partial_v h = \partial_v \partial_u h = \partial_v Y^v$, we find $b_1 = -\frac{1}{2}\mu$. Applying $\partial_i Y^v = \partial_i \partial_u h = \partial_u \partial_i h = \partial_u Y^i$, we deduce

$e_i = 0$ for $i = 1, \ldots, n$. Also, for $1 \le i \neq j \le n$, by applying $\partial_j Y^i = \partial_j \partial_i h = \partial_i \partial_j h = \partial_i Y^j$, we arrive at $c_{ij} = 0$. From $\partial_i Y^u = \partial_i \partial_v h = \partial_v \partial_i h = \partial_v Y^i$, we get $c_i = 0$ for $i = 1, \ldots, n$. Thus, we have

$$\begin{cases} \partial_v h = -\frac{1}{2}\mu u + b_2, \\ \partial_u h = -\frac{1}{2}\mu v + b_3, \\ \partial_i h = -\frac{1}{2}\mu x_i + \alpha_i, \quad 1 \le i \le n. \end{cases} \tag{3.26}$$

After an integration process of (3.26), we have (3.22). For the case (3.3), from the equation $\partial_u Y^u = \partial_u \partial_v h = \partial_v \partial_u h = \partial_v Y^v$, we arrive at $b_1 + \sum_{i=1}^n a_i x_i = -\frac{1}{2}\mu$. Using $\partial_i Y^u = \partial_i \partial_v h = \partial_v \partial_i h = \partial_v Y^i$, we deduce $a_l = 0$, which is a contradiction. Then, in this case, the Riemann soliton is not a gradient Riemann soliton.

Remark 3.1. *A vector field Y on an n-dimensional pseudo-Riemannian manifold (M, g) is called a Killing vector field if $\mathcal{L}_Y g = 0$ [31–33]. Since any Riemann soliton on $(\mathbb{R}^{n+2}, g_f)$ admits (3.5), we conclude any potential vector field of a Riemann soliton on $(\mathbb{R}^{n+2}, g_f)$ is a Killing vector field if $\mu = 0$, and $f = (pu + q)^2$ for some constants p, q.*

Remark 3.2. *A vector field Y on an n-dimensional pseudo-Riemannian manifold (M, g) is said to be a Ricci collineation vector field if $\mathcal{L}_Y S = 0$. From Theorem 3.1 and (2.4), if $Y^u = 0$ or $f = (pu + q)^2$ for some constants p, q, then any potential vector field of Riemann soliton on $(\mathbb{R}^{n+2}, g_f)$ is a Ricci collineation vector field.*

Remark 3.3. *A vector field Y on a pseudo-Riemannian manifold (M, g) is said to be a Ricci bi-conformal vector field [34] if there are two smooth functions α and β such that*

$$\mathcal{L}_Y g = \alpha g + \beta S, \quad \mathcal{L}_Y S = \alpha S + \beta g. \tag{3.27}$$

I recommend the papers [35–37] for the study of Ricci bi-conformal vector fields on different spacetimes. Also see [38–40]. From Theorem 3.1 and (3.27), any potential vector field of the Riemann soliton on $(\mathbb{R}^{n+2}, g_f)$ is a Ricci bi-conformal vector field for $\alpha = -\mu$ and $\beta = 0$ if $f = (pu + q)^2$ for some constants p, q.

4. Riemann solitons on $(\mathbb{R}^{n+2}, g_{cw})$

In this section, we investigate the Riemann solitons on $(\mathbb{R}^{n+2}, g_{cw})$.

Theorem 4.1. *A Cahen-Wallach space is a steady Riemann soliton where its potential vector field $Y = Y^u \partial_u + Y^v \partial_v + \sum_{i=1}^n Y^i \partial_i$ satisfies*

$$\begin{cases} Y^u = c_2, \\ Y^v = -\Big(b'(u) + \sum_{\substack{j=1 \\ j\neq i}}^n d'_{ij}(u) x_j\Big) x_i - \sum_{r=1}^n k_1 x_r \Big(\int b(u)du + \sum_{\substack{j=1 \\ j\neq r}}^n \int d_{ij}(u) du x_j\Big) \\ \qquad -k_1 u + \tilde{B}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \\ Y^i = b(u) + \sum_{\substack{j=1 \\ j\neq i}}^n d_{ij}(u) x_j, \quad i = 1, \ldots, n, \end{cases} \tag{4.1}$$

where

$$b(u) = a_1 e^{\sqrt{k_1}u} + a_2 e^{-\sqrt{k_1}u}, \qquad d_{ij}(u) = a_{ij1} e^{\sqrt{k_1}u} + a_{ij2} e^{-\sqrt{k_1}u}$$

for $k_1 > 0$*,*

$$b(u) = a_1 \sin(\sqrt{-k_1}u) + a_2 \cos(\sqrt{-k_1}u), \qquad d_{ij}(u) = a_{ij1} \sin(\sqrt{-k_1}u) + a_{ij2} \cos(\sqrt{-k_1}u)$$

for $k_1 < 0$*, and* $a_1, a_2, a_{ij1}, a_{ij2}$*, are constants such that* $a_{ij1} + a_{ji1} = 0$ *and* $a_{ij2} + a_{ji2} = 0$*. Also,* $\tilde{B}$ *is a smooth function.*

Proof. Using (3.4), $(\mathbb{R}^{n+2}, g_{cw}, \mu, Y)$ is a Riemann soliton if and only if

$$\begin{aligned}
2R_{uiui} &= 2\mu g_{uu}g_{ii} + g_{uu}(\mathcal{L}_Y g)(\partial_i, \partial_i) + g_{ii}(\mathcal{L}_Y g)(\partial_u, \partial_u), \ \ 1 \le i \le n,\\
2R_{vivi} &= g_{ii}(\mathcal{L}_Y g)(\partial_v, \partial_v), \ \ 1 \le i \le n,\\
2R_{uiuv} &= -g_{uv}(\mathcal{L}_Y g)(\partial_i, \partial_u) + g_{uu}(\mathcal{L}_Y g)(\partial_i, \partial_v), \ \ 1 \le i \le n,\\
2R_{uivi} &= 2\mu g_{uv}g_{ii} + g_{uv}(\mathcal{L}_Y g)(\partial_i, \partial_i) + g_{ii}(\mathcal{L}_Y g)(\partial_u, \partial_v), \ \ 1 \le i \le n,\\
2R_{uviv} &= -g_{uv}(\mathcal{L}_Y g)(\partial_v, \partial_i), \ \ 1 \le i \le n,\\
2R_{uiuj} &= g_{uu}(\mathcal{L}_Y g)(\partial_i, \partial_j), \ \ 1 \le i \ne j \le n,\\
2R_{uvuv} &= -2\mu g_{uu}^2 - 2g_{uv}(\mathcal{L}_Y g)(\partial_u, \partial_v).
\end{aligned}$$

Applying g_{cw} and (2.6) in the above equations, we respectively have

$$\begin{aligned}
&(\mathcal{L}_Y g)(\partial_u, \partial_u) = -2k_i - \mu \sum_{i=1}^{n} k_i x_i^2, \ \ 1 \le i \le n,\\
&(\mathcal{L}_Y g)(\partial_u, \partial_v) = -\mu,\\
&(\mathcal{L}_Y g)(\partial_u, \partial_i) = 0, \ \ 1 \le i \le n,\\
&(\mathcal{L}_Y g)(\partial_v, \partial_v) = 0,\\
&(\mathcal{L}_Y g)(\partial_v, \partial_i) = 0, \ \ 1 \le i \le n,\\
&(\mathcal{L}_Y g)(\partial_i, \partial_j) = 0, \ \ 1 \le i \ne j \le n,\\
&(\mathcal{L}_Y g)(\partial_i, \partial_i) = -\mu, \ \ 1 \le i \le n.
\end{aligned} \tag{4.2}$$

From the first equation in (4.2), one gets $k_i = k_1$ for $i = 2, \dots, n$. Applying (2.7) in the above equations, we respectively have

$$2\sum_{i=1}^{n} k_1 x_i Y^i + 2\left(\sum_{i=1}^{n} k_1 x_i^2\right)\partial_u Y^u + 2\partial_u Y^v = -2k_1 - \mu \sum_{i=1}^{n} k_1 x_i^2, \tag{4.3}$$

$$\left(\sum_{i=1}^{n} k_1 x_i^2\right)\partial_v Y^u + \partial_u Y^u + \partial_v Y^v = -\mu, \tag{4.4}$$

$$\left(\sum_{i=1}^{n} k_1 x_i^2\right)\partial_i Y^u + \partial_i Y^v + \partial_u Y^i = 0, \quad 1 \le i \le n, \tag{4.5}$$

$$2\partial_v Y^u = 0, \tag{4.6}$$

$$\partial_i Y^u + \partial_v Y^i = 0, \quad 1 \leq i \leq n, \tag{4.7}$$
$$\partial_i Y^j + \partial_j Y^i = 0, \quad 1 \leq i \neq j \leq n, \tag{4.8}$$
$$2\partial_i Y^i = -\mu, \quad 1 \leq i \leq n. \tag{4.9}$$

Now, we solve the above system of partial differential equations. Equation (4.6) yields

$$Y^u = A(u, x_1, \dots, x_n), \tag{4.10}$$

for some smooth function A. Inserting (4.10) in (4.4) one gets

$$Y^v = -(\mu + \partial_u A)v + B(u, x_1, \dots, x_n),$$

for some smooth function B. Replacing (4.10) in (4.7), it follows that

$$Y^i = -\partial_i A v + C_i(u, x_1, \dots, x_n), \quad i = 1, \dots, n \tag{4.11}$$

for some smooth functions C_i, $i = 1, \dots, n$. Applying (4.11) in (4.9), we deduce

$$-\partial^2_{ii} A v + \partial_i C_i = -\frac{1}{2}\mu, \quad i = 1, \dots, n. \tag{4.12}$$

Equation (4.12) is a polynomial with respect to v, then $\partial^2_{ii} A = 0$ and $\partial_i C_i = -\frac{1}{2}\mu$ for $i = 1, \dots, n$. Also, substituting (4.11) in (4.8), we obtain

$$-2\partial^2_{ij} A v + \partial_j C_i + \partial_i C_j = 0, \quad 1 \leq i \neq j \leq n.$$

Then, $\partial^2_{ij} A = 0$ and $\partial_j C_i + \partial_i C_j = 0$ for $1 \leq i \neq j \leq n$. From (4.5), we get

$$\left(k_1 \sum_{i=1}^{n} x_i^2\right) \partial_i A - 2\partial^2_{iu} A v + \partial_u C_i + \partial_i B = 0, \quad 1 \leq i \neq j \leq n. \tag{4.13}$$

Hence, $\partial^2_{iu} A = 0$ and $\left(k_1 \sum_{i=1}^{n} x_i^2\right) \partial_i A + \partial_u C_i + \partial_i B = 0$ for $i = 1, \dots, n$. Therefore, we can write

$$A = A_1(u) + \sum_{i=1}^{n} a_i x_i,$$

for some constants $a_1, \dots, a_n$ and function A_1. The Eq (4.3) leads to

$$\begin{aligned} \frac{1}{2} k_1 \left(2 + \mu \sum_{i=1}^{n} x_i^2\right) &= (\sum_{i=1}^{n} k_1 a_i x_i)v - \sum_{i=1}^{n} k_1 C_i x_i - \left(k_1 \sum_{i=1}^{n} x_i^2\right) A_1'(u) \\ &+ A_1''(u)v - \partial_u B. \end{aligned}$$

The last equation is a polynomial with respect to v, then $\sum_{i=1}^{n} k_1 a_i x_i + A_1''(u) = 0$, and

$$\sum_{i=1}^{n} k_1 C_i x_i + \left(k_1 \sum_{i=1}^{n} x_i^2\right) A_1'(u) + \partial_u B = -\frac{1}{2} k_1 \left(2 + \mu \sum_{i=1}^{n} x_i^2\right). \tag{4.14}$$

Since $\sum_{i=1}^{n} k_1 a_i x_i + A_1''(u) = 0$ is a polynomial with respect to x_i, we conclude that $a_i = 0$ for $i = 1, \ldots, n$ and $A_1(u) = c_1 u + c_2$ for some constants c_1 and c_2. Equation (4.13) yields $\partial_u C_i + \partial_i B = 0$. Taking the derivative of this relation with respect to x_i and using $\partial_i C_i = -\frac{1}{2}\mu$, we obtain $\partial_{ii}^2 B = 0$. Differentiating (4.14) with respect to x_i to obtain

$$k_1 \partial_i C_i x_i + k_1 C_i + 2k_1 c_1 x_i + \partial_{iu}^2 B = -k_1 \mu x_i.$$

Since $\partial_i C_i = -\frac{1}{2}\mu$, we deduce $k_1 C_i + 2k_1 c_1 x_i + \partial_{iu}^2 B = -\frac{1}{2}\mu x_i$. Taking the derivative of the last equation with respect to x_i, one gets $c_1 = 0$. Thus, we have

$$\sum_{i=1}^{n} k_1 C_i x_i + \partial_u B = -\frac{1}{2} k_1 \left(2 + \mu \sum_{i=1}^{n} x_i^2 \right). \tag{4.15}$$

Equations $\partial_j C_i + \partial_i C_j = 0$ and $\partial_i C_i = -\frac{1}{2}\mu$ for $1 \leq i \neq j \leq n$ yield $\partial_{jj}^2 C_i = \partial_{jk}^2 C_i = 0$ for $i \neq j, k$ and $j \neq k$. Therefore, we have

$$C_i = b(u) + \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij}(u) x_j, \quad i = 1, \ldots, n,$$

such that $d_{ij} + d_{ji} = 0$ for some smooth functions b and d_{ij}. From the relation $\partial_u C_i + \partial_i B = 0$, we arrive at

$$B = -\left(b'(u) + \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij}'(u) x_j \right) x_i + \bar{B}(u, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

for some smooth functions $\bar{B}$. Equation (4.15) implies that

$$\sum_{r=1}^{n} k_1 x_r \left(b(u) + \sum_{\substack{j=1 \\ j \neq r}}^{n} d_{ij}(u) x_j \right) - \left(b''(u) + \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij}''(u) x_j \right) x_i + \partial_u \bar{B} = -\frac{1}{2} k_1 \left(2 + \mu \sum_{i=1}^{n} x_i^2 \right).$$

This is a polynomial with respect to x_i, then $\mu = 0$,

$$\sum_{r=1}^{n} k_1 x_r \left(b(u) + \sum_{\substack{j=1 \\ j \neq r}}^{n} d_{ij}(u) x_j \right) + \partial_u \bar{B} = -k_1,$$

and

$$k_1 \left(b(u) + \sum_{\substack{j=1 \\ j \neq r}}^{n} d_{ij}(u) x_j \right) - \left(b''(u) + \sum_{\substack{j=1 \\ j \neq i}}^{n} d_{ij}''(u) x_j \right) = 0.$$

Also, the last equation is a polynomial with respect to x_j, then $k_1 b(u) - b''(u) = 0$ and $k_1 d_{ij}(u) - d_{ij}''(u) = 0$ for $1 \leq i \neq j \leq n$. If $k_1 > 0$, then

$$b(u) = a_1 e^{\sqrt{k_1} u} + a_2 e^{-\sqrt{k_1} u}, \qquad d_{ij}(u) = a_{ij1} e^{\sqrt{k_1} u} + a_{ij2} e^{-\sqrt{k_1} u}$$

and if $k_1 < 0$, then

$$b(u) = a_1 \sin(\sqrt{-k_1}u) + a_2 \cos(\sqrt{-k_1}u), \qquad d_{ij}(u) = a_{ij1} \sin(\sqrt{-k_1}u) + a_{ij2} \cos(\sqrt{-k_1}u)$$

for some constants $a_1, a_2, a_{ij1}, a_{ij2}$ such that $a_{ij1} + a_{ji1} = 0$ and $a_{ij2} + a_{ji2} = 0$. Therefore, we have (4.1).
This completes the proof of theorem.

Corollary 4.1. *If a Cahen-Wallach space admits a Riemann soliton then it becomes an ϵ-space.*

Now, we investigate gradient Riemann solitons on $(\mathbb{R}^{n+2}, g_{cw})$.

Corollary 4.2. *The Riemann soliton $(\mathbb{R}^{n+2}, g_{cw}, \mu, Y)$ is a gradient Riemann soliton with $Y = \nabla h$ if and only if $h = -\frac{1}{2}k_1u^2 + \tilde{b}u + a$ for some constants k_1, $\tilde{b}$ and a.*

Proof. From (2.5) and (4.1), we can conclude that any potential vector field of a Riemann soliton on $(\mathbb{R}^{n+2}, g_{cw})$ is as $\nabla h = \partial_v h\partial_u + \left(\partial_u h - (\sum_{i=1}^n k_1x_i^2)\partial_v h\right)\partial_v + \partial_i h\partial_i$, if and only if

$$\partial_u h = Y^v + \left(\sum_{i=1}^n k_1 x_i^2\right) Y^u, \qquad \partial_v h = Y^u, \qquad \partial_i h = Y^i, \quad i = 1, \ldots, n.$$

Equation

$$\partial_u Y^i = \partial_u \partial_i h = \partial_i \partial_u h = \partial_i Y^v + \left(\sum_{i=1}^n k_1 x_i^2\right) \partial_i Y^u + 2k_1 x_i Y^u$$

leads to $c_2 = b(u) = d_{ij}(u) = 0$ for $1 \leq i \neq j \leq n$. Also, the equation $\partial_j Y^i = \partial_j \partial_i h = \partial_i \partial_j h = \partial_i Y^j$ for $1 \leq i \neq j \leq n$ yields $\tilde{B} = \tilde{b}$ is a constant. Then $\partial_u h = -k_1 u + \tilde{b}$, $\partial_v h = \partial_i h = 0$, $i = 1, \ldots, n$.

Remark 4.1. *Theorem 4.1 leads to any potential vector field of the Riemann soliton $(\mathbb{R}^{n+2}, g_{cw})$ not being a Killing vector field, a Ricci bi-conformal vector field, or a Ricci collineation vector field, because $(\mathcal{L}_Y g_{cw})(\partial_u, \partial_u) = -2k_1 \neq 0$.*

5. Conclusions

In this paper, we study the Riemann solitons on Egorov and Cahen-Wallach symmetric spaces. We prove that the Egorov spaces admit a steady, shrinking, and expanding Riemann soliton, and the Cahen-Wallach symmetric spaces admit just a steady Riemann soliton. We prove any potential vector field of the Riemann soliton on Egorov spaces $(\mathbb{R}^{n+2}, g_f)$ is a Killing vector field if the Riemann soliton is steady and $f = (pu + q)^2$ for some constants p, q. Also, we conclude that any potential vector field of the Riemann soliton on $(\mathbb{R}^{n+2}, g_f)$ is a Ricci collineation vector field and a Ricci bi-conformal vector field with certain conditions. Also, we prove that any potential vector field of the Riemann soliton on Cahen-Wallach symmetric space is not a Killing vector field, a Ricci collineation vector field, or a Ricci bi-conformal vector field.

Author contributions

Shahroud Azami: Conceptualization, investigation, methodology, writing – original draft; Rawan Bossly: Conceptualization, investigation, methodology, writing – review & editing; Abdul Haseeb: Conceptualization, investigation, methodology, writing – review & editing. All authors have read and approved the final version of the manuscript for publication

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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