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Research article

A recursive filter for a class of two-dimensional nonlinear stochastic systems

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Abstract: A recursive filtering problem on minimum variance is investigated for a type of two-dimensional systems incorporating noise and a random parameter matrix in the measurement equation, along with random nonlinearity. It methodically describes random variables using statistical characteristics, placing a strong emphasis on the application of random multivariate analysis and computational techniques. A bidirectional time-sequence recursive filter is designed to achieve unbiasedness and reduce error variance effectively. This involves deriving the gain matrix through a completion of squares method and solving a complex difference equation with two independent variances. To facilitate the online implementation of this filter, various formulations and an algorithm are proposed. A numerical study demonstrates the effectiveness of the design in practical applications.

Keywords: two-dimensional nonlinear systems; filter; minimum variance; random parameter matrix; random nonlinearity

1. Introduction

During the past decades, two-dimensional (2D) systems have been extensively studied by virtue of their practical applications in broadcasting signals in two directions, crucial in fields such as image processing, heat diffusion phenomena and optics, manufacturing, multi-variable network realization, seismic data detection and analysis, and chemical processes.The applications are successful due to the existence of an interdependent two dimensional evolution process in the 2D systems. In practical applications, it is well known that the remote power system, where both voltage and current of the circuit vary with the time and space variables, can often be described by an approximate 2D system when taking appropriate sampling periods into account. 2D discrete-time systems are mathematically represented by difference equations involving two variables, with signals conveying along two independent directions. Unlike the states of traditional one-dimensional (1D) models that evolve along a single direction, the essential characteristic of states in 2D systems evolving along two independent directions significantly complicates the performance analysis and synthesis of these systems. Consequently, filter design for both deterministic and stochastic 2D systems has emerged as a focal point of numerous systematic studies. Building on the foundational concepts of 1D systems, substantial theoretical advancements have been made in addressing 2D filtering problems, leading to the development of several effective 2D filter algorithms that meet various practical requirements.

Several techniques have been developed to extend 1D Kalman filtering to two-dimensional case following its initial development reported by A. Hahibi [1]. For 2D linear systems, fundamental concepts have been introduced, and the algebraic realization of the spatial filtering problem has been addressed in references [2] and [3]. It is worth noting that 1D Kalman filtering techniques are no longer feasible when they are simply extended to the 2D case due to the inherently bidirectional evolution of 2D systems. The number of state variables for a 2D filter is proportional to an $L \times L$ (L is the duration of the filter) digital image, compared to L for the 1D Kalman filter, which has led to a limited number of approximate or recursive filter designs and related research outcomes for 2D systems. Approximate schemes for the 2D Kalman filter have been proposed to reduce the excessive computational load, providing a theoretical foundation for the 2D Kalman filter [4, 5]. A new recursively approximate filtering algorithm, paralleling the 1D Kalman filter, has been introduced for a stationary 2D random field model [6]. A polynomial algorithm of the optimal Kalman-Bucy filtering for a linear causal scalar system has been adopted [7]. Additionally, a recursive filter incorporating edge information has been designed for noisy nonhomogeneous images, where the filter combines a 1D predictor with a 1D fixed-interval smoother [8]. A recursive filter algorithm based on the 1D variable representation has been proposed, utilizing geometry and crosscut partition methods in 2D Fornasini-Marchesini II models [9].

The Kalman filtering algorithms for 2D systems described above have certain limitations, including restrictive assumptions and extensive computational requirements. These algorithms typically combine various 1D filters, smoothers, or predictors, which do not provide a systematic filtering framework for 2D scenarios. Consequently, developing a systematic framework for 2D filtering holds substantial theoretical and practical significance. To this end, a 2D Kalman filter has been successfully implemented for discrete-time linear systems, with a pioneering algorithm designed to have a modest computational load, as reported in [10]. Furthermore, to better capture the complexities of actual 2D systems, it is necessary to consider some factors due to sudden changes in the external environment and internal structural phenomena. A recursive 2D filter for shift-varying systems incorporating degraded measurements and stochastic nonlinearity has been developed [11], and a robust 2D filter has been designed for a class of 2D time-varying finite-horizon systems incorporating incomplete measurements and norm-bounded parameter uncertainties [12]. More recently, a recursive filter of locally minimum variance and a robust filter of the recursive structure have been respectively developed for 2D systems with dynamic quantization effects meeting random sensor failure and with bound variance noises in [13] and [14]. When 2D communication network suffers from degraded measurements and other constraints reflecting the real world, a robust filtering problem has been tackled for 2D amplify-and-forward relay systems [15]. When time-varying 2D systems with delays undergo hybrid cyberattacks, an ultimately bounded event-triggered 2D filter has been established with respect to 2D time-varying delays in [16]. Nonetheless, the above relevant filtering results for 2D systems have been pertained to measurement matrices of degraded measurements or known shift-varying constant matrices. The existing research does not adequately address 2D filtering results for more general 2D systems with measurement matrices covering degraded measurements and known shift-varying constant matrices, leaving a significant gap in the field. As such, it is of practical and theoretical significance to design a 2D recursive filter for the rather general case of measurement matrix: the random parameter matrix. The measurement matrices of degraded measurement in [11, 13] and [14, 15] or known shift-varying constant matrices in [10, 12, 16] are the special cases of the proposed measurement matrix. Our attention is to bridge this research deficit by developing filters for 2D systems that accommodate stochastic parameter matrices in measurements and incorporate stochastic nonlinearity.

Inspired by the studies mentioned above and the idea of decomposing stochastic parameter matrices and utilizing stochastic multivariate analysis and calculation for 1D nonlinear systems in [17], we aim to present a recursive filter minimizing error variance for discrete-time 2D nonlinear systems, incorporating a random parameter measurement matrix, and to design an algorithm with a modest computational burden for this filter. The proposed 2D filter, designed to ensure unbiasedness and minimize error variance, will be derived from the stochastic parameter matrix and stochastic nonlinearity. The algorithm is to effectively online realize our presented recursive filter, which can be numerically and iteratively computed by "scanning line by line". Compared to existing work, the considered system in the paper is more general and comprehensive, including not only stochastic state nonlinearity, but also the random matrix in the measurement. We employ the mathematical induction principle and stochastic analysis method of random variables in the analysis and design processes. This paper first establishes a systematic framework for a 2D recursive filter and specifically designs the filter for the state estimation problem of discrete-time 2D nonlinear systems incorporating random parameter matrices in measurement.

The rest of this paper is arranged as follows: The 2D filtering problem is described in Section 2, and a recursive 2D filter is designed in Section 3. The availability of the proposed filter is shown via a numerical example in Section 4. Some conclusions are provided in Section 5. Some used notations in the paper are listed as Table 1.

Table 1. Notation and its definition.

Notation	Definition
R^n	The n-dimensional Euclidean space
I	The identity matrix carrying appropriate dimensions
0	The zero matrix having appropriate dimensions
X^{-1}	Inverse of matrix X
X^T	Transpose of matrix X
$\mathbb{E}_r\{\cdot\}$	The mathematical expectation of stochastic variables
$Co(x, y)$	The covariance matrix of two random variables x and y
$[0\ L]$	$\{0, 1, 2, \cdots, L\}$
$\varphi[0\ L]$	$\{(q, r)\|q, r \in [0\ L]\}$

2. Problem statement

For a given positive integer L, consider a 2D shift-varying system

$$\begin{cases} x(q,r) = A_1(q,r-1)x(q,r-1) + A_2(q-1,r)x(q-1,r) \\ \qquad\qquad + g(x(q,r-1),\xi(q,r-1)) + g(x(q-1,r),\xi(q-1,r)) \\ \qquad\qquad + B_1(q,r-1)w(q,r-1) + B_2(q-1,r)w(q-1,r), \\ y(q,r) = C(q,r)x(q,r) + v(q,r), q,r \in [1\ L] \end{cases} \tag{2.1}$$

where $x(q,r) \in R^n$ and $y(q,r) \in R^m$ are the state and measurement output vectors, $v(q,r) \in R^m$ and $w(q,r) \in R^a$ are the measurement and process noises. Matrices $A_i(q,r) \in R^{n\times n}$ and $B_i(q,r) \in R^{n\times a}$ are known to be deterministic and time-varying for $i = 1,2$, $C(q,r) \in R^{m\times n}$ is a random parameter matrix with known statistical characteristics, which can be split into deterministic and random parts as in the approach [18] and denoted by $C(q,r) = \bar{C}(q,r)+\tilde{C}(q,r)$, $\mathbb{E}_r\{\tilde{C}(q,r)\} = 0$. The function $g(x(q,r),\xi(q,r))$ is nonlinear; it represents the stochastic nonlinearity.

For the system (2.1), we shall make the following assumptions.

Assumption 1. The noises $w(q,r)$ and $v(q,r)$ are mutually independent zero-mean stochastic processes with positive definite covariance matrices $R(q,r)$ and $Q(q,r)$.

Assumption 2. The random matrices $\tilde{C}(q,r)$ and $x(q,r)$ are independent.

Assumption 3. Function $g(x(q,r),\xi(q,r))$ has the same properties as in [11]:

$$g(0,\xi(q,r)) = 0, \tag{2.2}$$

$$\mathbb{E}_r\{g(x(s,t),\xi(s,t))|x(q,r)\} = 0, (s,t) \in \{(s_1,t_1)|s_1 > q \text{ or } t_1 > r\} \cup (q,r), \tag{2.3}$$

$$\mathbb{E}_r\{g(x(q,r),\xi(q,r))g^T(x(s,t),\xi(s,t))|x(q,r)\} \\ = \sum_{j=1}^{d} \Pi_j x^T(q,r)\Gamma_j x(q,r)\delta(q,s)\delta(r,t) \tag{2.4}$$

where $\xi(q,r) \in R^{n_\xi}$ is a random sequence with zero mean and variance $\sigma^2 I$, Π_j and $\Gamma_j (j \in [1\ \ d])$ are known matrices, n_ξ and d are given positive integers. $\xi(q,r), x(q,r)$, and $\tilde{C}(q,r)$ are mutually independent.

Assumption 4. Noises $v(q,r), w(q,r), \xi(q,r)$ and matrix $\tilde{C}(q,r)$ are mutually independent.

Assumption 5. $x(q,0)$ and $x(0,r)$ are set as the initial states and are independent of the above random variables. For $q,i,r,j \in [0\ L]$, the statistical traits are given:

$$\begin{aligned} &\mathbb{E}_r\{x(q,0)\} = u_1(q), \quad \mathbb{E}_r\{x(0,r)\} = u_2(r), \\ &Co\{x(q,0),x(i,0)\} = P(q,0)\delta(q,i), \\ &Co\{x(0,r),x(0,j)\} = P(0,r)\delta(r,j), \\ &Co\{x(q,0),x(0,r)\} = P(0,0)\delta(q,0)\delta(0,r) \end{aligned}$$

where $u_1(q), u_2(r), P(q,0)$, and $P(0,r)$ are known parameters, and $u_1(0) = u_2(0)$.

Remark 1. The system (2.1) under investigation is a rather general model that includes stochastic nonlinearity, noises, and the general case of measurement matrix: random parameter matrix. The measurement matrices of degraded measurements in [11, 13–15] or known shift-varying constant matrices in [10, 12, 16] are the special cases of the proposed measurement. A new model of measurement incorporating a random parameter matrix is proposed to characterize the phenomenon of random measurement.

A recursive bidirectional time-sequence filter is designed for (2.1) as follows:

$$\begin{cases}\hat{x}_p(q,r) = A_1(q,r-1)\hat{x}_u(q,r-1) + A_2(q-1,r)\hat{x}_u(q-1,r),\\ \hat{x}_u(q,r) = \hat{x}_p(q,r) + K(q,r)[y(q,r) - \bar{C}(q,r)\hat{x}_p(q,r)]\end{cases} \tag{2.5}$$

where $\hat{x}_p(q,r)$ and $\hat{x}_u(q,r)$ are the one-step prediction and the updated estimate of state $x(q,r)$, $K(q,r)$ is the filter gain matrix to be designed for $q, r \in [1\ L]$. The initial values of $\hat{x}_u(q,r)$ are $\hat{x}_u(q,0) = u_1(q)$ and $\hat{x}_u(0,r) = u_2(r)$ for $q, r \in [0\ L]$.

Remark 2. The recursive bidirectional time-sequence filter satisfies the essential characteristic of states in 2D systems evolving along two independent directions. This provides a systematic filtering framework for 2D scenarios.

Let us define $\tilde{x}_p(q,r) = x(q,r) - \hat{x}_p(q,r)$ and $\tilde{x}_u(q,r) = x(q,r) - \hat{x}_u(q,r)$ as the errors of the prediction and the estimation. Then, we obtain the following error dynamics from (2.1) and (2.5):

$$\begin{cases}\tilde{x}_p(q,r) = A_1(q,r-1)\tilde{x}_u(q,r-1) + A_2(q-1,r)\tilde{x}_u(q-1,r)\\ \qquad + g(x(q,r-1),\xi(q,r-1)) + g(x(q-1,r),\xi(q-1,r))\\ \qquad + B_1(q,r-1)w(q,r-1) + B_2(q-1,r)w(q-1,r),\\ \tilde{x}_u(q,r) = [I - K(q,r)\bar{C}(q,r)]\tilde{x}_p(q,r) - K(q,r)[\tilde{C}(q,r)x(q,r) + v(q,r)].\end{cases} \tag{2.6}$$

Our goal is to design the above filter (2.5) so that $\mathbb{E}\{\tilde{x}_u(q,r)\tilde{x}_u^T(q,r)\}$, which is the filter error variance, is minimized at each pair (q,r), for $(q,r) \in \varphi[0\ L]$, and to propose an algorithm running in a modest computational burden for this filter.

3. Main results

Our goal is to be achieved in this section. The gain parameter $K(q,r)$ is solved, and the recursive filter (2.5) for the 2D system (2.1) is designed to minimize the error variance. Then the online process of solving the filter is listed. Before obtaining the desired results, we shall introduce the following lemmas.

Lemma 1 ([19]). Let $A = (a_{ij})_{N_1\times N_2}$ and $B = (b_{ij})_{M_1\times M_2}$ be random matrices with $\tilde{A} = A - \mathbb{E}_r\{A\}$ and $\tilde{B} = B - \mathbb{E}_r\{B\}$. For any deterministic matrix $C = (c_{ij})_{N_2\times M_2}$, then the (s,t)-th entry of the matrix $\mathbb{E}_r\{\tilde{A}C\tilde{B}^T\}$, $s = 1,\cdots,N_1$, $t = 1,\cdots,M_1$, is given by

$$(\mathbb{E}_r\{\tilde{A}C\tilde{B}^T\})_{st} = \sum_{i=1}^{N_1}\sum_{j=1}^{M_2} Co(a_{si}, b_{jt})c_{ij}.$$

Lemma 2 ([20]). Let A be a random matrix and x be a random vector. If they are independent, then

$$\mathbb{E}_r\{Axx^TA^T\} = \mathbb{E}_r\{A\mathbb{E}_r\{xx^T\}A^T\}.$$

3.1. Solving the filter gain

Matrix $K(q,r)$ is to be solved according to the error variance minimized at each step in the subsection.

In order to facilitate the notation, let us define

$$
\begin{aligned}
P_p(q,r) &\triangleq \mathbb{E}_r\{\tilde{x}_p(q,r)\tilde{x}_p^T(q,r)\},\\
P_u(q,r) &\triangleq \mathbb{E}_r\{\tilde{x}_u(q,r)\tilde{x}_u^T(q,r)\},\\
X(q,r) &\triangleq \mathbb{E}_r\{x(q,r)x^T(q,r)\}.
\end{aligned}
$$

Then several conclusions are obtained based on Lemma 1 and Lemma 2 as below.

Theorem 1. Consider the 2D system (2.1) and the designed 2D filter (2.5) with initial values $\hat{x}_u(q,0) = u_1(q), q \in [0\ L]$, and $\hat{x}_u(0,r) = u_2(r), r \in [0\ L]$; it is unbiased, that is, $\mathbb{E}\{\tilde{x}_u(q,r)\} = 0$ for $(q,r) \in \varphi[0\ L]$.

Proof. The proof is given in Appendix A.

Theorem 2. Consider the 2D system (2.1); for $q, r \in [1\ L]$, the second-order moment $X(q,r)$ of state $x(q,r)$ has the following recursion:

$$
\begin{aligned}
X(q,r) \;=\;& A_1(q,r-1)X(q,r-1)A_1^T(q,r-1) + A_2(q-1,r)X(q-1,r)A_2^T(q-1,r)\\
&+A_1(q,r-1)\mathbb{E}_r\{x(q,r-1)x^T(q-1,r)\}A_2^T(q-1,r)\\
&+A_2(q-1,r)\mathbb{E}_r\{x(q-1,r)x^T(q,r-1)\}A_1^T(q,r-1)\\
&+\sum_{j=1}^{d}\Pi_j \mathrm{tr}\{(X(q,r-1)+X(q-1,r))\Gamma_j\}\\
&+B_1(q,r-1)R(q,r-1)B_1^T(q,r-1)\\
&+B_2(q-1,r)R(q-1,r)B_2^T(q-1,r).
\end{aligned}
\tag{3.1}
$$

Proof. The proof is given in Appendix B.

Theorem 3. The 2D second-order moment $P_p(q,r)$ of the prediction error for (2.1) has the following recursion:

$$
\begin{aligned}
P_p(q,r) \;=\;& A_1(q,r-1)P_u(q,r-1)A_1^T(q,r-1) + A_2(q-1,r)P_u(q-1,r)A_2^T(q-1,r)\\
&+A_1(q,r-1)\mathbb{E}_r\{\tilde{x}_u(q,r-1)\tilde{x}_u^T(q-1,r)\}A_2^T(q-1,r)\\
&+A_2(q-1,r)\mathbb{E}_r\{\tilde{x}_u(q-1,r)\tilde{x}_u^T(q,r-1)\}A_1^T(q,r-1)\\
&+\sum_{j=1}^{d}\Pi_j \mathrm{tr}\{(X(q,r-1)+X(q-1,r))\Gamma_j\}\\
&+B_1(q,r-1)R(q,r-1)B_1^T(q,r-1)\\
&+B_2(q-1,r)R(q-1,r)B_2^T(q-1,r)
\end{aligned}
\tag{3.2}
$$

for $(q,r) \in \varphi[1\ L]$.

Proof. The proof is given in Appendix C.

Theorem 4. Consider the system (2.1); the gain of filter (2.5) achieving the minimum error variance of the estimation $\hat{x}_u(q,r)$ is provided with

$$
K(q,r) = P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r) \tag{3.3}
$$

where

$$
\begin{aligned}
R_e(q,r) &= \bar{C}(q,r)P_p(q,r)\bar{C}^T(q,r) + Q(q,r) + \mathbb{E}_r\{\tilde{C}(q,r)X(q,r)\tilde{C}^T(q,r)\},\\
P_p(q,r) &= A_1(q,r-1)P_u(q,r-1)A_1^T(q,r-1) + A_2(q-1,r)P_u(q-1,r)A_2^T(q-1,r)\\
&\quad +A_1(q,r-1)\mathbb{E}_r\{\tilde{x}_u(q,r-1)\tilde{x}_u^T(q-1,r)\}A_2^T(q-1,r)\\
&\quad +A_2(q-1,r)\mathbb{E}_r\{\tilde{x}_u(q-1,r)\tilde{x}_u^T(q,r-1)\}A_1^T(q,r-1)\\
&\quad +\sum_{j=1}^{d}\Pi_j\mathrm{tr}\{(X(q,r-1)+X(q-1,r))\Gamma_j\}\\
&\quad +B_1(q,r-1)R(q,r-1)B_1^T(q,r-1) + B_2(q-1,r)R(q-1,r)B_2^T(q-1,r),
\end{aligned}
$$

and

$$
\begin{aligned}
X(q,r) &= A_1(q,r-1)X(q,r-1)A_1^T(q,r-1) + A_2(q-1,r)X(q-1,r)A_2^T(q-1,r)\\
&\quad +A_1(q,r-1)\mathbb{E}_r\{x(q,r-1)x^T(q-1,r)\}A_2^T(q-1,r)\\
&\quad +A_2(q-1,r)\mathbb{E}_r\{x(q-1,r)x^T(q,r-1)\}A_1^T(q,r-1)\\
&\quad +\sum_{j=1}^{d}\Pi_j\mathrm{tr}\{(X(q,r-1)+X(q-1,r))\Gamma_j\}\\
&\quad +B_1(q,r-1)R(q,r-1)B_1^T(q,r-1) + B_2(q-1,r)R(q-1,r)B_2^T(q-1,r)
\end{aligned}
$$

for $(q,r) \in [1\ L]$. The minimum estimation error variance is presented as

$$
P_u(q,r) = P_p(q,r) - K(q,r)\bar{C}(q,r)P_p(q,r). \tag{3.4}
$$

Proof. Becuse the noise $v(q,r)$ is independent of $\tilde{x}_p(q,r)$ and $x(q,r)$, it can be obtained that

$$
\begin{aligned}
P_u(q,r) &= [I-K(q,r)\bar{C}(q,r)]P_p(q,r)[I-K(q,r)\bar{C}(q,r)]^T\\
&\quad -[I-K(q,r)\bar{C}(q,r)]\mathbb{E}_r\{\tilde{x}_p(q,r)[\tilde{C}(q,r)x(q,r)+v(q,r)]^T\}K^T(q,r)\\
&\quad -K(q,r)\mathbb{E}_r\{[\tilde{C}(q,r)x(q,r)+v(q,r)]\tilde{x}_p^T(q,r)\}[I-K(q,r)\bar{C}(q,r)]^T\\
&\quad +K(q,r)\mathbb{E}_r\{[\tilde{C}(q,r)x(q,r)+v(q,r)][\tilde{C}(q,r)x(q,r)+v(q,r)]^T\}K^T(q,r)\\
&= [I-K(q,r)\bar{C}(q,r)]P_p(q,r)[I-K(q,r)\bar{C}(q,r)]^T\\
&\quad -[I-K(q,r)\bar{C}(q,r)]\mathbb{E}_r\{\tilde{x}_p(q,r)x^T(q,r)\tilde{C}^T(q,r)\}K^T(q,r)\\
&\quad -K(q,r)\mathbb{E}_r\{\tilde{C}(q,r)x(q,r)\tilde{x}_p^T(q,r)\}[I-K(q,r)\bar{C}(q,r)]^T\\
&\quad +K(q,r)\mathbb{E}_r\{\tilde{C}(q,r)X(q,r)\tilde{C}^T(q,r)\}K^T(q,r) + K(q,r)Q(q,r)K^T(q,r).
\end{aligned}
$$

Taking into account that

$$
\mathbb{E}_r\{\tilde{x}_p(q,r)x^T(q,r)\tilde{C}^T(q,r)\} = 0,\ \mathbb{E}_r\{\tilde{C}(q,r)x(q,r)\tilde{x}_p^T(q,r)\} = 0,
$$

and incorporating Assumption 3, we have

$$
\begin{aligned}
P_u(q,r) &= [I-K(q,r)\bar{C}(q,r)]P_p(q,r)[I-K(q,r)\bar{C}(q,r)]^T\\
&\quad +K(q,r)\mathbb{E}_r\{\tilde{C}(q,r)X(q,r)\tilde{C}^T(q,r)\}K^T(q,r) + K(q,r)Q(q,r)K^T(q,r)
\end{aligned}
$$

$$= P_p(q,r) - K(q,r)\bar{C}(q,r)P_p(q,r) - P_p(q,r)[K(q,r)\bar{C}(q,r)]^T$$
$$+K(q,r)\left[\bar{C}(q,r)P_p\bar{C}^T(q,r) + \mathbb{E}_r\{\tilde{C}(q,r)X(q,r)\tilde{C}^T(q,r)\} + Q(q,r)\right]K^T(q,r).$$

Then focus on the above term and perform a completion of squares; we obtain

$$\begin{aligned}
& P_u(q,r) \\
= & [I\ K(q,r)]\begin{bmatrix} P_p(q,r) & -P_p(q,r)\bar{C}^T(q,r) \\ -\bar{C}(q,r)P_p(q,r) & R_e(q,r) \end{bmatrix}\begin{bmatrix} I \\ K^T(q,r) \end{bmatrix} \\
= & [I\ K(q,r)]\begin{bmatrix} I & -P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r) \\ 0 & I \end{bmatrix}\begin{bmatrix} \Delta & 0 \\ 0 & R_e(q,r) \end{bmatrix} \\
& \times\begin{bmatrix} I & 0 \\ -R_e^{-1}(q,r)\bar{C}(q,r)P_p(q,r) & I \end{bmatrix}\begin{bmatrix} I \\ K^T(q,r) \end{bmatrix} \\
= & (K(q,r) - P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r))R_e(q,r)(K(q,r) - P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r))^T + \Delta
\end{aligned} \tag{3.5}$$

where

$$R_e(q,r) = \bar{C}(q,r)P_p(q,r)\bar{C}^T(q,r) + \mathbb{E}_r\{\tilde{C}(q,r)X(q,r)\tilde{C}^T(q,r)\} + Q(q,r),$$
$$\Delta = P_p(q,r) - \left(P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r)\right)R_e(q,r)\left(P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r)\right)^T.$$

Now we need to find the 2D matrix $K(q,r)$ that minimizes $P_u(q,r)$. Then $K(q,r)$ should be chosen as

$$K(q,r) = P_p(q,r)\bar{C}^T(q,r)R_e^{-1}(q,r).$$

Meanwhile the 2D filter error variance (3.5) reaches its minimal value

$$P_u(q,r) = P_p(q,r) - K(q,r)R_e(q,r)K^T(q,r) = P_p(q,r) - K(q,r)\bar{C}(q,r)P_p(q,r).$$

The derived 2D filter minimizes its error variance when $K(q,r)$ is chosen as (3.3). The proof is completed.

Remark 3. The 2D filter has a similar structure to the Kalman filter for 1D systems. It is observed that the cross-item $\mathbb{E}_r\{x(q-1,r)x^T(q,r-1)\}$ is involved in (3.1) and $\mathbb{E}_r\{\tilde{x}_u(q-1,r)\tilde{x}_u^T(q,r-1)\}$ is involved in (3.2), which dynamics need further analysis for completing the calculation process.

3.2. Filter calculation

In contrast with the traditional 1D filtering dynamics, whose variables evolve along a single direction, the information of 2D filtering dynamics transmits along two independent directions and the system with dynamics relies on two independent variables. It is easily observed that the recursions of $X(q,r)$ in (3.1) and $P_p(q,r)$ in (3.2) respectively accompany $\mathbb{E}_r\{x(q,r-1)x^T(q-1,r)\}$ and $\mathbb{E}_r\{\tilde{x}_u(q,r-1)\tilde{x}_u^T(q-1,r)\}$ due to the dynamical and structural complexity of $2D$ filters, which differs significantly from the filter of $1D$ systems. Therefore the two recursions should be further derived to facilitate the filter gain (3.3). By utilizing random multivariate analysis and calculation, it is obtained that for $q,r \in [2\ L]$,

$$\mathbb{E}_r\{x(q,r-1)x^T(q-1,r)\}$$

$$
\begin{aligned}
&= A_1(q,r-2)\mathbb{E}_r\{x(q,r-2)x^T(q-1,r-1)\}A_1^T(q-1,r-1)\\
&\quad +A_1(q,r-2)\mathbb{E}_r\{x(q,r-2)x^T(q-2,r)\}A_2^T(q-2,r)\\
&\quad +A_2(q-1,r-1)\mathbb{E}_r\{x(q-1,r-1)x^T(q-2,r)\}A_2^T(q-2,r)\\
&\quad +A_2(q-1,r-1)X(q-1,r-1)A_1^T(q-1,r-1)+\sum_{j=1}^{d}\Pi_j\mathrm{tr}\{X(q-1,r-1)\Gamma_j\}\\
&\quad +B_2(q-1,r-1)R(q-1,r-1)B_1^T(q-1,r-1) \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_r\{\tilde{x}_u(q,r-1)\tilde{x}_u^T(q-1,r)\}\\
= &[I-K(q,r-1)\bar{C}(q,r-1)\Big\{A_1(q,r-2)\mathbb{E}_r\{\tilde{x}_u(q,r-2)\tilde{x}_u^T(q-1,r-1)\}A_1^T(q-1,r-1)\\
&+A_1(q,r-2)\mathbb{E}_r\{\tilde{x}_u(q,r-2)\tilde{x}_u^T(q-2,r)\}A_2^T(q-2,r)\\
&+A_2(q-1,r-1)P_u(q-1,r-1)A_1^T(q-1,r-1)+\sum_{s=1}^{h}\Pi_s\mathrm{tr}\{X(q-1,r-1)\Gamma_s\}\\
&+A_2(q-1,r-1)\mathbb{E}_r\{\tilde{x}_u(q-1,r-1)\tilde{x}_u^T(q-2,r)\}A_2^T(q-2,r)\\
&+B_2(q-1,r-1)R(q-1,r-1)B_1^T(q-1,r-1)\Big\}[I-K(q-1,r)\bar{C}(q-1,r)]^T. \qquad (3.7)
\end{aligned}
$$

Repeating the same computation for $q,r\in[z\ L]$ ($z\in[2\ \ L-1]$), it follows

$$
\begin{aligned}
&\mathbb{E}_r\{x(q,r-z)x^T(q-z,r)\}\\
= &A_1(q,r-z-1)\mathbb{E}_r\{x(q,r-z-1)x^T(q-z,r-1)\}A_1^T(q-z,r-1)\\
&+A_1(q,r-z-1)\mathbb{E}_r\{x(q,r-z-1)x^T(q-z-1,r)\}A_2^T(q-z-1,r)\\
&+A_2(q-1,r-z)\mathbb{E}_r\{x(q-1,r-z)x^T(q-z,r-1)\}A_1^T(q-z,r-1)\\
&+A_2(q-1,r-z)\mathbb{E}_r\{x(q-1,r-z)x^T(q-z-1,r)\}A_2^T(q-z-1,r) \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_r\{\tilde{x}_u(q,r-z)\tilde{x}_u^T(q-z,r)\}\\
= &[I-K(q,r-z)\bar{C}(q,r-z)]\Big\{A_1(q,r-z-1)\\
&\times\mathbb{E}_r\{\tilde{x}_u(q,r-z-1)\tilde{x}_u^T(q-z,r-1)\}A_1^T(q-z,r-1)\\
&+A_1(q,r-z-1)\mathbb{E}_r\{\tilde{x}_u(q,r-z-1)\tilde{x}_u^T(q-z-1,r)\}A_2^T(q-z-1,r)\\
&+A_2(q-1,r-z)\mathbb{E}_r\{\tilde{x}_u(q-1,r-z)\tilde{x}_u^T(q-z,r-1)\}A_1^T(q-z,r-1)\\
&+A_2(q-1,r-z)\mathbb{E}_r\{\tilde{x}_u(q-1,r-z)\tilde{x}_u^T(q-z-1,r)\}A_2^T(q-z-1,r)\Big\}\\
&\times[I-K(q-z,r)\bar{C}(q-z,r)]^T. \qquad (3.9)
\end{aligned}
$$

It is observed from (3.6) and (3.7) that the covariance matrices $\mathbb{E}_r\{x(q,r-1)x^T(q-1,r)\}$ and $\mathbb{E}_r\{\tilde{x}_u(q,r-1)\tilde{x}_u^T(q-1,r)\}$ at (q,r) can be iteratively computed out by the information of the three neighbor points $(q,r-2),(q-1,r-1)$, and $(q-2,r)$ for $q,r\in[2\ L]$. From (3.8) and (3.9), it is shown that $\mathbb{E}_r\{x(q,r-z)x^T(q-z,r)\}$ and $\mathbb{E}_r\{\tilde{x}_u(q,r-z)\tilde{x}_u^T(q-z,r)\}$ can be obtained by the information of the four neighbor points $(q,r-z-1),(q-z,r-1),(q-z-1,r)$, and $(q-1,r-z)$ for

$q, r \in [z\ L]$ ($z \in [2\ L-1]$). For each (q, r), it is influenced by two points at a distance of k, one on its left and one below. These two points, in turn, are influenced by their respective left and below points. The iterative computation of the covariance matrices is based on the information from these neighboring points. Especially, $x(1, r)\tilde{x}_u(1, r)$ and $x(q, 1)\tilde{x}_u(q, 1)$ are influenced by their respective left neighboring and below neighboring points with the initial values $u_1(q)$ and $u_2(r)$.

Then combined with the given initial values, in terms of the established conclusions, the parameter $K(q, r)$ can be computed by solving recursions (3.1), (3.2), and (3.6)–(3.9). Finally the filter $\hat{x}_u(q, r)$ (2.5) is obtained for $q, r \in [0\ L]$. The process of solving the filter is shown as follows.

- Step 1. Give initial values $u(q), u(r)$, and $P_u(q, 0)$, and $P_u(0, r)$ for all $(q, r) \in \varphi[0\ L]$, and set $i = 1, j = 1$.
- Step 2. If $i \le L$ and $j \le L$, calculate $\hat{x}_p(i, j)$, $X(i, j)$, and $P_p(i, j)$ from the first equation of (2.5), (3.1) and (3.2), respectively; then compute matrix $K(i, j)$, filter $\hat{x}_u(i, j)$, and matrix $P_u(i, j)$ from (3.3), the second equation of (2.5), and (3.4), respectively; and go to the next step, otherwise step.
- Step 3. If $i \le L$ and $j \le L-1$, compute the items $\mathbb{E}_r\{x(i, j)x^T(i_0, i+j-i_0)\}$ and $\mathbb{E}_r\{\tilde{x}_u(i, j)\tilde{x}_u^T(i_0, i+j-i_0)\}$ via the formula (3.6)-(3.9) for ($i_0 \in [i+j-min\{i+l, L\}\quad i-1]$); set $j = j+1$ and return to Step 2, else go to Step 4.
- Step 4. If $i \le L$ and $j = L$, then set $i = i+1, j = 1$ and return to Step 2.
- Step 5. Stop.

Remark 4. It comes down to the fact that the computation of the recursive filter can be implemented line by line from left to right, and for each line from below to above. It is shown that the process of solving the filter (2.5) has been operated with a modest computational burden [10].

4. Simulation

In order to illustrate the effectiveness of the proposed filtering strategy, numerical simulations are performed by one example stemmed from monitoring a long transmission line in circuit systems [11] below.

Let $x(q, r) = (x^1(q, r)\ x^2(q, r))^T$ and $\xi(q, r) = (\xi^1(q, r)\ \xi^2(q, r))^T$ be the state and the noise. $v(q, r)$ and $w(q, r)$ are zero-mean Gaussian white noises with variance $R(q, r) = 0.025$ and $Q(q, r) = 0.125$, and set the initial value as $x(q, 0) = x(0, r) = 0$ and $\hat{x}_u(q, 0) = \hat{x}_u(0, r) = 0$. Parameters of system (2.1) are given as follows:

$$A_1(q, r) = \begin{bmatrix} -0.4 & 0.3\sin(3q) \\ -0.1 & 0.35 \end{bmatrix},\ A_2(q, r) = \begin{bmatrix} 0.3+\sin(4q) & -0.1 \\ 0.2-0.1\sin(0.8r) & 0.25 \end{bmatrix},$$
$$B_1(q, r) = \begin{bmatrix} 0.1 \\ 0.1e^{-r} \end{bmatrix},\ B_2(q, r) = \begin{bmatrix} 0.18-0.1e^{-4q} \\ 0.12 \end{bmatrix},\ \bar{C}(q, r) = [-0.3,\ 0.35].$$

The function

$$\begin{aligned} g(x(q, r), \xi(q, r)) \quad &= \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0.1sign(x^1(q, r))x^1(q, r)\xi^1(q, r) \\ &\quad +0.2sign(x^2(q, r))x^2(q, r)\xi^2(q, r)) \end{aligned}$$

where $\xi^1(q,r)$ and $\xi^2(q,r)$ are independent white noises with mean 0 and variance 1. $d = 1$, $u_1(q) = u_2(r) = 0$, $P_u(0,0) = 0.1I_2$, $P_u(q,0) = P_u(0,r) = 0.1I_2$ (I_2 is a 2×2 unit matrix), and

$$\Pi_j = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \Gamma_j = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

The simulations are fulfilled. Figures 1 and 2 show the development of the filter error $\tilde{x}_u(q,r)$, which k-th element is denoted as $\tilde{x}_u^k(q,r)$ $(k = 1, 2)$. It is obvious that the error of our designed filter decreases when the two independent variables q, r increase, even the error is closer to zero. The example has been shown that the designed algorithm is effective in dealing with the recursive 2D filtering problem.

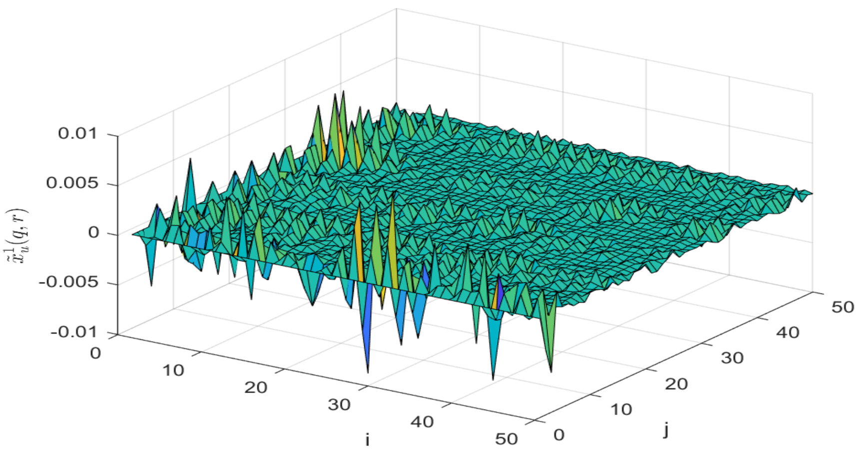

Figure 1. Estimation error $\tilde{x}_u^1(q,r)$ of state element $x_u^1(q,r)$.

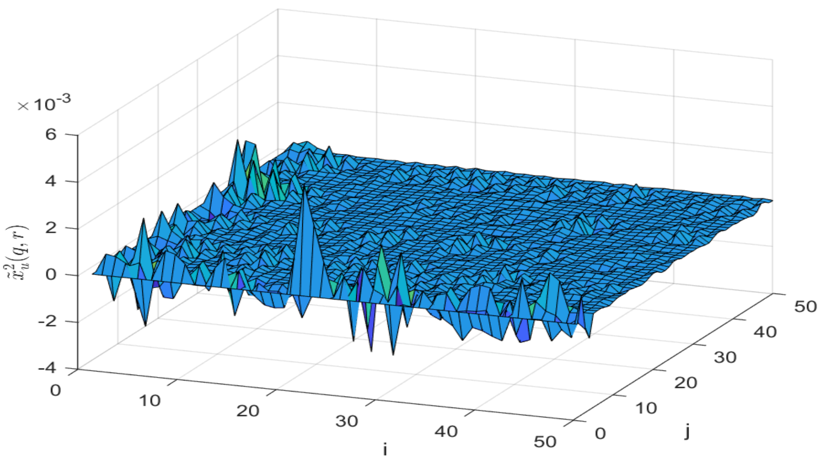

Figure 2. Estimation error $\tilde{x}_u^2(q,r)$ of state element $x_u^2(q,r)$.

5. Conclusions

The filtering problem for a 2D discrete-time system incorporating noise and stochastic parameter matrices in both state and measurement equations is investigated in this paper. It methodically describes random variables using statistical characteristics. The two-step 2D recursive filter satisfies the essential characteristic of states in 2D systems evolving along two independent directions. This provides a systematic filtering framework for 2D scenarios. The techniques used in the paper can solve some more complicated and generalized filtering or other problems of 2D stochastic systems.

Author contributions

Shulan Kong: Conceptualization, Writing—review and editing, Datacuration, Writing—original draft preparation, Supervision; Chengbin Wang: Conceptualization, Writing—original draft, Investigation, Yawen Sun: Project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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Appendix A. Proof of Theorem 1

The four steps of the mathematical induction are carried out as follows:

Step 1. In view of the initial conditions $\hat{x}_u(q,0) = u_1(q), \hat{x}_u(0,r) = u_2(r)$, and Assumption 5 for $q, r \in [0\ L]$

$$\hat{x}_u(q,0) = \mathbb{E}_r\{x(q,0)\} = u_1(q), \quad \hat{x}_u(0,r) = \mathbb{E}_r\{x(0,r)\} = u_2(r).$$

It is clear that

$$\begin{aligned}&\mathbb{E}_r\{\tilde{x}_u(q,0)\} = \mathbb{E}_r\{x(q,0)\} - \hat{x}_u(q,0) = 0,\\&\mathbb{E}_r\{\tilde{x}_u(0,r)\} = \mathbb{E}_r\{x(0,r)\} - \hat{x}_u(0,r) = 0, q, r \in [0\, L].\end{aligned}$$

Recalling Assumptions 1 and 5, it follows

$$\mathbb{E}_r\{\tilde{x}_p(1,1)\} = A_1(1,0)\mathbb{E}_r\{\tilde{x}_u(1,0)\} + A_2(0,1)\mathbb{E}_r\{\tilde{x}_u(0,1)\} = 0.$$

Since $\tilde{C}(q,r)$ is independent of $x(q,r)$ and $\mathbb{E}_r\{\tilde{C}(q,r)\} = 0$ based on (A2), and $\mathbb{E}_r\{v(q,r)\} = 0$ based on (A1), it follows from (2.6) and (2.1) that

$$\mathbb{E}_r\{\tilde{x}_u(1,1)\} = [I - K(1,1)C(1,1)]\,\mathbb{E}_r\{\tilde{x}_p(1,1)\} - K(1,1)\left[\mathbb{E}_r\{\tilde{C}(1,1)\}\mathbb{E}_r\{x(1,1)\} + \mathbb{E}_r\{v(1,1)\}\right] = 0.$$

Assume that $\mathbb{E}_r\{\tilde{x}_u(k_0,1)\} = 0$ and $\mathbb{E}_r\{\tilde{x}_u(1,l_0)\} = 0$ are true for given constants $k_0, l_0, 1 < k_0 < L, 1 < l_0 < L$. Then we have

$$\begin{aligned}&\mathbb{E}_r\{\tilde{x}_p(k_0+1,1)\} = A_1(k_0+1,0)\mathbb{E}_r\{\tilde{x}_u(k_0+1,0)\} + A_2(k_0,1)\mathbb{E}_r\{\tilde{x}_u(k_0,1)\} = 0,\\&\mathbb{E}_r\{\tilde{x}_p(1,l_0+1)\} = A_2(0,l_0+1)\mathbb{E}_r\{\tilde{x}_u(0,l_0+1)\} + A_1(1,l_0)\mathbb{E}_r\{\tilde{x}_u(1,l_0)\} = 0,\end{aligned}$$

and

$$\mathbb{E}_r\{\tilde{x}_u(k_0+1,1)\} = \mathbb{E}_r\{\tilde{x}_u(1,l_0+1)\} = 0.$$

Thus $\mathbb{E}_r\{\tilde{x}_u(m,1)\} = \mathbb{E}_r\{\tilde{x}_u(1,n)\} = 0$ and $\mathbb{E}_r\{\tilde{x}_p(m,1)\} = \mathbb{E}_r\{\tilde{x}_p(1,n)\} = 0$ for $m, n \in [1\ L]$.

Step 2. Assume inductively that $\mathbb{E}_r\{\tilde{x}_u(k_1,n)\} = 0$ and $\mathbb{E}_r\{\tilde{x}_u(m,k_2)\} = 0, \forall m, n \in [0\ L]$ is true for some given constants $k_1, k_2, 1 < k_1 < L, 1 < k_2 < L$.

Step 3. According to Step 2, note that $\mathbb{E}_r\{\tilde{x}_u(k_1,n)\} = 0$ and $\mathbb{E}_r\{\tilde{x}_u(k_1+1,n-1)\} = 0$ is true when take $m = k_1 + 1$ and $k_2 = n - 1$. It is obtained that

$$\mathbb{E}_r\{\tilde{x}_p(k_1+1,n)\} = A_1(k_1+1,n-1)\mathbb{E}_r\{\tilde{x}_u(k_1+1,n-1)\} + A_2(k_1,n)\mathbb{E}_r\{\tilde{x}_u(k_1,n)\} = 0.$$

From (2.6) we obtain that

$$\mathbb{E}_r\{\tilde{x}_u(k_1+1,n)\} = 0.$$

Similarly, it can be concluded that

$$\mathbb{E}_r\{\tilde{x}_u(m,k_2+1)\} = 0.$$

Step 4. Based on steps 1–3, we obtain $\mathbb{E}_r\{\tilde{x}_u(q,r)\} = 0$ for $(q,r) \in \varphi[0\ L]$.

Appendix B. Proof of Theorem 2

According to Assumptions 1 and 3, as well as the properties of $w(q,r)$, $v(q,r)$ and $\xi(q,r)$, it is obtained immediately that

$$\mathbb{E}_r\{x(q,r)w^T(s,t)\} = 0, \ \ \mathbb{E}_r\{g(x(q,r),\xi(q,r))w^T(s,t)\} = 0,$$
$$\mathbb{E}_r\{x(q,r)v^T(s,t)\} = 0, \ \ \mathbb{E}_r\{g(x(q,r),\xi(q,r))v^T(s,t)\} = 0,$$

and

$$\begin{aligned}
& \mathbb{E}_r\{x(q,r)g^T(x(s,t),\xi(s,t))\} \\
= \ & \mathbb{E}_r\{\mathbb{E}_r\{x(q,r)g^T(x(s,t),\xi(s,t))|x(q,r)\}\} \\
= \ & \mathbb{E}_r\{x(q,r)\mathbb{E}_r\{g^T(x(s,t),\xi(s,t))|x(q,r)\}\} = 0,
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E}_r\{g(x(q,r),\xi(q,r))g^T(x(s,t),\xi(s,t))\} \\
= \ & \mathbb{E}_r\{\mathbb{E}_r\{g(x(q,r),\xi(q,r))g^T(x(s,t),\xi(s,t))|x(q,r)\}\} \\
= \ & \sum_{j=1}^{d} \Pi_j \mathbb{E}_r\{x^T(q,r)\Gamma_j x(q,r)\}\delta(q,s)\delta(r,t) \\
= \ & \sum_{j=1}^{d} \Pi_j \mathrm{tr}\{X(q,r)\Gamma_j\}\delta(q,s)\delta(r,t)
\end{aligned}$$

for $(s,t) \in \{(s_0,t_0)|s_0 > q, or\ t_0 > r\} \cup (q,r)$. Then (3.1) can be computed by (2.1).

Appendix C. Proof of Theorem 3

Consider the prediction error $\tilde{x}_p(q,r)$ in (2.6) together with the following equations

$$\begin{aligned}
& \mathbb{E}_r[\tilde{x}_u(q,r)g^T(x(s,t),\xi(s,t))] \\
= \ & \mathbb{E}_r\{\tilde{x}_u(q,r)\mathbb{E}_r\{g^T(x(s,t),\xi(s,t))|\tilde{x}_u(q,r)\}\} \\
= \ & \mathbb{E}_r\{\tilde{x}_u(q,r)\mathbb{E}_r\{g^T(x(s,t),\xi(s,t))|x(q,r)\}\} = 0,
\end{aligned}$$

and

$$\mathbb{E}_r\{\tilde{x}_u(q,r)w^T(s,t)\} = \mathbb{E}_r\{x_u(q,r)w^T(s,t)\} - \mathbb{E}_r\{\hat{x}_u(q,r)w^T(s,t)\} = 0,$$
$$\mathbb{E}_r\{\tilde{x}_u(q,r)v^T(s,t)\} = \mathbb{E}_r\{x_u(q,r)v^T(s,t)\} - \mathbb{E}_r\{\hat{x}_u(q,r)v^T(s,t)\} = 0$$

for $(s,t) \in \{(s_0,t_0)|s_0 > q, or\ t_0 > r\} \cup (q,r)$. Then the second-order moment is given by

$$\begin{aligned}
P_p(q,r) \ = \ & A_1(q,r-1)P_u(q,r-1)A_1^T(q,r-1) + A_2(q-1,r)P_u(q-1,r)A_2^T(q-1,r) \\
& +A_1(q,r-1)\mathbb{E}_r\{\tilde{x}_u(q,r-1)\tilde{x}_u^T(q-1,r)\}A_2^T(q-1,r) \\
& +A_2(q-1,r)\mathbb{E}_r\{\tilde{x}_u(q-1,r)\tilde{x}_u^T(q,r-1)\}A_1^T(q,r-1)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{d}\Pi_j \mathrm{tr}\{(X(q,r-1)+X(q-1,r))\Gamma_s\}\\
&+B_1(q,r-1)R(q,r-1)B_1^T(q,r-1)+B_2(q-1,r)R(q-1,r)B_2^T(q-1,r)
\end{aligned}
$$

for $(q,r)\in\varphi[1\ L]$.