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*Research article*

# Vanishing magnetic field limits of solutions to the non-isentropic Chaplygin gas magnetogasdynamics equations

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**Abstract:** This paper studied the Riemann problem for the non-isentropic Chaplygin gas magnetogasdynamics equations and investigated the general asymptotic behavior of its Riemann solutions. Due to the influence of the source term in the equations, the Riemann solutions for the non-isentropic Chaplygin gas magnetogasdynamics equations are no longer self-similar. We performed the analysis after eliminating the source term by using a velocity transformation. When the Riemann initial data of density and velocity satisfied the condition $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$, as the reciprocal of magnetic flux density $\mu$ tended to zero, the Riemann solutions of the non-isentropic Chaplygin gas magnetogasdynamics equations converged to the delta shock solutions of the non-isentropic Chaplygin Euler equations. Otherwise, the Riemann solutions converged to a contact discontinuity of the non-isentropic Chaplygin Euler equations.

**Keywords:** non-isentropic Chaplygin gas magnetogasdynamics equations; source term; Riemann problem; vanishing magnetic field limits; delta shock
**Mathematics Subject Classification:** 35L03, 35L65, 35L67, 35Q31

## 1. Introduction

As one of the important components of partial differential equations, hyperbolic conservation laws play a crucial role in fields such as aerospace, meteorology, and chemical engineering. The Riemann problem was proposed by Riemann when studying shock tube experiments corresponding to the Euler equations. In 1957, for the Riemann problem, Lax [1] first constructed solutions under the assumptions that the equations are strictly hyperbolic and all the characteristics of the equations are genuinely nonlinear or linearly degenerate. Based on mass conservation and momentum conservation, the one-dimensional isentropic Euler equations can be derived as follows:

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + \left(\rho u^2 + P\right)_x = 0,\end{cases} \tag{1.1}$$

where $\rho$, $P$, and $u$ represent density, pressure, and velocity, respectively. For research related to the Eq (1.1), please refer to references [2, 3]. Based on the Eq (1.1), the non-isentropic Euler equations with energy conservation are

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + \left(\rho u^2 + P\right)_x = 0,\\ \left(\frac{\rho u^2}{2} + \rho e\right)_t + \left(\left(\frac{\rho u^2}{2} + \rho e + P\right)u\right)_x = 0.\end{cases} \tag{1.2}$$

Shelkovich, Nilsson, and Rozanova [4] used the variable $H$ to replace the product of $\rho$ and $e$, transforming the Eq (1.2) into

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + \left(\rho u^2 + P\right)_x = 0,\\ \left(\frac{\rho u^2}{2} + H\right)_t + \left(\left(\frac{\rho u^2}{2} + H + P\right)u\right)_x = 0,\end{cases} \tag{1.3}$$

where $H = \rho e \geq 0$ represents the internal energy. Under the condition of a negative state $(\rho_-, u_-, H_-)$, Pang [5] studied the Riemann problem for the Eq (1.3) with the equation of state $p = -\frac{1}{\rho}$, and obtained Riemann solutions that include both contact discontinuities and delta shock waves. For more research related to the Eq (1.3), please refer to references [6, 7]. Considering the effect of the source term, the non-isentropic Euler Eq (1.3) are transformed into

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + \left(\rho u^2 + P\right)_x = \beta\rho,\\ \left(\frac{\rho u^2}{2} + H\right)_t + \left(\left(\frac{\rho u^2}{2} + H + P\right)u\right)_x = \beta\rho u,\end{cases} \tag{1.4}$$

where $\beta$ is a constant. In 2019, Pang and Hu [8] studied the Riemann problem for the Eq (1.4) for Van der Waals gas with external forces being continuous functions of time, and provided explicit forms for rarefaction waves, shock waves, and contact discontinuities. In 2020, Pang, Ge [9] investigated the exact solutions of the Riemann problem for the Eq (1.4) for compressible ideal fluids and proved that, for $t > 0$, the solutions of the system exhibit vacuum phenomena and no longer possess self-similarity. Thus, the study of the Eq (1.4) has broad physical significance.

Magnetic fluids are materials that exhibit both the flow characteristics of liquids and the magnetic properties of solids. It has a wide range of applications in various demanding fields such as magnetic fluid seals, shock absorption, medical devices, sound modulation, optical displays, and magnetic fluid beneficiation. When the equation of state is influenced by magnetic fluids, researchers have focused more on studying the equations

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + \left(\rho u^2 + P + \frac{B^2}{2\mu'}\right)_x = 0.\end{cases} \tag{1.5}$$

Based on the Eq (1.5), considering that the magnetic fluid satisfies

$$\begin{cases} P = -\frac{M}{\rho^{\alpha}}, \quad 0 < \alpha < 1, \\ \frac{1}{2\mu'\mu^2} = N\frac{k^2\rho^2}{2\mu'}, \quad k > 0, \end{cases} \tag{1.6}$$

where $\mu^{'}$ and $\mu$ represent the magnetic permeability and the reciprocal of magnetic flux density, respectively, and $B = \frac{1}{\mu}$ with $M$ and $N$ being positive real numbers. In view of the vanishing magnetic field limits for the Riemann solutions for the Eqs (1.5) and (1.6), in 2022, Shao [10] demonstrated that as the magnetic field vanishes, certain Riemann solutions with two shocks converge to a delta shock solution of the Chaplygin gas equations, resulting in a density that becomes a weighted $\delta$-measure. Conversely, other Riemann solutions converge to a state characterized by two contact discontinuities, where the intermediate state remains nonvacuum. For more research on the Eq (1.5), please refer to references [11–13].

In contrast to the above studies, we investigate the Riemann problem for Eq (1.4), which describe Chaplygin gas affected by magnetic fields. The Riemann solutions of the non-isentropic Chaplygin gas magnetogasdynamics equations may include both elementary wave and delta shock wave. The delta shock wave is particularly significant in handling impulses and instantaneous events, and has important applications in fields such as signal processing and Fourier transform. After introducing magnetic fluids, we can discuss the Chaplygin gas model that is affected by the magnetic field. Physically, magnetic permeability is defined as $\mu^{'} = \frac{1}{\mu\xi}$, where $\mu^{'}$, $\mu$, $\xi$ represent magnetic permeability, the reciprocal of magnetic flux density and magnetic field intensity, respectively. Through the Eq (1.6) and definition of magnetic permeability, we get the equation of state which we investigate

$$P = \frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho}, \quad k_0 > 0, \tag{1.7}$$

where $\mu > 0$ is the reciprocal of magnetic flux density. This paper investigates the Riemann solutions of the Eqs (1.4) and (1.7) and their vanishing magnetic field limits. In 2019, Zhang, Pang and Wang [14] studied concentration and cavitation in the vanishing pressure limit of solutions to the generalized Chaplygin Euler equations of compressible fluid flow. In 2021, Zhang and Pang [15] studied the phenomena of concentration and cavitation by examining the vanishing pressure limit of solutions to the simplified isentropic relativistic Euler equations. In 2022, Peng and Wang [16] used the pressureless limit method to study the limit behavior of continuous solutions for the isentropic Euler equations. Their research indicated that during the pressureless limit process, for the isentropic Euler equations, the initial data of compressive continuous solutions converges to the mass concentrated solutions of the pressureless Euler equations. In 2023, Lei and Shao [17] constructively solved the Riemann problem for relativistic Euler equations using a logarithmic equation of state and proved that, as pressure vanishes, the Riemann solutions of the relativistic Euler equations converge to the Riemann solutions of the pressureless relativistic Euler equations. This demonstrates that the pressureless limit is an important method for studying the Riemann problem.

We study the Riemann problem of Eqs (1.4) and (1.7) with the initial data

$$(\rho, u, H)(0, x) = \begin{cases} (\rho_-, u_-, H_-), x < 0, \\ (\rho_+, u_+, H_+), x > 0, \end{cases} \tag{1.8}$$

where $\rho_i,\ H_i > 0,\ u_i,\ i = -, +$, are constants. By using the velocity transformation $u = v+\beta t$ introduced by Faccanoni and Mangeney [18], the source term is eliminated

$$\begin{cases}\rho_t + (\rho(v+\beta t))_x = 0,\\ (\rho v)_t + \left(\rho v\,(v+\beta t) + \frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho}\right)_x = 0,\\ \left(\frac{1}{2}\rho v^2 + H\right)_t + \left(\left(\frac{1}{2}\rho v^2 + H\right)(v+\beta t) + \left(\frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho}\right)v\right)_x = 0.\end{cases} \tag{1.9}$$

When $t = 0$, $u_\pm = v_\pm$. Meanwhile, the initial data (1.8) becomes

$$(\rho, v, H)(0, x) = \begin{cases}(\rho_-, v_-, H_-)\,, x < 0,\\ (\rho_+, v_+, H_+)\,, x > 0.\end{cases} \tag{1.10}$$

We obtain the solutions to the Eqs (1.7), (1.9), and (1.10). Then, we use characteristic analysis and phase plane analysis methods [19–21] to study the Riemann solutions to the systems (1.7), (1.9), and (1.10) and the limiting behavior of Riemann solutions as the reciprocal of magnetic flux density $\mu$ approaches zero.

The structure of this paper is arranged as follows: In the second section, we briefly review the Riemann solutions of the non-isentropic Chaplygin Euler equations. In the third section, we study the Riemann solutions of the non-isentropic Chaplygin gas magnetogasdynamics equations in both the phase plane and the physical plane. The fourth section analyzes the vanishing magnetic field limits of the Riemann solutions for the non-isentropic Chaplygin gas magnetogasdynamics equations in two cases $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$ and $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$, as the reciprocal of magnetic flux density $\mu$ approaches zero. The main conclusion is as follows.

**Theorem 1.** *When $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$, the Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) converges to the delta shock wave of the Eqs (1.4) and (1.8) with $P = -\frac{1}{\rho}$ as the reciprocal of magnetic flux density $\mu$ approaches zero. When $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$, the Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) converges to the contact discontinuity of the Eqs (1.4) and (1.8) with $P = -\frac{1}{\rho}$ as the reciprocal of magnetic flux density $\mu$ approaches zero.*

**Remark 1.** *In [5], Pang considered the delta shock wave and the contact discontinuity of (1.4) and (1.8) with $P = -\frac{1}{\rho}$ in two cases $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$ and $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$, respectively. The velocity $u_\delta(t)$ of delta shock wave satisfies the entropy condition*

$$v_+ - \frac{1}{\rho_+} < v_+ < v_+ + \frac{1}{\rho_+} \leq u_\delta(t) \leq v_- - \frac{1}{\rho_-} < v_- < v_- + \frac{1}{\rho_-}.$$

*We want to explore the limit behavior of the Riemann solutions for systems (1.9) and (1.10) with (1.7) as the reciprocal of magnetic flux density $\mu$ approaches zero. Notice that in (1.7), we see $\lim\limits_{\mu\to 0}\left(\frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho}\right) = -\frac{1}{\rho}$; thus, as $\mu$ tends to zero, the equation of state converges to the equation of state of the Chaplygin gas. The first and third eigenvalues of (1.9) with (1.7) are calculated as $\lambda_1(\rho, v) = v + \beta t - \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}$, $\lambda_3(\rho, v) = v + \beta t + \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}$, which are independent of the variable $H$; therefore, $\lim\limits_{\mu\to 0}\lambda_1(\rho_-, v_-) = v_- - \frac{1}{\rho_-}$, $\lim\limits_{\mu\to 0}\lambda_3(\rho_+, v_+) = v_+ + \frac{1}{\rho_+}$. Thus, as the reciprocal of magnetic flux density $\mu$ approaches zero, we consider the limit of Riemann solutions of systems (1.9) and (1.10) with (1.7) in cases $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$ and $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$ in Theorem 1.*

## 2. Riemann problem for non-isentropic Chaplygin Euler equations

This section briefly reviews the Riemann solutions of the non-isentropic Chaplygin Euler equations under the equation of state $p = -\frac{1}{\rho}$, and the equations satisfy the thermodynamic conditions

$$T\mathrm{d}s - \mathrm{d}e = P\mathrm{d}\left(\frac{1}{\rho}\right),$$

where $T = T(\rho, s)$ represents the temperature, and e and s represent the internal energy and entropy of the fluid, respectively. The physical region that satisfies the above thermodynamic conditions is

$$\Omega = \left\{(\rho, u, H) \mid \rho > 0, u \in \mathbb{R}, H \geq \frac{1}{2\rho}\right\}.$$

The expressions for the non-isentropic Chaplygin Euler equations are

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 - \frac{1}{\rho}\right)_x = \beta\rho, \\ \left(\frac{1}{2}\rho u^2 + H\right)_t + \left(\left(\frac{1}{2}\rho u^2 + H - \frac{1}{\rho}\right)u\right)_x = \beta\rho u, \end{cases} \tag{2.1}$$

where the state variable H ≥ 0 is the internal energy. After using the variable substitution $u = v+\beta t$ [18], for a given negative state $(\rho_-, v_-, H_-)$, Pang [22] solved the distribution of its Riemann solutions in the phase plane, as shown in Figure 1.

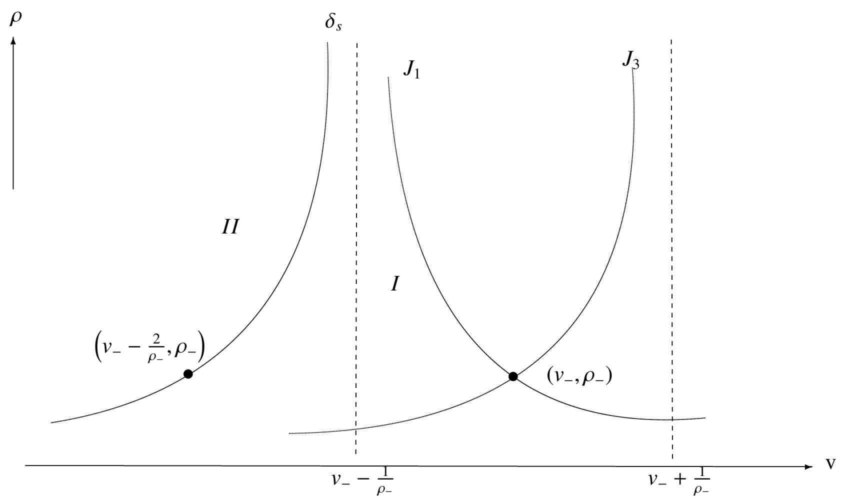


**Figure 1.** The Riemann solutions of Eqs (1.8) and (2.1) in the phase plane.

When the projection of $(\rho_+, v_+, H_+)$ is located in $I$, the contact discontinuities are as follows:

$$J_1 : \begin{cases} v - \frac{1}{\rho} = v_- - \frac{1}{\rho_-}, \\ \rho_- H - \rho H_- = \frac{\rho_-^2 - \rho^2}{2\rho\rho_-}, \end{cases} \tag{2.2}$$

$$J_2:\begin{cases} v=v_-, \quad \rho=\rho_-, \\ H\neq H_-, \end{cases} \tag{2.3}$$

$$J_3:\begin{cases} v+\frac{1}{\rho}=v_++\frac{1}{\rho_+}, \\ \rho_+H-\rho H_+=\frac{\rho_+^2-\rho^2}{2\rho\rho_+}. \end{cases} \tag{2.4}$$

Since the equations in differential form are no longer valid on the interrupted line of solutions and the equations in integral form still hold, we use delta shock to construct interrupted solutions of the equations. Mathematically, they are characterized by the delta functions appearing in the state variables. Physically, they represent the process of concentration of the mass and formation of the universe [23]. When the projection of $(\rho_+, v_+, H_+)$ is located in $II$, the delta shock wave should satisfy the following generalized Rankine-Hugoniot conditions [24]:

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t}=u_\delta(t)+\beta t, \\ \frac{\mathrm{d}\omega(t)}{\mathrm{d}t}=u_\delta(t)[\rho]-[\rho(v+\beta t)], \\ \frac{\mathrm{d}(\omega(t)u_\delta(t))}{\mathrm{d}t}=u_\delta(t)[\rho v]-\left[\rho v(v+\beta t)-\frac{1}{\rho}\right], \\ \frac{\mathrm{d}\left(\omega(t)u_\delta^2(t)/2+h(t)\right)}{\mathrm{d}t}=u_\delta(t)\left[\frac{\rho v^2}{2}+H\right]-\left[\left(\frac{\rho v^2}{2}+H\right)(v+\beta t)-\frac{v}{\rho}\right], \end{cases} \tag{2.5}$$

where $x(t)$ and $u_\delta(t)$ represent the position and velocity of the delta shock, and $h(t)$ and $\omega(t)$ are the weights of $\delta_s$ on the state variables $H$ and $\rho$, respectively, with $x(0)=0$, $u_\delta(0)=u_0$, $\omega(0)=0$, $h(0)=0$. Additionally, to ensure the uniqueness of the solutions, the following entropy condition must be satisfied:

$$\lambda_1(\rho_+,v_+)<\lambda_2(\rho_+,v_+)<\lambda_3(\rho_+,v_+)\leq u_\delta(t)\leq\lambda_1(\rho_-,v_-)<\lambda_2(\rho_-,v_-)<\lambda_3(\rho_-,v_-).$$

## 3. Riemann solutions of systems (1.9) and (1.10) with (1.7)

Next, we study the Riemann solutions of the systems (1.9) and (1.10) with (1.7). Since the system (1.4) is non-self-similar, we use the variable substitution $u=v+\beta t$ introduced by [18], then the system (1.4) can be transformed into the following form:

$$\begin{cases} \rho_t+(\rho(v+\beta t))_x=0, \\ (\rho v)_t+\left(\rho v(v+\beta t)+\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)_x=0, \\ \left(\frac{1}{2}\rho v^2+H\right)_t+\left(\left(\frac{1}{2}\rho v^2+H\right)(v+\beta t)+\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)v\right)_x=0, \end{cases} \tag{3.1}$$

with the following initial data:

$$(\rho,v,H)(0,x)=\begin{cases} (\rho_-,v_-,H_-), x<0, \\ (\rho_+,v_+,H_+), x>0, \end{cases} \tag{3.2}$$

where $\rho_i,\ H_i>0,\ v_i,\ i=-,+$, are given constants. Denote the matrix

$$A=\begin{pmatrix} 1 & 0 & 0 \\ v & \rho & 0 \\ \frac{1}{2}v^2 & \rho v & 1 \end{pmatrix},$$

$$B=\begin{pmatrix} v+\beta t & \rho & 0 \\ v\left(v+\beta t\right)+\mu k_0^2\rho+\frac{1}{\rho} & 2\rho v+\rho\beta t & 0 \\ \frac{1}{2}v^2\left(v+\beta t\right)+\mu k_0^2\rho v+\frac{1}{\rho^2}v & \frac{3}{2}\rho v^2+\rho v\beta t+H+\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho} & v+\beta t \end{pmatrix}.$$

Simplifying and writing the system (3.1) in the following matrix form,

$$A\begin{pmatrix} \rho \\ v \\ H \end{pmatrix}_t+B\begin{pmatrix} \rho \\ v \\ H \end{pmatrix}_x=0, \tag{3.3}$$

let $\lambda$ satisfy

$$det\left(\lambda A-B\right)=0,$$

the eigenvalues are calculated as

$$\lambda_1=v+\beta t-\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}},\quad \lambda_2=v+\beta t,\quad \lambda_3=v+\beta t+\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}},$$

then the right eigenvectors corresponding to $\lambda_1, \lambda_2, \lambda_3$ are

$$\vec{r}_1=\begin{pmatrix} -\rho \\ \sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}} \\ -H-\frac{1}{2}\mu k_0^2\rho^2+\frac{1}{\rho} \end{pmatrix},\qquad \vec{r}_2=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\qquad \vec{r}_3=\begin{pmatrix} \rho \\ \sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}} \\ H+\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho} \end{pmatrix},$$

and

$$\begin{aligned}
\nabla\lambda_1&=\left(-\frac{\mu k_0^2-\frac{1}{\rho^3}}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}},1,0\right),\\
\nabla\lambda_2&=\left(0,1,0\right),\\
\nabla\lambda_3&=\left(\frac{\mu k_0^2\rho-\frac{1}{\rho^3}}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}},1,0\right).
\end{aligned}$$

Thus, we know

$$\begin{aligned}
\nabla\lambda_2\cdot\vec{r}_2&=\left(0,1,0\right)\cdot\left(0,0,1\right)^T=0,\\
\nabla\lambda_1\cdot\vec{r}_1&=\left(-\frac{\mu k_0^2-\frac{1}{\rho^3}}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}},1,0\right)\cdot\left(-\rho,\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}},-H-\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)\right)^T\\
&=\frac{\mu k_0^2\rho-\frac{1}{\rho^3}}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}+\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}=\frac{\mu k_0^2\rho-\frac{1}{\rho^2}+2\mu k_0^2\rho+\frac{1}{\rho^2}}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}=\frac{3\mu k_0^2\rho}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}>0,
\end{aligned}$$

$$\nabla\lambda_3\cdot\vec{r}_3=\frac{\mu k_0^2\rho-\frac{1}{\rho^3}}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}+\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}=\frac{3\mu k_0^2\rho}{2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}>0,$$

therefore, the first and third eigenvalues are genuinely nonlinear, while the second eigenvalue is linearly degenerate. Given the negative state $(\rho_-, v_-, H_-)$, the rarefaction wave curve is obtained by solving the Riemann problem (1.9) and (1.10) with (1.7)

$$\overleftarrow{R}(\rho_-, v_-, H_-):\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}t}=\lambda_1(\rho,v)=v+\beta t+\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}},\\ v-v_-=-\int_{\rho_-}^{\rho}s^{-2}\left(\mu k_0^2s^3+1\right)^{\frac{1}{2}}\mathrm{d}s,\ \rho<\rho_-,\\ \frac{H}{\rho}-\frac{H_-}{\rho_-}=\frac{1}{2}\mu k_0^2\rho+\frac{1}{2\rho^2}-\left(\frac{1}{2}\mu k_0^2\rho_-+\frac{1}{2\rho_-^2}\right),\end{cases}\tag{3.4}$$

and

$$\overrightarrow{R}(\rho_-, v_-, H_-):\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}t}=\lambda_3(\rho,v)=v+\beta t+\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}},\\ v-v_-=\int_{\rho_-}^{\rho}s^{-2}\left(\mu k_0^2s^3+1\right)^{\frac{1}{2}}\mathrm{d}s,\ \rho>\rho_-,\\ \frac{H}{\rho}-\frac{H_-}{\rho_-}=\frac{1}{2}\mu k_0^2\rho+\frac{1}{2\rho^2}-\left(\frac{1}{2}\mu k_0^2\rho_-+\frac{1}{2\rho_-^2}\right).\end{cases}\tag{3.5}$$

We analyze the asymptotic trend of the projection of the rarefaction wave curve on the $(\rho, v)$ plane. For $\overleftarrow{R}(\rho_-, v_-, H_-)$ where $\rho<\rho_-$,

$$\frac{\mathrm{d}v}{\mathrm{d}\rho}=-\frac{\sqrt{\mu k_0^2+\frac{1}{\rho}}}{\rho}<0,$$
$$\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2}=\frac{\mu k_0^2\rho+\frac{4}{\rho^2}}{2\rho^2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}>0.$$

Thus, the graph of $v$ with respect to $\rho$ is convex downward, and $v=v_--\int_{\rho_-}^{\rho}s^{-2}\left(\mu k_0^2s^3+1\right)^{\frac{1}{2}}\mathrm{d}s$. As $\rho$ approaches $0^+$, $\lim_{s\to0^+}s^2\frac{\sqrt{\mu k_0^2s^3+1}}{s^2}=1>0$, so at this time, $\int_{\rho_-}^{\rho}s^{-2}\left(\mu k_0^2s^3+1\right)^{\frac{1}{2}}\mathrm{d}s$ diverges. For $\rho<\rho_-$, as $\rho$ approaches $0^+$, $v=v_--\int_{\rho_-}^{\rho}s^{-2}\left(\mu k_0^2s^3+1\right)^{\frac{1}{2}}\mathrm{d}s$ approaches $+\infty$.

For $\overrightarrow{R}(\rho_-, v_-, H_-)$ where $\rho>\rho_-$,

$$\frac{\mathrm{d}v}{\mathrm{d}\rho}=\frac{\sqrt{\mu k_0^2+\frac{1}{\rho}}}{\rho}>0,$$
$$\frac{\mathrm{d}^2v}{\mathrm{d}\rho^2}=-\frac{\mu k_0^2\rho+\frac{4}{\rho^2}}{2\rho^2\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}<0.$$

Thus, the graph of $v$ with respect to $\rho$ is concave upward, and $v = v_- + \int_{\rho_-}^{\rho} s^{-2}\left(\mu k_0^2 s^3 + 1\right)^{\frac{1}{2}} \mathrm{d}s$. As $\rho$ approaches $+\infty$, similarly, it can be shown that $v = v_- + \int_{\rho_-}^{\rho} s^{-2}\left(\mu k_0^2 s^3 + 1\right)^{\frac{1}{2}} \mathrm{d}s$ approaches $+\infty$. The solutions $(\rho_{br}, v_{br}, H_{br})$ at any point $(x, t)$ in the backward rarefaction wave satisfy

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = v_{br} + \beta t - \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}}, \\ x(0) = 0. \end{cases}$$

Integrating the above equation yields

$$x = v_{br} t + \frac{1}{2}\beta t^2 - \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}}\, t,$$

that is,

$$v_{br} - \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}} = \frac{x}{t} - \frac{1}{2}\beta t,$$

so

$$\int_{\rho_-}^{\rho_{br}} \frac{\sqrt{\mu k_0^2 \rho + \frac{1}{\rho^2}}}{\rho} \mathrm{d}\rho + \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}} = v_- - \frac{x}{t} + \frac{1}{2}\beta t. \tag{3.6}$$

From the second equation in (3.4), we have

$$v_{br} = \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}} + \frac{x}{t} - \frac{1}{2}\beta t.$$

Combining the above formula with (3.6),

$$v_- - \int_{\rho_-}^{\rho_{br}} \frac{\sqrt{\mu k_0^2 \rho + \frac{1}{\rho^2}}}{\rho} \mathrm{d}\rho = \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}} + \frac{x}{t} - \frac{1}{2}\beta t,$$

and integrating both sides yields

$$-\int_{\rho_-}^{\rho_{br}} \frac{\sqrt{\mu k_0^2 \rho + \frac{1}{\rho^2}}}{\rho} \mathrm{d}\rho = -v_- + \frac{x}{t} - \frac{1}{2}\beta t + \sqrt{\mu k_0^2 \rho_{br} + \frac{1}{\rho_{br}^2}}, \tag{3.7}$$

since $\frac{H_{br}}{\rho_{br}} - \frac{H_-}{\rho_-} = \left(\frac{1}{2}\mu k_0^2 \rho_{br} + \frac{1}{2\rho_{br}^2}\right) - \left(\frac{1}{2}\mu k_0^2 \rho_- + \frac{1}{2\rho_-^2}\right)$, that is,

$$H_{br} = \frac{\rho_{br}}{\rho_-} H_- + \rho_{br}\left(\frac{1}{2}\mu k_0^2 \rho_{br} + \frac{1}{2\rho_{br}^2}\right) - \rho_{br}\left(\frac{1}{2}\mu k_0^2 \rho_- + \frac{1}{2\rho_-^2}\right).$$

The above calculation shows that the solutions $(\rho_{br},\ v_{br},\ H_{br})$ at any point $(x,\ t)$ in the backward rarefaction wave satisfies

$$\begin{cases}\int_{\rho_-}^{\rho_{br}}\frac{\sqrt{\mu k_0^2\rho+\frac{1}{2\rho^2}}}{\rho}\mathrm{d}\rho+\sqrt{\mu k_0^2\rho_{br}+\frac{1}{\rho_{br}^2}}=v_--\frac{x}{t}+\frac{1}{2}\beta t,\\ v_{br}=\sqrt{\mu k_0^2\rho_{br}+\frac{1}{\rho_{br}^2}}+\frac{x}{t}-\frac{1}{2}\beta t,\\ H_{br}=\frac{\rho_{br}}{\rho_-}H_{br}+\rho_{br}\left(\frac{1}{2}\mu k_0^2\rho_{br}+\frac{1}{2\rho_{br}^2}\right)-\rho_{br}\left(\frac{1}{2}\mu k_0^2\rho_-+\frac{1}{2\rho_-^2}\right).\end{cases} \tag{3.8}$$

Similarly, it can be concluded that the solutions $\left(\rho_{fr},\ v_{fr},\ H_{fr}\right)$ at any point $(x,\ t)$ in the forward rarefaction wave satisfies

$$\begin{cases}\int_{\rho_-}^{\rho_{fr}}\frac{\sqrt{\mu k_0^2\rho+\frac{1}{2\rho^2}}}{\rho}\mathrm{d}\rho+\sqrt{\mu k_0^2\rho_{fr}+\frac{1}{\rho_{br}^2}}=-v_-+\frac{x}{t}-\frac{1}{2}\beta t,\\ v_{fr}=v_-+\int_{\rho_-}^{\rho_{fr}}\frac{\sqrt{\mu k_0^2\rho+\frac{1}{\rho^2}}}{\rho}\mathrm{d}\rho,\\ H_{fr}=\frac{\rho_{fr}}{\rho_-}H_-+\rho_{fr}\left(\frac{1}{2}\mu k_0^2\rho_{fr}+\frac{1}{2\rho_{fr}^2}\right)-\rho_{fr}\left(\frac{1}{2}\mu k_0^2\rho_-+\frac{1}{2\rho_-^2}\right).\end{cases} \tag{3.9}$$

Next, we will investigate the shock wave solutions. From the Eq (3.3), the Rankine-Hugoniot conditions are obtained as follows:

$$\begin{cases}-\sigma^{\mu}[\rho]+[\rho(v+\beta t)]=0,\\ -\sigma^{\mu}[\rho v]+\left[\rho v(v+\beta t)+\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right]=0,\\ -\sigma^{\mu}\left[\frac{1}{2}\rho v^2+H\right]+\left[\left(\frac{1}{2}\rho v^2+H\right)(v+\beta t)+\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)\right]=0,\end{cases} \tag{3.10}$$

where $\sigma^{\mu}(t)=\frac{\mathbf{d}x(t,\ \mu)}{\mathbf{d}t}$, and $[\rho]=\rho_+-\rho_-$ represents the jump discontinuity. For a negative state $(\rho_-, v_-, H_-)$, the shock wave curve for the Riemann problems (1.7), (1.9), and (1.10) is given by

$$\overleftarrow{S}(\rho_-, v_-, H_-):\begin{cases}\sigma_1^{\mu}=v_-+\beta t-\frac{\rho}{\rho-\rho_-}\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho}\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)\right)},\\ v-v_-=-\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)},\\ H=\frac{\rho}{\rho_-}H_-+\frac{\rho-\rho_-}{2\rho_-}\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)+\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right),\end{cases} \tag{3.11}$$

where $\lambda_1(\rho,\ v)\leq\sigma_1^{\mu}(t)\leq\lambda_1(\rho_-,\ v_-)$,

$$\overrightarrow{S}(\rho_-, v_-, H_-):\begin{cases}\sigma_3^{\mu}=v_-+\beta t-\frac{\rho}{\rho-\rho_-}\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho}\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)\right)},\\ v-v_-=-\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)},\\ H=\frac{\rho}{\rho_-}H_-+\frac{\rho-\rho_-}{2\rho_-}\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)+\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right),\end{cases} \tag{3.12}$$

where $\lambda_3(\rho,\ v)\leq\sigma_3^{\mu}(t)\leq\lambda_3(\rho_-,\ v_-)$. Next, we analyze the asymptotic behavior of the projection of the shock wave curve onto the $(\rho, v)$ plane. For $\overleftarrow{S}(\rho_-, v_-, H_-)$ under the condition $\rho>\rho_-$,

$$\frac{\mathrm{d}v}{\mathrm{d}\rho}=-\frac{1}{2}\left(\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)\right)^{-\frac{1}{2}},$$

$$\left(\frac{1}{\rho^2}\left(\left(\frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho} - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right) + \left(\frac{1}{\rho_- - \frac{1}{\rho}\left(\mu k_0^2\rho + \frac{1}{\rho^2}\right)}\right)\right)\right) < 0,$$

and as $\rho$ tends to $+\infty$, $v$ tends to $-\infty$. For $\overrightarrow{S}(\rho_-, v_-, H_-)$ under the condition $\rho < \rho_-$,

$$\frac{\mathrm{d}v}{\mathrm{d}\rho} = -\frac{1}{2}\left(\left(\frac{1}{\rho_-} - \frac{1}{\rho}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)\right)^{-\frac{1}{2}},$$
$$\left(\frac{1}{\rho^2}\left(\left(\frac{1}{2}\mu k_0^2\rho^2 - \frac{1}{\rho} - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right) + \left(\frac{1}{\rho_- - \frac{1}{\rho}\left(\mu k_0^2\rho + \frac{1}{\rho^2}\right)}\right)\right)\right) > 0,$$

and as $\rho$ tends to $0^+$, $v$ tends to $-\infty$. Since the shock wave curve and the rarefaction wave curve are tangent to each other at a second-order point in the phase plane $(v, \rho)$, their concavity and convexity are consistent.

Next, we consider the contact discontinuity. When $[\rho] = 0$, by the first equation of the Rankine-Hugoniot conditions $[\rho(v + \beta t)] = 0$, we know $\rho_+[v] = 0$, which implies $[v] = 0$. From the third equation of the Rankine-Hugoniot conditions,

$$\begin{aligned}
&-\sigma_2^{\mu}\left(\left(\frac{1}{2}\rho_+ v_+^2 + H_+\right) - \left(\frac{1}{2}\rho_- v_-^2 + H_-\right)\right) + \left(\left(\frac{1}{2}\rho_+ v_+^2 + H_+\right)(v + \beta t) - \left(\frac{1}{2}\rho_- v_-^2 + H_-\right)(v_- + \beta t)\right)\\
&+\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)v_+ - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)v_-\right)\\
=&-\sigma_2^{\mu}[H] + (H_+(v_+ + \beta t) - H_-(v_- + \beta t))\\
=&-\sigma_2^{\mu}[H] + (v_- + \beta t)[H] = 0.
\end{aligned}$$

Thus, a contact discontinuity curve is given by

$$J(\rho_-, v_-, H_-) : \begin{cases}\sigma_2^{\mu} = v_- + \beta t,\\ [\rho] = 0, \quad [v] = 0,\\ [H] \neq 0.\end{cases}$$

In summary, we have derived the elementary wave curves in the phase plane of Eqs (3.1) and (3.2). In the phase plane, given the negative state $(\rho_-, v_-, H_-)$, we can draw the corresponding curves based on the expressions for the rarefaction wave curve and the shock wave curve. The phase plane is divided into four regions, as shown in Figure 2.

If the projection of $(\rho_+, v_+, v_+)$ is located in I $(\rho_-, v_-)$, the Riemann solutions to the Eqs (3.1) and (3.2) is

$$(\rho, v, H)(t, x) = \begin{cases}(\rho_-, v_-, H_-), & x < x_1^-(t),\\ (\rho_{br}, v_{br}, H_{br}), & x_1^-(t) \leq x \leq x_1^+(t),\\ (\rho_*, v_*, H_{*1}), & x_1^+(t) < x < x_2(t),\\ (\rho_*, v_*, H_{*2}), & x_2(t) < x < x_3^-(t),\\ \left(\rho_{fr}, v_{fr}, H_{fr}\right), & x_3^-(t) \leq x \leq x_3^+(t),\\ (\rho_+, v_+, H_+), & x > x_3^+(t).\end{cases} \tag{3.13}$$

If $\overleftarrow{R}(\rho_-, v_-, H_-)$ has $\frac{\mathrm{d}x}{\mathrm{d}t} = v + \beta t - \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}$, that is, $x(t) = vt + \frac{1}{2}\beta t^2 - \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}t$, it can be concluded that

$$x_1^-(t) = \left(v_- - \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}\right)t + \frac{1}{2}\beta t^2,$$
$$x_1^+(t) = \left(v_* - \sqrt{\mu k_0^2\rho_* + \frac{1}{\rho_*^2}}\right)t + \frac{1}{2}\beta t^2.$$

Similarly, from $\overrightarrow{R}(\rho_+, v_+, H_+)$ we get $\frac{\mathrm{d}x}{\mathrm{d}t} = v + \beta t + \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}$, that is, $x(t) = \left(v + \sqrt{\mu k_0^2\rho + \frac{1}{\rho}}t + \frac{1}{2}\beta t^2\right)$. It can be concluded that

$$x_3^+(t) = \left(v_- + \sqrt{\mu k_0^2\rho_+ + \frac{1}{\rho_+^2}}\right)t + \frac{1}{2}\beta t^2,$$
$$x_3^-(t) = \left(v_* + \sqrt{\mu k_0^2\rho_* + \frac{1}{\rho_*^2}}\right)t + \frac{1}{2}\beta t^2.$$

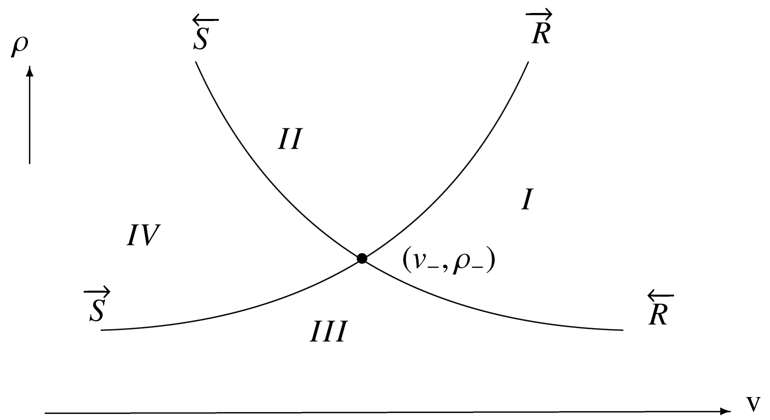


**Figure 2.** The Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) in the phase plane.

From $\overleftarrow{R}(\rho_-, v_-, H_-)$, we know that $H_{*1} = \rho_*\left(\frac{H_-}{\rho_-} - \left(\frac{1}{2}\mu k_0^2\rho_- + \frac{1}{2\rho_-^2}\right)\right) + \frac{1}{2}\mu k_0^2\rho_*^2 + \frac{1}{2\rho_*}$. Similarly, from $\overrightarrow{R}(\rho_+, v_+, H_+)$ we know that $H_{*2} = \rho_*\left(\frac{H_+}{\rho_+} - \left(\frac{1}{2}\mu k_0^2\rho_+ + \frac{1}{2\rho_+^2}\right)\right) + \frac{1}{2}\mu k_0^2\rho_*^2 + \frac{1}{2\rho_*}$, where $(\rho_*, v_*)$ is the solution to the following equations:

$$\begin{cases} v_* - v_- = \int_{\rho_-}^{\rho_*} -\frac{(\mu k_0^2 s^3+1)^{\frac{1}{2}}}{s^2}\mathrm{d}s, \\ v_* - v_+ = \int_{\rho_+}^{\rho_*} s^{-2}\left(\mu k_0^2 s^3 + 1\right)^{\frac{1}{2}}\mathrm{d}s, \end{cases} \tag{3.14}$$

and $(\rho_{br}, v_{br}, H_{br})$ and $\left(\rho_{fr}, v_{fr}, H_{fr}\right)$ are the states before and after the rarefaction wave, respectively. From $\sigma_2^\mu(t) = v_- + \beta t$, we know that $x_2(t) = v_* t + \frac{1}{2}\beta t^2$. The physical plane of the Riemann solutions for the Eqs (3.1) and (3.2) are shown in Figure 3.

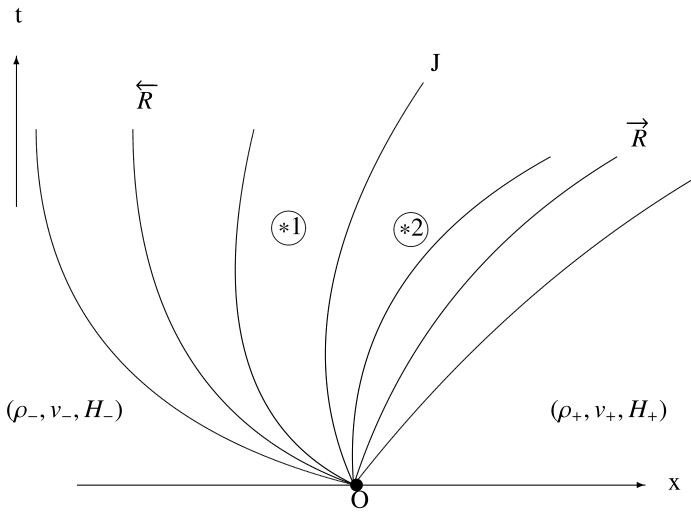


**Figure 3.** The Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) in the physical plane when $(\rho_+, v_+, H_+) \in \mathrm{I}$.

If the projection of $(\rho_+, v_+, H_+)$ is located in II $(\rho_-, v_-)$, the Riemann solutions to the Eqs (3.1) and (3.2) are

$$(\rho, v, H)(t, x) = \begin{cases} (\rho_-, v_-, H_-), & x < x_1(t), \\ (\rho_*, v_*, H_{*1}), & x_1(t) \leq x \leq x_2(t), \\ (\rho_*, v_*, H_{*2}), & x_2(t) < x < x_3^-(t), \\ \left(\rho_{fr}, v_{fr}, H_{fr}\right), & x_3^-(t) \leq x \leq x_3^+(t), \\ (\rho_+, v_+, H_+), & x > x_3^+(t), \end{cases} \tag{3.15}$$

where

$$\begin{aligned} x_1(t) &= \left(v_- - \frac{\rho_*}{\rho_{*-\rho_-}}\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)}\right)t + \frac{1}{2}\beta t^2, \\ x_2(t) &= v_* t + \frac{1}{2}\beta t^2, \\ x_3^+(t) &= \left(v_- + \sqrt{\mu k_0^2\rho_+ + \frac{1}{\rho_+^2}}\right)t + \frac{1}{2}\beta t^2, \\ x_3^-(t) &= \left(v_* + \sqrt{\mu k_0^2\rho_* + \frac{1}{\rho_*^2}}\right)t + \frac{1}{2}\beta t^2, \\ H_{*1} &= \frac{\rho_*}{\rho_-}H_- + \frac{\rho_* - \rho_-}{2\rho_-}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) + \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right), \\ H_{*2} &= \rho_*\left(\frac{H_+}{\rho_+}\left(\frac{1}{2}\mu k_0^2\rho_+ + \frac{1}{2\rho_+^2}\right)\right) + \frac{1}{2}\mu k_0^2\rho_*^2 + \frac{1}{2\rho_*}, \end{aligned}$$

and $(\rho_*, v_*)$ is the solution to the following equations:

$$\begin{cases} v_* - v_- = -\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)}, \\ v_* - v_+ = \int_{\rho_+}^{\rho_*} s^{-2}\left(\mu k_0^2 s^3 + 1\right)^{\frac{1}{2}} \mathrm{d}s. \end{cases}$$

Then the physical plane of the Riemann solutions for the Eqs (3.1) and (3.2) is shown in Figure 4. If the projection of $(\rho_+, v_+, H_+)$ is located in III $(\rho_-, v_-)$, the Riemann solutions to the Eqs (3.1) and (3.2) are

$$(\rho, v, H)(t, x) = \begin{cases} (\rho_-, v_-, H_-), & x < x_1^-(t), \\ (\rho_{br}, v_{br}, H_{br}), & x_1^-(t) \le x \le x_1^+(t), \\ (\rho_*, v_*, H_{*1}), & x_1^+(t) < x < x_2(t), \\ (\rho_*, v_*, H_{*2}), & x_2(t) < x < x_3^-(t), \\ \left(\rho_{fr}, v_{fr}, H_{fr}\right), & x_3^-(t) \le x \le x_3^+(t), \\ (\rho_+, v_+, H_+), & x > x_3^+(t), \end{cases} \tag{3.16}$$

where

$$\begin{aligned} x_1^-(t) &= \left(v_- - \sqrt{\mu k_0^2\rho + \frac{1}{\rho^2}}\right)t + \frac{1}{2}\beta t^2, \\ x_1^+(t) &= \left(v_* - \sqrt{\mu k_0^2\rho_* + \frac{1}{\rho_*^2}}\right)t + \frac{1}{2}\beta t^2, \\ x_2(t) &= v_* t + \frac{1}{2}\beta t^2, \\ x_3(t) &= \left(v_+ - \frac{\rho_*}{\rho_+ - \rho_*}\sqrt{\left(\frac{1}{\rho_+} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right)}\right)t + \frac{1}{2}\beta t^2, \end{aligned}$$

$$\begin{aligned} H_{*1} &= \rho_*\left(\frac{H_-}{\rho_-} - \left(\frac{1}{2}\mu k_0^2\rho_- + \frac{1}{2\rho_-^2}\right)\right) + \frac{1}{2}\mu k_0^2\rho_*^2 + \frac{1}{2\rho_*}, \\ H_{*2} &= \frac{\rho_*}{\rho_+}H_+ - \frac{\rho_* - \rho_+}{2\rho_+}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) + \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right), \end{aligned}$$

and $(\rho_*, v_*)$ is the solution to the following equations:

$$\begin{cases} v_* - v_- = \int_{\rho_-}^{\rho_*} -s^{-2}\left(\mu k_0^2 s^3 + 1\right)^{\frac{1}{2}} \mathrm{d}s, \\ v_+ - v_* = -\sqrt{\left(\frac{1}{\rho_+} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_-}\right)\right)}. \end{cases}$$

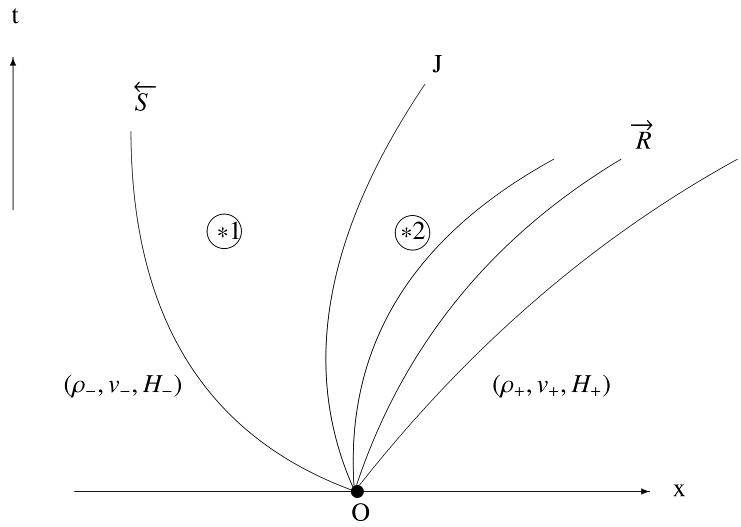


**Figure 4.** The Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) in the physical plane when $(\rho_+, v_+, H_+) \in$ II.

The physical plane of the Riemann solutions for the Eqs (3.1) and (3.2) are shown in Figure 5. If the projection of $(\rho_+, v_+, H_+)$ is located in IV $(\rho_-, v_-)$, the Riemann solutions to the Eqs (3.1) and (3.2) are

$$(\rho, v, H)(t, x) = \begin{cases} (\rho_-, v_-, H_-), & x < x_1(t), \\ (\rho_*, v_*, H_{*1}), & x_1(t) < x < x_2(t), \\ (\rho_*, v_*, H_{*2}), & x_2(t) < x < x_3(t), \\ (\rho_+, v_+, H_+), & x > x_3(t), \end{cases} \tag{3.17}$$

where

$$x_1(t) = \left( v_- - \frac{\rho_*}{\rho_{*-\rho_-}} \sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2 \rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2 \rho_-^2 - \frac{1}{\rho_-}\right)\right)} \right) t + \frac{1}{2}\beta t^2,$$
$$x_2(t) = v_* t + \frac{1}{2}\beta t^2,$$
$$x_3(t) = v_+ - \frac{\rho_*}{\rho_+ - \rho_*} \sqrt{\left(\frac{1}{\rho_+} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2 \rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2 \rho_+^2 - \frac{1}{\rho_+}\right)\right)},$$

$$H_{*1} = \frac{\rho_*}{\rho_-} H_- + \frac{\rho_* - \rho_-}{2\rho_-} \left(\left(\frac{1}{2}\mu k_0^2 \rho_*^2 - \frac{1}{\rho_*}\right) + \left(\frac{1}{2}\mu k_0^2 \rho_-^2 - \frac{1}{\rho_-}\right)\right),$$
$$H_{*2} = \frac{\rho_*}{\rho_+} H_+ - \frac{\rho_* - \rho_+}{2\rho_+} \left(\left(\frac{1}{2}\mu k_0^2 \rho_*^2 - \frac{1}{\rho_*}\right) + \left(\frac{1}{2}\mu k_0^2 \rho_+^2 - \frac{1}{\rho_+}\right)\right),$$

and $(\rho_*, v_*)$ is the solution to the following equations:

$$\begin{cases} v_* - v_- = -\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)}, \\ v_+ - v_* = -\sqrt{\left(\frac{1}{\rho_+} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right)}. \end{cases}$$

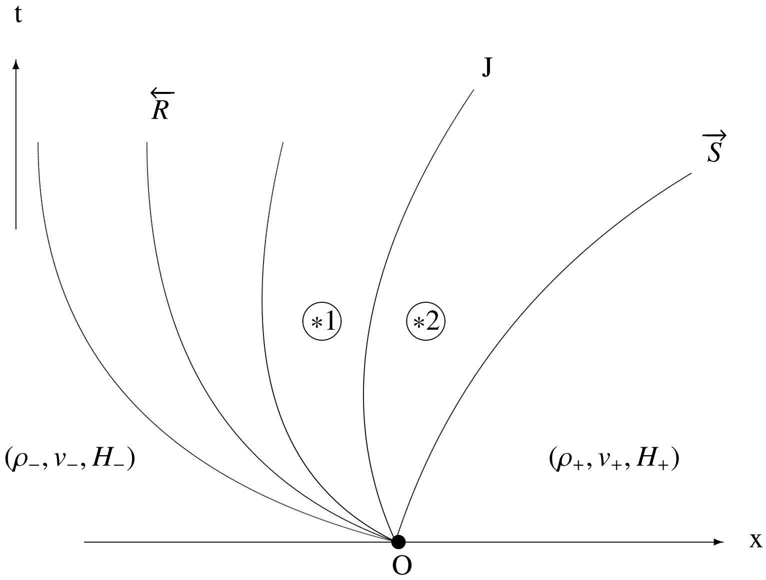


**Figure 5.** The Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) in the physical plane when $(\rho_+, v_+, H_+) \in$ III.

The physical plane of the Riemann solutions for the Eqs (3.1) and (3.2) is shown in Figure 6.

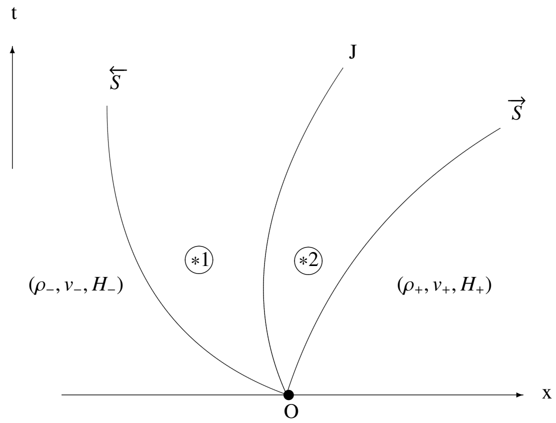


**Figure 6.** The Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) in the physical plane when $(\rho_+, v_+, H_+) \in$ IV.

## 4. The vanishing magnetic field limits of the Riemann solutions

In this section, we will consider the limiting behavior of the Riemann solutions of the Eqs (1.9) and (1.10) with (1.7). Through theoretical analyses, it is shown that the Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) can be converted to the Riemann solutions of the Eqs (1.8) and (2.1) with the disappearance of variable $\mu$ .

If $(v_-, \rho_-)$ is the negative state of the phase plane, we make a curve $v_- - \frac{1}{\rho_-} = v_+ + \frac{1}{\rho_+}$ (see Figure 7), which is the delta shock curve of Eqs (1.8) and (2.1).

In order to facilitate the study, the problem can be discussed in two parts:

(1) The appearance of delta shock

$$v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+},$$

(2) Formation of contact discontinuity

$$v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}.$$

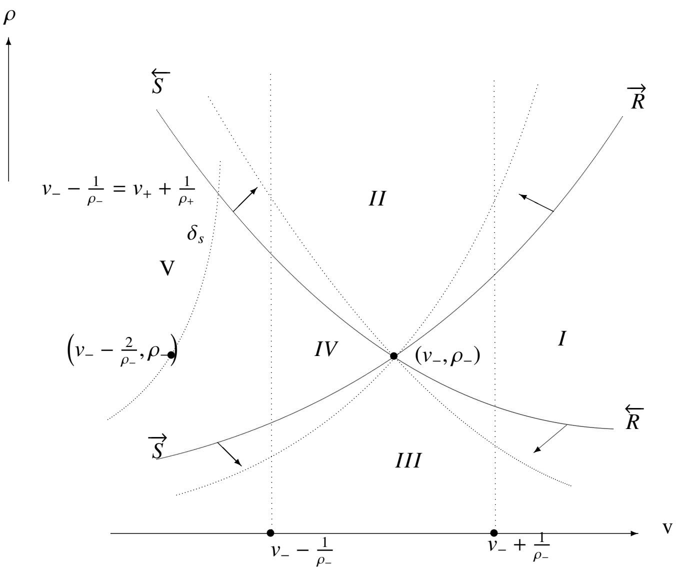


**Figure 7.** The Riemann solutions of (1.9) and (1.10) with (1.7) in the phase plane under $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$ as $\mu \to 0$.

### *4.1. Limit of Riemann solutions of the Eqs (3.1) and (3.2) when $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$*

**Lemma 1.** *Suppose that $(\rho_+, v_+, H_+)$ satisfies $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$. Then, there exists $\mu_0 > 0$ such that $0 < \mu \leq \mu_0$, $(\rho_+, v_+, H_+)$ always belongs to IV.*

*Proof.* From the second equation of the Eqs (3.11) and (3.12), we obtain

$$\begin{aligned} v &= v_- - \sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho^2-\frac{1}{\rho}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)} \\ &= v_- + \left(\frac{1}{\rho}-\frac{1}{\rho_-}\right)\sqrt{\frac{1}{2}\mu k_0^2\rho\rho_-\left(\rho+\rho_-\right)+1} \\ &= v_- + \left(\frac{1}{\rho}-\frac{1}{\rho_-}\right)\sqrt{\frac{1}{2}\mu k_0^2\rho^2\rho_- + \frac{1}{2}\mu k_0^2\rho\rho_-^2+1}. \end{aligned}$$

To differentiate $v$ with $\rho$, we have

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}\rho} &= -\frac{1}{\rho^2}\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2\rho+1}+\left(\frac{1}{\rho}-\frac{1}{\rho_-}\right)\cdot\frac{1}{2}\cdot\frac{\mu k_0^2\rho_-\rho+\frac{1}{2}\mu k_0^2\rho_-^2}{\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2+1}} \\ &= -\frac{2\rho_-\left(\frac{1}{\rho^2}\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2\rho+1}\right)}{2\rho_-\rho^2\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2+1}}+\frac{\rho\left(\rho_--\rho\right)\left(\mu k_0^2\rho_-\rho+\frac{1}{2}\mu k_0^2\rho_-^2\right)}{2\rho^2\rho_-\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2+1}} \\ &= -\frac{\mu k_0^2\rho_-^2\rho^2+\mu k_0^2\rho_-^3\rho+2\rho_-}{2\rho^2\rho_-\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2+1}}+\frac{\mu k_0^2\rho_-^2\rho^2+\frac{1}{2}\mu k_0^2\rho_-^3\rho-\mu k_0^2\rho_-\rho^3-\frac{1}{2}\mu k_0^2\rho_-^2\rho^2}{2\rho^2\rho_-\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2+1}} \\ &= -\frac{\frac{1}{2}\mu k_0^2\rho_-^3\rho+\frac{1}{2}\mu k_0^2\rho_-^2\rho^2+2\rho_-}{2\rho^2\rho_-\sqrt{\frac{1}{2}\mu k_0^2\rho_-\rho^2+\frac{1}{2}\mu k_0^2\rho_-^2+1}}. \end{aligned}$$

When $\rho > \rho_-$, $\mu$ is inversely proportional to $\frac{\mathrm{d}v}{\mathrm{d}\rho}$, we obtain

$$v_+ \le v_- - \sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_+}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)}.$$

Simplifying, we get

$$\mu \le \frac{2\rho_+\rho_-\left(v_--v_+\right)^2}{k_0^2\left(\rho_++\rho_-\right)\left(\rho_+-\rho_-\right)^2}-\frac{2}{k_0^2\left(\rho_++\rho_-\right)\rho_+\rho_-} =: \mu_0.$$

We have found the $\mu_0$ that satisfies the conditions, and the lemma is proved. When $(\rho_+, v_+, H_+) \in$ IV, the shock wave is given by

$$\overleftarrow{S}\left(\rho_-, v_-, H_-\right): \begin{cases} \sigma_1^\mu = v_- + \beta t - \frac{\rho_*}{\rho_*-\rho_-}\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)}, \\ v_* - v_- = -\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)},\ \rho_* > \rho_-, \\ H_{*1} = \frac{\rho_*}{\rho_-}H_- + \frac{\rho_*-\rho_-}{2\rho_-}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)+\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right), \end{cases}$$

$$\overrightarrow{S}(\rho_+, v_+, H_+):\begin{cases}\sigma_3^\mu = v_+ + \beta t - \frac{\rho_*}{\rho_+-\rho_*}\sqrt{\left(\frac{1}{\rho_+}-\frac{1}{\rho_*}\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)-\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)\right)\right)},\\ v_+ - v_* = -\sqrt{\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)-\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)\right)},\ \rho_* < \rho_+,\\ H_{*2} - \frac{\rho_*}{\rho_+}H_+ = \frac{\rho_*-\rho_+}{2\rho_+}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)+\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)\right),\end{cases}$$

then

$$H_{*1} = \frac{\rho_*}{\rho_-}H_- + \frac{\rho_*-\rho_-}{2\rho_-}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)+\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right), \tag{4.1}$$

$$H_{*2} = \frac{\rho_*}{\rho_+}H_+ - \frac{\rho_*-\rho_+}{2\rho_+}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)+\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)\right). \tag{4.2}$$

$$\begin{aligned}v_+ - v_- =& -\sqrt{\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)-\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)\right)\right)}\\ &-\sqrt{\left(\frac{1}{\rho_-}-\frac{1}{\rho_*}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)-\left(\frac{1}{2}\mu k_0^2\rho_-^2-\frac{1}{\rho_-}\right)\right)\right)}\\ =&\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\sqrt{\frac{1}{2}\mu k_0^2(\rho_+ + \rho_*)\rho_+\rho_* + 1} + \left(\frac{1}{\rho_*}-\frac{1}{\rho_-}\right)\sqrt{\frac{1}{2}\mu k_0^2(\rho_* + \rho_-)\rho_*\rho_- + 1}.\end{aligned}$$

As above, we have provided the expression for the intermediate state when the projection of the positive state $(\rho_+, v_+, H_+)$ on the phase plane falls into region IV. □

**Lemma 2.** *Suppose that $H_- > \frac{1}{2\rho_-}$, $H_+ > \frac{1}{2\rho_+}$, and the positive state $(\rho_+, v_+, H_+)$ satisfies the condition $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$. Then,*

$$\lim_{\mu\to 0}\rho_* = +\infty,$$
$$\lim_{\mu\to 0}H_{*1} = \lim_{\mu\to 0}H_{*2} = +\infty.$$

*Proof.* Suppose that $\lim_{\mu\to 0}\rho_* = C_0$, then we have

$$\lim_{\mu\to 0}(v_+ - v_-) = \lim_{\mu\to 0}\left(\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\sqrt{\frac{1}{2}\mu k_0^2(\rho_+ + \rho_*)\rho_+\rho_* + 1}\right) + \lim_{\mu\to 0}\left(\left(\frac{1}{\rho_*}-\frac{1}{\rho_-}\right)\sqrt{\frac{1}{2}\mu k_0^2(\rho_* + \rho_-)\rho_*\rho_- + 1}\right).$$

We find

$$v_+ + \frac{1}{\rho_+} = \frac{2}{C_0} + v_- - \frac{1}{\rho_-},$$

which implies that

$$v_+ + \frac{1}{\rho_+} > v_- - \frac{1}{\rho_-},$$

does not hold, so $\lim_{\mu\to 0}\rho_* = \infty$. To prove that $\lim_{\mu\to 0}H_{*1} = \lim_{\mu\to 0}H_{*2} = +\infty$, where $H_{*1}$ and $H_{*2}$ are given by Eqs (4.1) and (4.2),

$$\lim_{\mu\to 0}H_{*1} = \lim_{\mu\to 0}\left(\frac{\rho_*}{\rho_-}H_- + \frac{\mu k_0^2}{4\rho_-}\rho_*^3 - \frac{\mu k_0^2}{4}\rho_*^2 - \frac{1}{2\rho_-} + \frac{1}{2\rho_*} + \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\cdot\frac{\rho_*-\rho_-}{2\rho_-}\right) = \infty,$$

$$\lim_{\mu\to 0} H_{*2} = \lim_{\mu\to 0}\left(\frac{\rho_*}{\rho_+}H_+ + \frac{\mu k_0^2}{4\rho_+}\rho_*^3 - \frac{\mu k_0^2}{4}\rho_*^2 - \frac{1}{2\rho_+} + \frac{1}{2\rho_*} + \frac{\rho_* - \rho_+}{2\rho_+}\left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right) = \infty.$$

□

**Lemma 3.** *Suppose that* $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$. *Then,*

$$\lim_{\mu\to 0}\sigma_1^\mu = \lim_{\mu\to 0}\sigma_2^\mu = \lim_{\mu\to 0}\sigma_3^\mu = \sigma^\mu.$$

*Proof.*

$$\begin{aligned}
\lim_{\mu\to 0}\sigma_1^\mu &= \lim_{\mu\to 0}\left(v_- + \beta t - \frac{\rho_*}{\rho_* - \rho_-}\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)}\right)\\
&= v_- + \beta t - \lim_{\mu\to 0}\sqrt{\frac{\rho_*}{(\rho_* - \rho_-)}\cdot\frac{1}{\rho_-}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)}\\
&= v_- + \beta t - \lim_{\mu\to 0}\sqrt{\frac{\rho_*}{\rho_-(\rho_* - \rho_-)}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho^{-2} - \frac{1}{\rho_-}\right)\right)}\\
&= v_- + \beta t - \lim_{\mu\to 0}\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*(\rho_* + \rho_-) + \frac{1}{\rho_-^2}},\\
\lim_{\mu\to 0}\sigma_3^\mu &= v_+ + \beta t + \lim_{\mu\to 0}\sqrt{\frac{\rho_*}{(\rho_* - \rho_+)}\cdot\frac{1}{\rho_+}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right)}\\
&= v_+ + \beta t + \lim_{\mu\to 0}\sqrt{\frac{\rho_*}{\rho_+(\rho_* - \rho_+)}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right)}\\
&= v_+ + \beta t + \lim_{\mu\to 0}\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*(\rho_* + \rho_+) + \frac{1}{\rho_+^2}}.
\end{aligned}$$

Then,

$$\begin{aligned}
&v_+ - v_- = \left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)\sqrt{\frac{\mu k_0^2}{2\rho_+}\rho_*(\rho_+ + \rho_*) + \frac{1}{\rho_+}} = v_- + \rho_-\left(\frac{1}{\rho_*} - \frac{1}{\rho_-}\right)\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*(\rho_* + \rho_-) + \frac{1}{\rho_-^2}},\\
&v_+ + \beta t - \rho_+\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)\sqrt{\frac{\mu k_0^2}{2\rho_+}\rho_*(\rho_* + \rho_+) + \frac{1}{\rho_+^2}} = v_- + \beta t + \rho_-\left(\frac{1}{\rho_*} - \frac{1}{\rho_-}\right)\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*(\rho_* + \rho_-) + \frac{1}{\rho_-^2}}.
\end{aligned}$$

As $\mu \to 0$, we get

$$\begin{aligned}
&\lim_{\mu\to 0}\left(v_+ + \beta t - \rho_+\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)\sqrt{\frac{\mu k_0^2}{2\rho_+}\rho_*(\rho_* + \rho_+) + \frac{1}{\rho_+^2}}\right)\\
&= \lim_{\mu\to 0}\left(v_- + \beta t + \rho_-\left(\frac{1}{\rho_*} - \frac{1}{\rho_-}\right)\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*(\rho_* + \rho_-) + \frac{1}{\rho_-^2}}\right),
\end{aligned}$$

$$
\begin{aligned}
&v_+ + \beta t - \lim_{\mu\to 0}\left(\rho_+\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\sqrt{\frac{\mu k_0^2}{2\rho_+}\rho_*\left(\rho_*+\rho_+\right)\frac{1}{\rho_+^2}}\right)\\
=&v_- + \beta t + \lim_{\mu\to 0}\left(\rho_-\left(\frac{1}{\rho_*}-\frac{1}{\rho_-}\right)\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*\left(\rho_*+\rho_-\right)+\frac{1}{\rho_-^2}}\right),\\
&v_+ + \beta t + \lim_{\mu\to 0}\sqrt{\frac{\mu k_0^2}{2\rho_+}\rho_*\left(\rho_*+\rho_+\right)+\frac{1}{\rho_+^2}}\\
=&v_- + \beta t - \lim_{\mu\to 0}\sqrt{\frac{\mu k_0^2}{2\rho_-}\rho_*\left(\rho_*+\rho_-\right)+\frac{1}{\rho_-^2}}.
\end{aligned}
$$

Therefore, $\lim\limits_{\mu\to 0}\sigma_1^\mu = \lim\limits_{\mu\to 0}\sigma_3^\mu$. Also $\lim\limits_{\mu\to 0}\sigma_2^\mu = \lim\limits_{\mu\to 0}(v_* + \beta t)$, and substituting and calculating gives

$$
\begin{aligned}
v_+ - v_* &= -\sqrt{\left(\frac{1}{\rho_+}-\frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{\rho_*}\right)-\left(\frac{1}{2}\mu k_0^2\rho_+^2-\frac{1}{\rho_+}\right)\right)},\\
v_* &= v_+ + \sqrt{\frac{\rho_*-\rho_+}{\rho_*\rho_+}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2-\frac{1}{2}\mu k_0^2\rho_+^2\right)-\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\right)}\\
&= v_+ + \frac{\rho_*-\rho_+}{\rho_*}\sqrt{\frac{\mu k_0^2}{2\rho_+}\rho_*\left(\rho_*+\rho_+\right)+\frac{1}{\rho_+^2}}.
\end{aligned}
$$

Thus, $\lim\limits_{\mu\to 0}\sigma_1^\mu = \lim\limits_{\mu\to 0}\sigma_2^\mu = \lim\limits_{\mu\to 0}\sigma_3^\mu$. Suppose that $\lim\limits_{\mu\to 0}\sqrt{\mu\rho_*\left(\rho_*+\rho_+\right)} = \infty$ and $\lim\limits_{\mu\to 0}\sqrt{\mu\rho_*\left(\rho_*+\rho_-\right)} = \infty$. Then,

$$
\begin{aligned}
&\lim_{\mu\to 0}(v_+ - v_-) = \lim_{\mu\to 0}\left(\left(\frac{1}{\rho_*}-\frac{1}{\rho_+}\right)\sqrt{\frac{1}{2}\mu k_0^2\left(\rho_++\rho_*\right)\rho_+\rho_*+1}\right)+\left(\frac{1}{\rho_*}-\frac{1}{\rho_-}\right)\sqrt{\frac{1}{2}\mu k_0^2\left(\rho_-+\rho_*\right)\rho_-\rho_*+1}\\
&= -\lim_{\mu\to 0}\frac{1}{\rho_+}\sqrt{\frac{1}{2}\mu k_0^2\left(\rho_++\rho_*\right)\rho_+\rho_*+1}-\lim_{\mu\to 0}\frac{1}{\rho_-}\sqrt{\frac{1}{2}\mu k_0^2\left(\rho_-+\rho_*\right)\rho_-\rho_*+1}\\
&= -\infty,
\end{aligned}
$$

which implies that the assumption does not hold. In conclusion, $\lim\limits_{\mu\to 0}\sigma_1^\mu = \lim\limits_{\mu\to 0}\sigma_2^\mu = \lim\limits_{\mu\to 0}\sigma_3^\mu = \sigma^\mu$. □

**Lemma 4.** *Suppose that* $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$. *Then,*

$$
\begin{cases}
\lim\limits_{\mu\to 0}\int_{x_1(t,\ \mu)}^{x_2(t,\ \mu)}\rho_*\mathrm{d}x = \left(\sigma^\mu\left[\rho\right]-\left[\rho\left(v+\beta t\right)\right]\right)t,\\
\lim\limits_{\mu\to 0}\int_{x_1(t,\ \mu)}^{x_2(t,\ \mu)}\rho_*v_*\mathrm{d}x = \left(\sigma^\mu\left[\rho v\right]-\left[\rho v\left(v+\beta t\right)-\frac{1}{\rho}\right]\right)t,\\
\lim\limits_{\mu\to 0}\int_{x_1(t,\ \mu)}^{x_2(t,\ \mu)}\left(\frac{\rho_*v_*(v_*+\beta t)}{2}+H_*\right)\mathrm{d}x = \left(\sigma^\mu\left[\frac{\rho v(v+\beta t)}{2}+H\right]-\left[\left(\frac{\rho v(v+\beta t)}{2}+H-\frac{1}{\rho}\right)\left(v+\beta t\right)\right]\right)t,
\end{cases}
$$

*where* $x_1\left(t,\ \mu\right) = \int_0^t\sigma^\mu\left(\theta\right)d\theta,\ x_2\left(t,\ \mu\right) = \int_0^t\sigma^\mu\left(\theta\right)d\theta$.

*Proof.* From the first equation of the system (3.10), we have

$$\begin{cases}-\sigma_1^{\mu}(\rho_* - \rho_-) + (\rho_*(v_* + \beta t) - \rho_-(v + \beta t)) = 0,\\ -\sigma_2^{\mu}(\rho_+ - \rho_*) + (\rho_+(v_+ + \beta t) - \rho_*(v_* + \beta t)) = 0.\end{cases}$$

Adding both equations and taking the limit as $\mu \to 0$, by Lemma 3 we get

$$\lim_{\mu\to 0}\left(\sigma_2^{\mu} - \sigma_1^{\mu}\right)\rho_* = \sigma^{\mu}[\rho] - [\rho(v + \beta t)],$$

and integrating the above equation gives

$$\lim_{\mu\to 0}\int_{x_1(t,\,\mu)}^{x_2(t,\,\mu)} \rho_* \mathrm{d}x = (\sigma^{\mu}[\rho] - [\rho(v + \beta t)])\,t.$$

Similarly, from the second and third equations of (3.10) and by Lemma 3, we get

$$\begin{cases}\lim_{\mu\to 0}\int_{x_1(t,\,\mu)}^{x_2(t,\,\mu)} \rho_* v_* \mathrm{d}x = \left(\sigma^{\mu}[\rho v] - \left[\rho v(v + \beta t) - \frac{1}{\rho}\right]\right)t,\\ \lim_{\mu\to 0}\int_{x_1(t,\,\mu)}^{x_2(t,\,\mu)} \left(\frac{\rho_* v_*(v_* + \beta t)}{2} + H_*\right)\mathrm{d}x = \left(\sigma^{\mu}\left[\frac{\rho v(v+\beta t)}{2} + H\right] - \left[\left(\frac{\rho v(v+\beta t)}{2} + H - \frac{1}{\rho}\right)(v + \beta t)\right]\right)t.\end{cases}$$

In conclusion, Lemma 4 is proved. □

In summary, under the condition $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$, as $\mu \to 0$, the Riemann solutions of the non-isentropic Chaplygin gas magnetogasdynamics equations exhibits mass concentration, meaning it converges to the delta shock wave of the non-isentropic Chaplygin Euler equations (see Figures 7 and 8).

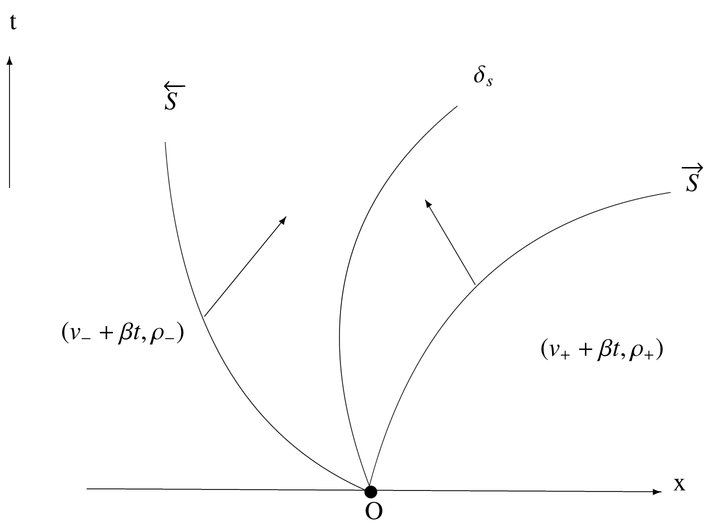


**Figure 8.** The Riemann solutions of (1.7), (1.9), and (1.10) in the physical plane under $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$ as $\mu \to 0$.

*4.2. Limit of Riemann solutions of the Eqs (3.1) and (3.2) when $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$*

**Case 1.** $(\rho_+, v_+, H_+) \in IV(\rho_-, v_-)$.

**Lemma 5.** *Suppose that $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$ and $(\rho_+, v_+, H_+) \in IV$. Then, as $\mu \to 0$, $\overleftarrow{S}(\rho_-, v_-, H_-)$ and $\overrightarrow{S}(\rho_+, v_+, H_+)$ converge to the Riemann solutions $J_1$ and $J_3$ of the non-isentropic Chaplygin Euler equations.*

*Proof.* From (4.1), we know $\lim\limits_{\mu\to 0}\rho_* = C_0$. Taking the limit as $\mu \to 0$ for the first equation in (3.11), we get

$$\begin{aligned}\lim_{\mu\to 0}\sigma_1^{\mu} &= \lim_{\mu\to 0}\left(v_- + \beta t - \frac{\rho_*}{\rho_* - \rho_-}\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)}\right)\\ &= v_- + \beta t - \lim_{\mu\to 0}\frac{\rho_*}{\rho_* - \rho_-}\cdot\frac{\rho_* - \rho_-}{\rho_-\cdot\rho_*}\\ &= v_- + \beta t - \frac{1}{\rho_-}.\end{aligned}$$

Likewise, the limit of the second equation in (3.11) is

$$\lim_{\mu\to 0}(v_* - v_-) = \lim_{\mu\to 0}\sqrt{\left(\frac{1}{\rho_-} - \frac{1}{\rho_*}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)},$$

and we know

$$v_* - \frac{1}{\rho_*} = v_- - \frac{1}{\rho_-}.$$

The limit of the third equation in (3.11) is

$$\lim_{\mu\to 0}(H_{*1}\rho_- - \rho_* H_-) = \lim_{\mu\to 0}\left(\frac{\rho_* - \rho_-}{2}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) + \left(\frac{1}{2}\mu k_0^2\rho_-^2 - \frac{1}{\rho_-}\right)\right)\right),$$

then

$$H_{*1}\rho_- - \rho_* H_- = \frac{\rho_-^2 - \rho_*^2}{2\rho_*\rho_-}.$$

Thus, as $\mu \to 0$, for the system (3.11) we have

$$\begin{cases}\lim\limits_{\mu\to 0}\sigma_1^{\mu} = v_- + \beta t - \frac{1}{\rho_-},\\ v_* - \frac{1}{\rho_*} = v_- - \frac{1}{\rho_-},\\ H_{*1}\rho_- - \rho_* H_- = \frac{\rho_-^2 - \rho_*^2}{2\rho_*\rho_-}.\end{cases}$$

Taking the limit as $\mu \to 0$ for the first equation in (3.12), we get

$$\lim_{\mu\to 0}\sigma_3^{\mu} = \lim_{\mu\to 0}\left(v_+ + \beta t - \frac{\rho_*}{\rho_+ - \rho_*}\sqrt{\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right) - \left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right)\right)}\right)$$

$$= v_+ + \beta t - \lim_{\mu\to 0} \frac{\rho_*}{\rho_+ - \rho_*} \sqrt{\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right) - \left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right)\right)}$$
$$= v_+ + \beta t + \frac{1}{\rho_+}.$$

Likewise, the limit of the second equation in (3.12) is

$$\lim_{\mu\to 0}(v_+ - v_*) = \lim_{\mu\to 0} \sqrt{\left(\frac{1}{\rho_*} - \frac{1}{\rho_+}\right)\left(\left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right) - \left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right)\right)},$$

and we can get

$$v_+ + \frac{1}{\rho_+} = v_* + \frac{1}{\rho_*}.$$

The limit of the third equation in (3.12) is

$$\lim_{\mu\to 0}(H_{*2}\rho_+ - \rho_* H_+) = \lim_{\mu\to 0}\left(\frac{\rho_* - \rho_+}{2}\left(\left(\frac{1}{2}\mu k_0^2\rho_*^2 - \frac{1}{\rho_*}\right) + \left(\frac{1}{2}\mu k_0^2\rho_+^2 - \frac{1}{\rho_+}\right)\right)\right),$$

and, similarly,

$$H_{*2}\rho_+ - \rho_* H_+ = \frac{\rho_+^2 - \rho_*^2}{2\rho_*\rho_+},$$

The limit of the Eq (3.12) is

$$\begin{cases} \lim\limits_{\mu\to 0}\sigma_3^\mu = v_+ + \beta t + \frac{1}{\rho_+}, \\ v_+ + \frac{1}{\rho_+} = v_* + \frac{1}{\rho_*}, \\ H_{*2}\rho_+ - \rho_* H_+ = \frac{\rho_+^2 - \rho_*^2}{2\rho_*\rho_+}. \end{cases}$$

In conclusion, when $(\rho_+, v_+, H_+)$ is located in $IV$ and as $\mu \to 0$, the shock wave $\overleftarrow{S}(\rho_-, v_-, H_-)$ and $\overrightarrow{S}(\rho_+, v_+, H_+)$ of the non-isentropic Chaplygin gas magnetogasdynamics equations converge to the contact discontinuities $J_1$ and $J_3$ of the non-isentropic Chaplygin Euler equations. □

**Case 2.** $(\rho_+, v_+, H_+) \in I(\rho_-, v_-)$.

**Lemma 6.** *Suppose that* $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$ *and* $(\rho_+, v_+, H_+) \in I$. *Then, as* $\mu \to 0$, *the rarefaction wave* $\overleftarrow{R}(\rho_-, v_-, H_-)$ *and* $\overrightarrow{R}(\rho_+, v_+, H_+)$ *converge to the Riemann solutions* $J_1$ *and* $J_3$ *of the non-isentropic Chaplygin Euler equations.*

*Proof.* Let $(\rho_*, v_*, H_{*1})$ and $(\rho_*, v_*, H_{*2})$ be the intermediate states connecting $R_1, J, R_3$, then we have

$$\overleftarrow{R}(\rho_-, v_-, H_-): \begin{cases} \frac{dx}{dt} = \lambda_1(\rho, v) = v_* + \beta t - \sqrt{\mu k_0^2\rho_+ + \frac{1}{\rho_+^2}}, \\ v_* - v_- = \int_{\rho_-}^{\rho_*} -\rho^{-2}\left(\mu k_0^2\rho^3 + 1\right)^{\frac{1}{2}} d\rho,\ \rho < \rho_-, \\ \frac{H_{*1}}{\rho_*} - \frac{H_-}{\rho_-} = \frac{1}{2}\mu k_0^2\rho_* + \frac{1}{2\rho_*^2} - \left(\frac{1}{2}\mu k_0^2\rho_- + \frac{1}{2\rho_-^2}\right), \end{cases}$$

$$\overrightarrow{R}(\rho_+, v_+, H_+): \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_3(\rho, v) = v_* + \beta t + \sqrt{\mu k_0^2 \rho_* + \frac{1}{\rho_*^2}}, \\ v_+ - v_* = \int_{\rho_*}^{\rho_+} \rho^{-2}\left(\mu k_0^2 \rho_+^3 + 1\right)^{\frac{1}{2}} \mathrm{d}\rho,\ \rho > \rho_-, \\ \frac{H_+}{\rho_+} - \frac{H_{*2}}{\rho_*} = \frac{1}{2}\mu k_0^2 \rho_+ + \frac{1}{2\rho_+^2} - \left(\frac{1}{2}\mu k_0^2 \rho_* + \frac{1}{2\rho_*^2}\right). \end{cases}$$

Taking the limit as $\mu \to 0$ for the first equation of $\overleftarrow{R}(\rho_-, v_-, H_-)$, we get

$$\lim_{\mu\to 0} \lambda_1 = \lim_{\mu\to 0}\left(v_* + \beta t - \sqrt{\mu k_0^2 \rho_*^2 + \frac{1}{\rho_*^2}}\right) = v_* + \beta t - \frac{1}{\rho_*}.$$

Likewise, the limit of the second equation of $\overleftarrow{R}(\rho_-, v_-, H_-)$ is

$$\lim_{\mu\to 0}(v_* - v_-) = \lim_{\mu\to 0} \int_{\rho_-}^{\rho_*} -\rho^{-2}\left(\mu k_0^2 \rho^3 + 1\right)^{\frac{1}{2}} \mathrm{d}\rho,$$

thus,

$$v_* - \frac{1}{\rho_*} = v_- - \frac{1}{\rho_-}.$$

The limit of the third equation of $\overleftarrow{R}(\rho_-, v_-, H_-)$ is

$$\lim_{\mu\to 0}\left(\frac{H_{*1}}{\rho_*} - \frac{H_-}{\rho_-}\right) = \lim_{\mu\to 0}\left(\left(\frac{1}{2}\mu k_0^2 \rho_* + \frac{1}{2\rho_*}\right) - \left(\frac{1}{2}\mu k_0^2 \rho_- + \frac{1}{2\rho_-^2}\right)\right),$$

and we know

$$\rho_- H_{*1} - \rho_* H_- = \frac{\rho_-^2 - \rho_*^2}{2\rho * \rho_-}.$$

Thus, as $\mu \to 0$, for $\overleftarrow{R}(\rho_-, v_-, H_-)$,

$$\begin{cases} \lim_{\mu\to 0} \lambda_1 = v_* + \beta t - \frac{1}{\rho_*}, \\ v_* - \frac{1}{\rho_*} = v_- - \frac{1}{\rho_-}, \\ \rho_- H_{*1} - \rho_* H_- = \frac{\rho_-^2 - \rho_*^2}{2\rho * \rho_-}. \end{cases}$$

Taking the limit as $\mu \to 0$ for the first equation of $\overrightarrow{R}(\rho_+, v_+, H_+)$, we get

$$\lim_{\mu\to 0} \lambda_3 = \lim_{\mu\to 0}\left(v_* + \beta t + \sqrt{\mu k_0^2 \rho_*^2 + \frac{1}{\rho_*^2}}\right) = v_* + \beta t + \frac{1}{\rho_*}.$$

Likewise, the limit of the second equation of $\overrightarrow{R}(\rho_+, v_+, H_+)$ is

$$\lim_{\mu\to 0}(v_+ - v_*) = \lim_{\mu\to 0} \int_{\rho_*}^{\rho_+} \rho^{-2}\left(\mu k_0^2 \rho^3 + 1\right)^{\frac{1}{2}} \mathrm{d}\rho,$$

thus

$$v_+ + \frac{1}{\rho_+} = v_* + \frac{1}{\rho_*}.$$

The limit of the third equation of $\overrightarrow{R}(\rho_+, v_+, H_+)$ is

$$\lim_{\mu\to 0}\left(\frac{H_+}{\rho_+} - \frac{H_{*2}}{\rho_{*2}}\right) = \lim_{\mu\to 0}\left(\left(\frac{1}{2}\mu k_0^2\rho_+ + \frac{1}{2\rho_+}\right) - \left(\frac{1}{2}\mu k_0^2\rho_* + \frac{1}{2\rho_*^2}\right)\right),$$

and we know

$$\rho_* H_+ - \rho_+ H_{*2} = \frac{\rho_*^2 - \rho_+^2}{2\rho_+\rho_*}.$$

Thus, as $\mu \to 0$, for $\overrightarrow{R}(\rho_+, v_+, H_+)$, we have

$$\begin{cases} \lim\limits_{\mu\to 0} \lambda_3 = v_* + \beta t + \frac{1}{\rho_*}, \\ v_* + \frac{1}{\rho_*} = v_- + \frac{1}{\rho_-}, \\ \rho_* H_+ - \rho_+ H_{*2} = \frac{\rho_*^2 - \rho_+^2}{2\rho_+\rho_*}. \end{cases}$$

In conclusion, when $(\rho_+, v_+, H_+)$ is located in $I$ and as $\mu \to 0$, the rarefaction wave $\overleftarrow{R}(\rho_-, v_-, H_-)$ and $\overrightarrow{R}(\rho_+, v_+, H_+)$ of the non-isentropic Chaplygin gas magnetogasdynamics equations converge to the contact discontinuities $J_1$ and $J_3$ of the non-isentropic Chaplygin Euler equations. □

**Case 3.** $(\rho_+, v_+, H_+) \in II(\rho_-, v_-)$.

**Lemma 7.** *Suppose that* $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$ *and* $(\rho_+, v_+, H_+) \in II$. *Then, as* $\mu \to 0$, $\overleftarrow{S}(\rho_-, v_-, H_-)$ *and* $\overrightarrow{R}(\rho_+, v_+, H_+)$ *converge to the Riemann solutions* $J_1$ *and* $J_3$ *of the non-isentropic Chaplygin Euler equations.*

*Proof.* If the conditions in the lemma hold, $\overleftarrow{S}(\rho_-, v_-, H_-)$ as $\mu \to 0$ is the same as in Case 1.

Thus, as $\mu \to 0$, for $\overleftarrow{S}(\rho_-, v_-, H_-)$, we have

$$\begin{cases} \lim\limits_{\mu\to 0} \sigma_1^\mu = v_- + \beta t - \frac{1}{\rho_-}, \\ v_* - v_- = v_- - \frac{1}{\rho_-}, \\ \rho_- H_{*1} - \rho_* H_- = \frac{\rho_-^2 - \rho_*^2}{2\rho_*\rho_-}. \end{cases}$$

Similarly, when $\mu \to 0$, the calculation for taking the limit of $\overrightarrow{R}(\rho_+, v_+, H_+)$ is the same as in Case 2. Thus, as $\mu \to 0$, for $\overrightarrow{R}(\rho_+, v_+, H_+)$, we have

$$\begin{cases} \lim\limits_{\mu\to 0} \lambda_3 = v_* + \beta t + \frac{1}{\rho_*}, \\ v_+ + \frac{1}{\rho_+} = v_* + \frac{1}{\rho_*}, \\ \rho_* H_+ - \rho_+ H_{*2} = \frac{\rho_*^2 - \rho_+^2}{2\rho_+\rho_*}. \end{cases}$$

In conclusion, when $(\rho_+, v_+, H_+)$ is located in $II$ and as $\mu \to 0$, the backward shock wave $\overleftarrow{S}(\rho_-, v_-, H_-)$ and the forward rarefaction wave $\overrightarrow{R}(\rho_+, v_+, H_+)$ of the non-isentropic Chaplygin gas magnetogasdynamics equations converge to the contact discontinuities $J_1$ and $J_3$ of the non-isentropic Chaplygin Euler equations. □

**Case 4.** $(\rho_+, v_+, H_+) \in III(\rho_-, v_-)$.

**Lemma 8.** *Suppose that* $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$ *and* $(\rho_+, v_+, H_+) \in III$. *Then, as* $\mu \to 0$, $\overleftarrow{R}(\rho_-, v_-, H_-)$ *and* $\overrightarrow{S}(\rho_+, v_+, H_+)$ *converge to the Riemann solutions* $J_1$ *and* $J_3$ *of the non-isentropic Chaplygin Euler equations.*

*Proof.* If the conditions in the lemma hold, $\overleftarrow{R}(\rho_-, v_-, H_-)$ as $\mu \to 0$ is the same as in Case 2.

Thus, as $\mu \to 0$, for $\overleftarrow{R}(\rho_-, v_-, H_-)$, we have

$$\begin{cases} \lim\limits_{\mu\to 0} \lambda_1 = v_* + \beta t - \frac{1}{\rho_*}, \\ v_* - \frac{1}{\rho_*} = v_- - \frac{1}{\rho_-}, \\ \rho_- H_{*1} - \rho_* H_- = \frac{\rho_-^2 - \rho_*^2}{2\rho_*\rho_-}. \end{cases}$$

Similarly, when $\mu \to 0$, the calculation for taking the limit of $\overrightarrow{S}(\rho_+, v_+, H_+)$ is the same as in Case 1. Thus, as $\mu \to 0$, for $\overrightarrow{S}(\rho_+, v_+, H_+)$, we have

$$\begin{cases} \lim\limits_{\mu\to 0} \sigma_3^\mu = v_+ + \beta t + \frac{1}{\rho_+}, \\ v_+ + \frac{1}{\rho_+} = v_* + \frac{1}{\rho_*}, \\ H_{*2}\rho_+ - \rho_* H_+ = \frac{\rho_+^2 - \rho_*^2}{2\rho_*\rho_+}. \end{cases}$$

In conclusion, when $(\rho_+, v_+, H_+)$ is located in $III$ and as $\mu \to 0$, the backward rarefaction wave $\overleftarrow{R}(\rho_-, v_-, H_-)$ and the forward shock wave $\overrightarrow{S}(\rho_+, v_+, H_+)$ of the non-isentropic Chaplygin gas magnetogasdynamics equations converge to the contact discontinuities $J_1$ and $J_3$ of the non-isentropic Chaplygin Euler equations. □

## 5. Discussion

In summary, the Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) converges to the delta shock wave of the Eqs (1.8) and (2.1) as the parameter $\mu$ approaches zero when $v_- - \frac{1}{\rho_-} \geq v_+ + \frac{1}{\rho_+}$. When $v_- - \frac{1}{\rho_-} < v_+ + \frac{1}{\rho_+}$, the Riemann solutions of the Eqs (1.9) and (1.10) with (1.7) converges to the contact discontinuity of the Eqs (1.8) and (2.1) as the parameter $\mu$ approaches zero.

## Author contributions

Jingye Zhao: Formal analysis, Writing-original draft, Writing-review & editing; Zonghua Wei: Visualization, Writing-original draft, Writing-review & editing; Jiahui Liu: Writing-original draft, Writing-review & editing; Yongqiang Fan: Conceptualization, Supervision, Project administration. All authors have read and approved the final version of the manuscript for publication.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

This work does not have any conficts of interest.

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