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Review

Nonclassical dynamical behavior of solutions of partial differential-difference equations

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Abstract: For partial differential-difference equations, a review of results regarding the relation between the type of the equation and dynamical properties of its solutions is provided. This includes the case of elliptic equations with timelike independent variables: Their solutions acquire dynamical properties (more exactly, behave as solutions of parabolic equations). The following approach to classify differential-difference equations into types, based on the property of differential-difference operators to be Fourier multipliers is applied in the following manner: An operator is treated to be elliptic if the real part of its symbol is positive, while the parabolic and hyperbolic types are defined correspondingly. It is shown that the proposed approach (being a natural extension of the classical notion of the ellipticity) is reasonable. On the other hand, fundamental novelties (compared with the classical theory of partial differential equations) occur as well. We provide conditions which guarantee the following results. For the half-space (half-plane) Dirichlet problem for elliptic equations, integral representations (of the Poisson type) of solutions are constructed, which are infinitely differentiable outside the boundary hyperplane (plane), and the asymptotic closeness of solutions (as the timelike independent variable unboundedly increases) takes place. For the Cauchy problem for parabolic equations, the same is valid, but we deal with the classical time instead of the timelike independent variable. For hyperbolic equations, multiparameter families of infinitely smooth global solutions are constructed. The said (sufficient) conditions restrict the sign of the real part of the symbol for the differential-difference operator with respect to spatial or spacelike independent variables. In a number of special cases, they might be weakened such that symbols with sign-changing real parts are admitted. The objective of the study is to observe the current stage of the classification issues for partial differential-difference equations in terms of the aforementioned approach: The ellipticity of a Fourier multiplier is defined by means of the sign (of there real part) of its symbol. Since both differential and translation operators are Fourier multipliers, methods of Fourier analysis are applicable in this study: We apply the Fourier transformation to the original partial differential-difference equation, solve the obtained ordinary differential equation, and apply the inverse Fourier transformation to the obtained solution. The main contribution obtained within this study is an efficient (workable) type concept for the fundamentally new extension of the class of partial differential equations, which is the class of partial differential-difference equations.

Keywords: classification; stationary equations; nonstationary equations; differential-difference equations; dynamical properties of solutions
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1. Introduction

Partial differential equations are classified as either elliptic, hyperbolic, or parabolic ones according to the ensembles of eigenvalues signs for their matrices of principle coefficients. This approach is formal, but, in general, it reflects the fundamental distinction between equations of different types. In particular, the difference between stationary and nonstationary equations is clearly visible: For hyperbolic and parabolic ones, a special independent variable is selected in a natural way, which is obviously interpreted as time, and properties of solutions (including issues related to kinds of problems well posed for those types of the equations) are heterogeneous, while no special independent variables are selected in the elliptic case. The above law is very general, but is not ultimate anyway. One of the most known examples of exclusions is the half-space Dirichlet problem for the Laplace equation (see, e.g., [1] and references therein): The equation remains to be elliptic, but the solutions qualitatively behave as solutions of parabolic equations, where the role of time is played by the only independent (spatial) variable varying along the positive semiaxis orthogonal to the boundary hyperplane. This allows one to call that selected independent variable the timelike one and to say about a dynamical behavior of solutions of a stationary equation.

Passing to functional-differential equations, i.e., to equations containing other operators (apart from differential ones) acting on the desired function, we see that their classification is an open issue now. The reason is that the diversity of such equations is much more rich than in the classical differential case. However, if we restrict the class of considered operators (contained in the investigated equations) to Fourier multipliers, then the following approach based on symbol signs of the operators is rather conventional: If the symbol of an operator L is negative, then it is natural to say that the operator L is elliptic, the operator $\frac{\partial}{\partial t} - L$ is parabolic, and the operator $\frac{\partial^2}{\partial t^2} - L$ is hyperbolic.

The above approach does not ensure a complete similarity with the classification of classical linear partial differential equations because symbols of differential operators are just polynomials and, therefore, this classical case is much more simple. For arbitrary Fourier multipliers, symbols might be quite different and, therefore, the property of the symbol to vanish at the origin and to be positive outside the origin might be not enough to exhaust all kinds of the ellipticity: Researchers are forced to consider nonnegative symbols, positive symbols, and symbols with positive infimums separately and, correspondingly, introduce notions of strictly elliptic and strongly elliptic operators apart from elliptic ones. However, in many cases, qualitative properties of solutions of classical partial differential equations, determined by their types, are inherited by their functional-differential generalizations (see, e.g., [2–5] and references therein). In the present paper, we restrict our consideration to the case of differential-difference equations (they are equations containing differential operators and translation ones) and provide a review of nonclassical dynamical properties of their solutions, established up to now.

2. Parabolic case

In this section, we compare the differential-difference parabolic case with the well-known classical Cauchy problem for the heat equation:

$$\sum_{j=1}^{n} u_{x_j x_j} - u_t = 0, \ x \in \mathbb{R}^n, \ t > 0, \tag{2.1}$$

$$u\Big|_{t=0} = u_0(x), \ x \in \mathbb{R}^n, \tag{2.2}$$

where the initial-value function u_0 is continuous and bounded. Actually, it is known that the well-posedness of problem (2.1)-(2.2) is preserved for much more broad classes of initial-value functions (they are called Tikhonov classes), but this extension is not considered here because the prototype class $C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ (by the way, it coincides with the simplest, or degenerate, Tikhonov class of zero index) is enough to clearly demonstrate specific parabolic properties of the problem and its solutions.

Thus, problem (2.1)-(2.2) is well-posed in the class of bounded functions and its unique (classical bounded) solution is represented by the convolution of the initial-value function and the Poissonian kernel $\mathcal{E}(x,t) = \dfrac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n}$:

$$u(x,t) = \mathcal{E}(x,t) * u_0(x) = \int\limits_{\mathbb{R}^n} \mathcal{E}(x-\xi,t) u_0(\xi) d\xi. \tag{2.3}$$

Besides, under the same assumptions about u_0, the following assertion is valid (see [6]):

Theorem (Repnikov–Eidel'man alternative)**.**

(a) Either there exists a real l such that $\lim\limits_{t\to\infty} u(x,t) = l$ for each x from $\mathbb{R}^n$ or $\lim\limits_{t\to\infty} u(x,t)$ exists for no x from $\mathbb{R}^n$.

(b) The above l exists if and only if

$$\lim_{R\to\infty} \frac{1}{|\{|y|<R\}|} \int\limits_{\{|y|<R\}} u_0(y) dy = l.$$

Now, let us pass to differential-difference equations.

2.1. Unique solvability

The class of differential-difference operators is reasonably decomposed into the following two subclasses: Equations with sums of differential and translation operators and equations with their products (superpositions).

2.1.1. Parabolic equations with sums

In the half-space $\mathbb{R}^n \times (0, \infty)$, consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{j=1}^{n} \sum_{k=1}^{m_j} a_{jk} u(x + b_{jk} h_j, t) \tag{2.4}$$

with the initial-value condition determined by (2.2), where $h_j = (h_{j1}, \ldots, h_{jn})$ are unit vectors mutually orthogonal (for $j = \overline{1, n}$) in $\mathbb{R}^n$, while all a_{jk} and b_{jk} are real constants.

The following assertion is valid (see, e.g., [7, Ch. 1]):

Theorem 2.1. *If $u_0 \in C(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, then there exists a unique classical solution of problems* (2.4) *and* (2.2)*, bounded in the layer $\{0 \leq t \leq T\}$ for each positive T. This solution is infinitely smooth outside the initial hyperplane and it is represented by relation* (2.3) *with the following kernel of the integral representation*:

$$\mathcal{E}(x, t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{-tG_1(\xi)} \cos\Big[x \cdot \xi - tG_2(\xi)\Big] d\xi, \tag{2.5}$$

where

$$G_{\left\{\substack{1\\2}\right\}}(\xi) = |\xi|^2 - \sum_{j=1}^{n} \sum_{k=1}^{m_j} a_{jk} \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\} b_{jk} h_j \cdot \xi.$$

The derivation of the kernel is based on the property of translation operators to be Fourier multipliers: Following [8], we formally apply the Fourier transformation with respect to x to problem (2.4),(2.2), which is a boundary-value problem for a *partial differential-difference* equation. This yields the problem

$$\frac{d\widehat{u}}{dt} = \left(-|\xi|^2 + \sum_{j=1}^{n} \sum_{k=1}^{m_j} a_{jk} \cos b_{jk} h_j \cdot \xi + i \sum_{j=1}^{n} \sum_{k=1}^{m_j} a_{jk} \sin b_{jk} h_j \cdot \xi\right) \widehat{u}, \ t \in (0, +\infty), \tag{2.6}$$

$$\widehat{u}(0; \xi) = \widehat{u_0}(\xi), \tag{2.7}$$

which is the Cauchy problem for a linear *ordinary* first-order *differential* equation with constant coefficients (depending on the n-dimensional parameter ξ). The solution of problem (2.6)-(2.7) is equal to

$$\widehat{u_0}(\xi) e^{\left(\sum\limits_{j=1}^{n} \sum\limits_{k=1}^{m_j} a_{jk} \cos b_{jk} h_j \cdot \xi + i \sum\limits_{j=1}^{n} \sum\limits_{k=1}^{m_j} a_{jk} \sin b_{jk} h_j \cdot \xi - |\xi|^2\right) t}.$$

Then we formally apply the inverse Fourier transformation to that solution. We obtain that

$$\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{ix\cdot\xi+\left(\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\cos b_{jk}h_j\cdot\xi+i\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\sin b_{jk}h_j\cdot\xi-|\xi|^2\right)t}\int\limits_{\mathbb{R}^n} u_0(z)e^{-iz\cdot\xi}dzd\xi$$

$$=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n} e^{i(x-z)\cdot\xi+\left(\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\cos b_{jk}h_j\cdot\xi+i\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\sin b_{jk}h_j\cdot\xi-|\xi|^2\right)t}d\xi dz$$

$$=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n} e^{i\left[(x-z)\cdot\xi+t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\sin b_{jk}h_j\cdot\xi\right]}e^{-t\left(|\xi|^2-\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\cos b_{jk}h_j\cdot\xi\right)}d\xi dz$$

$$=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n}\cos\left[(x-z)\cdot\xi+t\sum_{j=1}^{n}\sum_{k=1}^{m_j} a_{jk}\sin b_{jk}h_j\cdot\xi\right]e^{-t\left(|\xi|^2-\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\cos b_{jk}h_j\cdot\xi\right)}d\xi dz$$

$$+\frac{i}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n}\sin\left[(x-z)\cdot\xi+t\sum_{j=1}^{n}\sum_{k=1}^{m_j} a_{jk}\sin b_{jk}h_j\cdot\xi\right]e^{-t\left(|\xi|^2-\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\cos b_{jk}h_j\cdot\xi\right)}d\xi dz.$$

By virtue of the oddness of the function $\sin\left[(x-z)\cdot\xi+t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}\sin b_{jk}h_j\cdot\xi\right]$ with respect to each variable ξ_j, we obtain function (2.5).

All the actions undertaken within the above procedure (we change the order of the integrating, apply the direct and inverse Fourier transformations, and eliminate integrals of odd functions over symmetric regions) do not constitute a proof because no convergence of those integrals are proved. This all is done to obtain function (2.5). Now, we have to prove its well-definiteness, justify the legibility of its differentiation inside the integral, and to prove that it satisfies Eq (2.4) indeed.

2.1.2. Parabolic equations with products

Consider the equation

$$\frac{\partial u}{\partial t}=Lu:=\Delta u+\sum_{i=1}^{n}\sum_{j=1}^{m_i} a_{ij}\frac{\partial^2 u}{\partial x_i^2}(x_1,\dots,x_{i-1},x_i+b_{ij},x_{i+1},\dots,x_n,t),\tag{2.8}$$

where all a_{ij} and b_{ij} are real constants.

To investigate such equations, the following notion of strongly elliptic differential-difference operators (cf. the definition for bounded domains in [2, § 8]) is introduced:

Definition 2.1. We say that a differential-difference operator is *strongly elliptic* if there exists a positive constant C such that the real part of the operator is bounded from above by $-C|\xi|^2$.

For the last equation, the following unique-solvability theorem is valid (see, e.g., [7, Ch. 2]):

Theorem 2.2. *If* $u_0\in C(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$ *and the operator* $-L$ *from Eq* (2.8) *is strongly elliptic, then there exists a unique solution of problem* (2.8),(2.2) *in the sense of generalized functions. This solution is* *infinitely smooth outside the initial hyperplane and it is represented by relation* (2.3) *with kernel* (2.5) *of the integral representation, where*

$$G_{\{ {1 \atop 2} \}}(\xi) = \sum_{k,j=1}^{n} a_{kj}\xi_k^2 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} b_{kj}\xi_j.$$

2.2. Long-time behavior

To investigate dynamical properties of solutions in the case of problem (2.4),(2.2), we additionally (comparing with the assumptions of the unique solvability theorem) impose the strong ellipticity requirement for the operator $\mathcal{L} - \sum_{j=1}^{n}\sum_{k<m_j^0} a_{jk} I$, where the operator $\mathcal{L}$ acts as follows:

$$\mathcal{L}u = \Delta u + \sum_{j=1}^{n}\sum_{k<m_j^0} a_{jk}u(x + b_{jk}h_j, t),$$

$m_j^0 = m_j + 1$ if $a_{jk} < 0$ for each $k \in \overline{1, m_j}$, and $m_j^0 = \min_{a_{jk}>0} k$ otherwise.

This requirement is imposed under the assumption that the (finite) number sequence $\{a_{jk}\}_{k=1}^{m_j}$, $j = \overline{1, n}$, does not decrease (no generality is lost under this assumption).

Thus, under the specified requirement, the limit relation

$$\lim_{t\to+\infty}\left[e^{-t\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{m_j} a_{jk}} u(x,t) - w\left(\frac{x_1 + q_1 t}{p_1}, \dots, \frac{x_n + q_n t}{p_n}, t\right)\right] = 0 \qquad (2.9)$$

is satisfied for each x from $\mathbb{R}^n$, where $w(x, t)$ is the bounded solution of the Cauchy problem for Eq (2.1) with the initial-value function $u_0(p_1x_1, \dots, p_nx_n)$,

$$p_j = \sqrt{1 + \frac{1}{2}\sum_{k=1}^{m_j} a_{jk}b_{jk}^2}, \qquad \text{and} \qquad q_j = \sum_{k=1}^{m_j} a_{jk}b_{jk},\ j = \overline{1, n}.$$

Remark 2.1. Each p_j is well defined because the operator $\mathcal{L} - \sum_{j=1}^{n}\sum_{k<m_j^0} a_{jk} I$ is strongly elliptic, which implies the positivity of each of the above radicands.

For Eq (2.8), the following result about the long-time behavior is valid under the assumptions of Theorem 2.2:

the limit relation $\lim_{t\to+\infty}[u(x,t) - w(x,t)] = 0$ *holds for each x from* $\mathbb{R}^n$, *where* $w(x, t)$ *is the bounded solution of the Cauchy problem for the differential equation*

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n}\left(1 + \sum_{j=1}^{m_i} a_{ij}\right)\frac{\partial^2 u}{\partial x_i^2} \qquad (2.10)$$

with the initial-value function $u_0(x_1, \dots, x_n)$.

Remark 2.2. The strong ellipticity of the operator $-L$ guarantees the positivity of each coefficient $1 + \sum_{j=1}^{m_i} a_{ij}$. Therefore, Eq (2.10) is parabolic, i.e., the function $w(x, t)$ is well defined.

3. Hyperbolic case

At the moment, the following results about hyperbolic partial differential-difference equations are known.

3.1. Hyperbolic equations with sums

For equations with sums of differential operators and translation operators (or, which is the same, equations with nonlocal potentials), we select the case where all coefficients at nonlocal terms of the equation are nonnegative. For this case, the most general equation considered up to now is

$$u_{tt} - \Delta u + \sum_{k=1}^{m} a_k u(x - h_k, t) = 0, \tag{3.1}$$

where $h_1, \ldots, h_m$ are vectors from the space $\mathbb{R}^n$, $a_1, \ldots, a_m$ are real constants, and n and m are positive integers.

The following assertion is established in [9]:

Theorem 3.1. *If $a_k \geq 0$ for each $k \in \overline{1, m}$ and $a_0 h_0^2 < \frac{\pi^2}{4}$, where $a_0 = \sum\limits_{k=1}^{m} a_k$ and $h_0 = \max\limits_{k \in \overline{1,m}} |h_k|$, then each function*

$$\alpha F(x, t; \gamma) + \beta H(x, t; \gamma), \tag{3.2}$$

where α and β belong to $(-\infty, \infty)$, γ belongs to $\mathbb{R}^n$,

$$\begin{Bmatrix} F \\ H \end{Bmatrix}(x, t; \xi) = e^{\pm t G_1(\xi)} \sin\left[t G_2(\xi) \pm \varphi(\xi) \pm x \cdot \xi\right], \tag{3.3}$$

$$G_{\left\{\substack{1 \\ 2}\right\}}(\xi) = \rho(\xi) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} \varphi(\xi), \tag{3.4}$$

$$\varphi(\xi) = \frac{1}{2} \arctan \frac{b(\xi)}{|\xi|^2 + a(\xi)},$$

$$\rho(\xi) = \left(\left[|\xi|^2 + a(\xi)\right]^2 + b^2(\xi)\right)^{\frac{1}{4}},$$

and

$$\begin{Bmatrix} a \\ b \end{Bmatrix}(x) = \sum_{k=1}^{m} a_k \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} h_k \cdot x,$$

is an infinitely smooth global solution of Eq (3.1).

To prove this theorem, one just substitutes the functions F and H defined above to Eq (3.1), but the way to find those functions is to be clarified. The classical Gel'fand–Shilov scheme from [8] is applied in this case as well, but it cannot be completed: We have to stop prior to the step applying (formally) the inverse Fourier transformation because no convergence of the arising improper integral with respect to the dual variable ξ is guaranteed. Instead, we truncate the described procedure before the integrating with respect to that dual variable and treat it as an n-dimensional parameter (in Theorem 3.1, it is denoted by γ). The result of this procedure (modified this way against the procedure used for the proof of Theorem 2.1) is the function of variables x and t and parameters α, β, and γ, represented by (3.2).

It follows from the assumptions of Theorem 3.1 that the real part of the symbol of the operator $u \mapsto \Delta u - \sum_{k=1}^{m} a_k u(x - h_k, t)$ is less than zero. Thus, we deal with a natural generalization of the classical *differential* case so far: Once the assumptions of Theorem 3.1 are satisfied, it is reasonable to treat the investigated *differential-difference* equation as a hyperbolic one.

However, the requirement of the sign-constancy of the real part of the symbol of the differential-difference operator contained at the investigated equation can be taken off. At the moment, the most general case of the violation of the specified sign-constancy is the equation

$$u_{tt} - \Delta u - au(x + h, t) = 0, \tag{3.5}$$

where a is a real parameter and $h := (h_1, \ldots, h_n)$ is a vector parameter.

The following assertion is proved in [10]:

Theorem 3.2. *If α and β are real constants, while ξ from $\mathbb{R}^n$ is such that $|\xi|^2 - a \cos h \cdot \xi \neq 0$, then function* (3.2)*, where the functions F and H are defined by relation* (3.3)*,*

$$\rho(\xi) = \left[\left(|\xi|^2 - a \cos h \cdot \xi \right)^2 + a^2 \sin^2 h \cdot \xi \right]^{\frac{1}{4}},$$

$$\varphi(\xi) = \frac{1}{2} \begin{cases} \arctan \dfrac{a \sin h \cdot \xi}{|\xi|^2 - a \cos h \cdot \xi} \textit{ for } |\xi|^2 - a \cos h \cdot \xi > 0, \\ \arctan \dfrac{a \sin h \cdot \xi}{a \cos h \cdot \xi - |\xi|^2} \textit{ for } |\xi|^2 - a \cos h \cdot \xi < 0, \end{cases}$$

and

$$G_{\left\{ {1 \atop 2} \right\}}(\xi) = \begin{cases} \rho(\xi) \left\{ {\sin \atop \cos} \right\} \varphi(\xi) \textit{ for } |\xi|^2 - a \cos h \cdot \xi > 0, \\ \rho(\xi) \left\{ {\cos \atop \sin} \right\} \varphi(\xi) \textit{ for } |\xi|^2 - a \cos h \cdot \xi < 0, \end{cases}$$

satisfies Eq (3.5) *in $\mathbb{R}^{n+1}$ and is infinitely differentiable.*

Note that, unlike Theorem 3.1, the assumptions of Theorem 3.2 admit equations with sign-changing real parts of the symbol of the differential-difference operator.

3.2. Hyperbolic equations with products

For hyperbolic equations with superpositions of differential operators and translation operators, the most general (up to now) case is considered in [11]. This is the equation

$$u_{tt} - \Delta u - \sum_{j=1}^{n} a_j u_{x_j x_j}(x - h_j, t) = 0, \tag{3.6}$$

where $h_1, \ldots, h_n$ are vectors from the space $\mathbb{R}^n$, $a_1, \ldots, a_n$ are real constants, and n is a positive integer.

Prior to provide the result obtained for Eq (3.6), we have to introduce the following notation: $h_j =: (h_{j1}, \dots, h_{jn})$ for each $j = \overline{1, n}$;
$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix}(\xi) := \sum_{j=1}^{n} a_j \xi_j^2 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left(\sum_{k=1}^{n} h_{jk}\xi_k \right)$ are real-valued functions of variable ξ from $\mathbb{R}^n$.

The following assertion is valid (see [11]):

Theorem 3.3. *If* $|a_j| < 1$ *for each* $j = \overline{1, n}$*, then both functions* (3.3)*, where* $G_1(\xi)$ *and* $G_2(\xi)$ *are the functions defined by relations* (3.4)*,*

$$\varphi(\xi) = \frac{1}{2} \arctan \frac{\beta(\xi)}{|\xi|^2 + \alpha(\xi)},$$

and

$$\rho(\xi) = \left(\left[|\xi|^2 + \alpha(\xi) \right]^2 + \beta^2(\xi) \right)^{\frac{1}{4}},$$

are infinitely smooth global solutions of Eq (3.6) *for each* ξ *from* $\mathbb{R}^n$.

Note that, under the assumption of Theorem 3.3, the inequality $|\xi|^2 + \alpha(\xi) > 0$ holds outside the origin. Therefore, we are under the sign-constancy restriction for the real part of the symbol again.

4. Elliptic case

Nonclassical dynamical properties of elliptic problems in anisotropic unbounded domains (the prototype example of the half-space Dirichlet problem for the Laplace equation is provided in the introduction above) take place in the differential-difference case as well. In this section, we review results for the half-space Dirichlet problem for elliptic differential-difference equations. As in the classical theory of differential equations, qualitative properties of solutions substantially depend on whether the boundary-value function of the problem belongs to L_∞ or to L_1. Consider these two classes of problems separately.

4.1. Problems with bounded boundary-value function

The most general equation with sums of differential operators and translation operators, investigated up to now, is

$$u_{xx} + u_{yy} - \sum_{j=1}^{m} a_j u(x + h_j, y) = 0, \tag{4.1}$$

where m is a positive integer, $a = (a_1, \dots, a_m) \in \mathbb{R}^m$, and $h = (h_1, \dots, h_m) \in \mathbb{R}^m$. If we consider it in the half-plane $\{x \in (-\infty, +\infty),\ y \in (0, +\infty)\}$ and set the boundary-value condition

$$u\Big|_{y=0} = u_0(x) \tag{4.2}$$

on the line $\{x \in (-\infty, +\infty)\}$, where $u_0 \in C(\mathbb{R}) \cap L_\infty(\mathbb{R})$, then the following result is valid (see [12]):

Theorem 4.1. *Let* $\sum_{j=1}^{m} a_j \geq 0$ *and there exist a positive* a_0 *such that*

$$\xi^2 + \sum_{j=1}^{m} a_j \cos h_j \xi > a_0 \tag{4.3}$$

for each real ξ.

Then the function

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathcal{E}(x-\xi, y) u_0(\xi) d\xi, \tag{4.4}$$

where

$$\mathcal{E}(x,y) = \int_0^{\infty} e^{-yG_2(\xi)} \cos\left[x\xi - yG_1(\xi)\right] d\xi, \tag{4.5}$$

$G_1(\xi)$ *and* $G_2(\xi)$ *are the functions defined by relations* (3.4)*,*

$$\theta(\xi) = \frac{1}{2} \arctan \frac{\sum\limits_{j=1}^{m} a_j \sin h_j \xi}{\xi^2 + \sum\limits_{j=1}^{m} a_j \cos h_j \xi}$$

and

$$\rho(\xi) = \left[\left(\xi^2 + \sum_{j=1}^{m} a_j \cos h_j \xi \right)^2 + \left(\sum_{j=1}^{m} a_j \sin h_j \xi \right)^2 \right]^{\frac{1}{4}},$$

satisfies problem (4.1)-(4.2) *in the classical sense.*

Note that the functions $\rho(\xi)$ and $\theta(\xi)$ are well defined by virtue of condition (4.3).

The most general equation with products of differential operators and translation operators, investigated up to now, is

$$u_{xx} + \sum_{k=1}^{m} a_k u_{xx}(x+h_k, y) + u_{yy} = 0. \tag{4.6}$$

In [13], the following result is proved for problem (4.6),(4.2):

Theorem 4.2. *Suppose that there exists a positive constant C such that the inequality*

$$1 + \sum_{k=1}^{m} a_k \cos h_k \xi \geq C \tag{4.7}$$

holds on the real line. Then function (4.4)*, where*

$$\mathcal{E}(x,y) = \int_0^{\infty} e^{-yG_1(\xi)} \cos\left[x\xi - yG_2(\xi)\right] d\xi, \tag{4.8}$$

$G_1(\xi) = \xi\sqrt{\dfrac{\varphi(\xi)+a(\xi)+1}{2}}$, $G_2(\xi) = \xi\sqrt{\dfrac{\varphi(\xi)-a(\xi)-1}{2}}$, $\varphi(\xi) = \left[a^2(\xi)+b^2(\xi)+2a(\xi)+1\right]^{\frac{1}{2}}$, *and* $\begin{Bmatrix} a \\ b \end{Bmatrix}(\xi) = \sum_{k=1}^{m} a_k \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} h_k\xi$, *satisfies problem* (4.6),(4.2) *in the sense of distributions*, *it satisfies Eq* (4.6) *in the classical sense in the half-plane* $(-\infty,+\infty)\times(0,+\infty)$, *and the following limit relation holds for each real* x:

$$\lim_{y\to+\infty}[u(x,y)-v(x,y)] = 0, \tag{4.9}$$

where $v(x,y)$ *is the bounded solution of the partial differential equation*

$$(a_0+1)u_{xx}+u_{yy} = 0, \tag{4.10}$$

satisfying condition (4.2), *while* $a_0 = \sum_{k=1}^{m} a_k$.

Note that the functions $G_1(\xi)$ and $G_2(\xi)$ are well defined by virtue of condition (4.7) and the ellipticity of Eq (4.10) and, therefore, the unique solvability of problem (4.10),(4.2) is guaranteed by condition (4.7) as well.

4.2. Problems with summable boundary-value function

In this section, the independent variable x is n-dimensional and it is assumed that the boundary-value function u_0 from condition (4.2) belongs to $L_1(\mathbb{R}^n)$. Thus, condition (4.2) is set $L_1(\mathbb{R}^n)$, and the Dirichlet problem in $\mathbb{R}^n\times(0,\infty)$ is considered for the following two equations (see [14, 15]):

$$\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}(x,y)-\sum_{k=1}^{m}a_k u(x+h_k,y)+\frac{\partial^2 u}{\partial y^2}(x,y) = 0, \tag{4.11}$$

where m is a positive integer, $a_1,\dots,a_m$ are nonnegative constants, and $h_k = (h_{k1},\dots,h_{kn})$, $k\in\overline{1,m}$, are vectors from $\mathbb{R}^n$ with real coordinates, and

$$\sum_{j=1}^{n}u_{x_jx_j}(x,y)+u_{yy}(x,y)+\sum_{j=1}^{n}a_j u_{x_jx_j}(x+h_j,y) = 0, \tag{4.12}$$

where $a_1,\dots,a_n$ are real constants, $h_j = (h_{j1},\dots,h_{jn})$, $j=\overline{1,n}$, are arbitrary vectors from $\mathbb{R}^n$.

For Eq (4.11), we assume that

$$\max_{k\in\overline{1,m}}|h_k|\max\left\{\sum_{k=1}^{m}a_k,\ \sqrt{\sum_{k=1}^{m}a_k}\right\} < \frac{\pi}{2}. \tag{4.13}$$

For Eq (4.12), we assume that

$$\max_{j=\overline{1,n}}|a_j| < 1. \tag{4.14}$$

In both cases, the problem has a solution expressed by relation (2.3), where the kernel $\mathcal{E}$ is expressed by relation (2.5), and $G_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(\xi) = \rho(\xi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\varphi(\xi)$.

For problem (4.11),(4.2), i.e., for the case of sums of operators, we have

$$\theta(\xi) = \frac{1}{2}\arctan\frac{\sum\limits_{j=1}^{m} a_j \sin h_j \cdot \xi}{\xi^2 + \sum\limits_{j=1}^{m} a_j \cos h_j \cdot \xi}$$

and

$$\rho(\xi) = \left[\left(|\xi|^2 + \sum_{j=1}^{m} a_j \cos h_j \cdot \xi\right)^2 + \left(\sum_{j=1}^{m} a_j \sin h_j \cdot \xi\right)^2\right]^{\frac{1}{4}}.$$

For problem (4.12),(4.2), i.e., for the case of superpositions of operators, we have

$$\theta(\xi) = \frac{1}{2}\arctan\frac{\sum\limits_{j=1}^{n} a_j\xi_j^2 \sin h_j \cdot \xi}{|\xi|^2 + \sum\limits_{j=1}^{n} a_j\xi_j^2 \cos h_j \cdot \xi}$$

and

$$\rho(\xi) = \left[\left(|\xi|^2 + \sum_{j=1}^{n} a_j\xi_j^2 \cos h_j \cdot \xi\right)^2 + \left(\sum_{j,k=1}^{n} a_{jk}\xi_j^2 \sin h_j \cdot \xi\right)^2\right]^{\frac{1}{4}}.$$

In both cases, the obtained solution is infinitely differentiable in the *open* half-space and satisfies the boundary-value condition at least in the sense of distributions.

In both cases, the following uniform estimate is valid for the constructed solution and each its derivative:

$$|D^l u(x,y)| \leq \frac{C\|u_0\|_1}{y^{n+l}}, \tag{4.15}$$

where l is an arbitrary positive integer and the left-hand side denotes an arbitrary partial derivative of order l of the function $u(x,y)$, while the constant C depends neither on n nor on l.

5. Discussion

For equations containing sums of differential and translation operators with respect to spatial variables, the question arises that whether it is possible to pose problems natural for the parabolic case is solved without the analysis of the nonlocal part of the equation: If the equation obtained after the eliminating of all nonlocal terms (note that this is a *differential* equation) is parabolic, then the Cauchy problem (for the original *differential-difference* equation) is uniquely solvable in the class of bounded functions and the solution is infinitely smooth outside the initial hyperplane.

Once we pass to the investigation of qualitative properties (more exactly, the long-time behavior) of solutions, the situation is changed substantially: The nonlocal part of the equation matters. To establish those properties, we construct a special differential-difference operator, using parameters of the nonlocal part of the investigated equation, and require that operator to be strongly elliptic. Note that relation (2.9) obtained under this requirement is an asymptotical closeness (proximity) theorem quite typical for classical differential equations of the parabolic type (see, e.g., [16–23]): A so-called etalon function such that its properties (behavior) are known (in our case, this is a weighted solution of the Cauchy problem for the heat equation with a prescribed bounded initial-value function), and we prove that the difference between this etalon function and the investigated solution vanishes as $t \to \infty$. The qualitative novelty of the nonlocal case is that the said difference vanishes as the independent variable of the investigated solution and the independent variable of the etalon function tend to infinity along *different* ways: For the investigated solution, this is a line orthogonal to the initial hyperplane; for the etalon function, this is a line forming a nonzero (in general) angle with the former line. This angle is totally determined by the parameters (coefficients and translation vectors) of the nonlocal terms of the investigated equation.

Remark 5.1. The fact that the etalon function is to be weighted is not caused by the nonlocal nature of the problem. The reason is that the nonlocal part of the investigated equation is actually a *nonlocal potential*. In the classical case of parabolic differential equations, the presence of potential terms leads to the arising of weights in the etalon functions as well.

For equations containing superpositions of differential and translation operators with respect to spatial variables, the strong ellipticity condition imposed on the said superposition ensures all parabolic properties: The unique solvability of the Cauchy problem, the infinite smoothness of the Poisson-like representation of the solution outside the initial hyperplane, and the asymptotical closeness of solutions. Again, the etalon function is the solution of the Cauchy problem for an (etalon) parabolic differential equation, where the said etalon equation is constructed as follows: We take the investigated differential-difference equation and assign all translation vectors to be equal to the zero vector.

Remark 5.2. In general, (asymptotical) closeness theorems are stronger than stabilization one: They provide an information about the solution behavior even in the case where no stabilization takes place. However, it is possible to impose additional restrictions on the nonlocal terms of the equation to guarantee the validity of the Repnikov–Eidel'man stabilization criterion (see [7, Corollaries 1.6.1 and 2.5.1]).

If we treat the ellipticity of differential-difference operators in the sense of the sign-constancy of the real part of the symbol, i.e., if we extend the classical ellipticity notion in a natural way, then all results obtained for the case of products (superpositions) of differential operators and translation operators are valid for hyperbolic differential-difference equations: Multiparameter families of global infinitely smooth solutions are constructed explicitly. For the case of sums of differential operators and translation operators, this achievement is substantially extended: The above families are constructed even if the sign-constancy condition is not satisfied, i.e., it suffices to require the ellipticity only for the *differential* operator acting with respect to the spatial variables.

In both cases, the smoothness issue is to be clarified: In the classical theory of *differential* equations, infinitely smooth solutions are typical for elliptic and parabolic problems (e.g., solving the Laplace equation, we restore a harmonic, i.e., infinitely smooth, solution from its summable trace), but resolving operators of hyperbolic problems do not possess this smoothing property. Moreover, this smoothing property is not preserved even for the elliptic case in the nonlocal theory (see [24, 25] and later results in that direction): Even if the input data such as right-hand side or (and) boundary-value function is smooth, the smoothness of the solution itself might be broken; only the smoothness of the solution in subdomains is guaranteed by the smoothness of the input data. Thus, the existence of families of infinitely smooth solutions presented in Section 3 above should be explained. We do that as follows. Functions (3.2) presented above are global solutions, i.e., solutions in the whole space, of homogeneous equations. Such solutions do exist in the classical hyperbolic theory as well. For example, solving a mixed problem in a cylindrical domain by means of the separation variables method, we construct a function series. Each term of this series is an infinitely differentiable global solution of the *equation*, but it does not solve the investigated *problem*; it solves only the equation. The whole series satisfies the problem, but it is not infinite differentiable anymore. However, each its term is infinitely differentiable.

The approach to take into account nonlocal parts of partial differential-difference operators, defining their ellipticity, is shown to be reasonable not only for the general theory. It turns out that it "catches" the following specific (but quite important) effect: If we consider the equation in a heterogeneous region, where one of independent variables varies only on the positive semiaxis, then it plays the role of time in the sense that solutions acquire dynamical properties. Moreover, they are quite close to properties of solutions of *parabolic* equations. Note that all the assumptions about the nonlocal parts of elliptic equations, imposed in Section 4, i.e., conditions (4.3), (4.7), (4.13), and (4.14), are ellipticity conditions in the sense stated in the introduction above: We take the whole differential-difference operator of the equation and bound its sign (or the sign of its real part). The results demonstrate the fundamental similarity with the classical theory. In particular, estimate (4.9) is a theorem on the asymptotical closeness of solutions, typical for the theory of elliptic *differential* equations (see, e.g., [26–28]).

The dependence of dynamical properties of solutions (including the case of steady-state equations) on the type of the equation is preserved for *differential-difference* equations, but the diversity of this phenomenon becomes much more rich than in the classical theory of partial *differential* equations.

The approach to treat the ellipticity of partial differential-difference operators as the negativity of (the real parts of) their symbols and to classify types of partial differential-difference equations correspondingly demonstrates its reasonability from the viewpoint of dynamical properties, but there are important cases where it suffices to assume only the ellipticity of the differential operator with respect to the spatial variables.

Thus, the proposed concept of types of partial *differential-difference* equations is consistent and it inherits the following very important property from the classical theory of partial *differential* equations: Dynamical properties vary from type to type and they substantially correspond to physics interpretations of the investigated equations.

6. Conclusions

The dependence of dynamical properties of solutions (including the case of steady-state equations) on the type of the equation is preserved for *differential-difference* equations, but the diversity of this phenomenon becomes much more rich than in the classical theory of partial *differential* equations.

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6.1. Important directions of further investigations

6.1.1. Multidimensional elliptic case with (essentially) bounded boundary-value data

To the case where x is a vector independent variable, extensions of results of Section 4.1 independent variable would be quite important for the completion of the theory of half-space elliptic problems. Note that, in the classical case of *differential* equations, results of this kind are valid regardless the dimension of the spacelike (tangential) independent variable x. The same takes place in the *nonlocal* parabolic case (covered by Section 2). Thus, there are reasons to expect that the fact that the nonlocal elliptic case is valid only for the scalar independent variable x yet is caused just by technical difficulties. In the classical elliptic case, the kernel of the integral representation is radially symmetric with respect to x. In the nonlocal parabolic case, we have the factorability with respect to the spatial independent variable. No such symmetries (helping the researchers) are available in the nonlocal elliptic case, but the problem nature is to be the same.

6.1.2. Geometrical interpretations of hyperbolicity conditions

Analysing the assumptions of Theorem 3.2, we see that, for hyperbolic differential-difference equations, it is quite important to describe sign-constancy sets for real parts of symbols of differential-difference operators. A global infinitely smooth solution is constructed both in the case where the parameter ξ belongs to the positivity set D_+ of that real part and to its negativity set D_-, but its expression depends on whether the said parameter is located in D_+ or D_-. Thus, the description problem for the above sets of differential-difference operators becomes very timely. Moreover, there are reasons to expect that the solution of the specified problem might help to find an appropriate notion of characteristic manifolds for hyperbolic differential-difference equations. Up to the knowledge of the author, no such a notion is used now.

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Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

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