AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 1779–1791.
DOI:10.3934/math.2025081
Received: 23 September 2024
Revised: 12 January 2025
Accepted: 16 January 2025
Published: 24 January 2025

Research article

The construction conditions of a Hilbert-type local fractional integral operator and the norm of the operator

Ling Peng[1,*] **and Qiong Liu**[2]

[1] School of Medical Information and Engineering, Hunan University of Medicine, Huaihua 418000, Hunan, China

[2] School of Science and Teacher Education, Shaoyang University, Shaoyang 422000, Hunan, China

* **Correspondence:** Email: water16547749@163.com.

Abstract: The parameterized local fractional singular integral operator $T^{(\tau)}$ is defined on the space $L_p^{\tau\mu}(\mathbb{R}_+^{\tau})$ as $T^{(\tau)} : L_p^{\tau\mu}(\mathbb{R}_+^{\tau}) \to L_p^{\tau\nu(1-p)}(\mathbb{R}_+^{\tau})$, $T^{(\tau)}(f_\tau)(y) = {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}} f_\tau(x)\Big], y \in \mathbb{R}_+$. By employing the weight function method and analysis techniques on the fractal real line number set $\mathbb{R}_+^{\tau}$, a general Hilbert-type local fractional integral inequality has been established, thereby demonstrating the boundedness of the defined integral operator. Through optimization of parameters, it was determined that the necessary and sufficient condition for the constant factor in this general Hilbert-type local fractional inequality to be the best possible is that the power parameters σ and σ_1 satisfy $\sigma+\sigma_1 = \beta-\alpha$. Consequently, the formula for calculating the operator norm has been derived.

Keywords: Hilbert-type local fractional integral operator; weight function; the best constant factor; operator norm
Mathematics Subject Classification: 26D15, 47A05

1. Introduction

Let $(p, q)(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$ be a conjugate exponential pair, and let $f(x)$ and $g(y)$ be two non-negative real functions, satisfying $0 < \int_0^{+\infty} f^p(x)dx < +\infty, 0 < \int_0^{+\infty} f^q(y)dy < +\infty$. We have the famous Hardy-Hilbert integral inequality as follows [1]:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\frac{\pi}{p})}\Big\{\int_0^{+\infty} f^p(x)\mathrm{d}x\Big\}^{\frac{1}{p}}\Big\{\int_0^{+\infty} g^q(y)\mathrm{d}y\Big\}^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. We define the singular integral operator $T(f)(y) = \int_0^{+\infty} \frac{1}{x+y} f(x)dx$, then (1) can be represented by this operator and its norm as [1]

$$(Tf, g) < ||T|| ||f||_p ||g||_q.$$

It is precisely because of the deep connection between (1) and the operator that it has important applications in the field of analysis and partial differential equations [1, 2], so the in-depth study of (1) has attracted wide attention and achieved rich research results. This includes these works [3–6], which comprehensively and systematically summarized and elaborated on some research results. In addition, in recent years, Liu [7–9] has achieved some new research results on mixed kernel Hilbert-type integral inequalities; Rassias [10] and Huang [11] established some inverse Hilbert-type integral inequalities in the whole plane or involving the derivative function of a higher order; Yang [12] obtained a Hardy Hilbert-type integral inequality involving a multiple upper bound function and a high-order derivative function; and the equivalent conditions for the optimal matching parameters of the generalized homogeneous kernel multiple integration operator obtained by Hong [13], as well as some new half-discrete Hilbert-type inequalities [14, 15].

In 2011, Yang [16] established the fractal real line number set $\mathbb{R}^\tau$ with a fractal dimension $\tau(0 < \tau \leq 1)$ based on the real number set $\mathbb{R}$; the local fractional derivative and local fractional integral were defined on the real number set $\mathbb{R}^\tau$. Subsequently, a relatively complete theory of Young's Local Fractional Calculus (YLFC) was developed [17–19]. YLFC has been applied as a mathematical tool in fields such as physics, materials science, etc. [20–22]. In 2017, Liu et al. [23, 24] incorporated Hilbert-type integral inequalities into the fractal set $\mathbb{R}^\tau$ and extended several basic Hilbert-type integral inequalities. In 2021, they [25] established the Yang-Hardy-Hilbert local fractional integral inequality. They also studied the Hilbert-type local fractional integral inequalities with integral kernels of Mittag-Leffler function [26], hyperbolic cotangent function [27], and the abstract homogeneous function [28], respectively. During this period, Baleanu et al. [29–31] obtained a class of local fractional Hilbert-type inequalities via Cantor-type spherical coordinates and Hilbert-type local fractional integral inequalities of some other integral kernels.

In this paper, we introduced multiple parameters, $\alpha, \beta, \sigma, \sigma_1$, and constructed the following parameterized integral kernel function:

$$k_\tau(x, y) := \frac{|x - y|^{\tau\alpha}}{(x + y)^{\tau\beta}}.$$

Firstly, we established a general Hilbert-type local fractional integral inequality with the above integral kernel using the weight function method. This general Hilbert-type integral inequality helps us prove the boundedness of the defined operator. Still, its constant factor may not be the best possible, and we cannot obtain the formula for calculating the operator norm. Then, by optimizing the parameter conditions, we provide the necessary and sufficient condition for the existence of a Hilbert-type local fractional integral inequality with the best constant factor, which is that the parameters satisfy $\sigma + \sigma_1 = \beta - \alpha$, and the formula for calculating the norm of the defined operator is obtained.

2. Preliminary knowledge

Some of the operational properties and axioms of the elements in the fractal real number set $\mathbb{R}^\tau$, as well as the theory of YLFC on $\mathbb{R}^\tau$, can be found in references [16, 17, 27].

Definition 1. [16,17,25] Suppose that $[a,b]$ is a finite interval of the real line $\mathbb{R}$, $f_\tau(x) \in C_\tau[a,b]$(local fractional continuous function in [a,b]), we define the local fractional integral of f_τ by

$$ {}_a\mathscr{Y}_b^{(\tau)}(f_\tau(x)) = \frac{1}{\Gamma(\tau+1)} \int_a^b f_\tau(x)(\mathrm{d}x)^\tau = \frac{1}{\Gamma(\tau+1)} \lim_{t\to 0} \sum_{i=1}^{N} f_\tau(x_i)(\Delta x_i)^\tau, $$

with $\Delta x_i = x_i - x_{i-1}(i = 1, \cdots N)$ and $t = \max_{1\le i\le N}\{\Delta x_i\}$, and $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ is partition of interval $[a,b]$. Here, it follows that ${}_a\mathscr{Y}_b^{(\tau)}(f_\tau(x)) = 0^\tau$ if $a = b$, ${}_a\mathscr{Y}_b^{(\tau)}(f_\tau(x)) = -{}_b\mathscr{Y}_a^{(\tau)}(f_\tau(x))$ if $a < b$.

If $b = \infty$, the generalized integral is defined by

$$ {}_a\mathscr{Y}_\infty^{(\tau)}(f_\tau(x)) = \frac{1}{\Gamma(1+\tau)} \int_a^\infty f_\tau(x)(\mathrm{d}x)^\tau := \lim_{T\to\infty} {}_a\mathscr{Y}_T^{(\tau)}(f_\tau(x)). $$

If $[a,b]$ and $[c,d]$ are finite or infinite intervals of $\mathbb{R}$, $F_\tau(x,y)$ is a local fractional continuous function on the rectangle domains $\Omega = \{(x,y)|a < x < b, c < y < d\}$, the double local fractional integral of $F_\tau(x,y)$ is marked as

$$ {}_a\mathscr{Y}_b^{(\tau)}[\,{}_c\mathscr{Y}_d^{(\tau)}(F_\tau(x,y))] = \frac{1}{\Gamma^2(1+\tau)} \int_\Omega F_\tau(x,y)(\mathrm{d}x)^\tau(\mathrm{d}y)^\tau. $$

Definition 2. [32] We define some generalized special functions on Young's fractal set.

(i). If $u, v > 0$, the generalized beta function is defined as

$$ B_\tau(u,v) := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{t^{\tau(u-1)}}{(1+t)^{\tau(u+v)}}\Big]. $$

(ii). If $Re(\gamma_3) > Re(\gamma_2) > 0, |\arg(1-z)| < \pi$, the generalized hypergeometric function is defined as

$$ F_\tau(\gamma_1,\gamma_2,\gamma_3,z) := \frac{{}_0\mathscr{Y}_1^{(\tau)}\Big[t^{\tau(\gamma_2-1)}(1-t)^{\tau(\gamma_3-\gamma_2-1)}(1-zt)^{-\tau\gamma_1}\Big]}{B_\tau(\gamma_2,\gamma_3-\gamma_2)}. \tag{2} $$

Definition 3. Let $0 < \tau \le 1$ be a fractal dimension. We construct the normed Lebesgue fractal space and a specific integral kernel function $k_\tau(x,y)$ as follows:

$$ L_p^{\tau\mu}(\mathbb{R}_+^\tau) := \left\{ f_\tau(x) : ||f_\tau||_{p,\tau\mu} = \left\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}[x^{\tau\mu}|f_\tau(x)|^p] \right\}^{\frac{1}{p}} < +\infty \right\}, $$

$$ L_q^{\tau\nu}(\mathbb{R}_+^\tau) := \left\{ g_\tau(y) : ||g_\tau||_{q,\tau\nu} = \left\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}[y^{\tau\nu}|g_\tau(y)|^q] \right\}^{\frac{1}{q}} < +\infty \right\}, $$

$$ k_\tau(x,y) := \frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}, x, y \in \mathbb{R}_+. $$

Definition 4. Assume that $f_\tau \in L_p^{\tau\mu}(\mathbb{R}_+^\tau), g_\tau \in L_q^{\tau\nu}(\mathbb{R}_+^\tau)$, we define the Hilbert-type local fractional integral operator by

$$ T^{(\tau)} : L_p^{\tau\mu}(\mathbb{R}_+^\tau) \to L_p^{\tau\nu(1-p)}(\mathbb{R}_+^\tau), $$

$$ T^{(\tau)}(f_\tau)(y) := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[k_\tau(x,y)f_\tau(x)\Big] = {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)\Big], y \in \mathbb{R}_+. \tag{3} $$

The formal inner product of $T^{(\tau)}f_\tau$ and g_τ is

$$\left(T^{(\tau)}f_\tau, g_\tau\right) := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)g_\tau(y)\Big]\Big\}. \tag{4}$$

The norm of $T^{(\tau)}$ is

$$\|T^{(\tau)}\| := \sup_{f_\tau(\neq 0^\tau)\in L_p^{\tau\mu}(\mathbb{R}_+^\tau)}\frac{\|T^{(\tau)}f_\tau\|_{p,\tau\nu(1-p)}}{\|f_\tau\|_{p,\tau\mu}}. \tag{5}$$

Lemma 1. If $\sigma > 0, \alpha > -1, \sigma < \beta - \alpha$, defining the following local fractional integral:

$$\mathscr{M}_\tau(\alpha,\beta,\sigma) := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|1-t|^{\tau\alpha}t^{\tau(\sigma-1)}}{(1+t)^{\tau\beta}}\Big],$$

then, we have

$$\mathscr{M}_\tau(\alpha,\beta,\sigma) = B_\tau(\sigma,1+\alpha)F_\tau(\beta,\sigma,1+\alpha+\sigma,-1)+B_\tau(\beta-\alpha-\sigma,1+\alpha)F_\tau(\beta,\beta-\alpha-\sigma,1+\beta-\sigma,-1). \tag{6}$$

Proof. By (2), we obtain

$$\begin{aligned}
\mathscr{M}_\tau(\alpha,\beta,\sigma) &= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|1-t|^{\tau\alpha}t^{\tau(\sigma-1)}}{(1+t)^{\tau\beta}}\Big] \\
&= {}_0\mathscr{Y}_{1}^{(\tau)}\Big[\frac{(1-t)^{\tau\alpha}t^{\tau(\sigma-1)}}{(1+t)^{\tau\beta}}\Big] + {}_1\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{(t-1)^{\tau\alpha}t^{\tau(\sigma-1)}}{(1+t)^{\tau\beta}}\Big] \\
&\overset{\text{setting } \frac{1}{t}=u \text{ for the second integral}}{=\!=\!=} {}_0\mathscr{Y}_{1}^{(\tau)}\Big[\frac{(1-t)^{\tau\alpha}t^{\tau(\sigma-1)}}{(1+t)^{\tau\beta}}\Big] + {}_0\mathscr{Y}_{1}^{(\tau)}\Big[\frac{(1-u)^{\tau\alpha}u^{\tau(\beta-\alpha-\sigma-1)}}{(1+u)^{\tau\beta}}\Big] \\
&= B_\tau(\sigma,1+\alpha)F_\tau(\beta,\sigma,1+\alpha+\sigma,-1) + B_\tau(\beta-\alpha-\sigma,1+\alpha)F_\tau(\beta,\beta-\alpha-\sigma,1+\beta-\sigma,-1).
\end{aligned}$$

Lemma 2. If $0 < \sigma(\sigma_1) < \beta - \alpha, \alpha > -1$, defining the following weight functions:

$$\omega_\tau(\alpha,\beta,\sigma,\sigma_1,x) := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}x^{\tau\sigma}y^{\tau(\sigma_1-1)}}{(x+y)^{\tau\beta}}\Big], x\in\mathbb{R}_+,$$

$$\widetilde{\omega}_\tau(\alpha,\beta,\sigma,\sigma_1,y) := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}x^{\tau(\sigma-1)}y^{\tau\sigma_1}}{(x+y)^{\tau\beta}}\Big], y\in\mathbb{R}_+,$$

we have

$$\begin{aligned}
\omega_\tau(\alpha,\beta,\sigma,\sigma_1,x) &= x^{\tau(\sigma+\sigma_1+\alpha-\beta)}\mathscr{M}_\tau(\alpha,\beta,\sigma_1), \\
\widetilde{\omega}_\tau(\alpha,\beta,\sigma,\sigma_1,y) &= y^{\tau(\sigma+\sigma_1+\alpha-\beta)}\mathscr{M}_\tau(\alpha,\beta,\sigma).
\end{aligned} \tag{7}$$

Proof. Setting $\frac{y}{x} = t$, we find that

$$\begin{aligned}
\omega_\tau(\alpha,\beta,\sigma,\sigma_1,x) &= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}x^{\tau\sigma}y^{\tau(\sigma_1-1)}}{(x+y)^{\tau\beta}}\Big] \\
&= x^{\tau(\sigma+\sigma_1+\alpha-\beta)}\cdot {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|1-t|^{\tau\alpha}t^{\tau(\sigma_1-1)}}{(1+t)^{\tau\beta}}\Big] \\
&= x^{\tau(\sigma+\sigma_1+\alpha-\beta)}\mathscr{M}_\tau(\alpha,\beta,\sigma_1).
\end{aligned}$$

Similarly, we have $\widetilde{\omega}_\tau(\alpha,\beta,\sigma,\sigma_1,y) = y^{\tau(\sigma+\sigma_1+\alpha-\beta)}\mathscr{M}_\tau(\alpha,\beta,\sigma)$. □

Lemma 3. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f_\tau, g_\tau \geq 0^\tau, f_\tau \in L_p^{\tau\mu}(\mathbb{R}_+^\tau), g_\tau \in L_q^{\tau\nu}(\mathbb{R}_+^\tau), k_\tau(x,y) > 0^\tau$, and there exists a positive fractal real number M^τ, such that the Hilbert-type local fractional integral inequality

$$J_\tau := {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}[k_\tau(x,y) f_\tau(x) g_\tau(y)]\Big\} \leq M^\tau \|f_\tau\|_{p,\tau\mu}\|g_\tau\|_{q,\tau\nu} \tag{8}$$

holds, then we have the following equivalent inequality:

$$H_\tau := \Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau\nu(1-p)}\big({}_0\mathscr{Y}_{+\infty}^{(\tau)} k_\tau(x,y) f_\tau(x)\big)^p\Big]\Big\}^{\frac{1}{p}} \leq M^\tau \|f_\tau\|_{p,\tau\mu}, \tag{9}$$

where $J_\tau, H_\tau \in \mathbb{R}_+^\tau$.
Proof. We set the following a local fractional continuous function as

$$g_\tau(y) := y^{\tau\nu(1-p)}\Big[{}_0\mathscr{Y}_{+\infty}^{(\tau)}(k_\tau(x,y) f_\tau(x))\Big]^{p-1}, y \in \mathbb{R}_+.$$

"(8) ⇒ (9)". Letting (8) be holds, then we have

$$\begin{aligned}
0 < \|g_\tau\|_{q,\tau\nu}^q &= {}_0\mathscr{Y}_{+\infty}^{(\tau)}[y^{\tau\nu} g_\tau^q(y)] \\
&= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau\nu+\tau\nu(1-p)q}\big({}_0\mathscr{Y}_{+\infty}^{(\tau)} k_\tau(x,y) f_\tau(x)\big)^p\Big] \\
&= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau\nu(1-p)}\big({}_0\mathscr{Y}_{+\infty}^{(\tau)} k_\tau(x,y) f_\tau(x)\big)^p\Big](= H_\tau^p) \\
&= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{\Big[y^{\tau\nu(1-p)}\big({}_0\mathscr{Y}_{+\infty}^{(\tau)} k_\tau(x,y) f_\tau(x)\big)^{p-1}\Big]\Big[{}_0\mathscr{Y}_{+\infty}^{(\tau)} k_\tau(x,y) f_\tau(x)\Big]\Big\} \\
&= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}[k_\tau(x,y) f_\tau(x) g_\tau(y)]\Big\}(= J_\tau) \\
&\leq M^\tau \|f_\tau\|_{p,\tau\mu}\|g_\tau\|_{q,\tau\nu} = M^\tau \|f_\tau\|_{p,\tau\mu} \cdot H_\tau^{p-1},
\end{aligned}$$

by the above expression, we obtain (9).

"(9) ⇒ (8)". Letting (9) be holds, by Hölder local fractional integral inequality [19] and (9), we find that

$$\begin{aligned}
J_\tau &= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}[k_\tau(x,y) f_\tau(x) g_\tau(y)]\Big\} \\
&= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{\Big[y^{\frac{\tau\nu(1-p)}{p}}\, {}_0\mathscr{Y}_{+\infty}^{(\tau)}(k_\tau(x,y) f_\tau(x))\Big]\Big[y^{\frac{\tau\nu(p-1)}{p}} g_\tau(y)\Big]\Big\} \\
&\leq H_\tau \cdot \Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}(y^{\tau\nu} g_\tau^q(y))\Big\}^{\frac{1}{q}} = H_\tau \|g_\tau\|_{q,\tau\nu} \\
&\leq M^\tau \|f_\tau\|_{p,\tau\mu}\|g_\tau\|_{q,\tau\nu}.
\end{aligned}$$

The above expression is (8). In summary, (8) is equivalent to (9).

3. Main results

Theorem 1. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > -1, 0 < \sigma(\sigma_1) < \beta - \alpha$, $f_\tau(x), g_\tau(y) \geq 0^\tau$, satisfying

$$f_\tau(x) \in L_p^{\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}(\mathbb{R}_+^\tau), \quad g_\tau(y) \in L_q^{\tau[q(1-\sigma_1)-1+\sigma+\sigma_1+\alpha-\beta]}(\mathbb{R}_+^\tau),$$

Then we have the following pair of equivalent inequalities:

$${}_0\mathscr{Y}_{+\infty}^{(\tau)}\left\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}} f_\tau(x) g_\tau(y)\Big]\right\}$$

$$< \mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1+\sigma+\sigma_1+\alpha-\beta]}, \tag{10}$$

$$\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau[p(\sigma_1-1)+(p-1)(\beta-\alpha-\sigma-\sigma_1+1)]}\Big({}_0\mathscr{Y}_{+\infty}^{(\tau)}\big(\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)\big)\Big)^p\Big]\Big\}^{\frac{1}{p}}$$
$$< \mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}. \tag{11}$$

Proof. Using the weight function method based on the Hardy interpolation problem and the Hölder double local fractional integral inequality with weighted [28], by Lemma 2, we have

$$\begin{aligned}
&{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)g_\tau(y)\Big]\Big\}\\
&= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}\Big[x^{\frac{\tau(1-\sigma)}{q}}y^{\frac{\tau(\sigma_1-1)}{p}}\Big]\Big[x^{\frac{\tau(\sigma-1)}{q}}y^{\frac{\tau(1-\sigma_1)}{p}}\Big]f_\tau(x)g_\tau(y)\Big]\Big\}\\
&\le \Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[{}_0\mathscr{Y}_{+\infty}^{(\tau)}\big(\frac{x^{\tau(p-1)(1-\sigma)}y^{\tau(\sigma_1-1)}|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau^p(x)\big)\Big]\Big\}^{\frac{1}{p}}\times\\
&\quad\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[{}_0\mathscr{Y}_{+\infty}^{(\tau)}\big(\frac{x^{\tau(\sigma-1)}y^{\tau(q-1)(\sigma_1-1)}|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}g_\tau^q(y)\big)\Big]\Big\}^{\frac{1}{q}}\\
&= \Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\omega_\tau(\alpha,\beta,\sigma,,\sigma_1,x)x^{\tau[p(1-\sigma)-1]}f_\tau^p(x)\Big]\Big\}^{\frac{1}{p}}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\varpi_\tau(\alpha,\beta,\sigma,\sigma_1,y)y^{\tau[q(1-\sigma_1)-1]}g_\tau^q(y)\Big]\Big\}^{\frac{1}{q}}\\
&= \mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma)\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[x^{\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}f_\tau^p(x)\Big]\Big\}^{\frac{1}{p}}\times\\
&\quad\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau[q(1-\sigma_1)-1+\sigma+\sigma_1+\alpha-\beta]}g_\tau^q(y)\Big]\Big\}^{\frac{1}{q}}\\
&= \mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1+\sigma+\sigma_1+\alpha-\beta]}.
\end{aligned} \tag{12}$$

Assume that the " $\le$ " in (12) takes the form of the equality for a $y \in \mathbb{R}_+$, then according to the conclusion of the weighted Hölder local fractional inequality, there are constants A^τ and B^τ that are not all zero, such that

$$A^\tau x^{\tau(p-1)(1-\sigma)}y^{\tau(\sigma_1-1)}f_\tau^p(x) = B^\tau x^{\tau(\sigma-1)}y^{\tau(q-1)(1-\sigma_1)}g_\tau^q(y) \text{ a.e. in } \mathbb{R}_+^2$$

is valid. let us assume that $A^\tau \neq 0^\tau$, then there is a $y_0 \in \mathbb{R}_+$, such that

$$x^{\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]]}f_\tau^p(x) = \Big[\tfrac{B^\tau}{A^\tau}y_0^{\tau q(1-\sigma_1)}g_\tau^q(y_0)\Big]x^{\tau(1+\beta-\alpha-\sigma-\sigma_1)} \text{ a.e. in } \mathbb{R}_+$$

is valid, which contradicts $0^\tau < {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{x^{\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}f_\tau^p(x)\Big\} < +\infty$. So, (12) takes the strict unequal form. According to Lemma 3, (10) is equivalent to (11). □

Theorem 2. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > -1, 0 < \sigma(\sigma_1) < \beta - \alpha$, $f_\tau(x) \in L_p^{\tau[p(1-\sigma)-1]}(\mathbb{R}_+^\tau), g_\tau(y) \in L_q^{\tau[q(1-\sigma_1)-1]}(\mathbb{R}_+^\tau)$. If and only if $\sigma + \sigma_1 = \beta - \alpha$, the following pair of equivalent inequalities

$${}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)g_\tau(y)\Big]\Big\} < \mathscr{M}_\tau(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1]}, \tag{13}$$

$$\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau[p\sigma_1-1]}\Big({}_0\mathscr{Y}_{+\infty}^{(\tau)}\big(\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)\big)\Big)^p\Big]\Big\}^{\frac{1}{p}} < \mathscr{M}_\tau(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1]} \tag{14}$$

hold.

Proof. By $\sigma + \sigma_1 = \beta - \alpha$ and Lemma 1, we have

$$\begin{aligned}
&\mathscr{M}_\tau(\alpha,\beta,\sigma_1) = {}_0\mathscr{Y}^{(\tau)}_{+\infty}\Big[\frac{|1-t|^{\tau\alpha}t^{\tau(\sigma_1-1)}}{(1+t)^{\tau\beta}}\Big]\\
&= B_\tau(\sigma_1, 1+\alpha)F_\tau(\beta,\sigma_1,1+\alpha+\sigma_1,-1) + B_\tau(\beta-\alpha-\sigma_1,1+\alpha)F_\tau(\beta,\beta-\alpha-\sigma_1,1+\beta-\sigma_1,-1)\\
&= B_\tau(\beta-\alpha-\sigma,1+\alpha)F_\tau(\beta,\beta-\alpha-\sigma,1+\beta-\sigma,-1) + B_\tau(\sigma,1+\alpha)F_\tau(\beta,\sigma,1+\alpha+\sigma,-1)\\
&= \mathscr{M}_\tau(\alpha,\beta,\sigma),
\end{aligned}$$

hence, we have $\mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma) = \mathscr{M}_\tau(\alpha,\beta,\sigma)$. Substituting $\sigma+\sigma_1+\alpha-\alpha = 0$ and $\mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma) = \mathscr{M}_\tau(\alpha,\beta,\sigma)$ into (10) and (11) respectively, we obtain (13) and (14) accordingly.

On the other hand, hypotheses (13) and (14) are valid and let $\varsigma = \sigma+\sigma_1+\alpha-\beta$. When $\varsigma > 0$, for $0 < \varepsilon < \varsigma$, setting

$$f_\tau(x) = \begin{cases} x^{\tau(\sigma-1-\frac{\varepsilon}{p})}, & x\in[1,+\infty)\\ 0^\tau, & x\in(0,1)\end{cases}, \quad g_\tau(y) = \begin{cases} y^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}, & y\in[1,+\infty)\\ 0^\tau, & y\in(0,1)\end{cases},$$

we have

$$\begin{aligned}
&\|f_\tau\|_{p,\tau[p(1-\sigma)-1]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1]}\\
&= \Big\{{}_0\mathscr{Y}^{(\tau)}_{+\infty}\Big[x^{\tau[p(1-\sigma)-1]}f_\tau^p(x)\Big]\Big\}^{\frac{1}{p}}\Big\{{}_0\mathscr{Y}^{(\tau)}_{+\infty}\Big[y^{\tau[q(1-\sigma_1)-1]}f_\tau^p(x)\Big]\Big\}^{\frac{1}{q}}\\
&= \Big\{{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[x^{-\tau(1+\varepsilon)}\Big]\Big\}^{\frac{1}{p}}\Big\{{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[y^{-\tau(1+\varepsilon)}\Big]\Big\}^{\frac{1}{q}}\\
&= \frac{1^\tau}{\varepsilon^\tau}.
\end{aligned} \tag{15}$$

Setting $\frac{y}{x} = u$, and noting the fact that $x \in [1,+\infty)$, we obtain

$$\begin{aligned}
J_\tau &= {}_0\mathscr{Y}^{(\tau)}_{+\infty}\Big\{{}_0\mathscr{Y}^{(\tau)}_{+\infty}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)g_\tau(y)\Big]\Big\}\\
&= {}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big\{x^{\tau(\sigma-1-\frac{\varepsilon}{p})}\,{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[\frac{|x-y|^{\tau\alpha}y^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}}{(x+y)^{\tau\beta}}\Big]\Big\}\\
&= {}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big\{x^{-\tau(1+\varepsilon-\varsigma)}\,{}_{\frac{1}{x}}\mathscr{Y}^{(\tau)}_{+\infty}\Big[\frac{(u-1)^{\tau\alpha}u^{\tau(\sigma_1-\frac{\varepsilon}{q}-1)}}{(1+u)^{\tau\beta}}\Big]\Big\}\\
&\geq {}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big\{x^{-\tau(1+\varepsilon-\varsigma)}\,{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[\frac{(u-1)^{\tau\alpha}u^{\tau(\sigma_1-\frac{\varepsilon}{q}-1)}}{(1+u)^{\tau\beta}}\Big]\Big\}\\
&= {}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big\{x^{-\tau(1+\varepsilon-\varsigma)}\,{}_0\mathscr{Y}^{(\tau)}_{1}\Big[\frac{(1-t)^{\tau\alpha}t^{\tau(\beta-\alpha-\sigma_1+\frac{\varepsilon}{q}-1)}}{(1+t)^{\tau\beta}}\Big]\Big\}\\
&\xlongequal{\text{by (2)}} B_\tau(\beta-\alpha-\sigma_1+\frac{\varepsilon}{q},1+\alpha)F_\tau(\beta,\beta-\alpha-\sigma_1+\frac{\varepsilon}{q},1+\beta-\sigma_1+\frac{\varepsilon}{q},-1)\,{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[x^{-\tau(1+\varepsilon-\varsigma)}\Big]\\
&\triangleq (M_1^*)^\tau\,{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[x^{-\tau(1+\varepsilon-\varsigma)}\Big].
\end{aligned}$$

Based on the above expression, by (13) and (15), we find that

$$(M_1^*)^\tau\,{}_1\mathscr{Y}^{(\tau)}_{+\infty}\Big[x^{-\tau(1+\varepsilon-\varsigma)}\Big] \leq J_\tau$$

$$< \mathscr{M}_\tau(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1]} = \frac{\mathscr{M}_\tau(\alpha,\beta,\sigma)}{\varepsilon^\tau}.$$

In view of $\varsigma - \varepsilon > 0$, we get ${}_1\mathscr{Y}_{+\infty}^{(\tau)}\Big[x^{-\tau(1+\varepsilon-\varsigma)}\Big] = \infty$. Hence, we arrived at a contradictory conclusion, which is that $+\infty \le J_\tau < +\infty$.

When $\varsigma < 0$, for $0 < \varepsilon < -\varsigma$, setting

$$f_\tau(x) = \begin{cases} x^{\tau(\sigma-1+\frac{\varepsilon}{p})}, & x \in (0,1] \\ 0^\tau, & x \in (1,+\infty) \end{cases}, \quad g_\tau(y) = \begin{cases} y^{\tau(\sigma_1-1+\frac{\varepsilon}{q})}, & y \in (0,1] \\ 0^\tau, & y \in (1,+\infty) \end{cases},$$

we have

$$\begin{aligned} &\|f_\tau\|_{p,\tau[p(1-\sigma)-1]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1]} \\ &= \Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[x^{\tau[p(1-\sigma)-1]}f_\tau^p(x)\Big]\Big\}^{\frac{1}{p}}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau[q(1-\sigma_1)-1]}f_\tau^p(x)\Big]\Big\}^{\frac{1}{q}} \\ &= \Big\{{}_0\mathscr{Y}_{1}^{(\tau)}\Big[x^{-\tau(1-\varepsilon)}\Big]\Big\}^{\frac{1}{p}}\Big\{{}_0\mathscr{Y}_{1}^{(\tau)}\Big[y^{-\tau(1-\varepsilon)}\Big]\Big\}^{\frac{1}{q}} \\ &= \frac{1^\tau}{\varepsilon^\tau}. \end{aligned} \tag{16}$$

Setting $\frac{y}{x} = u$, and noting the fact that $x \in (0,1]$, we obtain

$$\begin{aligned} J_\tau &= {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{{}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}f_\tau(x)g_\tau(y)\Big]\Big\} \\ &= {}_0\mathscr{Y}_{1}^{(\tau)}\Big\{x^{\tau(\sigma-1+\frac{\varepsilon}{p})}\,{}_0\mathscr{Y}_{1}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}y^{\tau(\sigma_1-1+\frac{\varepsilon}{q})}}{(x+y)^{\tau\beta}}\Big]\Big\} \\ &= {}_0\mathscr{Y}_{1}^{(\tau)}\Big\{x^{-\tau(1-\varepsilon-\varsigma)}\,{}_0\mathscr{Y}_{\frac{1}{x}}^{(\tau)}\Big[\frac{(1-u)^{\tau\alpha}u^{\tau(\sigma_1+\frac{\varepsilon}{q}-1)}}{(1+u)^{\tau\beta}}\Big]\Big\} \\ &\ge {}_0\mathscr{Y}_{1}^{(\tau)}\Big\{x^{-\tau(1-\varepsilon-\varsigma)}\,{}_0\mathscr{Y}_{1}^{(\tau)}\Big[\frac{(1-u)^{\tau\alpha}u^{\tau(\sigma_1+\frac{\varepsilon}{q}-1)}}{(1+u)^{\tau\beta}}\Big]\Big\} \\ &\overset{\text{by (2)}}{=\!=} B_\tau(\sigma_1+\frac{\varepsilon}{q},1+\alpha)F_\tau(\beta,\sigma_1+\frac{\varepsilon}{q},1+\beta+\sigma_1+\frac{\varepsilon}{q},-1)\,{}_0\mathscr{Y}_{1}^{(\tau)}\Big[x^{-\tau(1-\varepsilon-\varsigma)}\Big] \\ &\triangleq (M_2^*)^\tau\,{}_0\mathscr{Y}_{1}^{(\tau)}\Big[x^{-\tau(1-\varepsilon-\varsigma)}\Big]. \end{aligned}$$

Based on the above expression, by (13) and (16), we find that

$$\begin{aligned} &(M_2^*)^\tau\,{}_0\mathscr{Y}_{1}^{(\tau)}\Big[x^{-\tau(1-\varepsilon-\varsigma)}\Big] \le J_\tau \\ &\qquad < \mathscr{M}_\tau(\alpha,\beta,\sigma)\|f_\tau\|_{p,\tau[p(1-\sigma)-1]}\|g_\tau\|_{q,\tau[q(1-\sigma_1)-1]} = \frac{\mathscr{M}_\tau(\alpha,\beta,\sigma)}{\varepsilon^\tau}. \end{aligned}$$

In view of $\varsigma + \varepsilon < 0$, we get ${}_0\mathscr{Y}_{1}^{(\tau)}\Big[x^{-\tau(1-\varepsilon-\varsigma)}\Big] = \infty$. So, we also arrived at a contradictory conclusion that $+\infty \le J_\tau < +\infty$. In summary, we have $\varsigma = 0$, that is $\sigma + \sigma_1 = \beta - \alpha$. □

Theorem 3. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > -1, 0 < \sigma(\sigma_1) < \beta - \alpha$, $f_\tau(x) \in L_p^{\tau[p(1-\sigma)-1]}(\mathbb{R}_+^\tau), g_\tau(y) \in L_q^{\tau[q(1-\sigma_1)-1]}(\mathbb{R}_+^\tau)$. If and only if $\sigma + \sigma_1 = \beta - \alpha$, the constant factor $\mathscr{M}_\tau(\alpha,\beta,\sigma)$ of (13) and (14) is the best possible.

Proof. According to Theorem 2, inequalities (13) and (14) are only valid when the condition $\sigma + \sigma_1 = \beta - \alpha$. Therefore, we only need to prove that the constant $\mathscr{M}_\tau(\alpha,\beta,\sigma)$ of (13) and (14) is the best possible. Assuming this proposition is incorrect, there exists a positive fractal real number $K^\tau(< \mathscr{M}_\tau(\alpha,\beta,\sigma)$, and when K^τ is used instead of the constant factor $\mathscr{M}_\tau(\alpha,\beta,\sigma)$ of (13), (13) still holds. Letting ε, δ be two sufficiently small positive numbers, we define the following expression:

$$\widetilde{f}_\tau(\varepsilon,x) = \begin{cases} x^{\tau(\sigma-1-\frac{\varepsilon}{p})}, & x \in [1,+\infty) \\ 0^\tau, & x \in (0,1) \end{cases}, \quad \widetilde{g}_\tau(\varepsilon,y) = \begin{cases} y^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}, & y \in [\delta,+\infty) \\ 0^\tau, & y \in (0,\delta) \end{cases},$$

we can easily obtain a expression, which is

$$\widetilde{H}_\tau^{(\varepsilon,\delta)} \cdot \varepsilon^\tau = \varepsilon^\tau \cdot \Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[x^{\tau[p(1-\sigma)-1]}\widetilde{f}_\tau^p(\varepsilon,x)\Big]\Big\}^{\frac{1}{p}} \Big\{ {}_\delta\mathscr{Y}_{+\infty}^{(\tau)}\Big[y^{\tau[q(1-\sigma_1)-1]}\widetilde{g}_\tau^q(\varepsilon,y)\Big]\Big\}^{\frac{1}{q}} = \Big(\frac{1^\tau}{\delta^{\tau\varepsilon}}\Big)^{\frac{1}{q}}. \tag{17}$$

In addition, noting $\sigma + \sigma_1 + \alpha - \beta = 0$ and $x \in [1,+\infty)$, by Fubini's theorem [33], we have,

$$\begin{aligned}
\widetilde{J}_\tau^{(\varepsilon,\delta)} \cdot \varepsilon^\tau &= \varepsilon^\tau \cdot {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}\widetilde{f}_\tau(\varepsilon,x)\widetilde{g}_\tau(\varepsilon,y)\Big]\Big\} \\
&= \varepsilon^\tau \cdot {}_1\mathscr{Y}_{+\infty}^{(\tau)}\Big\{x^{\tau(\sigma-1-\frac{\varepsilon}{p})}\, {}_\delta\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{y^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}}\Big]\Big\} \\
&\xlongequal{\text{setting } \frac{y}{x}=u} \varepsilon^\tau \cdot {}_1\mathscr{Y}_{+\infty}^{(\tau)}\Big\{x^{\tau(-1-\varepsilon)}\, {}_{\frac{\delta}{x}}\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{u^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}|1-u|^{\tau\alpha}}{(1+u)^{\tau\beta}}\Big]\Big\} \\
&= {}_{\frac{\delta}{x}}\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{u^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}|1-u|^{\tau\alpha}}{(1+u)^{\tau\beta}}\Big] \\
&\geq {}_\delta\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{u^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}|1-u|^{\tau\alpha}}{(1+u)^{\tau\beta}}\Big].
\end{aligned}$$

Based on the inequality above and (17), we obtain

$${}_\delta\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{u^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}|1-u|^{\tau\alpha}}{(1+u)^{\tau\beta}}\Big] \leq \widetilde{J}_\tau^{(\varepsilon,\delta)} \cdot \varepsilon^\tau < K^\tau \cdot \widetilde{H}_\tau^{(\varepsilon,\delta)} \cdot \varepsilon^\tau = K^\tau \cdot \frac{1^\tau}{(\delta^{\tau\varepsilon})^{\frac{1}{q}}}.$$

Performing limit operations on both sides of the above expression, by Fatou's Lemma [33], we have

$$\mathscr{M}_\tau(\alpha,\beta,\sigma) = \mathscr{M}_\tau(\alpha,\beta,\sigma_1) = \lim_{\delta\to0^+}\Big(\lim_{\varepsilon\to0^+} {}_\delta\mathscr{Y}_{+\infty}^{(\tau)}\Big[\frac{u^{\tau(\sigma_1-1-\frac{\varepsilon}{q})}|1-u|^{\tau\alpha}}{(1+u)^{\tau\beta}}\Big]\Big) \leq K^\tau,$$

which contradicts the previous assumption that $K^\tau < \mathscr{M}_\tau(\alpha,\beta,\sigma)$. Therefore, the constant factor $\mathscr{M}_\tau(\alpha,\beta,\sigma)$ of (13) is the best possible. According to the equivalence property, it is known that $\mathscr{M}_\tau(\alpha,\beta,\sigma)$ is also the best constant factor for (14). □

Theorem 4. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha > -1, 0 < \sigma(\sigma_1) < \beta - \alpha, f_\tau(x) \in L_p^{\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}(\mathbb{R}_+^\tau), g_\tau(y) \in L_q^{\tau[q(1-\sigma_1)-1+\sigma+\sigma_1+\alpha-\beta]}(\mathbb{R}_+^\tau)$, the operator $T^{(\tau)}$ is defined according to Definition 4, we have

(i) $T^{(\tau)}$ is a bounded operator of $L_p^{\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}(\mathbb{R}_+^\tau) \to L_p^{\tau[p(\sigma_1-1)+(p-1)(\beta-\alpha-\sigma-\sigma_1+1)]}(\mathbb{R}_+^\tau)$. In other words, there exists a fractal constant M^τ, such that

$$||T^{(\tau)}f_\tau||_{p,\tau[p(\sigma_1-1)+(p-1)(\beta-\alpha-\sigma-\sigma_1+1)]} \leq M^\tau||f_\tau||_{p,\tau[p(\sigma_1-1)+(p-1)(\beta-\alpha-\sigma-\sigma_1+1)]}. \tag{18}$$

(ii) If and only if $\sigma + \sigma_1 = \beta - \alpha$, the norm of the fractal operator $T^{(\tau)}$ is

$$\|T^{(\tau)}\| = \mathscr{M}_\tau(\alpha,\beta,\sigma). \tag{19}$$

(iii) We have the following pair of operator-equivalent inequalities with norm

$$(T^{(\tau)} f_\tau, g_\tau) < \|T^{(\tau)}\| \|f_\tau\|_{p,\tau[p(1-\sigma)-1]} \|g_\tau\|_{q,\tau[q(1-\sigma_1)-1]}, \tag{20}$$

$$\|T^{(\tau)}(f_\tau)\|_{p,\tau(p\sigma_1-1)} < \|T^{(\tau)}\| \|f_\tau\|_{p,\tau[p(1-\sigma)-1]}. \tag{21}$$

Proof. (i). Based on Definitions 4 and (11), for any fractal constant $M^\tau \geq \mathscr{M}_\tau^{\frac{1}{p}}(\alpha,\beta,\sigma_1)\mathscr{M}_\tau^{\frac{1}{q}}(\alpha,\beta,\sigma)$, the inequality

$$\left\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)} \Big[y^{\tau[p(\sigma_1-1)+(p-1)(\beta-\alpha-\sigma-\sigma_1+1)]} \Big({}_0\mathscr{Y}_{+\infty}^{(\tau)} \big(\frac{|x-y|^{\tau\alpha}}{(x+y)^{\tau\beta}} f_\tau(x) \big) \Big)^p \Big] \right\}^{\frac{1}{p}}$$
$$< M^\tau \|f_\tau\|_{p,\tau[p(1-\sigma)-1+\sigma+\sigma_1+\alpha-\beta]}$$

holds, it is equivalent to (18) being valid.

(ii). Because the constant factor $\mathscr{M}_\tau(\alpha,\beta,\sigma)$ of (14) is the best possible, by (5) and $\sigma + \sigma_1 = \beta - \alpha$, we obtain $\|T^{(\tau)}\| = \mathscr{M}_\tau(\alpha,\beta,\sigma)$.

(iii). From (13) and (14), and in conjunction with Definition 4, we derive (20) and (21). □

4. Applications

We choose suitable parameter values in (13) and compute the operator norm using (19) to derive some simple Hilbert-type fractional integral inequalities.

Example 1. By setting $\alpha = 0, \beta = 1, \sigma = \frac{1}{q}, \sigma_1 = \frac{1}{p}$ in (13), based on Definitions 3, (19), and (6), we obtain $k_\tau(x,y) = \frac{1^\tau}{x^\tau + y^\tau}$ and $\|T^{(\tau)}\| = B_\tau(\frac{1}{q},1)F_\tau(1,\frac{1}{q},1+\frac{1}{q},-1) + B_\tau(\frac{1}{p},1)F_\tau(1,\frac{1}{p},1+\frac{1}{p},-1)$. Hence, we further have

$${}_0\mathscr{Y}_{+\infty}^{(\tau)} \left\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)} \Big[\frac{f_\tau(x) g_\tau(y)}{x^\tau + y^\tau} \Big] \right\}$$
$$< \left\{ B_\tau(\frac{1}{q},1)F_\tau(1,\frac{1}{q},1+\frac{1}{q},-1) + B_\tau(\frac{1}{p},1)F_\tau(1,\frac{1}{p},1+\frac{1}{p},-1) \right\} \|f_\tau\|_p \|g_\tau\|_q. \tag{22}$$

Continuing to take $\tau = 1$ in (22), we derive (1). So, (22) is a generalization of (1) on the real fractal set $\mathbb{R}^\tau$.

Example 2. By setting $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}, \sigma = \frac{1}{q}, \sigma_1 = \frac{1}{p}$ in (13), based on Definitions 3, (19), and (6), we obtain $k_\tau(x,y) = \frac{1^\tau}{\sqrt{|x^{2\tau} - y^{2\tau}|}}$ and $\|T^{(\tau)}\| = B_\tau(\frac{1}{q},\frac{1}{2})F_\tau(\frac{1}{2},\frac{1}{q},\frac{1}{2}+\frac{1}{q},-1) + B_\tau(\frac{1}{p},\frac{1}{2})F_\tau(\frac{1}{2},\frac{1}{p},\frac{1}{2}+\frac{1}{p},-1)$. Hence, we further have

$${}_0\mathscr{Y}_{+\infty}^{(\tau)} \left\{ {}_0\mathscr{Y}_{+\infty}^{(\tau)} \Big[\frac{f_\tau(x) g_\tau(y)}{\sqrt{|x^{2\tau} - y^{2\tau}|}} \Big] \right\}$$
$$< \left\{ B_\tau(\frac{1}{q},\frac{1}{2})F_\tau(\frac{1}{2},\frac{1}{q},\frac{1}{2}+\frac{1}{q},-1) + B_\tau(\frac{1}{p},\frac{1}{2})F_\tau(\frac{1}{2},\frac{1}{p},\frac{1}{2}+\frac{1}{p},-1) \right\} \|f_\tau\|_p \|g_\tau\|_q. \tag{23}$$

Continuing to take $\tau = 1, p = q = 2$ in (23), we have $\|T^{(\tau)}\| = B(\frac{1}{4}, \frac{1}{2})$. Hence, we obtain a Hilbert-type integral inequality on the set $\mathbb{R}$, which is

$$\int_0^\infty \int_0^\infty \frac{1}{\sqrt{|x^2 - y^2|}} f(x)g(y)dxdy < B(\frac{1}{4}, \frac{1}{2})\|f\|_2\|g\|_2.$$

Author contributions

L. P.: Investigation, Validation, Writing–original draft; Q. L.: Conceptualization, Formal analysis, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors thank the reviewers for their helpful suggestions on how to improve the work. The author would like to thank the Scientific Research Project of Hunan Provincial Department of Education, No. 19A455; Hunan Provincial Natural Science Foundation Project (2020JJ4554, 2022JJ30548) for supporting this work.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, 2 Eds., Cambridge: Cambridge University Press, 1967.
2. D. S. Mintrinovic, J. E. Pecaric, A. M. Kink, *Inequalities involving functions and their integrals and derivertives*, Dordrecht: Springer, 1991. https://doi.org/10.1007/978-94-011-3562-7
3. B. C. Yang, *The norm of operator and Hilbert-type inequalities*, (Chinese), Science Press, 2009.
4. T. Batbold, M. Krnić, J. Pečarić, P. Vuković, *Further development of Hilbert-type inequalities*, Zagreb: Element, 2017.
5. B. C. Yang, M. T. Rassias, *On Hilbert-type and Hardy-type integral inequalities and applications*, Switzerland: Springer, 2019. https://doi.org/10.1007/978-3-030-29268-3
6. Y. Hong, B. He, *Theory and applications of Hilbert-type inqualities*, (Chinese), Beijing: Science Press, 2023.
7. Q. Liu, W. B. Sun, A Hilbert-type integral inequality with the mixed kernel of multi-parameters, *C. R. Math.*, **351** (2013), 605–611. https://doi.org/10.1016/j.crma.2013.09.001

8. Q. Liu, A Hilbert-type integral inequality under configuring free power and its applications, *J. Inequal. Appl.*, **2019** (2019), 91. https://doi.org/10.1186/s13660-019-2039-1

9. Q. Liu, On a mixed Kernel Hilbert-type integral inequality and its operator expressions with norm, *Math. Method. Appl. Sci.*, **44** (2021), 593–604. https://doi.org/10.1002/mma.6766

10. M. T. Rassias, B. C. Yang, A. Raigorodskii, On a more accurate reverse Hilbert-type inequality in the whole plane, *J. Math. Inequal.*, **14** (2020), 1359–1374. https://doi.org/10.7153/jmi-2020-14-88

11. X. S. Huang, B. C. Yang, C. M. Huang, On a reverse Hardy-Hilbert-type integral inequality involving derivative function of higher order, *J. Inequal. Appl.*, **2023** (2023), 60. https://doi.org/10.1186/s13660-023-02971-9

12. B. C. Yang, M. T. Rassias, A new Hardy-Hilbert-type integral inequality involving one mulitiple upper limit function and one derivative function of higher order, *Axioms*, **12** (2023), 499. https://doi.org/10.3390/axioms12050499

13. Y. Hong, Q. Chen, Equivalence condition for the best matching parameters of multiple integral operator with generalized homogeneous kernel and applications, (Chinese), *Scientia Sinica Mathematica*, **53** (2023), 717–728. https://doi.org/10.1360/SSM-2021-0149

14. Y. Hong, Y. R. Zong, B. C. Yang, A more accurate half-discrete multidimension Hilbert-type inequality involving one multiple upper limit function, *Axioms*, **12** (2023), 211. https://doi.org/10.3390/axioms12020211

15. X. Y. Huang, S. H. Wu, B. C. Yang, A Hardy-Hilbert-type inequality involving modified coefficients and partial sums, *AIMS Math.*, **7** (2022), 6294–6310. https://doi.org/10.3934/math.2022350

16. X. J. Yang, *Local fractional functional analysis and its applications*, Hong Kong: Asian Academic Publisher Limited, 2011.

17. X. J. Yang, *Advanced local fractional calculus and its applications*, New York: World Science Publisher, 2012.

18. X. J. Yang, D. Baleanu, H. M. Srivastava, *Local fractional integral transforms and their applications*, New York: Academic Press, 2015. https://doi.org/10.1016/C2014-0-04768-5

19. G.-S. Chen, H. M. Srivastava, P. Wang, W. Wei, Some further generalizations of Hölder's inequality and related results on fractal space, *Abstr. Appl. Anal.*, **2014** (2014), 832802. https://doi.org/10.1155/2014/832802

20. X. J. Yang, F. Gao, H. M. Srivastava, Exact traveling wave solutions for the local fractional two-dimensional Burgers-type equations, *Comput. Math. Appl.*, **73** (2017), 203–210. https://doi.org/10.1016/j.camwa.2016.11.012

21. Y. Y. Feng, X. J. Yang, J. G. Liu, Z. Q. Chen, New perspective aimed at local fractional order memristor model on cantor sets, *Fractals*, **29** (2021), 21500011. https://doi.org/10.1142/S0218348X21500110

22. X. J. Yang, L. L. Geng, Y. R. Fan, New special functions applied to represent the weierstrass-nandelbrot function, *Fractals*, **32** (2024), 2340113. https://doi.org/10.1142/S0218348X23401138

23. Q. Liu, W. B. Sun, A Hilbert-type fractal integral inequality and its applications, *J. Inequal. Appl.*, **2017** (2017), 83. https://doi.org/10.1186/s13660-017-1360-9

24. Q. Liu, D. Z. Chen, A Hilbert-type integral inequality on the fractal space, *Integr. Trans. Spec. Funct.*, **28** (2017), 772–780. https://doi.org/10.1080/10652469.2017.1359588

25. Y. D. Liu, Q. Liu, Generalization of Yang-Hardy-Hilbert's integral inequality on the fractal set $\mathbb{R}_+^\alpha$, *Fractals*, **30** (2021), 22500177. https://doi.org/10.1142/S0218348X22500177

26. Q. Liu, A Hilbert-type fractional integral inequality with the kernel of Mittag-Leffler function and its applications, *Math. Inequal. Appl.*, **21** (2018), 729–737. https://doi.org/10.7153/mia-2018-21-52

27. Y. D. Liu, Q. Liu, A Hilbert-type local fractional integral inequality with the kernel of a hyperbolic cosecant function, *Fractals*, **32** (2024), 24400280. https://doi.org/10.1142/S0218348X24400280

28. Y. D. Liu, Q. Liu, The structural features of Hilbert-type local fractional integral inequalities with abstract homogeneous kernel and its applications, *Fractals*, **28** (2020), 2050111. https://doi.org/10.1142/S0218348X2050111X

29. D. Baleanu, M. Krnić, P. Vuković, A class of fractal Hilbert-type inequalities obtained via Cantor-type spherical coordinates, *Math. Method. Appl. Sci.*, **44** (2021), 6195–6208. https://doi.org/10.1002/mma.7180

30. P. Vuković, Some local fractional Hilbert-type inequalities, *Fractal Fract.*, **7** (2023), 205. https://doi.org/10.3390/fractalfract7020205

31. T. Batbold, M. Krnić, P. Vuković, A unified approach to fractal Hilbert-type inequalities, *J. Inequal. Appl.*, **2019** (2019), 117. https://doi.org/10.1186/s13660-019-2076-9

32. Z. S. Huang, D. R. Guo, *An introduction to special function*, (Chinese), Beijing: Beijing University Press, 2000.

33. J. C. Kuang, *Introduction to real analysis*, (Chinese), Changsha: Hunan Education Press, 2000.

AIMS Press