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Research article

On the list injective coloring of planar graphs without a 4^--cycle intersecting with a 5^--cycle

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Abstract: An injective coloring of a graph G is a vertex coloring such that a pair of vertices obtain distinct colors if there is a path of length two between them. It is proved in this paper that $\chi_i^l(G) \leq \Delta+4$ if $\Delta \geq 12$ when G does not have a 4^--cycle intersecting with a 5^--cycle. Our result improves a previous result of Cai et al. in 2023, who showed that $\chi_i^l(G) \leq \Delta + 4$ when $\Delta \geq 12$ and G has disjoint 5^--cycles.

Keywords: injective coloring; face; discharging method; planar graph; cycle
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1. Introduction

Let G be a finite, simple, and planar graphs throughout this paper. A 2-*distance k-coloring* of G is a mapping $c : V(G) \rightarrow \{1, 2, \cdots, k\}$ such that the vertices whose distance is at most two receive distinct colors. The 2-*distance chromatic number* is the least integer k such that G has a 2-distance k-coloring, denoted by $\chi_2(G)$. In 1977, Wegner [15] first defined the 2-distance coloring and proposed the following conjecture:

Conjecture 1.1. *Let G be a planar graph with maximum degree Δ. Then*

$$\chi_2(G) \leq \begin{cases} 7 & if\ \Delta = 3; \\ \Delta + 5 & if\ 4 \leq \Delta \leq 7; \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1 & if\ \Delta \geq 8. \end{cases}$$

Recently, Kim and Lian [10] showed that each subcubic planar graph G has $\chi_2(G) \leq 7$ if $g(G) \geq 6$. Then Yu et al. [16] proved that every planar graph G has $\chi_2(G) \leq 18$ if $\Delta \leq 5$, and $\chi_2(G) \leq 4\Delta - 3$ if $\Delta \geq 6$.

An *injective k-coloring* of G is a mapping $c : V(G) \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ if there is a path of length two between u and v. The *injective chromatic number* of G is the smallest positive integer k such that G is injectively k-colorable, denoted by $\chi_i(G)$. So it is clear that $\chi_i(G) \leq \chi_2(G)$. Give a list assignment, $L = \{L(v) : v \in V(G)\}$. A list injective coloring of G is an injective coloring of G such that $c(v) \in L(v)$ for each vertex $v \in G$. An injective *L-coloring* is an injective coloring such that $c(v) \in L(v)$ for any vertex v of G. Moreover, G is *injectively k-choosable* if G has an injective L-coloring for any L with $|L(v)| \geq k$. The *injective choosability number* of G is the smallest positive integer k such that G is injectively k-choosable, denoted by $\chi_i^l(G)$.

In 2002, Hahn et al. [9] first defined the concept of injective coloring, and they applied it to the theory of error-correcting codes. If the injective chromatic number of the hypercube Q_n had been shown to be exponential in n, then there would have been consequences for some complexity concerns on random access machines. They gave a slightly smaller upper bound: $\chi_i(G) \leq \Delta^2 - \Delta + 1$. The upper bound is strengthened to $\chi_i(G) \leq \Delta^2 - \Delta$ if $\Delta \geq 3$ by Chen et al. [7]. In 2010, Lužar [12] gave the following conjecture by studying Conjecture 1.1:

Conjecture 1.2. *Let G be a planar graph with maximum degree Δ.*

$$\chi_i(G) \leq \begin{cases} 5 & if\ \Delta = 3; \\ \Delta + 5 & if\ 4 \leq \Delta \leq 7; \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1 & if\ \Delta \geq 8. \end{cases}$$

Many researchers have done many studies on this conjecture. For every K_4-minor-free graph G, Chen et al. [7] showed $\chi_i(G) \leq \left\lceil \frac{3}{2}\Delta \right\rceil$. They conjectured $\chi_i(G) \leq \left\lceil \frac{3}{2}\Delta \right\rceil$ for every planar graph G, but Lužar and Škrekovski [13] proved it was wrong.

Also, there are many results about planar graph G with girth restrictions.

Theorem 1.3. *(Lužar et al. [14]) Let G be a planar graph and $\Delta \leq 3$.*

(1) *If $g(G) \geq 19$, then $\chi_i(G) \leq 3$;*
(2) *If $g(G) \geq 10$, then $\chi_i(G) \leq 4$;*
(3) *If $g(G) \geq 7$, then $\chi_i(G) \leq 5$.*

Theorem 1.4. *Suppose that planar graph G has $g(G) \geq 6$.*

(1) *[6] $\chi_i^l(G) \leq \Delta + 3$;*
(2) *[3] If $\Delta \geq 8$, then $\chi_i^l(G) \leq \Delta + 2$;*
(3) *[1] If $\Delta \geq 24$, then $\chi_i^l(G) \leq \Delta + 1$.*

Theorem 1.5. *Suppose that planar graph G has $g(G) \geq 5$.*

(1) [5] $\chi_i^l(G) \leq \Delta + 6$;
(2) [2] *If $\Delta \geq 11$, then $\chi_i^l(G) \leq \Delta + 4$;*
(3) [8] *If $\Delta \geq 2339$, then $\chi_i(G) \leq \Delta + 1$.*

More recently, Li et al. [11] proved that if G is a planar graph with $\Delta \geq 22$ that has no intersecting 4-cycles or triangles, then $\chi_i^l(G) \leq \Delta + 4$. Cai et al. [4] showed that $\chi_i^l(G) \leq \Delta + 4$ if G is a planar graph with $\Delta \geq 12$ that has disjoint 5^--cycles. We strengthen the result of Cai et al. [4] and allow that G has 5-cycles intersecting 5-cycles by proving Theorem 1.6.

Theorem 1.6. *If G is a planar graph with $\Delta \geq 12$ that has no 4^--cycles intersecting with 5^--cycles, then $\chi_i^l(G) \leq \Delta + 4$.*

We define $N(v) = \{v_1, v_2, \cdots, v_k\}$ and $D(v) = \sum_{1\le i\le k} d(v_i)$ for a k-vertex v. The number of k-neighbors of v is denoted by $n_k(v)$. For a 2^+-vetex v, if $D(v) \ge \Delta + 4 + d(v)$, then v is called a *heavy* vertex; otherwise, v is called a *light* vertex. A $k(l)$-vertex is a k-vertex that has l 2-neighbors. For a path xwy, if $d(w) = 2$, then we say x and y are $fake\text{-}adjacent$. The number of k-faces that are incident with v, denoted $m_k(v)$. We say that 2-vertex v is of *Class one*(*resp.*, *Class two*, *Class three*, *Class four*, *Class five*) if $m_3(v) = 1$(*resp.*, $m_4(v) = 1$, $m_5(v) = 2$, $m_5(v) = m_{6^+}(v) = 1$, $m_{6^+}(v) = 2$). If a 3(1)-vertex w has a Class one 2-neighbor, then w is called a *strong* 3(1)-vertex; otherwise, w is called a *weak* 3(1)-vertex. The number of strong 3(1)-neighbors of w is denoted by $n_{st}(w)$. For k-vertex v, we define $f_1, f_2, \cdots, f_k$ as being incident with v. If two cycles have a common vertex, then we say they are intersecting with each other.

Observation. If v is a Class one 2-vertex, then $m_{7^+}(v) = 1$; if v is a Class two 2-vertex, then $m_{6^+}(v) = 1$.

2. Structural properties of critical graphs

With the intention of proving Theorem 1.6, we suppose instead that G is a counterexample with the fewest edges, which indicates that $\chi_i^l(G) > \Delta + 4$ and $\chi_i^l(H) \le \Delta + 4$ for any $H \subset G$. For a partial vertex coloring c of G and each vertex v, the forbidden color set is denoted by $F(v)$, and L denotes an arbitrary list assignment with $|L(v)| \ge \Delta + 4$.

Lemma 2.1. *There are no adjacent light vertices.*

Proof. Suppose that u and v are distinct light vertices and $uv \in E(G)$. By G having the fewest edges, $G - uv$ is injective L-choosable. Decolor u and v. Clearly, $|F(v)| \le D(u) - d(u) \le \Delta + 3$ and $|F(u)| \le D(u) - d(u) \le \Delta + 3$. So recolor u and v by $c(u) \in L(u) - F(u)$ and $c(v) \in L(v) - F(v)$. Then G has an injective L-coloring, a contradiction. □

It is easy to know the following corollaries from Lemma 2.1.

Corollary 2.2. *There are no adjacent* 2*-vertices, and* $\delta(G) \ge 2$.

Corollary 2.3. *Suppose* $3 \le d(v) \le 5$. *If* $n_2(v) \ge 1$, *then* v *is a heavy vertex and* $n_2(v) \le d(v) - 2$.

Lemma 2.4. *Let* v *be a 3(1)-vertex and* u *be a 5-vertex. If* $n_2(u) \ge 1$, *then* $uv \notin E(G)$.

Proof. Assume the assertion of the lemma is false that u is adjacent to v. Suppose that v_1 is the 2-neighbor of v and u_1 is the 2-neighbor of u. By the choice of G, $G - vv_1$ is injective L-choosable. Remove the colors of v, u_1, and v_1. Clearly, $|F(v)| \le \Delta + 3$. Note that v_1 and u_1 are light vertices, which indicates that $|F(v_1)| \le D(v_1) - d(v_1) \le \Delta + 3$ and $|F(u_1)| \le D(u_1) - d(u_1) \le \Delta + 3$. Thereby, we recolor v, u_1, and v_1 in sequence, a contradiction. □

Lemma 2.5. *Suppose that* $d(v) = 6$ *and* v_1 *is a 2-neighbor of* v. *If* $m_4(v_1) = 1$, *then* v *is a heavy vertex.*

Proof. Assume to the contrary that v is a light vertex. By G having the fewest edges, $G - vv_1$ has an injective L-coloring. Decolor v and v_1. Clearly, $|F(v_1)| \le 5 + \Delta - 2 = \Delta + 3$. Since v is a light vertex, we have $|F(v)| \le D(v) - d(v) \le \Delta + 3$. So we can recolor v_1 and v in sequence, a contradiction. □

Lemma 2.6. *Let $f = [v_1'v_1vv_2v_2']$ be a 5-face with $d(v) = 6$, $d(v_1) = d(v_2') = 2$, and $d(v_2) = 3$. Then v is a heavy vertex.*

Proof. Assume to the contrary that v is a light vertex. Clearly, $G - vv_1$ has an injective L-coloring. Decolor v, v_1, and v_2'. Clearly, $|F(v_1)| \leq d(v) - 1 + \Delta - 2 = \Delta + 3$. It follows from v and v_2' being light vertices that we can recolor v_1, v, and v_2' in sequence, a contradiction. □

Lemma 2.7. *Let v be a 3(1)-vertex. If v_1 is a Class one 2-neighbor, then $D(v) \geq \Delta + 8$.*

Proof. Suppose to the contrary that $D(v) \leq \Delta + 7$. By Corollary 2.3, we need to consider that $D(v) = \Delta + 7$. It follows from G having the fewest edges that $G - vv_1$ has an injective L-coloring. Decolor v and v_1. Obviously, $|F(v)| \leq D(v) - d(v) - 1 \leq \Delta + 3$. Notice that v_1 is a light vertex. So, we recolor v and v_1 in sequence, a contradiction. □

3. Proof of Theorem 1.6

Note that G has no 4^--cycles intersect with 5^--cycles. According to Euler's formula $|V(G)|+|F(G)|-|E(G)| = 2$, and $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$, we derive the following equation:

$$\sum_{v\in V(G)} (d(v) - 6) + \sum_{f\in F(G)} (2d(f) - 6) = -12.$$

Then we construct the weight function $\omega(v) = d(v) - 6$ for each $v \in V(G)$ and $\omega(f) = 2d(f) - 6$ for each $f \in F(G)$, which means that $\sum_{x\in V(G)\bigcup F(G)} \omega(x) = -12$. In this section, we get a new weight function $\omega'(x)$ by assigning the weight. Thereby, we have the following contradiction:

$$0 \leq \sum_{x\in V(G)\bigcup F(G)} \omega'(x) = \sum_{x\in V(G)\bigcup F(G)} \omega(x) = -12.$$

It shows that G does not exist, so Theorem 1.6 is proved. Then $\tau(u \to v)$ shows the weight that u transfers to v, and $\tau(u \to f \to v)$ denotes the weight that u transfers to v by f, where $u, v \in V(G)$ and $f \in F(G)$. Next, we introduce two face types of *configuration A* and *configuration B*. We define the number of configuration A-face(*resp.*, B-face) contain v as $m_A(v)$(*resp.*, $m_B(v)$).

configuration A-face: Suppose $f = [v_1'v_1vv_2v_2']$ with $6 \leq d(v) \leq 8$, $2 \leq d(v_i) \leq 3$, and $d(v_i') \geq 10 (i = 1, 2)$ (See Figure 1(a)). *configuration B*-face: Suppose $f = [v_1'v_1vv_2v_2']$ with $d(v) = 9$, $d(v_i) = 2$ and $d(v_i') \geq 9 (i = 1, 2)$ (See Figure 1(b)).

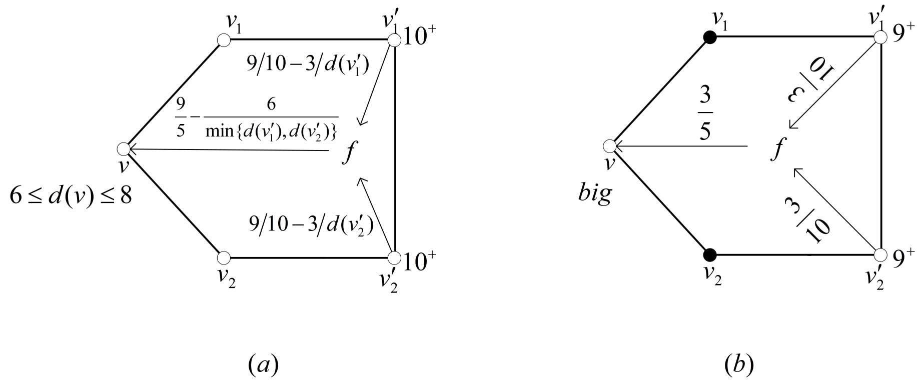

Figure 1. Discharging rule R8.

The discharging rules

R1 Let f be a 4^+-face. Then $\tau(f \to incident\ vertices) = 2 - \frac{6}{d(f)}$.

R2 Let $uv \in E(G)$ and $d(u) \geq 3$. If v is a Class one(*resp.*, Class two, Class three, Class four, Class five) 2-vertex, then $\tau(u \to v) = \frac{10}{7}(resp., \frac{5}{4}, \frac{6}{5}, \frac{11}{10}, 1)$.

R3 Let $d(v) = 3$ and $uv \in E(G)$.

R3.1 Suppose that v is a light 3(0)-vertex and u is a heavy vertex with $3 \leq d(u) \leq 7$. Then $\tau(u \to v) = \frac{1}{3}$.

R3.2 Suppose that v is a heavy 3(0)-vertex. If $d(u) = 4(resp., 5 \leq d(u) \leq 7)$, then $\tau(u \to v) = \frac{1}{30}(resp., \frac{1}{3})$.

R3.3 Suppose that v is a weak 3(1)-vertex. Assume $m_3(v) = 1$, if $d(u) = 5(resp., 6, 7, 8, 9)$, then $\tau(u \to v) = \frac{7}{10}(resp., \frac{3}{4}, \frac{4}{5}, \frac{9}{10}, \frac{11}{10})$. Assume $m_{4^+}(v) = 3$, if $d(u) = 5(resp., 6, 7, 8, 9)$, then $\tau(u \to v) = \frac{1}{2}(resp., \frac{11}{20}, \frac{3}{5}, \frac{9}{10}, 1)$.

R3.4 Suppose that v is a strong 3(1)-vertex. If $d(u) = 6(resp., 7, 8, 9)$, then $\tau(u \to v) = \frac{7}{8}(resp., \frac{12}{13}, 1, \frac{8}{7})$. If $d(u) \geq 10$, then $\tau(u \to v) = 2 - \frac{7}{d(u)}$.

R4 Let $d(v) = 4$ and $uv \in E(G)$. If $d(u) = 5(resp., 6 \leq d(u) \leq 7)$, then $\tau(u \to v) = \frac{1}{30}(resp., \frac{2}{7})$.

R5 Suppose $8 \leq d(u) \leq 9$ such that $uv \in E(G)$.

R5.1 If $d(v) = 3, 4$ except for 3(1)-vertex, then $\tau(u \to v) = \frac{9}{10}$ when $d(u) = 8$ and $\tau(u \to v) = 1$ when $d(u) = 9$.

R5.2 If $d(v) = 5$, then $\tau(u \to v) = \frac{3}{5}$.

R5.3 If $d(v) = 6$, then $\tau(u \to v) = \frac{3}{5}$ when $d(u) = 9$.

R6 Suppose $3 \leq d(v) \leq 8$ except for strong 3(1)-vertex. If $d(u) \geq 10$ such that $uv \in E(G)$, then $\tau(u \to v) = \frac{9}{5} - \frac{6}{d(u)} \geq \frac{6}{5}$.

R7 Let $d(u) \geq 10$ and $6 \leq d(v) \leq 8$. If u is fake-adjacent to v by a Class five 2-vertex, then $\tau(u \to v) = \frac{4}{5} - \frac{6}{d(u)}$.

R8 Suppose $f = [v_1'v_1vv_2v_2']$. After R1~R6, the 9-vertex v is called *big* 9-vertex if $\omega'(v) < 0$.

R8.1 If f is a configuration A-face, then $\tau(v_i' \to f) = \frac{9}{10} - \frac{3}{d(v_i')}(i = 1, 2)$ through $v_1'v_2'$ and $\tau(f \to v) \geq \frac{9}{5} - \frac{6}{\min\{d(v_1'), d(v_2')\}}$ (See Figure 1(a)).

R8.2 If v is a big vertex and f is a configuration B-face, then $\tau(v_i' \to f) = \frac{3}{10}$ through $v_1'v_2'$ and f transfers $\frac{3}{5}$ to each big 9-vertex equally (See Figure 1(b)).

Firstly, we check $\omega'(v)$ for each $v \in V(G)$.

Case 1. $d(v) = 2$ and then $\omega(v) = -4$.

If v is a Class one 2-vertex, then $m_3(v) = m_{7^+}(v) = 1$, which means that $\sum_{i=1}^{2} \tau(f_i \to v) \geq \frac{8}{7}$ by R1. Hence, $\omega'(v) \geq -4 + \frac{8}{7} + \frac{10}{7} \times 2 = 0$ by R2. If v is a Class two 2-vertex, then $m_4(v) = m_{6^+}(v) = 1$, which indicates that $\sum_{i=1}^{2} \tau(f_i \to v) \geq \frac{1}{2} + 1 = \frac{3}{2}$ by R1. So $\omega'(v) \geq -4 + \frac{3}{2} + \frac{5}{4} \times 2 = 0$ by R2. Next, consider that v is a Class three, Class four, or Class five 2-vertex. It resembles the above arguments that can be obtained that $\omega'(v) \geq -4 + \min\{\frac{6}{5} \times 2 + \frac{4}{5} \times 2, \frac{11}{10} \times 2 + \frac{4}{5} + 1, 2 + 2\} = 0$ by R1, R2.

Case 2. $d(v) = 3$ and then $\omega(v) = -3$. Let $N(v) = \{v_1, v_2, v_3\}$ with $d(v_1) \leq d(v_2) \leq d(v_3)$.

Subcase 2.1. Suppose $n_2(v) = 0$. If $m_{4^-}(v) = 1$, then $m_{6^+}(v) = 2$; otherwise, $m_{5^+}(v) = 3$. Then $\sum_{i=1}^{3} \tau(f_i \to v) \geq \min\{2, \frac{4}{5} \times 3\} = 2$ by R1. Consider that v is a light vertex, which means that $v_i (i = 1, 2, 3)$ are heavy vertices by Lemma 2.1. According to R3.1, R5.1, R6, $\sum_{i=1}^{3} \tau(v_i \to v) \geq \frac{1}{3} \times 3 = 1$. Then $\omega'(v) \geq -3 + 2 + 1 = 0$. Otherwise, consider that v is a heavy vertex. Suppose $n_3(v) = 0$. If $n_4(v) = 0$, then $n_{5^+}(v) = 3$; if $n_4(v) \geq 1$, then either $n_{9^+}(v) \geq 1$ or $n_7(v) = n_8(v) = 1$, which means that $\sum_{i=1}^{3} \tau(v_i \to v) \geq \min\{\frac{1}{3} \times 3, \frac{1}{30} + 1, \frac{1}{30} + \frac{1}{3} + \frac{9}{10}\} = 1$ by R3.2, R5.1, R6. Hence, $\omega'(v) \geq -3 + 2 + 1 = 0$. Suppose $d(v_1) = 3$, which indicates that $\tau(v \to v_1) \leq \frac{1}{3}$ by R3.1. If $d(v_2) = 4$, then $d(v_3) \geq 12$; if $5 \leq d(v_2) \leq 7$, then $d(v_3) \geq 9$; otherwise, $d(v_2), d(v_3) \geq 8$. So $\sum_{i=2}^{3} \tau(v_i \to v) \geq \min\{\frac{1}{30} + (\frac{9}{5} - \frac{6}{12}), \frac{1}{3} + 1, \frac{9}{10} \times 2\} = \frac{4}{3}$ by R3.2, R5.1, R6. Furthermore, $\omega'(v) \geq -3 + 2 - \frac{1}{3} + \frac{4}{3} = 0$.

Subcase 2.2. Suppose $d(v_1) = 2$. By Corollary 2.3, $d(v_2) + d(v_3) \geq \Delta + 5$. Consider that v is a weak 3(1)-vertex. Suppose $m_3(v) = 1$. If $d(v_2) = 5$, then $d(v_3) \geq 12$; if $d(v_2) = 6$, then $d(v_3) \geq 11$; if $d(v_2) = 7$, then $d(v_3) \geq 10$; otherwise, $d(v_2) \geq 8$ and $d(v_3) \geq 9$. Therefore, $\sum_{i=2}^{3} \tau(v_i \to v) \geq \min\{\frac{7}{10} + (\frac{9}{5} - \frac{6}{12}), \frac{3}{4} + (\frac{9}{5} - \frac{6}{11}), \frac{4}{5} + (\frac{9}{5} - \frac{6}{10}), \frac{9}{10} + \frac{11}{10}\} = 2$ by R3.3, R5, R6. Thereby, $\omega'(v) \geq -3 + 2 - 1 + 2 = 0$ by R1, R2. If $m_{4^+}(v) = 3$, then $-\tau(v \to v_1) + \sum_{i=1}^{3} \tau(f_i \to v) \geq \min\{-\frac{5}{4} + \frac{1}{2} + 2, -\frac{6}{5} + \frac{4}{5} \times 3\} = \frac{6}{5}$ by R1, R2. So $\omega'(v) \geq -3 + \frac{6}{5} + \min\{\frac{1}{2} + (\frac{9}{5} - \frac{6}{12}), \frac{11}{20} + (\frac{9}{5} - \frac{6}{11}), \frac{3}{5} + (\frac{9}{5} - \frac{6}{10}), \frac{9}{10} + 1\} = 0$ by R3.3, R6. Next, consider that v is a strong 3(1)-vertex. By Lemma 2.7, $d(v_2) + d(v_3) \geq \Delta + 6$. Moreover, $-\tau(v \to v_1) + \sum_{i=1}^{3} \tau(f_i \to v) \geq -\frac{10}{7} + \frac{8}{7} + 1 = \frac{5}{7}$ by R1, R2. Thus, $\omega'(v) \geq -3 + \frac{5}{7} + \min\{\frac{7}{8} + (2 - \frac{7}{12}), \frac{12}{13} + (2 - \frac{7}{11}), 1 + (2 - \frac{7}{10}), \frac{8}{7} + \frac{8}{7}\} = 0$ by R3.4.

Case 3. $d(v) = 4$ and then $\omega(v) = -2$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$ with $d(v_1) \leq d(v_2) \leq d(v_3) \leq d(v_4)$.

Subcase 3.1. Suppose $n_2(v) = 0$. If $m_{4^-}(v) = 1$, then $m_{6^+}(v) = 3$; otherwise, $m_{5^+}(v) = 4$. So $\sum_{i=1}^{4} \tau(f_i \to v) \geq \min\{3, \frac{4}{5} \times 4\} = 3$ by R1. If v is a heavy vertex, then $n_3(v) \leq 3$. Note that $n_{11^+}(v) \geq 1$ if $n_3(v) = 3$. Therefore, $-\tau(v \to 3\text{-}neighbors) + \tau(11^+\text{-}neighbor \to v) \geq \min\{-\frac{1}{3} \times 3 + (\frac{9}{5} - \frac{6}{11}), -\frac{1}{3} \times 2\} = -\frac{2}{3}$ by R3.1, R6. Hence, $\omega'(v) \geq -2 + 3 - \frac{2}{3} = \frac{1}{3}$. Otherwise, suppose that v is a light vertex, which implies that $v_i (1 \leq i \leq 4)$ are not light 3-vertices by Lemma 2.1. Clearly, $\omega'(v) \geq -2 + 3 - \frac{1}{30} \times 4 = \frac{13}{15}$ by R3.2.

Subcase 3.2. Suppose $d(v_1) = 2$. By Corollary 2.3, $d(v_2) + d(v_3) + d(v_4) \geq \Delta + 6$. According to R1, R2, if v_1 is a Class one 2-vertex, then $-\tau(v \to v_1) + \sum_{i=1}^{4} \tau(f_i \to v) \geq -\frac{10}{7} + \frac{8}{7} + 2 = \frac{12}{7}$; if v_1 is a Class two 2-vertex, then $-\tau(v \to v_1) + \sum_{i=1}^{4} \tau(f_i \to v) \geq -\frac{5}{4} + \frac{1}{2} + 3 = \frac{9}{4}$; if $m_{5^+}(v) = 4$, then

$-\tau(v \to v_1) + \sum\limits_{i=1}^{4} \tau(f_i \to v) \geq -\frac{6}{5} + \frac{4}{5} \times 4 = 2$; otherwise $m_{6^+}(v_1) = 2$ and $m_{4^-}(v) = 1$, then $-\tau(v \to v_1) + \sum\limits_{i=1}^{4} \tau(f_i \to v) \geq -1 + 3 = 2$. Therefore, $-\tau(v \to v_1) + \sum\limits_{i=1}^{4} \tau(f_i \to v) \geq \min\{\frac{12}{7}, \frac{9}{4}, 2, 2\} = \frac{12}{7}$. If $n_3(v) = 0$, then $n_{6^+}(v) \geq 1$; if $n_3(v) = 1$, then $n_{8^+}(v) \geq 1$; if $n_3(v) = 2$, then $n_{12^+}(v) = 1$. This implies that $-\tau(v \to 3\text{-}neighbors) + \tau(6^+\text{-}neighbors \to v) \geq \min\{\frac{2}{7}, -\frac{1}{3} + \frac{9}{10}, -\frac{1}{3} \times 2 + (\frac{9}{5} - \frac{6}{12})\} = \frac{2}{7}$ by R3.1, R4, R5.1, R6. Moreover, $\omega'(v) \geq -2 + \frac{12}{7} + \frac{2}{7} = 0$.

Subcase 3.3. Suppose $d(v_1) = d(v_2) = 2$, which means that $d(v_3) + d(v_4) \geq \Delta + 4$ by Corollary 2.3. According to R1, R2, $-\tau(v \to v_1) - \tau(v \to v_2) + \sum\limits_{i=1}^{4} \tau(f_i \to v) \geq \min\{-\frac{10}{7} - 1 + \frac{8}{7} + 2, -\frac{5}{4} \times 2 + \frac{1}{2} + 3, -\frac{6}{5} \times 2 + \frac{4}{5} \times 4, -2 + 3\} = \frac{5}{7}$. If $d(v_3) = 4$, then $d(v_4) \geq 12$; if $d(v_3) = 5$, then $d(v_4) \geq 11$; if $6 \leq d(v_3) \leq 7$, then $d(v_4) \geq 9$; otherwise, $d(v_3), d(v_4) \geq 8$. This indicates that $\sum\limits_{i=3}^{4} \tau(v_i \to v) \geq \min\{\frac{9}{5} - \frac{6}{12}, \frac{1}{30} + (\frac{9}{5} - \frac{6}{11}), \frac{2}{7} + 1, \frac{9}{10} \times 2\} = \frac{9}{7}$ by R4, R5.1, R6. So $\omega'(v) \geq -2 + \frac{5}{7} + \frac{9}{7} = 0$.

Claim 3.1. *Let* $5 \leq d(v) \leq 7$. *Note that* v *is adjacent to at most one weak 3(1)-vertex that is incident with a 3-face.*

Case 4. $d(v) = 5$ and then $\omega(v) = -1$.

Subcase 4.1. Suppose $n_2(v) = 0$. According to R1, $\sum\limits_{i=1}^{5} \tau(f_i \to v) \geq \min\{4, 4 + \frac{1}{2}, \frac{4}{5} \times 5\} = 4$. Therefore, $\omega'(v) \geq -1 + 4 - \frac{7}{10} - \frac{1}{2} \times 4 = \frac{3}{10}$ by R3.3 and Claim 3.1. Suppose $n_2(v) = 1$, which implies that v is a heavy vertex by Corollary 2.3 and $n_{3(1)}(v) = 0$ by Lemma 2.4. Then $n_3(v) + n_4(v) \leq 3$. According to R1, R2, $-\tau(v \to 2\text{-}neighbor) + \sum\limits_{i=1}^{5} \tau(f_i \to v) \geq \min\{-\frac{10}{7} + \frac{8}{7} + 3, -\frac{5}{4} + \frac{1}{2} + 4, -\frac{6}{5} + \frac{4}{5} \times 5, -1 + 4\} = \frac{19}{7}$. Note that $n_{10^+}(v) = 1$ if $n_3(v) = 3$. Therefore, $-\tau(v \to 3\text{-}neighbors) - \tau(v \to 4\text{-}neighbors) + \tau(10^+\text{-}neighbors \to v) \geq \min\{-\frac{1}{3} \times 3 + (\frac{9}{5} - \frac{6}{10}), -\frac{1}{3} \times 2 - \frac{1}{30} \times 2\} = -\frac{11}{15}$ by R3.1, R3.2, R4, R6. Hence, $\omega'(v) \geq -1 + \frac{19}{7} - \frac{11}{15} = \frac{103}{105}$.

Subcase 4.2. Suppose $n_2(v) = 2$. This implies that v is a heavy vertex by Corollary 2.3, and $n_{3(1)}(v) = 0$ by Lemma 2.4. Then $n_3(v) + n_4(v) \leq 2$. According to R1, R2, $-\tau(v \to 2\text{-}neighbors) + \sum\limits_{i=1}^{5} \tau(f_i \to v) \geq \min\{-\frac{10}{7} - 1 + \frac{8}{7} + 3, -\frac{5}{4} \times 2 + \frac{1}{2} + 4, -\frac{6}{5} \times 2 + \frac{4}{5} \times 5, -2 + 4\} = \frac{8}{5}$. Note that $n_{11^+}(v) = 1$ if $n_3(v) = 2$. Moreover, $-\tau(v \to 3\text{-}neighbors) - \tau(v \to 4\text{-}neighbors) + \tau(11^+\text{-}neighbor \to v) \geq \min\{-\frac{1}{3} \times 2 + (\frac{9}{5} - \frac{6}{11}), -\frac{1}{3} - \frac{1}{30} \times 2\} = -\frac{2}{5}$ by R3.1, R4, R6. Thus, $\omega'(v) \geq -1 + \frac{8}{5} - \frac{2}{5} = \frac{1}{5}$. Suppose $n_2(v) = 3$, which means that v is a heavy vertex by Corollary 2.3, and $n_{3(1)}(v) = 0$ by Lemma 2.4. Then $n_3(v) + n_4(v) \leq 1$. Clearly, $-\tau(v \to 2\text{-}neighbors) + \sum\limits_{i=1}^{5} \tau(f_i \to v) \geq \min\{-\frac{10}{7} - 2 + \frac{8}{7} + 3, -\frac{5}{4} \times 2 - 1 + \frac{1}{2} + 4, -\frac{6}{5} \times 3 + \frac{4}{5} \times 5, -3 + 4\} = \frac{2}{5}$ by R1, R2. If $n_3(v) = n_4(v) = 0$, then $n_{8^+}(v) \geq 1$; if $n_3(v) = 1$, then $n_{12^+}(v) = 1$; if $n_4(v) = 1$, then $n_{11^+}(v) = 1$. Moreover, $\omega'(v) \geq -1 + \frac{2}{5} + \min\{\frac{3}{5}, -\frac{1}{3} + (\frac{9}{5} - \frac{6}{12}), -\frac{1}{30} + (\frac{9}{5} - \frac{6}{11})\} = 0$ by R3.1, R4, R5.2, R6.

Claim 3.2. *Suppose* $d(v) \geq 6$, $m_{5^+}(v) = d(v)$, *and* $n_{st}(v) = t$. *If* $1 \leq t \leq d(v) - 1$, *then* $m_{6^+}(v) \geq t + 1$, *w.l.o.g.,* $m_{6^+}(v) \geq t$ *for* $t \geq 0$.

Case 5. $d(v) = 6$ and then $\omega(v) = 0$.

Claim 3.3. *Consider that v is a light vertex. Suppose that w is a 2-neighbor of v and u is a 3(1)-neighbor of v. Then $m_{4^-}(w) = 0$ and uv is not incident with 3-face. If v is incident with a configuration A-face f contains 2-neighbors and 3(1)-neighbor of v, then $\tau(f \to v) \geq \frac{69}{55}$.*

Proof. By Lemma 2.5, $m_4(w) = 0$. If $m_3(w) = 1$, then there is a Δ-vertex in $N(v)$ by Lemma 2.1, which means v is a heavy vertex, a contradiction. Suppose that uv is incident with 3-face. Then there is a $(\Delta-1)^+$-vertex in $N(v)$, which implies that v is a heavy vertex, a contradiction. Next, consider that v is incident with the configuration A-face f and f is incident with a 2-neighbor and a 3(1)-neighbor of v. By R8.1, it is easy to know that $\tau(f \to v) \geq \frac{9}{5} - \frac{6}{11} = \frac{69}{55}$. □

Subcase 5.1. Suppose $n_2(v) = 0$. If v is a heavy vertex, then $n_3(v) \leq 5$. According to R1, $\sum_{i=1}^{6} \tau(f_i \to v) \geq \min\{\frac{4}{5} \times 6, 5\} = \frac{24}{5}$. Therefore, $\omega'(v) \geq \frac{24}{5} - 5 \times \frac{7}{8} - \frac{2}{7} = \frac{39}{280}$ by R3.4, R4. Otherwise, consider that v is a light vertex. If $m_{4^-}(v) = 1$, then $n_{st}(v) \leq 4$ by Claim 3.2. Then $\omega'(v) \geq 5 - 4 \times \frac{7}{8} - \frac{3}{4} - \frac{11}{20} = \frac{1}{5}$ by R1, R3.3, R3.4 and Claim 3.1. Consider that $m_{5^+}(v) = 6$. If $n_{st}(v) \leq 4$, then $\omega'(v) \geq \frac{4}{5} \times 6 - 4 \times \frac{7}{8} - \frac{11}{20} \times 2 = \frac{1}{5}$ by R1, R3.3, R3.4. If $n_{st}(v) \geq 5$, then $m_{6^+}(v) = 6$ by Claim 3.2. It follows that $\omega'(v) \geq 6 - 6 \times \frac{7}{8} = \frac{3}{4}$ by R1, R3.4.

Subcase 5.2. Suppose $n_2(v) = 1$. If v is a heavy vertex, then either $n_3(v) + n_4(v) \leq 4$ or $n_4(v) = 5$, which derives that $\tau(v \to 3\text{-}neighbors) + \tau(v \to 4\text{-}neighbors) \leq \max\{\frac{7}{8} \times 4, \frac{2}{7} \times 5\} = \frac{7}{2}$ by R3.4, R4. According to R1, R2, $-\tau(v \to 2\text{-}neighbor) + \sum_{i=1}^{6} \tau(f_i \to v) \geq \min\{-\frac{10}{7} + \frac{8}{7} + 4, -\frac{5}{4} + \frac{1}{2} + 5, -\frac{6}{5} + \frac{4}{5} \times 6, -1 + 5\} = \frac{18}{5}$. Therefore, $\omega'(v) \geq \frac{18}{5} - \frac{7}{2} = \frac{1}{10}$. Consider that v is a light vertex. If $m_{4^-}(v) = 1$, then $n_{st}(v) \leq 3$ by Claim 3.2 and Claim 3.3, and $\tau(v \to 2\text{-}neighbor) = 1$ by R2 and Claim 3.3. So $\omega'(v) \geq -1 + 5 - \frac{7}{8} \times 3 - \frac{3}{4} - \frac{11}{20} = \frac{3}{40}$ by R1, R3.3, R3.4, and Claim 3.1. Consider that $m_{5^+}(v) = 6$. Suppose $n_{st}(v) = t \leq 5$. Then $-\tau(v \to 2\text{-}neighbor) + \sum_{i=1}^{6} \tau(f_i \to v) \geq -\frac{6}{5} + t + \frac{4}{5}(6-t) = \frac{1}{5}t + \frac{18}{5}$ by R1, R2 and Claim 3.2. Thereby, $\omega'(v) \geq \frac{1}{5}t + \frac{18}{5} - \frac{7}{8}t - \frac{11}{20}(5-t) = -\frac{1}{8}t + \frac{17}{20} \geq \frac{9}{40}$ by R3.3, R3.4.

Subcase 5.3. Suppose $n_2(v) = 2$. If v is a heavy vertex, then $n_3(v) + n_4(v) \leq 3$. According to R1, R2, $-\tau(v \to 2\text{-}neighbors) + \sum_{i=1}^{6} \tau(f_i \to v) \geq \min\{-\frac{10}{7} - 1 + \frac{8}{7} + 4, -\frac{5}{4} \times 2 + \frac{1}{2} + 5, -\frac{6}{5} \times 2 + \frac{4}{5} \times 6, -2 + 5\} = \frac{12}{5}$. Note that $n_{9^+}(v) = 1$ if $n_3(v) = 3$. Clearly, $\omega'(v) \geq \frac{12}{5} + \min\{-\frac{7}{8} \times 3 + \frac{3}{5}, -\frac{7}{8} \times 2 - \frac{2}{7}\} = \frac{51}{140}$ by R3.4, R4, R5.3. Otherwise, consider that v is a light vertex. By Claim 3.3 and R2, $\tau(v \to 2\text{-}neighbors) = 2$. If $m_3(v) = 1$, then $n_{3(1)}(v) \leq 2$ by Claim 3.3. Then $\omega'(v) \geq 5 - 2 - \frac{7}{8} \times 2 - \frac{1}{3} \times 2 = \frac{7}{12}$ by R1, R2, R3.2, R3.4. If $m_4(v) = 1$, then $n_{st}(v) \leq 2$ by Claim 3.2 and Claim 3.3. Obviously, $\omega'(v) \geq -2 + 5 + \frac{1}{2} - \frac{7}{8} \times 2 - \frac{11}{20} \times 2 = \frac{13}{20}$ by R1, R2, R3.3, R3.4, and Claim 3.1. Finally, consider that $m_{5^+}(v) = 6$. Suppose $n_{st}(v) = t \leq 4$. If $t = 0$, then $\omega'(v) \geq -\frac{6}{5} \times 2 + \frac{4}{5} \times 6 - \frac{11}{20} \times 4 = \frac{1}{5}$ by R1, R2, R3.3; if $1 \leq t \leq 3$, then $\omega'(v) \geq -\frac{6}{5} \times 2 + t + 1 + \frac{4}{5}(5-t) - \frac{7}{8}t - \frac{11}{20}(4-t) = -\frac{1}{8}t + \frac{2}{5} \geq \frac{1}{40}$ by R1, R2, R3.3, R3.4 and Claim 3.2; if $t = 4$, then $m_{6^+}(v) \geq 5$ and v has two 2-neighbors of Class four or Class five. Moreover, $\omega'(v) \geq \min\{-\frac{11}{10}, -1\} \times 2 + 5 + \frac{4}{5} - \frac{7}{8} \times 4 = \frac{1}{10}$ by R1, R2, R3.4.

Subcase 5.4. Suppose $n_2(v) = 3$. If v is a heavy vertex, then $n_3(v) + n_4(v) \leq 2$. Clearly, $n_{10^+}(v) = 1$ if $n_3(v) = 2$. Hence, $\omega'(v) \geq -\frac{6}{5} \times 3 + \frac{4}{5} \times 6 + \min\{-\frac{7}{8} \times 2 + (\frac{9}{5} - \frac{6}{10}), -\frac{7}{8} - \frac{2}{7}\} = \frac{11}{280}$ by R1, R2, R3.4, R4, R6. Otherwise, consider that v is a light vertex. If $m_3(v) = 1$, then $n_{3(1)}(v) \leq 1$ by Claim 3.3. Then $\omega'(v) \geq 5 - 3 - \frac{7}{8} - 2 \times \frac{1}{3} = \frac{11}{24}$ by R1, R2, R3.2, R3.4 and Claim 3.3. If $m_4(v) = 1$, then $n_{st}(v) \leq 1$ by Claim 3.2 and Claim 3.3. So $\omega'(v) \geq 5 + \frac{1}{2} - 3 - \frac{7}{8} - \frac{11}{20} \times 2 = \frac{21}{40}$ by R1, R2, R3.3, R3.4, and Claim 3.3. Consider that $m_{5^+}(v) = 6$. Suppose $n_{st}(v) = t \leq 3$. If $m_A(v) \geq 1$, then $\omega'(v) \geq -\frac{6}{5} \times 3 + t + \frac{4}{5}(6-t) + \frac{69}{55} - \frac{7}{8}t -$

$\frac{11}{20}(3-t) = -\frac{1}{8}t + \frac{177}{220} \geq \frac{189}{440}$ by R1, R2, R3.3, R3.4, Claim 3.2 and Claim 3.3. Consider that $m_A(v) = 0$. Suppose $n_{st}(v) = 0$. If $n_{3(1)}(v) = 0$, then $\omega'(v) \geq -\frac{6}{5}\times 3 + \frac{4}{5}\times 6 - \frac{1}{3}\times 3 = \frac{1}{5}$ by R1, R2, R3.2. If $n_{3(1)}(v) \geq 1$, then $m_{6^+}(v) \geq 2$ and v has at least two 2-neighbors of Class four or Class five by Lemma 2.6. Hence, $\omega'(v) \geq -\frac{6}{5} - \frac{11}{10}\times 2 + 2 + \frac{4}{5}\times 4 - \frac{11}{20}\times 3 = \frac{3}{20}$ by R1, R2, R3.3. Consider that $n_{st}(v) = 1$. Then $m_{6^+}(v) \geq 2$ by Claim 3.2. Note that $m_{6^+}(v) \geq 4$ if $n_{3(1)}(v) = 2$; $m_{6^+}(v) \geq 5$ if $n_{3(1)}(v) = 3$ by Lemma 2.6. Therefore, $-\tau(v \to 3\text{-}neighbors) + \sum_{i=1}^{6} \tau(f_i \to v) \geq -\frac{7}{8} + \min\{-\frac{11}{20} - \frac{1}{3} + 4 + \frac{4}{5}\times 2, -\frac{11}{20}\times 2 + 5 + \frac{4}{5}, -\frac{1}{3}\times 2 + 2 + \frac{4}{5}\times 4\} = \frac{439}{120}$ by R1, R3.2, R3.3. Hence, $\omega'(v) \geq -\frac{6}{5}\times 3 + \frac{439}{120} = \frac{7}{120}$ by R1, R3.4. If $n_{st}(v) \geq 2$, then $m_{6^+}(v) \geq 4$ and v has one Class five 2-neighbor and two 2-neighbors of Class four or Class five. Note that v is fake-adjacent a Δ-vertex by Class five 2-vertex, which indicates that v receives at least $\frac{4}{5} - \frac{6}{12} = \frac{3}{10}$ by R7. Moreover, $\omega'(v) \geq -1 + \min\{-\frac{11}{10}, -1\}\times 2 + 4 + \frac{4}{5}\times 2 - \frac{7}{8}\times 3 + \frac{3}{10} = \frac{3}{40}$ by R1, R2, R3.4.

Subcase 5.5. Suppose $n_2(v) = 4$. If v is a heavy vertex, then $n_3(v) + n_4(v) \leq 1$. Note that $n_{11^+}(v) = 1$ if $n_3(v) = 1$; $n_{10^+}(v) = 1$ if $n_4(v) = 1$. So $\omega'(v) \geq -\frac{6}{5}\times 4 + \frac{4}{5}\times 6 + \min\{-\frac{7}{8} + (\frac{9}{5} - \frac{6}{11}), -\frac{2}{7} + (\frac{9}{5} - \frac{6}{10}), 0\} = 0$ by R1, R2, R3.4, R4, R6. Consider that v is a light vertex. If $m_3(v) = 1$, then $n_{3(1)}(v) = 0$ by Claim 3.3. So $\omega'(v) \geq 5 - 4 - \frac{1}{3}\times 2 = \frac{1}{3}$ by R1, R2, R3.2. If $m_4(v) = 1$, then $n_{st}(v) = 0$ by Claim 3.2 and Claim 3.3. Then $\omega'(v) \geq 5 + \frac{1}{2} - 4 - \frac{11}{20}\times 2 = \frac{2}{5}$ by R1, R2, R3.3. Next, consider that $m_{5^+}(v) = 6$. If $m_A(v) \geq 2$, then $\omega'(v) \geq -\frac{6}{5}\times 4 + \frac{4}{5}\times 6 + \frac{69}{55}\times 2 - \frac{7}{8}\times 2 = \frac{167}{220}$ by R1, R2, R3.4, and Claim 3.3. Consider that $m_A(v) = 1$. Note that $m_{6^+}(v) \geq 3$ if $n_{st}(v) \geq 1$. Thus, $\omega'(v) \geq -\frac{6}{5}\times 4 + \frac{69}{55} + \min\{\frac{4}{5}\times 3 + 3 - \frac{7}{8}\times 2, \frac{4}{5}\times 6 - \frac{11}{20}\times 2\} = \frac{23}{220}$ by R1, R2, R3.3, R3.4, and Claim 3.3. Finally, suppose $m_A(v) = 0$. If $n_{3(1)}(v) = 0$, then $m_{6^+}(v) \geq 2$ and v has either at least one Class five 2-neighbor or four Class four 2-neighbors. Hence, $\omega'(v) \geq \min\{-\frac{6}{5}\times 3 - 1 + 2 + \frac{4}{5}\times 4 - \frac{1}{3}\times 2 + (\frac{4}{5} - \frac{6}{12}), -\frac{11}{10}\times 4 + 2 + \frac{4}{5}\times 4 - \frac{1}{3}\times 2\} = \frac{2}{15}$ by R1, R2, R3.2, R7. If $n_{3(1)}(v) = 1$, then $m_{6^+}(v) \geq 4$ and v has at least two Class five 2-neighbors by Lemma 2.6. Thus, $\omega'(v) \geq -\frac{6}{5}\times 2 - 2 + 4 + \frac{4}{5}\times 2 - \frac{7}{8} + (\frac{4}{5} - \frac{6}{12})\times 2 - \frac{1}{3} = \frac{71}{120}$ by R1, R2, R3.2, R3.4, R7. If $n_{3(1)}(v) = 2$, then $m_{6^+}(v) \geq 5$ and v has four Class five 2-neighbors by Lemma 2.6. So $\omega'(v) \geq -4 + 5 + \frac{4}{5} - \frac{7}{8}\times 2 + (\frac{4}{5} - \frac{6}{12})\times 4 = \frac{5}{4}$ by R1, R2, R3.4, R7.

Subcase 5.6. Suppose $n_2(v) = 5$. If v is a heavy vertex, then $n_{12^+}(v) = 1$. This shows that $\omega'(v) \geq -\frac{6}{5}\times 5 + \frac{4}{5}\times 6 + (\frac{9}{5} - \frac{6}{12}) = \frac{1}{10}$ by R1, R2, R6. Consider that v is a light vertex, which implies that $m_{5^+}(v) = 6$ by Claim 3.3. If $m_A(v) \geq 2$, then $\omega'(v) \geq -\frac{6}{5}\times 5 + \frac{4}{5}\times 6 + \frac{69}{55}\times 2 - \frac{7}{8} = \frac{191}{440}$ by R1, R2, R3.4, and Claim 3.3. Consider that $m_A(v) = 1$. This implies that $m_{6^+}(v) \geq 3$ and v has at least one Class five 2-neighbor. It follows from R1, R2, R3.4, R7, and Claim 3.3 that $\omega'(v) \geq -1 - 4\times\frac{6}{5} + 3 + 3\times\frac{4}{5} - \frac{7}{8} + (\frac{4}{5} - \frac{6}{12}) + \frac{69}{55} = \frac{123}{440}$. Finally, suppose $m_A(v) = 0$, which means that $m_{6^+}(v) \geq 4$ and v has at least three Class five 2-neighbors. So $\omega'(v) \geq -3 - \frac{6}{5}\times 2 + 4 + \frac{4}{5}\times 2 - \frac{7}{8} + (\frac{4}{5} - \frac{6}{12})\times 3 = \frac{9}{40}$ by R1, R2, R3.4, R7.

Subcase 5.7. Suppose $n_2(v) = 6$. Obviously, v is a light vertex. Then $m_{5^+}(v) = 6$ by Claim 3.3. If $m_A(v) \geq 2$, then $\omega'(v) \geq -\frac{6}{5}\times 6 + \frac{4}{5}\times 6 + \frac{69}{55}\times 2 = \frac{6}{55}$ by R1, R2 and Claim 3.3. If $m_A(v) \leq 1$, then $m_{6^+}(v) \geq 5$ and v has at least four Class five 2-neighbors. Hence, $\omega'(v) \geq -4 - \frac{6}{5}\times 2 + 5 + \frac{4}{5} + (\frac{4}{5} - \frac{6}{12})\times 4 = \frac{3}{5}$ by R1, R2, R7.

Case 6. $d(v) = 7$ and then $\omega(v) = 1$.

Claim 3.4. *If light v is incident with a configuration A-face f and f is incident with 2-neighbors and 3(1)-neighbors of v, then $\tau(f \to v) \geq \frac{9}{5} - \frac{6}{10} = \frac{6}{5}$.*

Subcase 6.1. Suppose $n_2(v) = 0$. Clearly, $\omega'(v) \geq 1 + \min\{6, \frac{4}{5}\times 7\} - \frac{12}{13}\times 7 = \frac{9}{65}$ by R1, R3.4. Suppose $n_2(v) = k \geq 1$. If v is a heavy vertex, then $n_2(v) + n_3(v) \leq 6$. If $1 \leq k \leq 2$, then $\omega'(v) \geq 1 + \min\{-\frac{10}{7} - (k-1) + \frac{8}{7} + 5, -\frac{5}{4}k + \frac{1}{2} + 6, -\frac{6}{5}k + \frac{4}{5}\times 7, -k + 6\} - \frac{12}{13}(6-k) - \frac{2}{7} = -\frac{18}{65}k + \frac{353}{455} \geq \frac{101}{455}$ by R1, R2, R3.4, R4. Consider that $3 \leq k \leq 6$. If $m_{4^-}(v) = 1$, then $\omega'(v) \geq 1 + \min\{-\frac{10}{7} - (k-1) + \frac{8}{7} +$

$5, -\frac{5}{4}\times 2-(k-2)+\frac{1}{2}+6, -k+6\}-\frac{12}{13}(6-k)-\frac{2}{7}=-\frac{1}{13}k+\frac{81}{91}\geq\frac{3}{7}$ by R1, R2, R3.4, R4. Next, suppose $m_{5^+}(v)=7$. If $3\leq k\leq 4$, then $\omega'(v)\geq 1-\frac{6}{5}k+(6-k)+\frac{4}{5}(k+1)-\frac{12}{13}(6-k)-\frac{2}{7}=-\frac{31}{65}k+\frac{899}{455}\geq\frac{31}{455}$ by R1, R2, R3.4, R4, and Claim 3.2. Assume that $k=5$, we have $n_{10^+}(v)=1$ if $n_3(v)=1$. Then $\omega'(v)\geq 1-\frac{6}{5}\times 5+\frac{4}{5}\times 7+\min\{-\frac{12}{13}+(\frac{9}{5}-\frac{6}{10}), -\frac{2}{7}\}=\frac{11}{35}$ by R1, R2, R3.4, R4, R6. If $k=6$, then $n_{11^+}(v)=1$. So $\omega'(v)\geq 1-\frac{6}{5}\times 6+\frac{4}{5}\times 7+(\frac{9}{5}-\frac{6}{11})=\frac{36}{55}$ by R1, R2, R3.4, R6.

Subcase 6.2. Consider that v is a light vertex. If $m_3(v)=1$, then $n_{3(1)}(v)\leq 6-k$. Then $\omega'(v)\geq 1+\min\{-\frac{10}{7}-(k-1)+\frac{8}{7}+5, -k+6\}-\frac{12}{13}(6-k)-\frac{1}{3}=-\frac{1}{13}k+\frac{230}{273}\geq\frac{83}{273}$ by R1, R2, R3.2, R3.4. If $m_4(v)=1$, then $\omega'(v)\geq 1+\min\{-\frac{5}{4}-(k-1)+\frac{1}{2}+6, -\frac{5}{4}\times 2-(k-2)+\frac{1}{2}+6, -k+6\}-\frac{12}{13}(7-k)=-\frac{1}{13}k+\frac{7}{13}\geq 0$ by R1, R2, R3.4. Consider that $m_{5^+}(v)=7$. If $1\leq k\leq 3$, then $\omega'(v)\geq 1-\frac{6}{5}k+\min\{\frac{4}{5}\times 7-\frac{3}{5}(7-k), (7-k)+\frac{4}{5}k-\frac{12}{13}(7-k)\}=-\frac{31}{65}k+\frac{20}{13}\geq\frac{7}{65}$ by R1, R2, R3.4 and Claim 3.2. Suppose $4\leq k\leq 5$. If $m_A(v)\geq 1$, then $\omega'(v)\geq -\frac{31}{65}k+\frac{20}{13}+\frac{6}{5}\geq\frac{23}{65}$ by Claim 3.4. Suppose $m_A(v)=0$. Consider that $k=4$. Note that $m_{6^+}(v)\geq 2$ if $n_{st}(v)=1$; $m_{6^+}(v)\geq 5$ if $n_{st}(v)\geq 2$. Therefore, $-\tau(v\rightarrow 3\text{-}neighbors)+\sum_{i=1}^{7}\tau(f_i\rightarrow v)\geq\min\{-\frac{12}{13}-\frac{3}{5}\times 2+2+\frac{4}{5}\times 5, -\frac{12}{13}\times 3+5+\frac{4}{5}\times 2, -\frac{3}{5}\times 3+\frac{4}{5}\times 7\}=\frac{19}{5}$ by R1, R3.3, R3.4. So $\omega'(v)\geq 1-\frac{6}{5}\times 4+\frac{19}{5}=0$ by R2. If $k=5$, then $m_{6^+}(v)\geq 3$ and v has either at least two Class five 2-neighbors or one Class five 2-neighbor and four Class four 2-neighbors. Therefore, $\omega'(v)\geq 1+3+\frac{4}{5}\times 4-\frac{12}{13}\times 2+\min\{-\frac{6}{5}\times 3-2+(\frac{4}{5}-\frac{6}{11})\times 2, -\frac{11}{10}\times 4-1+(\frac{4}{5}-\frac{6}{11})\}=\frac{149}{715}$ by R1, R2, R3.4, R7. Finally, suppose $6\leq k\leq 7$. If $m_A(v)\geq 2$, then $\omega'(v)\geq -\frac{31}{65}k+\frac{20}{13}+\frac{6}{5}\times 2\geq\frac{3}{5}$ by Claim 3.4. Next, consider that $m_A(v)=1$. If $k=6$, then $m_{6^+}(v)\geq 4$ and v has at least two Class five 2-neighbors. Clearly, $\omega'(v)\geq 1-2-\frac{6}{5}\times 4+4+\frac{4}{5}\times 3-\frac{12}{13}+(\frac{4}{5}-\frac{6}{11})\times 2+\frac{6}{5}=\frac{991}{715}$ by R1, R2, R3, R7, and Claim 3.4. If $k=7$, then $m_{6^+}(v)\geq 6$ and v has at least five Class five 2-neighbors. Obviously, $\omega'(v)\geq 1-5-\frac{6}{5}\times 2+6+\frac{4}{5}+(\frac{4}{5}-\frac{6}{11})\times 5+\frac{6}{5}=\frac{158}{55}$ by R1, R2, R3.4, R7, and Claim 3.4. Finally, consider that $m_A(v)=0$. If $k=6$, then $m_{6^+}(v)\geq 5$ and v has at least four Class five 2-neighbors. Hence, $\omega'(v)\geq 1-4-\frac{6}{5}\times 2+\frac{4}{5}\times 2+5-\frac{12}{13}+(\frac{4}{5}-\frac{6}{11})\times 4=\frac{926}{715}$ by R1, R2, R3.4, R7. If $k=7$, then $m_{6^+}(v)=7$. Then $\omega'(v)\geq 1-7+7=1$ by R1, R2.

Case 7. $d(v)=8$ and then $\omega(v)=2$.

Claim 3.5. *If light v is incident with a configuration A-face f and f is incident with 2-neighbors of v, then $\tau(f\rightarrow v)\geq\frac{9}{5}-\frac{6}{10}=\frac{6}{5}$.*

If $m_{4^-}(v)=1$, then $\omega'(v)\geq 2+\min\{-\frac{10}{7}-7+\frac{8}{7}+6, -\frac{5}{4}\times 2-6+\frac{1}{2}+7, -8+7\}=\frac{5}{7}$ by R1, R2. Consider that $m_{5^+}(v)=8$. Suppose $n_2(v)=k\leq 8$. If $k\leq 4$, then $\omega'(v)\geq 2-\frac{6}{5}k+\min\{\frac{4}{5}\times 8-\frac{9}{10}(8-k), (8-k)+\frac{4}{5}k-(8-k)\}=-\frac{3}{10}k+\frac{6}{5}\geq 0$ by R1, R2, R3.4, R5.1. Next, consider that $k\geq 5$.

Consider that v is a heavy vertex. If $k=5$, then either $n_3(v)+n_4(v)+n_5(v)\leq 2$ or $n_4(v)\leq 1$ and $n_4(v)+n_5(v)=3$. Moreover, $\omega'(v)\geq 2-\frac{6}{5}\times 5+\frac{4}{5}\times 8+\min\{-1\times 2, -\frac{9}{10}-\frac{3}{5}\times 2\}=\frac{3}{10}$ by R1, R2, R3.4, R5.1. If $k=6$, then $n_3(v)+n_4(v)+n_5(v)\leq 1$. So $\omega'(v)\geq 2-\frac{6}{5}\times 6+\frac{4}{5}\times 8+\min\{-1, -\frac{9}{10}\}=\frac{1}{5}$ by R1, R2, R3.4, R5.1. If $k=7$, then $n_{10^+}(v)=1$. Thus, $\omega'(v)\geq 2-\frac{6}{5}\times 7+\frac{4}{5}\times 8+(\frac{9}{5}-\frac{6}{10})=\frac{6}{5}$ by R1, R2, R6. Consider that v is a light vertex. If $m_A(v)\geq 1$, then $\omega'(v)\geq 2+\frac{4}{5}\times 8-\frac{6}{5}\times 8+\frac{6}{5}=0$ by R1, R2 and Claim 3.5. Next, consider that $m_A(v)=0$. If $k=5$, then $m_{6^+}(v)\geq 2$ and v has either at least one Class five 2-vertex or four Class four 2-vertex. It follows from R1, R2, R3.4 that $\omega'(v)\geq 2+\min\{-\frac{6}{5}\times 4-1, -\frac{6}{5}-\frac{11}{10}\times 4\}+2+\frac{4}{5}\times 6-1\times 3=0$. If $k\geq 6$, then $m_{6^+}(v)\geq 4$ and v has at least two Class five 2-neighbors. So $\omega'(v)\geq 2-\frac{6}{5}\times 6-2+\frac{4}{5}\times 4+4+(\frac{4}{5}-\frac{6}{10})\times 2=\frac{2}{5}$ by R1, R2, R7.

Case 8. $d(v)=9$ and then $\omega(v)=3$.

If $m_{4^-}(v)=1$, then $\omega'(v)\geq 3+\min\{-\frac{10}{7}+\frac{8}{7}+7-\frac{8}{7}\times 8, -\frac{5}{4}\times 2+\frac{1}{2}+8-\frac{8}{7}\times 7, -\frac{8}{7}\times 9+8\}=\frac{4}{7}$ by R1, R2, R3.4. Consider that $m_{5^+}(v)=9$. Suppose $n_2(v)=k\leq 8$. If $k\leq 6$, then $\omega'(v)\geq$

$3-\frac{6}{5}k+\min\left\{\frac{4}{5}\times 9-(9-k),(9-k)+\frac{4}{5}k-\frac{8}{7}(9-k)\right\} = -\frac{1}{5}k+\frac{6}{5} \geq 0$ by R1, R2, R3.4, R5.1, and Claim 3.2. Next, consider $7 \leq k \leq 9$.

Suppose that v is a heavy vertex. If $k = 7$, then either $n_3(v)+n_4(v)+n_5(v)+n_6(v) \leq 1$ or $n_5(v)+n_6(v) = 2$. Clearly, $\omega'(v) \geq 3-\frac{6}{5}\times 7+\frac{4}{5}\times 9+\min\left\{-\frac{8}{7},-\frac{3}{5}\times 2\right\} = \frac{3}{5}$ by R1, R2, R3.4, R5.2. If $k = 8$, then $n_{9^+}(v) = 1$. Thereby, $\omega'(v) \geq 3-\frac{6}{5}\times 8+\frac{4}{5}\times 9 = \frac{3}{5}$ by R1, R2. Consider that v is a light vertex. If v is not a big vertex, then $\omega'(v) \geq 0$. Next, suppose that v is a big vertex. For configuration B-face (See Figure 1(b)), $\omega'(v'_i) \geq 3-\frac{6}{5}\times 8+\frac{4}{5}\times 9 = \frac{3}{5}(i = 1,2)$ by R1~R7. So configuration B-face has just one big vertex. If $m_B(v) \geq 1$, then $\omega'(v) \geq 3+\frac{4}{5}\times 9-\frac{6}{5}\times 9+\frac{3}{5} = 0$ by R1, R2, R8.2. Suppose $m_B(v) = 0$. If $k = 7$, then $m_{6^+}(v) \geq 5$. Obviously, $\omega'(v) \geq 3+\frac{4}{5}\times 4+5-\frac{6}{5}\times 7-\frac{8}{7}\times 2 = \frac{18}{35}$ by R1, R2, R3.4. If $k = 8$, then $m_{6^+}(v) \geq 7$. So $\omega'(v) \geq 3+\frac{4}{5}\times 2+7-\frac{6}{5}\times 8-\frac{8}{7} = \frac{6}{7}$. If $k = 9$, then $m_{6^+}(v) = 9$, which indicates that $\omega'(v) \geq 3+9-9 = 3$.

Case 9. $d(v) = k \geq 10$ and then $\omega(v) = k-6$. Note that $2-\frac{7}{k} > \frac{9}{5}-\frac{6}{k} \geq \frac{6}{5}$.

If $m_{4^-}(v) = 1$, then $\omega'(v) \geq k-6+\min\{-\frac{10}{7}+k-2+\frac{8}{7}-(2-\frac{7}{k})(k-1),-\frac{5}{4}+k-1+\frac{1}{2}-(2-\frac{7}{k})(k-1),k-1-(2-\frac{7}{k})k\} = 0$ by R1, R2, R3.4. Next, consider that $m_{5^+}(v) = k$. Suppose $n_{st}(v) = t \geq 0$. Moreover, $\omega'(v) \geq k-6+\frac{4}{5}(k-t)+t-(2-\frac{7}{k})t-(\frac{9}{5}-\frac{6}{k})(k-t) = \frac{1}{k}t \geq 0$ by R1, R3.4, R6, and Claim 3.2. From the above argument, it is clear that $\omega'(v'_i) \geq \frac{9}{5}-\frac{6}{d(v'_i)}(i = 1,2)$ for configuration A-face (See Figure 1(a)) by R1~R7.

Next, it follows from the above argument that $\omega'(f) \geq 0$ if f is a configuration A-face or configuration B-face, which derives that $\omega'(f) \geq 0$ for each $f \in F(G)$.

Therefore, Theorem 1.6 is proved.

4. Conclusions

It is difficult to consider $\chi_i(G)$ and $\chi_i^l(G)$ for planar graph G that has $g(G) \geq 4$. So we consider the case of G where it does not have a 4^--cycle intersecting with a 5^--cycle. In addition, we will try to explore whether there exists a constant C such that $\chi_i^l(G) \leq \Delta + C$ for G has disjoint 4^--cycles.

Author contributions

Yuehua Bu: Conceptualization; Hongrui Zheng: Writing-original & draft; Hongrui Zheng and Hongguo Zhu: Writing-review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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