Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 1826–1841.
DOI: 10.3934/math.2025084
Received: 11 November 2024
Revised: 11 January 2025
Accepted: 16 January 2025
Published: 24 January 2025

Research article

Blow-up criterion for the 3-D inhomogeneous incompressible viscoelastic rate-type fluids with stress-diffusion

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Abstract: In this paper, we investigate the blow-up criterion of an evolutionary partial differential equation system controlling the flows of incompressible viscoelastic rate-type fluids with stress-diffusion, where the extra stress tensor describing the elastic response of the fluid is purely spherical. By utilizing this criterion, the global well-posedness of the system can be readily obtained. Despite being a physical simplification, this model exhibits features that necessitate novel mathematical approaches to tackle the technically complex structure of the associated internal energy, as well as the more complicated forms of the corresponding entropy and energy fluxes. The paper provides the first rigorous proof of the existence of a global solution to the model under small initial data.

Keywords: viscoelastic fluids; Friedrich's method; Lions-Aubin compactness principle; energy method
Mathematics Subject Classification: 76A15, 35Q35, 35D30

1. Introduction

In this paper, we investigate the global well-posedness of the following system:

$$\begin{cases} u_t + u\cdot\nabla u - \Delta u + \nabla\Pi + \sigma \mathrm{div}\,(\nabla b\otimes\nabla b - \frac{1}{2}|\nabla b|^2\mathbb{I}) = 0, \\ b_t + u\cdot\nabla b + \frac{1}{\nu}(e'(b) - \sigma\Delta b) = 0, \\ \mathrm{div}\, u = 0, \\ (u,b)(t,x)|_{t=0} = (u_0,b_0)(x), \end{cases} \tag{1.1}$$

where $u = (u^1, u^2, u^3)$ and $b = b(x,t)$ represent the velocity field of the fluid and the spherical component of the elastic strain tensor, respectively. The scalar pressure function is denoted by $\Pi = \Pi(x,t)$. Generally, the viscosity coefficients ν and σ are positive constants, and $e(\cdot)$ denotes a smooth convex function of b. Furthermore, it is assumed that the second derivative of e with respect to b, which is denoted as $e''(b)$, is bounded by a constant C.

The system we study in this paper controls the motion of non-Newtonian fluids, which is described by a simplified viscoelastic rate-type model incorporating a stress-diffusion component. It retains many qualitative characteristics of more complex viscoelastic rate-type models, which makes it an important tool for studying the mathematical properties of non-Newtonian fluids. In this paper, we present a proof regarding the blow-up criterion applicable to this model. In the context of differential equations, when we say that the solution of an equation containing the "time" variable undergoes "blow-up", it typically means that the domain of definition of the solution is finite, and some kind of "undesirable" situation occurs at the endpoint of the time interval: The solution may tend towards infinity or it may lose smoothness (which can cause the differential equation to lose physical meaning), among other possibilities. This is an important and challenging phenomenon. As time progresses, whether it is displacement, velocity, heat, field intensity, or any other form of response, they will all tend towards infinity.

On the other hand, the presence of the stress-diffusion term in the governing equations is significant for several reasons. First, it enhances the qualitative mathematical properties of the governing equations to a certain degree. Second, the existence of the diffusion term has a profound impact on the dynamical behavior predicted by the system of governing equations, as exemplified by its utilization in modeling the shear banding phenomenon. There are two interpretations of the stress-diffusion term: One views it as a consequence of a nonlocal energy storage mechanism, while the other sees it as a result of a nonlocal entropy production mechanism. These different interpretations of the stress-diffusion mechanism lead to distinct evolution equations for temperature. Hence, understanding the thermodynamic foundation of the derived models is crucial when examining the nonlinear stability of equilibrium resting states. Moreover, in the current model, the presence of the additional stress tensor disrupts the original structure outlined in [1]. Growth strain induces stress, which, in turn, affects the diffusion process through chemical potential and diffusivity. Consequently, substantial modifications must be made to this part of the theory.

This model combines features of the Oldroyd-B and Giesekus models, and incorporates diffusion terms. Moreover, the fluid type studied in this investigation is characterized by its elastic response, which can be represented by spherical strain (a scalar multiplier of the identity tensor), specifically a scalar multiplication of the identity tensor. A detailed derivation process of the mathematical model can be found in the appendix of [2] (carrying on the developments outlined in [3–5]). Although mathematically complex, the model is capable of capturing the complex microstructure of fluids, which is essential for predicting and simulating the flow of non-Newtonian fluids. The rate-type fluid models with stress-diffusion are popular in the modeling of shear and vorticity banding phenomena; see, for example, the reviews in [6–8]. The results of this model provide profound insights into the behavior of viscoelastic fluids and offer mathematical tools and theoretical foundations for fluid dynamics problems in engineering and scientific applications. This task is particularly interesting if one considers fluid models that include terms that are associated with the elastic properties of the material.

When $b = 0$, the system reduces to the incompressible Navier-Stokes equations. The theoretical investigation of these equations was initiated by the foundational works of Leray [9] and Hopf [10] on the global existence of a weak solution. Regarding the local well-posedness of strong solutions, Ciga and Miyakawa [11], Kato [12], and others researched it using semigroup theory. Abidi, Gui, and

Zhang [13] explored the global well-posedness of solutions in Besov spaces for systems with a bounded density. More recently, the authors of [14] also investigated the global existence and uniqueness of solution to two-dimensional (2-D) inhomogeneous incompressible Navier-Stokes equations in critical spaces.

Moreover, in the case where $\sigma = 0$, our Eq (1.1) can be conceptualized as the coupling of two systems: The inhomogeneous Navier-Stokes equation governing a fluid with constant density, and a transport equation with the damped term $e'(b)$. As we mentioned above, research on the three-dimensional (3-D) incompressible Navier-Stokes equation is indeed extensive, with numerous studies, such as [1, 15, 16] and others. On the other hand, for when $\sigma > 0$, Wang and Zhang [17] investigated the global well-posedness of the 2-D Boussinesq system incorporating the temperature-dependent viscosity and thermal diffusivity. Abidi and Zhang [18] explored the global well-posedness of a 3-D Boussinesq system with variable viscosity.

For the system we study in this paper, M. Bulíček, J. Málek, Vit Průsă, and E. Süli have conducted some related research in recent years. They [3] established the long-time and large-data existence of a weak solution to the evolutionary spatially periodic problem associated with the set of governing equations for compressible fluids with variable density in 3-D space. M. Bathory, M. Bulíček, and J. Málek [19] explored a global-in-time and large-data existence theory, within the context of weak solutions, to a class of homogeneous incompressible rate-type viscoelastic fluids confined in a closed 3-D container.

Compared with these reference papers, our primary focus is on the simplified model of viscoelastic fluids with stress-diffusion, which was derived in [2]. In this model, we specifically consider the scenario where the extra stress tensor, which describes the elastic response of the fluid, is purely spherical. Furthermore, we have incorporated the divergence-free condition to investigate the incompressible situation. In this paper, we have rigorously proven, for the first time, the existence and uniqueness of the local solution to the system using the energy method. Additionally, we have obtained global well-posedness under the assumption of small initial data through the application of a blow-up criterion.

Our main result reads as follows:

Theorem 1.1. *Assume that when* $s > \frac{3}{2}$ *and the initial data* (u_0, b_0) *satisfy* $u_0 \in H^s(\mathbb{R}^3)$, $b_0 \in H^{s+1}(\mathbb{R}^3)$ *and* $\operatorname{div} u_0 = 0$, *the following hold:*
(1) (Local existence) There exists a small time $T^* > 0$ *such that* (1.1) *has a unique strong solution* (u, b) *on* $(0, T^*)$ *satisfying*

$$u \in C(0, T^*; H^s) \cap L^2(0, T^*; H^{s+1}), \quad b \in C(0, T^*; H^{s+1}) \cap L^2(0, T^*; H^{s+2}).$$

(2) (Blow-up criterion) T^* *is the finite blow-up time of* (u, b) *if and only if*

$$\int_0^{T^*} (\|\nabla u\|_{L^2}^4 + \|\nabla^2 b\|_{L^2}^4) d\tau = +\infty.$$

(3) (Global existence) If there exists a small constant $\varepsilon > 0$ *such that*

$$\|u_0\|_{H^s} + \|b_0\|_{H^{s+1}} < \varepsilon,$$

then $T^* = \infty$.

The remainder of this paper is structured as follows. In Section 2, we introduce preliminary knowledge that will be subsequently used throughout the text. Next, the primary focus of the subsequent sections is on proving Theorem 1.1. In Section 3, we utilize Friedrich's method to construct approximate solutions for the system in (1.1). By employing a priori estimates to the approximate system, we establish the existence and uniqueness of the local strong solution for (1.1). In Section 4, we derive a blow-up criterion by a contradiction argument to determine whether the local solution exhibits blow-up behavior. Finally, we demonstrate the global existence of the solution under the assumption that the initial data are sufficiently small, thereby completing the proof of (1.1).

Let us complete this section with the notations we are going to use in the following.
(1) If u, v are two functions, let us set $u := v$ to define u as equal to v.
(2) For X being a Banach space and I being an interval of $\mathbb{R}$, $1 \leq q \leq \infty$, we denote by $L^q(I;X)$ the set of measurable function on I with values in X. Sometimes, for convenience in this paper, we write it as $L^q(X)$.
(3) For $m \in \mathbb{N}$, $f_i \in L^p$ $(i = 1, 2, ..., m)$, we write $\|(f_1, f_2, ..., f_m)\|_{L^p} := \sum_{i=1}^{m} \|f_i\|_{L^p}$.
(4) For $a \lesssim b$, we mean that there exists a uniform constant C, which may be different in different lines, such that $a \leq Cb$.
(5) For $m > 1$, $x \in \mathbb{R}^n$, and a vector-valued function $u(x) = (u^1, u^2, ..., u^m)$, we define

$$\nabla u := \begin{pmatrix} u^1_{x_1} & \cdots & u^1_{x_n} \\ \vdots & \ddots & \vdots \\ u^1_{x_1} & \cdots & u^m_{x_n} \end{pmatrix},$$

$$\Delta u := (\sum_{i=1}^{n} u^1_{x_i x_i}, \sum_{i=1}^{n} u^2_{x_i x_i}, ..., \sum_{i=1}^{n} u^m_{x_i x_i}).$$

2. Preliminary

First, in order to facilitate the construction of an approximate equation for the system in (1.1), we introduce two projection operators P_n and P here.

Definition 2.1. *(see [20]) Let us define the Helmholtz projection operator P with a divergence-free condition by*

$$P = (\delta_{ij} + R_i R_j)_{1 \leq i,j \leq 3},$$

where R_j denotes the Riesz transform and satisfies $\mathcal{F}(R_j f)(\xi) = -\frac{i\xi_j}{|\xi|}\mathcal{F}f(\xi)$.

Definition 2.2. *(see [21]) The projection operator P_n is defined as follows:*

$$P_n f(x) = \mathcal{F}^{-1}(1_{B_n})(\xi)\mathcal{F}f(\xi)(x),$$

where the Fourier operator $\mathcal{F}f(\xi) := \int_{\mathbb{R}^3} f(x)e^{-ix\xi}dx$, and the characteristic function 1_{B_n} is defined on the surface of a sphere B_n centered at the origin with a radius n.

Clearly, based on the definitions of P_n and P provided above, it is straightforward to deduce that $P_nP = PP_n$ and $P_n^2 = P_n$. The proofs of these statements are omitted here for brevity.

Next, we recall some basic definitions and properties related to the Littlewood-Paley theory (see [21] for more details). Assume that ψ and θ are two functions in $C^\infty(\mathbb{R}^3)$ with

$$Supp\,\hat{\psi} \subseteq \{\frac{3}{4} \le |\xi| \le \frac{8}{3}\}, \quad Supp\,\hat{\theta} \subseteq \{|\xi| \le \frac{4}{3}\},$$

and $\hat{\theta}(\xi) + \sum_{j\ge 0} \hat{\psi}(2^{-j}\xi) = 1$ $(\forall\, \xi \in \mathbb{R}^3)$.

Definition 2.3. *(see [21]) We define the Littlewood-Paley operator Δ_j in $\mathbb{R}^3$ as follows:*

$$\Delta_j f(x) := \psi * f(x) = \int_{\mathbb{R}^3} \psi_j(x-y)f(y)dy; \quad \psi_j(x) := 2^{3j}\psi(2^j x), \ j \ge 0;$$

$$\Delta_{-1} f(x) := \theta * f(x); \quad S_j f(x) := \sum_{k=-1}^{j-1} \Delta_k f(x).$$

In particular, utilizing the definition of the Littlewood-Paley operator provided above, we can re-describe the Sobolev space $H^s(\mathbb{R}^3)$.

Definition 2.4. *(see [22]) Let $s \in \mathbb{R}$; the Sobolev space $H^s(\mathbb{R}^3)$ can then be defined as*

$$H^s(\mathbb{R}^3) := \{u \in \mathcal{D}'(\mathbb{R}^3) : \sum_{j\ge -1} 2^{2js}\|\Delta_j u\|_{L^2}^2 < \infty\}.$$

We set the norm as

$$\|u\|_{H^s}^2 := \sum_{j\ge -1} 2^{2js}\|\Delta_j u\|_{L^2}^2.$$

Definition 2.5. *(see [22]) We define the space $\tilde{L}^\infty(0,T;H^s)$ by the norm:*

$$\|u\|_{\tilde{L}^\infty(0,T;H^s)}^2 := \sum_{j\ge -1} 2^{2js}\|\Delta_j u\|_{L^\infty(0,T;L^2)}^2 < \infty.$$

Next, we introduce some lemmas that will be used in the following sections.

Lemma 2.1. *(see [23]) Assume that $k \in \mathbb{N}$, $1 \le p \le q \le \infty$, and $j \ge 0$, α is a multi-index (i.e., an element of $\mathbb{N}^3$), $\forall f \in L^p(\mathbb{R}^3)$, and there exists a positive constant C which is independent of j such that*

$$\|\partial^\alpha \Delta_j f\|_{L^q} + \|\partial^\alpha S_j f\|_{L^q} \le C2^{j|\alpha| + 3j(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.$$

Lemma 2.2. *(see [24]) Let $s > 0$ and $f \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$; $F(\cdot)$ is a smooth function on $\mathbb{R}$ and $F(0) = 0$. The following thus holds*

$$\|F(f)\|_{H^s} \lesssim (1 + \|f\|_{L^\infty})^{\lfloor s\rfloor+1}\|f\|_{H^s}.$$

Lemma 2.3. *(see [25]) Let* $s > 0$ *and set* $\Lambda^s := \sqrt{-\Delta}$, *then*

$$\|u(x)\|_{H^s}^2 = \int_{\mathbb{R}^3} (\Lambda^s u)^2(x)dx,$$

and for $\forall f, g \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, *we have*

$$\|\Lambda^s(fg)\|_{L^2} \lesssim \|f\|_{L^\infty}\|\Lambda^s g\|_{L^2} + \|g\|_{L^\infty}\|\Lambda^s f\|_{L^2}.$$

Lemma 2.4. *(see [26]) (Lions-Aubin lemma) Assume that* $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$, *where* X, Z *are reflexible spaces, and* X *is dense in* Z, *then let* $W := \{u \in L^{p_0}(0,T;X),\ u_t \in L^{p_1}(0,T;X),$ *and* $1 < p_0, p_1 \le \infty\}$. *It then holds that* $W \hookrightarrow\hookrightarrow L^{p_0}(0,T;Y)$.

3. Local well-posedness

This section is devoted to the proof of the local well-posedness of Eq (1.1). To clearly illustrate the process, we will divide it into five steps.

Step 1. Construct approximate equations.

We use the Friedrich's method to construct the approximate solutions. Recall the projector operator P_n and the Helmholtz projection operator P stated in Definitions 2.1 and 2.2, respectively, and then utilize them to introduce the following approximate system for (1.1):

$$\begin{cases} \partial_t u_n + P_n P(P_n u_n \cdot \nabla P_n u_n) - P_n \Delta P_n u_n + \sigma P_n P(\nabla P_n b_n \Delta P_n b_n) = 0, \\ \partial_t b_n + P_n(P_n u_n \cdot \nabla P_n b_n) + \frac{1}{\nu} P_n(e'(P_n b_n) - \sigma \Delta P_n b_n) = 0, \\ (u_n, b_n)(t,x)|_{t=0} = (P_n u_0, P_n b_0)(x), \end{cases} \tag{3.1}$$

where we used the fact that

$$\operatorname{div}(\nabla b \otimes \nabla b - \frac{1}{2}|\nabla b|^2 \mathbb{I}) = \nabla b \Delta b. \tag{3.2}$$

Step 2. Energy estimates to approximate equations.

First, we take the scalar product of $(3.1)_1$ with u_n and integrate the resulting expression over $\mathbb{R}^3$, which gives

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \|\nabla P_n u_n\|_{L^2}^2 + \sigma \int_{\mathbb{R}^3} \nabla P_n b_n \cdot \Delta P_n b_n \cdot P_n u_n \, dx = 0. \tag{3.3}$$

Next, by multiplying $(3.1)_2$ by $-\sigma \Delta b_n$ and integrating in space, we have

$$\begin{aligned} &\frac{\sigma}{2}\frac{d}{dt}\|\nabla b_n\|_{L^2}^2 + \frac{\sigma^2}{\nu}\|\nabla^2 P_n b_n\|_{L^2}^2 + \frac{\sigma}{\nu}\int_{\mathbb{R}^3} e''(P_n b_n)|\nabla P_n b_n|^2 \, dx \\ &= -\sigma \int_{\mathbb{R}^3} P_n u_n \cdot \nabla P_n b_n \cdot \Delta P_n b_n. \end{aligned} \tag{3.4}$$

Hence, by summing up (3.3) and (3.4), we get

$$\frac{d}{dt}(\frac{1}{2}\|u_n\|_{L^2}^2 + \frac{\sigma}{2}\|\nabla b_n\|_{L^2}^2) + \|\nabla P_n u_n\|_{L^2}^2 + \frac{\sigma^2}{\nu}\|\nabla^2 P_n b_n\|_{L^2}^2 \le 0. \tag{3.5}$$

By then integrating it over $[0, T_n]$, one eventually gets

$$\begin{aligned} &u_n \in L^\infty(0, T_n; L^2), \quad \nabla P_n u_n \in L^2(0, T_n; L^2), \\ &\nabla b_n \in L^\infty(0, T_n; L^2), \quad \nabla^2 P_n b_n \in L^2(0, T_n; L^2). \end{aligned} \tag{3.6}$$

In addition, taking the L^2 inner product of the b_n equation with $P_n(e'(P_n b_n))$ gives

$$\frac{d}{dt}\|P_n(e(b_n))\|_{L^1} + \frac{1}{\nu}\|P_n(e'(P_n b_n))\|_{L^2}^2 \le \frac{1}{2\nu}\|P_n(e'(P_n b_n))\|_{L^2}^2 + C\|\Delta P_n(b_n)\|_{L^2}^2.$$

Next, doing integration by parts and integrating it over $[0, T_n]$, which, along with (3.6) shows that

$$P_n(e(P_n b_n)) \in L^\infty(0, T_n; L^1), \quad P_n(e'(P_n b_n)) \in L^2(0, T_n; L^2). \tag{3.7}$$

Therefore, from (3.6) and (3.7), we observe that each term in $(3.1)_1$ and $(3.1)_2$ is bounded in $L^2(\mathbb{R}^3)$. Consequently, it follows from the Cauchy-Lipschitz theorem that there exists a unique solution $(u_n, b_n) \in C(0, T_n; L^2)$ to the system in (3.1). Due to the uniqueness of the solution, one has $P_n u_n = u_n$ and $P_n b_n = b_n$. Hence the approximate system in (3.1) simplifies to the following form:

$$\begin{cases} \partial_t u_n + P_n P(u_n \cdot \nabla u_n) - \Delta u_n + \sigma P_n(\nabla b_n \Delta b_n) = 0, \\ \partial_t b_n + P_n(u_n \cdot \nabla b_n) + \frac{1}{\nu} P_n(e'(b_n) - \sigma \Delta b_n) = 0, \\ (u_n, b_n)(t, x)|_{t=0} = (P_n u_0, P_n b_0)(x). \end{cases} \tag{3.8}$$

Moreover, the inequalities in (3.6) and (3.7) can also be reformulated as follows:

$$\begin{aligned} &u_n \in L^\infty(0, T_n; L^2) \cap L^2(0, T_n; \dot{H}^1), \quad b_n \in L^\infty(0, T_n; H^1) \cap L^2(0, T_n; H^2), \\ &e(b_n) \in L^\infty(0, T_n; L^1), \qquad\qquad\qquad e'(b_n) \in L^2(0, T_n; L^2). \end{aligned} \tag{3.9}$$

In what follows, we focus on estimating $\|u_n\|_{H^s}$ and $\|b_n\|_{H^{s+1}}$.

Applying the operator Λ^s to $(3.8)_1$ and taking the L^2 inner product of the resulting equation with $\Lambda^s u_n$, followed by integrating it over $\mathbb{R}^3$, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u_n\|_{L^2}^2 + \int_{\mathbb{R}^3} (\Lambda^s(u_n \cdot \nabla u_n) \cdot \Lambda^s u_n + |\Lambda^s \nabla u_n|^2 + \sigma \Lambda^s(\nabla b_n \Delta b_n) \cdot \Lambda^s u_n) dx = 0. \tag{3.10}$$

Similarly, we perform the same operation on the b_n equation. Applying the Λ^s operator to $(3.8)_2$ and multiplying the resulting expression by $\Lambda^s b_n$, $-\Lambda^s \Delta b_n$, by integrating them over $\mathbb{R}^3$, we then have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s b_n\|_{L^2}^2 + \int_{\mathbb{R}^3} (\Lambda^s(u_n \cdot \nabla b_n)\Lambda^s b_n + \frac{1}{\nu}\Lambda^s e'(b_n)\Lambda^s b_n + \frac{\sigma}{\nu}|\Lambda^s \nabla b_n|^2) dx = 0 \tag{3.11}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s \nabla b_n\|_{L^2}^2 + \int_{\mathbb{R}^3} (- \Lambda^s(u_n \cdot \nabla b_n)\Lambda^s \Delta b_n - \frac{1}{\nu}\Lambda^s e'(b_n)\Lambda^s \Delta b_n + \frac{\sigma}{\nu}|\Lambda^s \Delta b_n|^2) dx = 0. \tag{3.12}$$

Summing Eqs (3.10)–(3.12) gives

$$\frac{1}{2}\frac{d}{dt}\|(\Lambda^s u_n, \Lambda^s b_n, \Lambda^s \nabla b_n)\|_{L^2}^2 + (\|\Lambda^s \nabla u_n\|_{L^2}^2 + \frac{\sigma}{\nu}\|\Lambda^s \nabla b_n\|_{L^2}^2 + \frac{\sigma}{\nu}\|\Lambda^s \Delta b_n\|_{L^2}^2)$$

$$= \int_{\mathbb{R}^3} \Big[-\Lambda^s(u_n \cdot \nabla u_n) \cdot \Lambda^s u_n - \sigma \Lambda^s(\nabla b_n \Delta b_n) \cdot \Lambda^s u_n - \Lambda^s(u_n \cdot \nabla b_n) \Lambda^s b_n$$
$$- \frac{1}{\nu} \Lambda^s e'(b_n) \Lambda^s b_n + \Lambda^s(u_n \cdot \nabla b_n) \Lambda^s \Delta b_n + \frac{1}{\nu} \Lambda^s e'(b_n) \Lambda^s \Delta b_n \Big] dx.$$

According to Hölder's inequality, Young's inequality, and Lemmas 2.2 and 2.3, we can estimate each term on the right side of the above inequality:

$$\int_{\mathbb{R}^3} -\Lambda^s(u_n \cdot \nabla u_n) \cdot \Lambda^s u_n dx \le |\int_{\mathbb{R}^3} \Lambda^s(u_n \otimes u_n) \Lambda^s \nabla u_n dx| \le \|\Lambda^s(u_n \otimes u_n)\|_{L^2} \|\Lambda^s \nabla u_n\|_{L^2}$$
$$\le 2\|u_n\|_{L^\infty} \|\Lambda^s u_n\|_{L^2} \|\Lambda^s \nabla u_n\|_{L^2} \le \frac{1}{4} \|\Lambda^s \nabla u_n\|_{L^2}^2 + C\|u_n\|_{L^\infty}^2 \|\Lambda^s u_n\|_{L^2}^2,$$
$$\int_{\mathbb{R}^3} -\sigma \Lambda^s(\nabla b_n \Delta b_n) \cdot \Lambda^s u_n dx \le \sigma |\int_{\mathbb{R}^3} \Lambda^s(\nabla b_n \otimes \nabla b_n) \Lambda^s \nabla u_n dx|$$
$$\le \frac{1}{4} \|\Lambda^s \nabla u_n\|_{L^2}^2 + C\|\nabla b_n\|_{L^\infty}^2 \|\Lambda^s \nabla b_n\|_{L^2}^2,$$
$$\int_{\mathbb{R}^3} -\Lambda^s(u_n \cdot \nabla b_n) \Lambda^s b_n dx \le |\int_{\mathbb{R}^3} \Lambda^s(u_n b_n) \Lambda^s \nabla b_n dx|$$
$$\le \frac{\sigma}{2\nu} \|\Lambda^s \nabla b_n\|_{L^2}^2 + C\|u_n\|_{L^\infty}^2 \|\Lambda^s b_n\|_{L^2}^2 + C\|b_n\|_{L^\infty}^2 \|\Lambda^s u_n\|_{L^2}^2,$$
$$\int_{\mathbb{R}^3} -\frac{1}{\nu} \Lambda^s e'(b_n) \Lambda^s b_n dx \le |\int_{\mathbb{R}^3} \Lambda^s e'(b_n) \Lambda^s b_n dx| \le CJ(\|b_n\|_{L^\infty}) \|\Lambda^s b_n\|_{L^2}^2,$$
$$\int_{\mathbb{R}^3} \Lambda^s(u_n \cdot \nabla b_n) \Lambda^s \Delta b_n dx \le |\int_{\mathbb{R}^3} \Lambda^s(u_n \cdot \nabla b_n) \Lambda^s \Delta b_n dx|$$
$$\le \frac{\sigma}{2\nu} \|\Lambda^s \Delta b_n\|_{L^2}^2 + C\|u_n\|_{L^\infty}^2 \|\Lambda^s \nabla b_n\|_{L^2}^2 + C\|\nabla b_n\|_{L^\infty}^2 \|\Lambda^s u_n\|_{L^2}^2,$$
$$\int_{\mathbb{R}^3} \frac{1}{\nu} \Lambda^s e'(b_n) \Lambda^s \Delta b_n dx \le \frac{1}{\nu} |\int_{\mathbb{R}^3} \Lambda^s e'(b_n) \Lambda^s \Delta b_n dx| \le \frac{1}{\nu} |\int_{\mathbb{R}^3} \Lambda^{s+1} e'(b_n) \Lambda^s \nabla b_n dx|$$
$$\le CJ(\|b_n\|_{L^\infty}) \|\Lambda^s \nabla b_n\|_{L^2}^2,$$

where $J(\|b_n\|_{L^\infty}) = (1 + \|b_n\|_{L^\infty})^{\lfloor s \rfloor + 2}$.

Hence, plugging these estimates into the above inequality yields

$$\frac{d}{dt}(\|u_n\|_{H^s}^2 + \|b_n\|_{H^s}^2 + \|\nabla b_n\|_{H^s}^2) + (\|\nabla u_n\|_{H^s}^2 + \|\nabla b_n\|_{H^s}^2 + \|\nabla^2 b_n\|_{H^s}^2)$$
$$\lesssim (J(\|b_n\|_{L^\infty}) + \|u_n\|_{L^\infty}^2 + \|b_n\|_{L^\infty}^2 + \|\nabla b_n\|_{L^\infty}^2)(\|u_n\|_{H^s}^2 + \|b_n\|_{H^s}^2 + \|\nabla b_n\|_{H^s}^2),$$

which, together with Grönwall's inequality, gives rise to

$$E_n(t) \lesssim E_n(0) \exp \int_0^t \mathcal{H}_n(\tau) d\tau, \tag{3.13}$$

where we defined

$$E_n(t) := \|u_n(t)\|_{H^s}^2 + \|b_n(t)\|_{H^{s+1}}^2 + \int_0^t (\|u_n\|_{H^{s+1}}^2 + \|b_n\|_{H^{s+2}}^2)(\tau) d\tau,$$
$$E_n(0) := \|P_n u_0\|_{H^s}^2 + \|P_n b_0\|_{H^{s+1}}^2,$$

$$\mathcal{H}_n(t) := J(\|b_n\|_{L^\infty}) + \|u_n\|_{L^\infty}^2 + \|b_n\|_{L^\infty}^2 + \|\nabla b_n\|_{L^\infty}^2.$$

Subsequently, assume that T_n is the maximal existence time of the approximate system in (3.8). Furthermore, there exists a uniform lower bound for T_n, which ensures that the limit of time is in a certain interval. We omit the detailed proof here; the readers can refer to [27]. In addition, we set

$$\widetilde{T_n} := \sup_t \{t \in [0, T_n) : E_n(t) \leq 4E_n(0)\}.$$

Since $J(\cdot)$ is an increasing function and $s > \frac{3}{2}$, it follows from the Sobolev embedding theorem that when $t \in [0, \widetilde{T_n})$,

$$\begin{aligned} E_n(t) &\leq E_n(0)\exp\{\int_0^t [J(\|b_n\|_{H^s}) + \|u_n\|_{H^s}^2 + \|b_n\|_{H^s}^2 + \|\nabla b_n\|_{H^s}^2]d\tau\} \\ &\leq E_n(0)\exp\{\int_0^t [J(E_n) + E_n]d\tau\} \\ &\leq E_n(0)\exp\{J(4E_n(0))t + 4E_n(0)t\}. \end{aligned}$$

Clearly, $\exp\{J(4E_n(0))t+4E_n(0)t\}$ is an increasing and continuous function with respect to t. Therefore, we can choose $T^* \in [0, \widetilde{T_n})$ to be sufficiently small such that $\exp\{J(4E_n(0))t + 4E_n(0)t\} < 2$. Consequently, for any $t \in [0, T^*]$, one has

$$E_n(t) \leq 2E_n(0), \tag{3.14}$$

which implies that

$$u_n \in L^\infty(0, T^*; H^s) \cap L^2(0, T^*; H^{s+1}), \quad b_n \in L^\infty(0, T^*; H^{s+1}) \cap L^2(0, T^*; H^{s+2}). \tag{3.15}$$

Finally, with the help of Lemma 2.3 and Sobolev's embedding theorem, it follows from $(3.8)_1$ that

$$\begin{aligned} &\|\partial_t u_n\|_{H^{s-1}}^2 \leq \|u_n \cdot \nabla u_n\|_{H^{s-1}}^2 + \|\nabla^2 u_n\|_{H^{s-1}}^2 + \|\nabla b_n \Delta b_n\|_{H^{s-1}}^2 \\ &\leq \|u_n\|_{L^\infty}^2 \|\nabla u_n\|_{H^{s-1}}^2 + \|\nabla u_n\|_{L^\infty}^2 \|u_n\|_{H^{s-1}}^2 + \|u_n\|_{H^{s+1}}^2 + \|\nabla b_n\|_{L^\infty}^2 \|\Delta b_n\|_{H^{s-1}}^2 \\ &\quad + \|\Delta b_n\|_{L^\infty}^2 \|\nabla b_n\|_{H^{s-1}}^2 \\ &\leq \|u_n\|_{H^s}^2 \|u_n\|_{H^{s+1}}^2 + \|u_n\|_{H^{s+1}}^2 + \|\nabla b_n\|_{H^s}^2 \|b_n\|_{H^{s+1}}^2 + \|b_n\|_{H^{s+2}}^2 \|b_n\|_{H^s}^2, \end{aligned}$$

which, combined with (3.15), shows that

$$\partial_t u_n \in L^2(0, T^*; H^{s-1}). \tag{3.16}$$

Similarly, performing the same operation on $(3.8)_2$ yields

$$\partial_t b_n \in L^2(0, T^*; H^s). \tag{3.17}$$

Thus, we have completed the desired estimates for the approximate solutions.

Step 3. The existence of a local solution for the system in (1.1).

Based on the estimates of (3.15) to (3.17) derived in Step 2 , the Lions-Aubin compactness theorem guarantees the existence of the subsequence (u_{n_k}, b_{n_k}) of (u_n, b_n) and a function $(u, b) \in$

$L^\infty(0,T^*;H^s) \cap L^2(0,T^*;H^{s+1}) \times L^\infty(0,T^*;H^{s+1}) \cap L^2(0,T^*;H^{s+2})$ such that for any $s' < s$, when $k \to \infty$,

$$\begin{aligned} u_{n_k} &\to u \qquad in \quad L^2(0,T^*;H^{s'+1}_{loc}), \\ b_{n_k} &\to b \qquad in \quad L^2(0,T^*;H^{s'+2}_{loc}). \end{aligned}$$

Additionally, it is straightforward to observe that (u,b) satisfies Eq (1.1) by taking the limit of the approximate equations; thus we have proved the uniqueness of the system in (1.1).

Step 4. Continuity of the solution.

To establish the continuity of the solution, we apply the operator Δ_j to both sides of (3.5) and then integrate it on $[0,T^*]$. Subsequently, multiplying the integrated expression by 2^{2js} and ultilizing Definition 2.5 gives rise to

$$||u_n||_{\tilde{L}^\infty(0,T^*;H^s)} + ||b_n||_{\tilde{L}(0,T^*;H^{s+1})} \leq C.$$

Therefore, we easily infer the improved estimates for (u,b) that

$$||u||_{\tilde{L}^\infty(0,T^*;H^s)} + ||b||_{\tilde{L}^\infty(0,T^*;H^{s+1})} \leq C. \tag{3.18}$$

Meanwhile, Definition 2.4 indicates that there exists a positive integer N such that

$$\sum_{j=N}^{\infty} 2^{2js}||\Delta_j u||^2_{L^\infty(0,T^*;L^2)} \leq \frac{\varepsilon}{4}.$$

Hence, for any $t \in (0,T^*)$ and any δ such that $t+\delta \in (0,T^*)$, one obtains

$$\begin{aligned} &||u(t+\delta) - u(t)||^2_{H^s} \\ =&\sum_{j=-1}^{N} 2^{2js}||\Delta_j u(t+\delta) - \Delta_j u(t)||^2_{L^2} + \sum_{j=N}^{\infty} 2^{2js}||\Delta_j u(t+\delta) - \Delta_j u(t)||^2_{L^2} \\ \leq&\sum_{j=-1}^{N} 2^{2js}||\int_t^{t+\delta} \partial_\tau \Delta_j u(\tau)d\tau||^2_{L^2} + \frac{\varepsilon}{2} \\ \leq&\sum_{j=-1}^{N} 2^{2js}|\delta|\int_t^{t+\delta} ||\partial_\tau u(\tau)||^2_{L^2}d\tau + \frac{\varepsilon}{2} \\ \leq&2N2^{2sN}||\partial_t u||^2_{L^2(0,T^*;L^2)}|\delta| + \frac{\varepsilon}{2}, \end{aligned}$$

where we have used Hölder's inequality and Lemma 2.1.

Taking δ which is small enough that $2N2^{2sN}|\delta|||\partial_t u(t)||^2_{L^2(0,T^*;L^2)} \leq \frac{\varepsilon}{2}$, one can easily check that for any $\varepsilon > 0$, a $\delta > 0$ exists such that

$$||u(t+\delta) - u(t)||^2_{H^s} \leq \varepsilon,$$

which yields

$$u \in C(0,T^*;H^s). \tag{3.19}$$

On the other hand, by the same argument, the fact $b \in \tilde{L}^\infty(0,T^*;H^{s+1})$ ensures that $b \in C(0,T^*;H^{s+1})$. Therefore, the continuity of the solution (u,b) to Eq (1.1) has been proved.

Step 5. Uniqueness of the solution.

Let $(u^{(1)},b^{(1)},\Pi^{(1)})$ and $(u^{(2)},b^{(2)},\Pi^{(2)})$ be two solutions of Eq (1.1) with the same initial data. We denote $\tilde{u} := u^{(2)} - u^{(1)}$, $\tilde{b} := b^{(2)} - b^{(1)}$, and $\tilde{\Pi} := \Pi^{(2)} - \Pi^{(1)}$; $(\tilde{u},\tilde{b},\tilde{\Pi})$ then solves the following equation:

$$\begin{cases} \tilde{u}_t + \tilde{u}\cdot\nabla u^{(2)} + u^{(1)}\cdot\nabla\tilde{u} - \Delta\tilde{u} + \nabla\tilde{\Pi} + \sigma\nabla b^{(2)}\Delta\tilde{b} + \sigma\nabla\tilde{b}\Delta b^{(1)} = 0,\\ \tilde{b}_t + \tilde{u}\cdot\nabla b^{(2)} + u^{(1)}\cdot\nabla\tilde{b} + \frac{1}{\nu}(e'(b^{(2)}) - e'(b^{(1)})) - \frac{\sigma}{\nu}\Delta\tilde{b} = 0,\\ \operatorname{div}\tilde{u} = 0,\\ \tilde{u} = 0, \quad \tilde{b} = 0. \end{cases} \tag{3.20}$$

As we proved in Step 2, we can apply the standard L^2 energy estimate to $\tilde{u}$ and $\tilde{b}$, which gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2 + \sigma\|\nabla\tilde{b}\|_{L^2}^2) + (\|\nabla\tilde{u}\|_{L^2}^2 + \frac{\sigma}{\nu}\|\nabla\tilde{b}\|_{L^2}^2 + \frac{\sigma^2}{\nu}\|\Delta\tilde{b}\|_{L^2}^2)\\ \le &\int_{\mathbb{R}^3}\Big[\tilde{u}\cdot\nabla u^{(2)}\cdot\tilde{u} + \sigma\nabla\tilde{b}\Delta b^{(1)}\cdot\tilde{u} + \tilde{u}\cdot\nabla b^{(2)}\cdot\tilde{b} + u^{(1)}\cdot\nabla\tilde{b}\tilde{b} + \frac{1}{\nu}(e'(b^{(2)}) - e'(b^{(1)}))\tilde{b}\\ &\quad + \sigma u^{(1)}\cdot\nabla\tilde{b}\Delta\tilde{b} + \frac{\sigma}{\nu}(e'(b^{(2)}) - e'(b^{(1)}))\Delta\tilde{b}\Big]dx. \end{aligned}$$

Combining integration by parts with Gagliardo-Nirenberg inequality, Hölder's inequality, and Young's inequality, we can estimate the right side of the inequality above as follows:

$$\begin{aligned} &\int_{\mathbb{R}^3}\tilde{u}\cdot\nabla u^{(2)}\cdot\tilde{u}dx \le \|\nabla u^{(2)}\|_{L^\infty}\|\tilde{u}\|_{L^2}^2,\\ &\int_{\mathbb{R}^3}\sigma\nabla\tilde{b}\Delta b^{(1)}\cdot\tilde{u}dx \le \|\nabla\tilde{b}\|_{L^2}\|\Delta b^{(1)}\|_{L^\infty}\|\tilde{u}\|_{L^2} \le \frac{\sigma}{6\nu}\|\nabla\tilde{b}\|_{L^2}^2 + \|\Delta b^{(1)}\|_{L^\infty}^2\|\tilde{u}\|_{L^2}^2,\\ &\int_{\mathbb{R}^3}\tilde{u}\cdot\nabla b^{(2)}\cdot\tilde{b}dx \le \|\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\tilde{u}\|_{L^2}^{\frac{1}{2}}\|\nabla b^{(2)}\|_{L^2}\|\nabla\tilde{b}\|_{L^2}\\ &\qquad\qquad \le \frac{1}{2}\|\nabla\tilde{u}\|_{L^2}^2 + \frac{\sigma}{6\nu}\|\nabla\tilde{b}\|_{L^2}^2 + C\|\nabla b^{(2)}\|_{L^2}^4\|\tilde{u}\|_{L^2}^2,\\ &\int_{\mathbb{R}^3}u^{(1)}\cdot\nabla\tilde{b}\tilde{b}dx \le \|u^{(1)}\|_{L^\infty}\|\nabla\tilde{b}\|_{L^2}\|\tilde{b}\|_{L^2} \le \frac{\sigma}{6\nu}\|\nabla\tilde{b}\|_{L^2}^2 + C\|u^{(1)}\|_{L^\infty}^2\|\tilde{b}\|_{L^2}^2,\\ &\int_{\mathbb{R}^3}\frac{1}{\nu}(e'(b^{(2)}) - e'(b^{(1)}))\tilde{b}dx \le e''(\xi)\|\tilde{b}\|_{L^2}^2 \le C\|\tilde{b}\|_{L^2}^2,\\ &\int_{\mathbb{R}^3}\sigma u^{(1)}\cdot\nabla\tilde{b}\Delta\tilde{b}dx \le \|u^{(1)}\|_{L^\infty}\|\nabla\tilde{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2} \le \|u^{(1)}\|_{L^\infty}^2\|\nabla\tilde{b}\|_{L^2}^2 + \frac{\sigma^2}{4\nu}\|\Delta\tilde{b}\|_{L^2}^2,\\ &\int_{\mathbb{R}^3}\frac{\sigma}{\nu}(e'(b^{(2)}) - e'(b^{(1)}))\Delta\tilde{b}dx \le \|e''(\xi)\tilde{b}\|_{L^2}\|\Delta\tilde{b}\|_{L^2} \le \frac{\sigma^2}{4\nu}\|\Delta\tilde{b}\|_{L^2}^2 + C\|\tilde{b}\|_{L^2}^2, \end{aligned}$$

where ξ is a function between $b^{(1)}$ and $b^{(2)}$. Putting all these estimates into the former inequality and utilizing the Sobolev embedding theorem, we can then rewrite the inequality as follows:

$$\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}^2) + \|\nabla\tilde{u}\|_{L^2}^2 + \|\nabla\tilde{b}\|_{L^2}^2 + \|\Delta\tilde{b}\|_{L^2}^2$$

$$\leq C(\|u^{(2)}\|_{H^{s+1}} + \|b^{(1)}\|_{H^{s+2}}^2 + \|u^{(1)}\|_{H^s}^2 + \|b^{(2)}\|_{H^1}^4 + 1)(\|\tilde{u}\|_{L^2}^2 + \|\tilde{b}\|_{L^2}^2 + \sigma\|\nabla\tilde{b}\|_{L^2}^2).$$

Note that

$$\int_0^t (\|u^{(2)}\|_{H^{s+1}} + \|b^{(1)}\|_{H^{s+2}}^2 + \|u^{(1)}\|_{H^s}^2 + \|b^{(2)}\|_{H^1}^4 + 1)(\tau)d\tau \leq C(t).$$

Grönwall's inequality then yields

$$\sup_{t>0}(\|\tilde{u}(t)\|_{L^2}^2 + \|\tilde{b}(t)\|_{L^2}^2 + \|\nabla\tilde{b}(t)\|_{L^2}^2) \leq \exp(C(t))\Big(\|\tilde{u}_0\|_{L^2}^2 + \|\tilde{b}_0\|_{L^2}^2 + \|\nabla\tilde{b}_0\|_{L^2}^2\Big).$$

Therefore, one can easily conclude that

$$\tilde{u} = 0, \quad \tilde{b} = 0, \quad \nabla\tilde{b} = 0.$$

Hence the uniqueness of the solution has been proved.

4. Blow-up criteria

In this section, we prove the blow-up criterion of the system in (1.1) by a contradiction argument. First of all, let $T^* < \infty$ be the maximal existence time of the solution and assume that

$$\int_0^{T^*} (\|\nabla u\|_{L^2}^4 + \|\nabla^2 b\|_{L^2}^4)d\tau < +\infty. \tag{4.1}$$

Taking the L^2 inner product of $(1.1)_1$ with $-\Delta u$, which, along with Hölder's and the Gagliardo-Nirenberg inequality, yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq |\int_{\mathbb{R}^3}(u\cdot\nabla u\cdot\Delta u + \sigma\nabla b\Delta b\cdot\Delta u)dx| \\
&\leq \|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla^2 u\|_{L^2}^{\frac{3}{2}} + \sigma\|\nabla^2 b\|_{L^2}^{\frac{3}{2}}\|\nabla^3 b\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2} \\
&\leq \frac{1}{4}\|\nabla^2 u\|_{L^2}^2 + \frac{\sigma}{4\nu}\|\nabla^3 b\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|\nabla^2 b\|_{L^2}^6,
\end{aligned}$$

Next, by applying the Laplace operator Δ to $(1.1)_2$ and multiplying the resulting expression by Δb, and then integrating the resulting expression over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta b\|_{L^2}^2 + \frac{\sigma}{\nu}\|\nabla^3 b\|_{L^2}^2 = -\int_{\mathbb{R}^3}[\Delta(u\cdot\nabla b)\cdot\Delta b + \frac{1}{\nu}\Delta e'\cdot\Delta b]dx \\
&= -\int_{\mathbb{R}^3}(\Delta u\cdot\nabla b\cdot\Delta b + u\cdot\nabla\Delta b\cdot\Delta b + 2\nabla u\cdot\nabla^2 b\cdot\Delta b - \frac{1}{\nu}\nabla e'\nabla^3 b)dx.
\end{aligned}$$

As we computed in the previous example, each term on the right-hand side of the inequality can also be estimated as follows:

$$-\int_{\mathbb{R}^3}\Delta u\cdot\nabla b\cdot\Delta b dx \leq 2\|\nabla^2 u\|_{L^2}\|\nabla^2 b\|_{L^2}^{\frac{3}{2}}\|\nabla^3 b\|_{L^2}^{\frac{1}{2}} \leq \frac{\sigma}{8\nu}\|\nabla^3 b\|_{L^2}^2 + \frac{1}{4}\|\nabla^2 u\|_{L^2}^2 + C\|\nabla^2 b\|_{L^2}^6,$$

$$-\int_{\mathbb{R}^3} u\cdot\nabla\Delta b\cdot\Delta b dx \le \|\nabla u\|_{L^2}\|\nabla^3 b\|_{L^2}^{\frac{3}{2}}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}} \le \frac{\sigma}{8\nu}\|\nabla^3 b\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla^2 b\|_{L^2}^2,$$
$$-2\int_{\mathbb{R}^3} \nabla u\cdot\nabla^2 b\cdot\Delta b dx \le \|\nabla u\|_{L^2}\|\nabla^3 b\|_{L^2}^{\frac{3}{2}}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}} \le \frac{\sigma}{8\nu}\|\nabla^3 b\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla^2 b\|_{L^2}^2,$$
$$\frac{1}{\nu}\int_{\mathbb{R}^3} \nabla e'\nabla^3 b dx \le C\|\nabla b\|_{L^2}\|\nabla^3 b\|_{L^2} \le \frac{\sigma}{8\nu}\|\nabla^3 b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2.$$

Hence, it follows from (3.9) and a simple simplification that

$$\frac{d}{dt}\|(\nabla u,\Delta b)\|_{L^2}^2 + \|(\Delta u,\nabla^3 b)\|_{L^2}^2 \lesssim (\|\nabla u\|_{L^2}^4 + \|\nabla^2 b\|_{L^2}^4)(\|\nabla u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2) + 1.$$

Then, by virtue of Grönwall's inequality and (4.1), one deduces that

$$u\in L^\infty(0,T^*;H^1)\cap L^2(0,T^*;H^2),\quad b\in L^\infty(0,T^*;H^2)\cap L^2(0,T^*;H^3). \tag{4.2}$$

Furthermore, the Sobolev embedding theorem gives

$$\|u\|_{L^2(L^\infty)} \le \|u\|_{L^2(H^2)},\quad \|b\|_{L^\infty(L^\infty)} \le \|b\|_{L^\infty(H^2)},\quad \|\nabla b\|_{L^2(L^\infty)} \le \|\nabla b\|_{L^2(H^2)}. \tag{4.3}$$

Thus, with (4.2) and (4.3), by using the same method as employed in proving Eq (3.13), one eventually finds that when $s > \frac{3}{2}$,

$$\begin{aligned}\|(u,b,\nabla b)\|_{H^s}^2(T^*) &\le \|(u_0,b_0,\nabla b_0)\|_{H^s}^2 \exp\{\int_0^{T^*}[J(\|b\|_{H^2}) + \|u\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\nabla b\|_{H^2}^2]d\tau\}\\ &\le C.\end{aligned} \tag{4.4}$$

The result implies that the solution can be extended even when $t = T^*$, which contradicts the definition of T^*. Therefore, we have

$$\int_0^{T^*}(\|\nabla u\|_{L^2}^4 + \|\nabla^2 b\|_{L^2}^4)d\tau = +\infty. \tag{4.5}$$

Finally, the converse is readily obvious from (4.4).

5. Global existence

First, for simplicity, we define

$$E(t) := \|u(t)\|_{H^s}^2 + \|b(t)\|_{H^{s+1}}^2 + \int_0^t(\|\nabla u\|_{H^s}^2 + \|b\|_{H^{s+2}}^2(\tau)d\tau,$$

$$E(0) := \|u_0\|_{H^s}^2 + \|b_0\|_{H^{s+1}}^2.$$

Assuming that the solution (u,b) blows up at T^* and $T^* < \infty$, by utilizing the same proof of (3.14), one can find that when $t\in[0,T^*)$,

$$E(t) \le 2E(0) \le 2\varepsilon,$$

then

$$\int_0^{T^*}(\|\nabla u\|_{L^2}^4 + \|\nabla^2 b\|_{L^2}^4)d\tau \le 4\varepsilon^2 < +\infty.$$

On the basis of the blow-up criterion we proved in Section 4, we conclude that $T^* = \infty$. Therefore, the proof of Theorem 1.1 is completed.

6. Conclusions

In this paper, we studied a streamlined model that shares numerous similarities with the more complicated viscoelastic rate-type models commonly used in modeling fluids with complex microstructures. We have proven the blow-up result for this fluid, which implies the global well-posedness of the 3-D incompressible viscoelastic rate-type fluid system. In other words, this model has been mathematically proven to be thermodynamically consistent, and the blow-up criteria implies that these models are capable of describing the long-term behavior of fluids under small initial conditions and external forces. There are still many issues to explore concerning this equation, such as studying the stability of the solution or the existence of the smooth solution in the case of variable density, among others. These results will provide important preliminary insights into the mathematical properties of those more complex and practically significant models of non-Newtonian fluids used in applications.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author would like to thank the anonymous referees for their suggestions which made the paper more readable.

Conflict of interest

The author has no relevant financial or non-financial interests to disclose. The author has no competing interests to declare that are relevant to the content of this article.

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