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Research article

Planar Turán number of double star $S_{3,4}$

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Abstract: Planar Turán number, denoted by $\mathrm{ex}_{\mathcal{P}}(n, H)$, is the maximum number of edges in an n-vertex planar graph which does not contain H as a subgraph. Ghosh, Győri, Paulos and Xiao initiated the topic of the planar Turán number for double stars. There were two double stars $S_{3,4}$ and $S_{3,5}$ that remained unknown. In this paper, we give the exact value of $\mathrm{ex}_{\mathcal{P}}(n, S_{3,4})$.

Keywords: Turán-type problem; planar graph; double star; extremal graphs
Mathematics Subject Classification: 05C35

1. Introduction

All graphs considered in this paper are finite, simple and planar. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ are the vertex set and edge set. We use $v(G)$, $e(G)$, $\delta(G)$ and $\Delta(G)$ to denote number of vertices, number of edges, minimum degree and maximum degree, respectively. For any subset $S \subset V(G)$, the subgraph induced on S is denoted by $G[S]$. We denote by $G\backslash S$ the subgraph induced on $V(G)\backslash S$. If $S = \{v\}$, we simply write $G\backslash v$. We use $e[S, T]$ to denote the number of edges between S and T, where S, T are subsets of $V(G)$.

Given a graph H, a graph G is called H-free if it does not contain H as a subgraph. One of the most classical problems in extremal graph theory is to determine $\mathrm{ex}(n, H)$, which is the maximum number of edges in an n-vertex H-free graph. In 1941, Turán [1] gave the exact value of $\mathrm{ex}(n, K_r)$ and the extremal graph, where K_r is a complete graph on r vertices. Erdős-Stone [2] Theorem extends this to the case for all non-bipartite graphs H and shows that $ex(n, H) = (1 - \frac{1}{\chi(H)-1})\binom{n}{2} + o(n^2)$, where $\chi(H)$ denotes the chromatic number of H. This latter result has been called the 'fundamental theorem of extremal graph theory'.

In 2016, Dowden [3] initiated the study of Turán-type problems when host graphs are planar graphs. We use $\mathrm{ex}_{\mathcal{P}}(n, H)$ to denote the maximum number of edges in an n-vertex H-free planar graph. Dowden studied the planar Turán number of C_4 and C_5, where C_k is a cycle on k vertices. Ghosh, Győri, Martin, Paulos and Xiao [4] gave the exact value for C_6. Shi, Walsh and Yu [5], and independently Győri, Li and Zhou [6] gave the exact value for C_7. The planar Turán number of C_k is still unknown for $k \geq 8$. Cranston, Lidický, Liu and Shantanam [7] first gave both lower and upper bound for general cycles, while Lan and Song [8] improved the lower bound. Recently, Shi, Walsh and Yu [9] improved the upper bound, while Győri, Varga and Zhu [10] gave a new construction and improved the lower bound. Lan, Shi and Song [11] gave a sufficient condition for graphs with planar Turán number $3n - 6$. We refer the interested readers to more results on paths, theta graphs and other graphs [12–18].

In 2022, Ghosh, Győri, Paulos and Xiao [19] studied the planar Turán number for some double stars. A double star $S_{k,l}$ is the graph obtained by taking an edge uv with k vertices joining u and l vertices joining v. Moreover, they gave the exact value for $S_{2,2}$ and $S_{2,3}$. Later, Xu, Hu and Zhang [20] improved the upper bound for $S_{2,5}$. Xu and Shao [21] gave the exact value of $S_{2,4}$. Xu, Zhou, Li, Yan [22] determined the planar Turán number for all balanced double stars, solving a conjecture proposed by Győri et al. Here we give the exact value of $\mathrm{ex}_{\mathcal{P}}(n, S_{3,4})$.

Theorem 1.1. *Let G be an n-vertex $S_{3,4}$-free planar graph. Then $e(G) \leq \frac{5}{2}n$ with equality when $n \equiv 0 \ (mod\ 12)$.*

The paper is organized as follows. We prove the theorem in Section 4. In the next section, some necessary definitions are provided.

2. Preliminaries

Let G be an $S_{3,4}$-free planar graph. For any vertex v, the degree of v discussed here is the degree of v in G. Next, we introduce more notations needed in the proof.

A $\boldsymbol{k}$**-**$\boldsymbol{l}$ edge is an edge whose end vertices are of degree k and l in G. A $\boldsymbol{k_1}$**-**$\boldsymbol{k_2}$**-**$\cdots$**-**$\boldsymbol{k_s}$ path is a path containing s vertices of degree k_1, k_2, $\cdots$, k_s in G, respectively. An alternating $\boldsymbol{k}$**-**$\boldsymbol{l}$**-**$\boldsymbol{k}$ path is a path of even length, where the degrees of these vertices alternate between k and l, and no two vertices of degree k are adjacent to each other. An alternating k-l-k path is called maximal if it is not contained in any other alternating k-l-k path.

A $\boldsymbol{k^+}$**-**$\boldsymbol{l^-}$ star is a copy of star graph such that the central vertex has degree at least k and the leaves have degree at most l in G. Moreover, a k-l^- star is a copy of star graph where the central vertex has degree exactly k and the leaves have degree at most l. Similarly, we can define the k^--l star and the k-l^+ star.

For a vertex v in G, the open neighborhood of v, denoted by $N(v)$, is the set of vertices in G adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. We define analogously for any $S \subset V(G)$ the open neighborhood $N(S) = \bigcup\limits_{v \in S} N(v)$ and the closed neighborhood $N[S] = \bigcup\limits_{v \in S} N[v]$.

Definition 2.1. *A star-block B in G is a subgraph induced on the closed neighborhood of one of the following vertex sets: (i) a single vertex of degree at least* 7*; (ii) vertices of a* 6-6 *edge; (iii) vertices of a maximal alternating* 6-4-6 *path; (iv) a single vertex of degree* 6*.*

For example, a 6-5^- star is a star-block which can be induced on the closed neighborhood of a single vertex of degree 6.

Indeed, for any vertex v of degree at least 6, we can determine that v is contained in a unique star-block which is the first one by checking in the order of (i), (ii), (iii), (iv). Note that there does not exist a 6-5-6 path, a 6-5-5-6 path, a 6-5-4-6 path or a 6-5-5-5-6 path. Otherwise, we can find an $S_{3,4}$. Thus, each vertex of degree at least 6 must lie in a specific star-block defined above.

Definition 2.2. *Let H be a subgraph of G. The star-block base $\mathcal{B}$ of H is the set consisting of star-blocks satisfying $V(H) = \bigcup_{B\in\mathcal{B}} V(B)$. If there are two star-blocks $B, B' \in \mathcal{B}$ with $V(B) \cap V(B') \neq \emptyset$, then the common vertices are called shared vertices.*

It is noticed that for any shared vertex v, we have $2 \leq d(v) \leq 4$. In fact, if $d(v) = 1$, v can only belong to one star-block. If $d(v) \geq 5$, then there exists a 6-5-6 path, a 6-5-5-6 path, a 6-5-4-6 path or a 6-5-5-5-6 path, a contradiction.

Given two subgraphs $H, H' \subset G$, we use $\boldsymbol{H + H'}$ to denote the subgraph induced on $V(H) \cup V(H')$. If $V(H') = \{v\}$, we simply write $H+v$. The edge weight of H is defined as $\boldsymbol{w_0(H)} := e(H) + \frac{1}{2}(e[H, G\backslash H]) = \frac{1}{2}\sum_{v\in V(H)} d(v)$.

Definition 2.3. *Let $G = G_1 + G_2$. If $\mathcal{B}$ is a star-block base of G_1 and any vertex in $V(G_2)$ has degree at most 5 in G, then we say G has a star-block cover. Let $d_{\mathcal{B}}(v)$ be the number of star-blocks containing v in base $\mathcal{B}$. For any star-block $B \in \mathcal{B}$, let $s_i(B)$ denote the number of shared vertices of degree i in B for $i = 2, 3, 4$ and $s(B)$ be the total number of all shared vertices in B.*

It is easy to see that G must have a star-block cover. In fact, after identifying the unique star-block for each vertex of degree at least 6, G_2 can be induced on the remaining vertices of degree at most 5 in G. Given a star-block cover $G = G_1 + G_2$ and a star-block base $\mathcal{B}$, we have $e(G) = w_0(G) = w_0(G_1) + w_0(G_2)$. Furthermore, let v be a shared vertex. If $d(v) = 4$, it can be checked that there is only one possibility, which is that v is shared by two subgraphs each induced on a 6-5 edge, and v is connected to each subgraph with exactly two edges. If $d(v) = 3$, there are two scenarios: it is shared by either three or two star-blocks. Here we use $s'_3(B), s''_3(B)$ to denote the number of vertices of degree 3 in B shared by three or two star-blocks of $\mathcal{B}$, respectively. Obviously, $s_3(B) = s'_3(B) + s''_3(B)$.

Definition 2.4. *Let G have a star-block cover and a star-block base $\mathcal{B}$. For any $B \in \mathcal{B}$, the modified weight of B, denoted by $\boldsymbol{w(B)}$, is defined as*

$$\boldsymbol{w(B)} := w_0(B) + \frac{s_3(B)}{2} + \frac{s_4(B)}{4} + \mathbf{1}_{s'_3(B)},$$

where the characteristic function

$$\mathbf{1}_{s'_3(B)} = \begin{cases} 1 & \text{if } s'_3(B) \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For the sake of convenience in subsequent discussion, we categorize the star-blocks into three types:

- $\mathcal{B}_0 := \{B \in \mathcal{B} \mid s(B) = 0\}$,
- $\mathcal{B}_1 := \{B \in \mathcal{B} \mid s(B) \geq 1 \text{ and } s'_3(B) = 0\}$,
- $\mathcal{B}_2 := \{B \in \mathcal{B} \mid s(B) \geq 1 \text{ and } s'_3(B) \geq 1\}$.

We show that for any star-block B, $w(B)$ has the same upper bound.

Lemma 2.1. *Let G be an n-vertex $S_{3,4}$-free planar graph with $\delta(G) \geq 3$. If G has no 3-3 edge, then there exists a star-block cover $G = G_1 + G_2$ such that $\mathcal{B}$ is a base of G_1. For any star-block $B \in \mathcal{B}$, $w(B) \leq \frac{5}{2}v(B)$ if $s'_3(B) \geq 1$ and $w(B) < \frac{5}{2}v(B)$ if $s'_3(B) = 0$.*

By using this lemma, we can deduce the following lemma and identify the corresponding extremal graphs.

Lemma 2.2. *Let G be an n-vertex $S_{3,4}$-free connected planar graph with $\delta(G) \geq 3$. If G contains no copy of* 3-3 *edge, then $e(G) \leq \frac{5}{2}n$.*

This lemma is the key to prove our main result and will be proved in Section 4.

3. Proof of Lemma 2.1

For any vertex of degree at least 6 in G, we first determine the star-block containing it by checking in the order of single vertex of degree at least 7, 6-6 edge, maximal alternating 6-4-6 path, single vertex of degree 6. Then we obtain a star-block cover $G = G_1 + G_2$ and a star-block base $\mathcal{B}$. We show that each star-block $B \in \mathcal{B}$ or its variation satisfies the corresponding upper bound.

Now we consider each star-block in turn. For the sake of simplicity, we use s, s_3, s_4, s_3', s_3'' to replace $s(B), s_3(B), s_4(B), s_3'(B), s_3''(B)$, respectively. Recall that $s_3 = s_3' + s_3''$.

Case 1. B contains a vertex of degree at least 7.

Let u be the vertex of the maximum degree in B.

Case 1.1. B is a 8^+-3 star.

If there is a vertex $v \in N(u)$ with $d(v) \geq 4$, we find an $S_{3,4}$ easily. So all neighbors of u have degree 3. Since G contains no 3-3 edge, there does not exist any edge between the vertices in $N(u)$. Hence

$$\begin{aligned} w(B) &\leq \frac{1}{2}\sum_{v\in B} d(v) + \frac{1}{2}\cdot d(u) + \mathbf{1}_{s_3'} \\ &\leq \frac{1}{2}(d(u) + 3\cdot d(u)) + \frac{d(u)}{2} + 1 \\ &< \frac{5}{2}\cdot (d(u)+1). \end{aligned}$$

Case 1.2. B is a 7-3^+ star.

If $v \in N(u)$ with $d(v) \geq 4$, then we have $N(v)\backslash\{u\} \subset N(u)\backslash\{v\}$. Otherwise, an $S_{3,4}$ is found.

Let T_1, T_2 denote the sets of the vertices of degree 3 in $N(u)$ that have exactly one or two neighbors in $G\backslash B$, respectively. Let $t_1 = |T_1|$ and $t_2 = |T_2|$.

Note that for any vertex $v \in N(u)\backslash\{T_1 \cup T_2\}$, it can be checked that $N(v)\backslash\{u\} \subset N(u)$. Hence we have

$$\begin{aligned} w(B) &\leq (3(8 - t_1 - t_2) - 6) + (2t_1 + t_2 + \frac{1}{2}t_1 + t_2) + \frac{1}{2}(t_1 + t_2) + \mathbf{1}_{s_3'} \\ &= 18 - \frac{1}{2}t_2 + \mathbf{1}_{s_3'} \\ &\leq 19 < \frac{5}{2}\cdot 8. \end{aligned}$$

Case 2. B contains a 6-6 edge.

Let u, v be the two adjacent vertices of degree 6. There exist at least 4 triangles sitting on uv, otherwise we find an $S_{3,4}$.

Case 2.1. There are five common neighbors of u, v.

Let $S_1 = \{a_1, a_2, a_3, a_4, a_5\}$, where the vertices are common neighbors of u and v, as shown in Figure 1(a). Let $H_1 = G[S_1]$. It is easy to see that $V(B) = S_1 \cup \{u, v\}$.

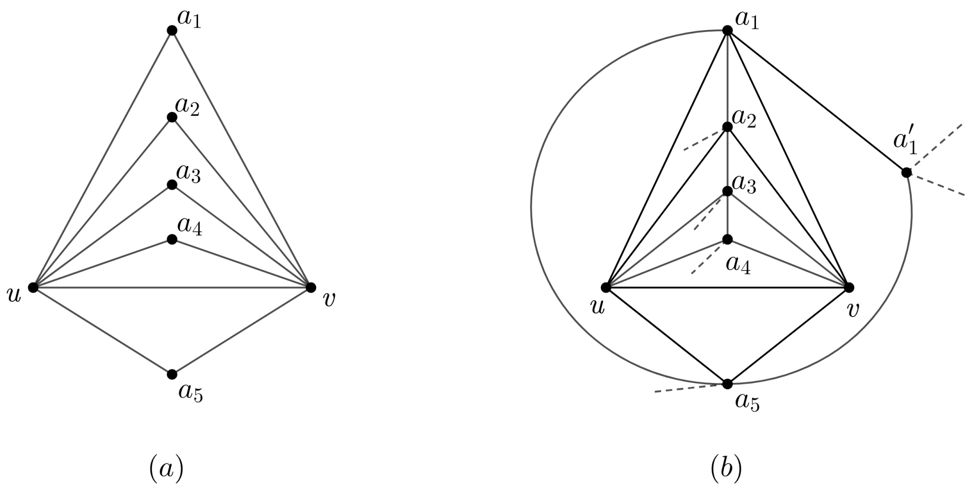

Figure 1. (a) A 6-6 edge uv with five common neighbors. (b) $G[B]$ is the maximal planar graph on 7 vertices.

Note that the vertices in S_1 can form a path of length at most 4 and each vertex in S_1 has at most one neighbor in $V(G)\backslash B$, otherwise an $S_{3,4}$ is found. It follows that $e(H_1) \leq 4$ and $e[H_1, G\backslash B] \leq 5$. Moreover, for any $x \in S_1$, if x is a shared vertex, we have $d(x) = 3$. Otherwise, there also exists an $S_{3,4}$. This means $s_4 = 0$. Furthermore, the shared vertices are covered by exactly two star-blocks. Thus, $s_3' = 0$ and $\mathbf{1}_{s_3'} = 0$.

Hence, we have

$$\begin{aligned} w(B) &= w_0(B) + \frac{s_3}{2} + \frac{s_4}{4} + \mathbf{1}_{s_3'} \\ &= (3 \cdot (7 - s_3) - 6 + \frac{5 - s_3}{2}) + (2s_3 + \frac{s_3}{2}) + \frac{s_3}{2} \\ &= 17\frac{1}{2} - \frac{s_3}{2} \\ &\leq \frac{5}{2} \cdot 7. \end{aligned}$$

The equality holds when $s_3(B) = 0$. Besides, $G[B]$ is the maximal planar graph on 7 vertices and each vertex in S_1 has exactly one neighbor outside. A more detailed discussion demonstrates the existence of a variation of this 6-6 edge star-block which has a strictly smaller weight except that there exists a vertex of degree 3 shared by three star-blocks.

Note that $d(a_1) = d(a_2) = d(a_3) = 5$. There is an edge $a_1a_1' \in E(G)$, where a_1' is a vertex in $G\backslash B$. It is clear that $d(a_1') \leq 4$, otherwise an $S_{3,4}$ is found. Let $B' = B + a_1'$ and replace B by B'. Here, a_1' is at the distance 2 from u and B' should be considered as a variation of B. Moreover, it does not affect our discussion, so we still refer to B' as a star-block. Unless it causes ambiguity, we will not explain it further. Then we obtain a new star-block base, denoted by $\mathcal{B}'$.

If $d(a'_1) = 4$, then either $a'_1 a_2$ or $a'_1 a_5$ is an edge. Otherwise, we find an $S_{3,4}$. The subgraph is shown in Figure 1(b). Without loss of generality, we assume that $a'_1 a_5 \in E(G)$. Then $w(B') = w_0(B) + \frac{d(a'_1)}{2} + \frac{s_4(B')}{4} = 17\frac{1}{2} + \frac{4}{2} + \frac{1}{4} = 19\frac{3}{4} < \frac{5}{2} \cdot 8$.

If $d(a'_1) = 3$ and $d_{\mathcal{B}'}(a'_1) \leq 2$, then it follows that $w(B') = 17\frac{1}{2} + \frac{3}{2} + \frac{1}{2} = 19\frac{1}{2} < \frac{5}{2} \cdot 8$.

If $d(a'_1) = 3$ and $d_{\mathcal{B}'}(a'_1) = 3$, a'_1 is not adjacent to any vertex in B. There are two vertices a'_2, a'_5 such that $a_2 a'_2, a_5 a'_5 \in E(G)$. Let $B'' = B' + a'_2 + a'_5$ and $\mathcal{B}''$ be the new base. Through a simple discussion of the different degrees of vertices a'_2 and a'_5, we can deduce that $w(B'') \leq 19\frac{1}{2} + \mathbf{1}_{s'_3(B'')} + \frac{4}{2} \cdot 2 + \frac{1}{4} \cdot 2 = \frac{5}{2} \cdot 10$.

Case 2.2. There are four common neighbors of u, v.

Let $S_1 = \{a_1, a_2, a_3, a_4\}$, where the vertices are these four common neighbors. Let b_1 be the vertex only adjacent to u and b_2 be the vertex only adjacent to v, as shown in Figure 2(a).

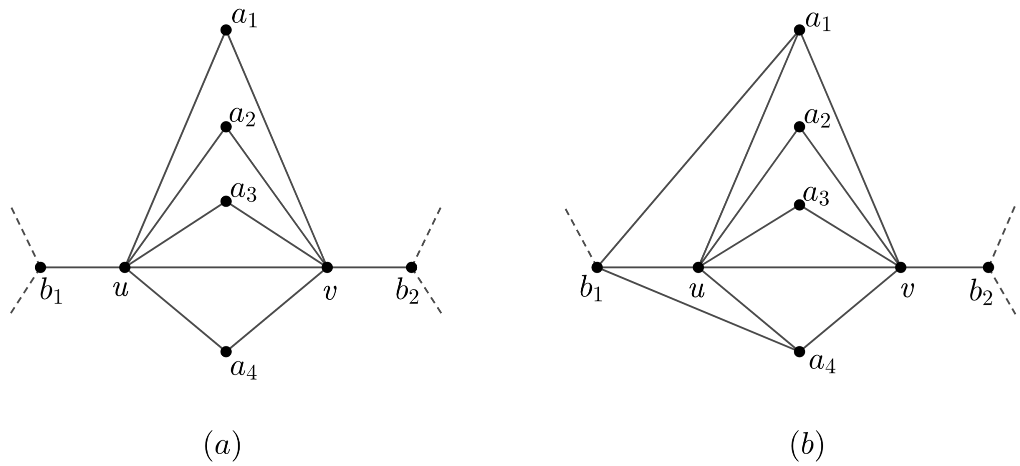

Figure 2. (a) A 6-6 edge uv with four common neighbors. (b) $d(b_1) = 4, d(b_2) = 3$.

Similarly, if some vertex in S_1 is shared, then the degree of the vertex must be equal to 3 by $\delta(G) \geq 3$. Moreover, if some vertex in S_1 is adjacent to either b_1 or b_2, then this vertex cannot have any neighbor in $G \backslash B$. It can be also checked that $d(b_1), d(b_2) \leq 4$, otherwise G contains an $S_{3,4}$. We use s_0 to denote the number of shared vertices in S_1.

(I) $d(b_1) = d(b_2) = 3$.

(i) $d_{\mathcal{B}}(b_1) = 3, d_{\mathcal{B}}(b_2) \leq 3$.

Note that each vertex in S_1 has at most one neighbor in $G \backslash B$ and is not adjacent to b_1. Hence, $d(a_1), d(a_2) \leq 5$ and $d(a_3), d(a_4) \leq 4$. It follows that $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0+2}{2} + \mathbf{1}_{s'_3} \leq \frac{36-s_0}{2} + \frac{s_0+2}{2} + 1 = \frac{5}{2} \cdot 8$.

(ii) $d_{\mathcal{B}}(b_1) = 2, d_{\mathcal{B}}(b_2) \leq 2$.

The vertex b_1 has at least one neighbor outside. If b_1 has no neighbor in S_1, then $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0+2}{2} \leq \frac{36-s_0}{2} + \frac{s_0+2}{2} = 19 < \frac{5}{2} \cdot 8$. Assume b_1 has exactly one neighbor in S_1. If $N(b_1) \cap N(b_2) \cap S_1 = \emptyset$, then the degrees of vertices in S_1 are discussed in a similar way. We have $w(B) \leq \frac{36-s_0}{2} + \frac{s_0+2}{2} < \frac{5}{2} \cdot 8$. If $N(b_1) \cap N(b_2) \cap S_1 \neq \emptyset$, then any vertex in S_1 can be the common neighbor of b_1, b_2. For all these subcases, it can be checked that $\sum_{i=1}^{4} d(a_i) \leq 19$. Hence, we have $w(B) \leq \frac{37-s_0}{2} + \frac{s_0+2}{2} = 19\frac{1}{2} < \frac{5}{2} \cdot 8$.

(iii) $d_{\mathcal{B}}(b_1) = d_{\mathcal{B}}(b_2) = 1$.

If b_1, b_2 have no common neighbor in S_1, then $\sum_{i=1}^{4} d(a_i) \leq 19$. Otherwise, it is obtained that $\sum_{i=1}^{4} d(a_i) \leq 20$. Thus, we have $w(B) \leq \frac{38-s_0}{2} + \frac{s_0}{2} = 19 < \frac{5}{2} \cdot 8$.

(II) $d(b_1) = 4, d(b_2) = 3$.

Note that b_1 is not shared and b_1 has two neighbors, say a_1, a_4, in S_1, as shown in Figure 2(b). Moreover, a_1, a_4 have no neighbor in $G \backslash B$, otherwise, an $S_{3,4}$ is found.

(i) $d_{\mathcal{B}}(b_2) = 3$.

Since the vertex b_2 has no neighbor in S_1, we have $\sum_{i=1}^{4} d(a_i) \leq 18$. Then $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0+1}{2} + \mathbf{1}_{s_3'} \leq \frac{37-s_0}{2} + \frac{s_0+1}{2} + 1 = \frac{5}{2} \cdot 8$.

(ii) $d_{\mathcal{B}}(b_2) = 2$.

The vertex b_2 may be adjacent to some vertex in S_1. Then we obtain that $\sum_{i=1}^{4} d(a_i) \leq 19$. It follows that $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0+1}{2} \leq \frac{38-s_0}{2} + \frac{s_0+1}{2} = 19\frac{1}{2} < \frac{5}{2} \cdot 8$.

(iii) $d_{\mathcal{B}}(b_2) = 1$.

Similarly, we have $\sum_{i=1}^{4} d(a_i) \leq 20$. Hence, $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0}{2} \leq \frac{39-s_0}{2} + \frac{s_0}{2} = 19\frac{1}{2} < \frac{5}{2} \cdot 8$.

(III) $d(b_1) = d(b_2) = 4$.

Note that b_1, b_2 are not shared now and each has two neighbors in S_1. We show that $\sum_{i=1}^{4} d(a_i) \leq 20$. It follows $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0}{2} \leq \frac{40-s_0}{2} + \frac{s_0}{2} = \frac{5}{2} \cdot 10$.

Now we show that there also exists a variation of this star-block such that weight of the variation satisfies the upper bound. Here, the equality holds when b_1, b_2 have two common neighbors in S_1. There are two possible planar embeddings, as shown in Figure 3.

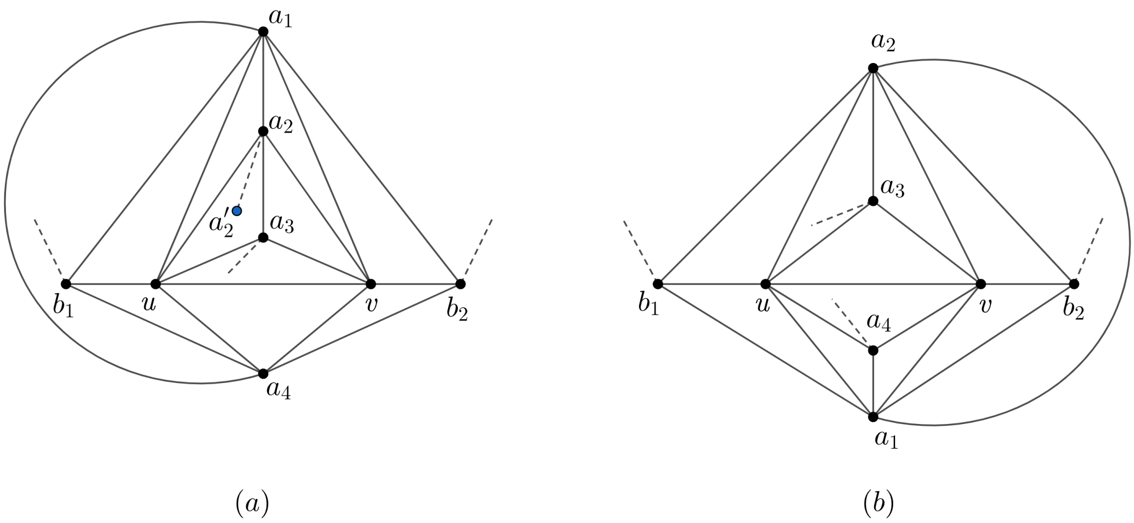

Figure 3. Two possible planar embeddings.

For the first planar embedding, a_1, a_4 are these two common vertices and $d(a_2) = 5$. There is an edge $a_2a_2' \in E(G)$. It is easy to see that $d(a_2') \leq 4$. Let $B' = B + a_2'$ and $\mathcal{B}'$ be the new base by replacing B by B'.

If $d(a_2') = 4$, then $a_2'a_3$ is also an edge in G. It follows that $w(B') \leq w_0(B) + \frac{d(a_2')}{2} + \frac{1}{4} = 22\frac{1}{4} < \frac{5}{2} \cdot 9$.

If $d(a_2') = 3$ and $a_2'a_3 \in E(G)$, it follows that $w(B') \leq w_0(B) + \frac{d(a_2')}{2} + \frac{1}{2} < \frac{5}{2} \cdot 9$.

Assume that $d(a_2') = 3$ and $a_2'a_3 \notin E(G)$. If $d_{\mathcal{B}'}(a_2') \leq 2$, then $w(B') \leq w_0(B) + \frac{d(a_2')}{2} + \frac{1}{2} < \frac{5}{2} \cdot 9$.

If $d_{\mathcal{B}'}(a_2') = 3$, then a_2' is not adjacent to a_3. There exist edges $a_3a_3', b_1b_1' \in E(G)$ such that $d(a_3'), d(b_1') \leq 4$. Let $B'' = B' + a_3' + b_1'$ and $\mathcal{B}''$ be the new base by replacing B' by B''. We have $w(B'') \leq w(B') + \frac{4+4}{2} + \frac{2}{4} = \left(w(B) + \frac{d(a_2')}{2} + \frac{1}{2} + \mathbf{1}_{s_3'(B')}\right) + \frac{4+4}{2} + \frac{2}{4} = \frac{5}{2} \cdot 11$.

For the second planar embedding, a_1, a_2 are the two common neighbors and $d(a_3) = d(a_4) = 4$. There are two different vertices a_3', a_4' in $G \backslash B$ such that $a_3a_3', a_4a_4' \in E(G)$. It can be shown that $d(a_3'), d(a_4') \leq 4$. Otherwise an $S_{3,4}$ is contained. Note that a_3' is not adjacent to other vertices in B. If $d(a_3') = 4$, then a_3' can not be shared. By symmetry, if $d(a_4') = 4$, it is not shared either. Let $B' = B + a_3' + a_4'$ and $\mathcal{B}'$ be the new base by replacing B by B'.

If $d(a_3') = d(a_4') = 4$, then $w(B') \leq w_0(B) + \frac{d(a_3')+d(a_4')}{2} = 24 < \frac{5}{2} \cdot 10$.

If $d(a_3') = 3$ and $d(a_4') = 4$, then $w(B') \leq w_0(B) + \frac{d(a_3')+d(a_4')}{2} + \frac{1}{2} + \mathbf{1}_{s_3'} \leq \frac{5}{2} \cdot 10$.

If $d(a_3') = d(a_4') = 3$, then $w(B') \leq w_0(B) + \frac{d(a_3')+d(a_4')}{2} + \frac{2}{2} + \mathbf{1}_{s_3'} \leq \frac{5}{2} \cdot 10$.

Case 3. B contains a maximal alternating 6-4-6 path.

For a 6-4-6 path, there are four possible planar embeddings, as shown in Figure 4. Let u, v, w be the three vertices with $d(u) = d(w) = 6$.

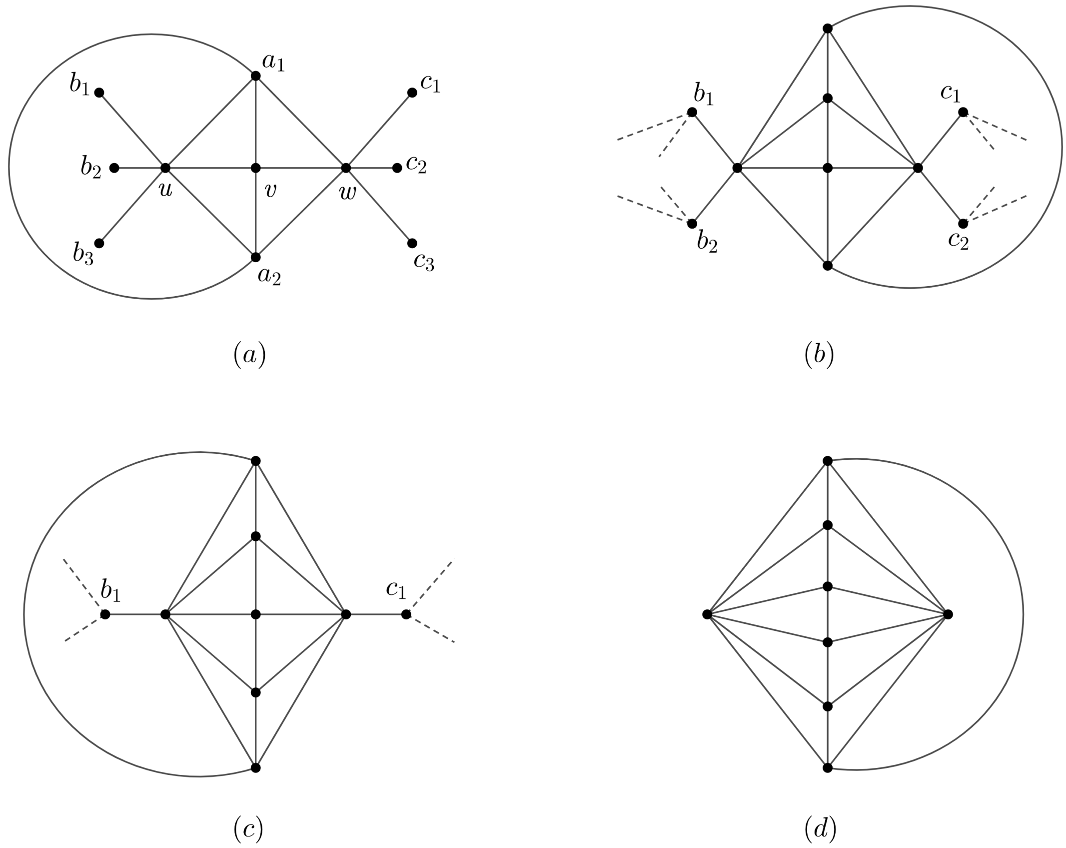

Figure 4. Four planar embeddings for 6-4-6 path.

For the first planar embedding, let a_1, a_2 be the remaining common neighbors of u and w. Note that a_1, a_2 cannot have any neighbor in $G\backslash B$. It is possible that $a_1a_2 \in E(G)$.

A maximal alternating 6-4-6 path constructed by this planar embedding is shown in Figure 5. For $i = 1, 2, 3$, let b_i, c_i be the vertices only adjacent to the ends of the path. Note that for such vertices, if the degree is 4, each vertex has at most one neighbor in $G\backslash B$ and cannot be shared. Moreover, the vertex, say b_1, must be adjacent to both b_2 and b_3. Let s_0 be the number of shared vertices among these six vertices.

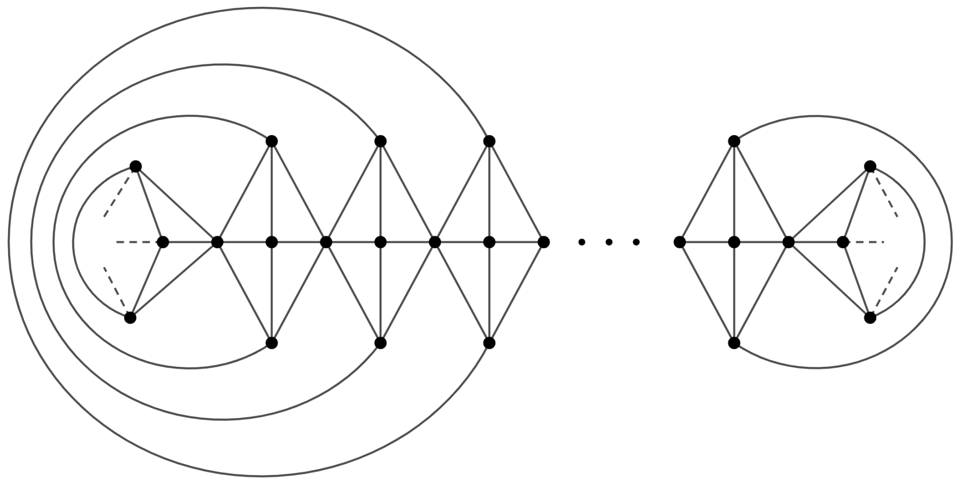

Figure 5. A maximal alternating 6-4-6 path.

Here we use k to denote the number of copies of 6-4-6 path contained in this star-block. It follows that $v(B) = 4k + 7$. Then $w(B) \leq 9k + 3 + (\frac{1}{2}\sum_{i=1}^{3}(b_i + c_i) - \frac{s_0}{2}) + \frac{s_0}{2} + 1 = 9k + 16 < \frac{5}{2} \cdot (4k + 7)$.

For the second planar embedding, let b_1, b_2, c_1, c_2 be the four vertices only adjacent to the ends of the path. Obviously, $d(b_1) = d(b_2) = d(c_1) = d(c_2) = 3$. However, $b_1b_2, c_1c_2 \notin E(G)$. Otherwise there exists a 3-3 edge. It can be obtained that $w(B) \leq \frac{1}{2}\sum_{v\in B} d(v) + \frac{1}{2} \cdot 4 + \mathbf{1}_{s_3'} = 23 < \frac{5}{2} \cdot 10$.

For the third planar embedding, let b_1, c_1 be the two vertices only adjacent to the ends of the path. We have $d(b_1) = d(c_1) = 3$ and $w(B) \leq \frac{1}{2}\sum_{v\in B} d(v) + \frac{1}{2} \cdot 2 + \mathbf{1}_{s_3'} = 21 < \frac{5}{2} \cdot 9$.

For the fourth planar embedding, it is a connected component on 8 vertices. It follows that $w(B) = w_0(B) \leq (3 \cdot 8 - 6) < \frac{5}{2} \cdot 8$.

Case 4. B is a 6-5^- star.

Without loss of generality, we assume that there does not exist any copy of 6-6 edge or 6-4-6 path in G.

Case 4.1. There exists $v \in N(u)$ such that $d(v) = 5$.

Let uv be the 6-5 edge and u be the vertex of degree 6. There exist at least 3 triangles sitting on the edge uv, otherwise an $S_{3,4}$ is found.

Case 4.1.1. There are four common neighbors of u, v.

Let $S_1 = \{a_1, a_2, a_3, a_4\}$ be the set of vertices which are adjacent to u, v. Let b_1 be the vertex only adjacent to u, as shown in Figure 6(a). Obviously, $d(b_1) \leq 4$. Otherwise, G contains an $S_{3,4}$. We also use s_0 to denote the number of shared vertices in S_1.

(I) $d(b_1) = 3$.

If $d_{\mathcal{B}}(b_1) = 3$, then b_1 has no neighbor in S_1. This implies $\sum_{i=1}^{4} d(a_i) \leq 5 + 5 + 4 + 4 = 18$. Hence, $w(B) = \frac{1}{2}\sum_{v\in B} d(v) + \frac{s_0+1}{2} + \mathbf{1}_{s_3'} \leq \frac{32-s_0}{2} + \frac{s_0+1}{2} + 1 = \frac{5}{2} \cdot 7$.

If $d_{\mathcal{B}}(b_1) = 2$, the vertex b_1 can have at most one neighbor in S_1. Then we have $\sum_{i=1}^{4} d(a_i) \leq 19$. Hence, $w(B) = \frac{1}{2} \sum_{v \in B} d(v) + \frac{s_0+1}{2} \leq \frac{33-s_0}{2} + \frac{s_0+1}{2} = 17 < \frac{5}{2} \cdot 7$.

If $d_{\mathcal{B}}(b_1) = 1$, then b_1 may have two neighbors in S_1. This means that $\sum_{i=1}^{4} d(a_i) \leq 20$. It follows that $w(B) \leq \frac{34-s_0}{2} + \frac{s_0}{2} = 17 < \frac{5}{2} \cdot 7$

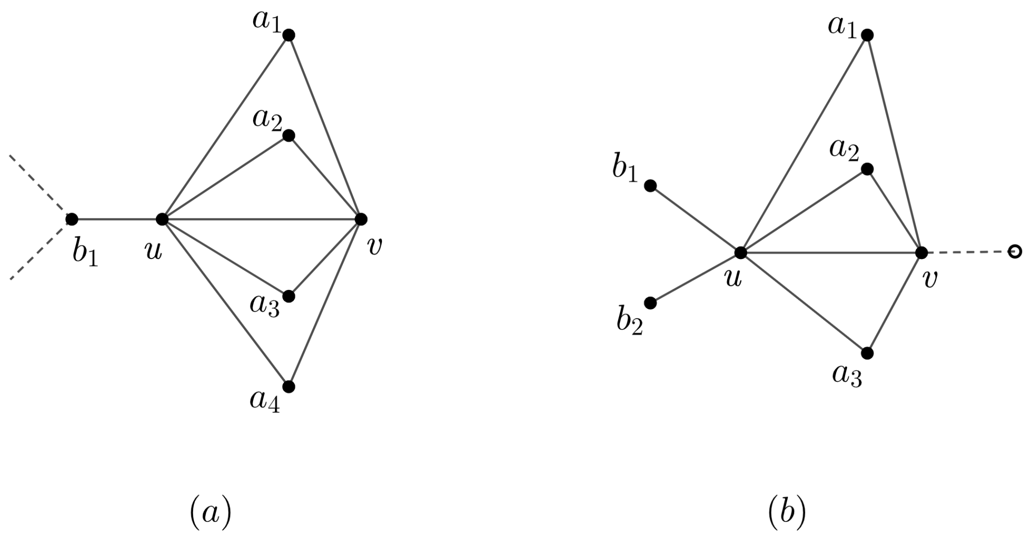

Figure 6. (a) A 6-5 star with four common neighbors; (b) A 6-5 star with three common neighbors.

(II) $d(b_1) = 4$.

The vertex b_1 has exactly two neighbors in S_1 and b_1 is not shared. Since there does not exist a 6-6 edge, all neighbors of u have degree at most 5. Furthermore, if $d(a_3) = 5$, b_1 is adjacent to a_3. By symmetry, a_3, a_2 cannot both be vertices of degree 5. This means $\sum_{i=1}^{4} d(a_i) \leq 19$. It follows that $w(B) \leq \frac{34-s_0}{2} + \frac{s_0}{2} < \frac{5}{2} \cdot 7$.

Case 4.1.2. There are three common neighbors of u, v.

Let $S_1 = \{a_1, a_2, a_3\}$ be the set of vertices which are adjacent to u, v and $S_2 = \{b_1, b_2\}$ be the set of vertices only adjacent to u, as shown in Figure 6(b). Note that each vertex in S_1 has at most one neighbor in $G \backslash N(u)$. If $d(b_1) \geq 4$, b_1 can have at most one neighbor in $G \backslash N(u)$. The vertex b_2 is the same as well. This means $s_4(B) = 0$. If some vertex is shared by three star-blocks, then this vertex belongs to S_2.

Since there is no 6-6 edge, we have $\sum_{i=1}^{3} d(a_i) \leq 15$. A simple discussion reveals that $\sum_{i=1}^{2} d(b_i) \leq 9$, with equality when $b_1b_2 \in E(G)$. Assume that $\sum_{i=1}^{3} d(a_i) = 15$. This implies $d(a_1) = d(a_2) = d(a_3) = 5$. Then b_1, b_2 must be in different regions and $b_1b_2 \notin E(G)$. Thus, $\sum_{i=1}^{3} d(a_i) + \sum_{j=1}^{2} d(b_j) \leq 23$.

If there is no shared vertex in B, we have $s_3(B) = 0$. It follows $w(B) = \frac{1}{2}(\sum_{i=1}^{3} d(a_i) + \sum_{j=1}^{2} d(b_j) + d(u) + d(v)) \leq 17 < \frac{5}{2} \cdot 7$.

Now assume that $s_3(B) > 0$. Let s_a, s_b denote the number of shared vertices in S_1, S_2 respectively.

If a_1 is a shared vertex, then $d(a_1) = 3$ and $d_{\mathcal{B}}(a_1) \leq 2$. We have $a_1a_2, a_1a_3 \notin E(G)$. This implies $d(a_2)+d(a_3) \leq 9-s_a$ and $\sum_{j=1}^{2} d(b_j) \leq 8-s_b$. It follows that $w(B) \leq \frac{1}{2} \sum_{x\in B} d(x)+\frac{1}{2}(s_a+s_b)+1_{s_3'(B)} = \frac{33}{2} < \frac{5}{2}\cdot 7$.

If a_2 is a shared vertex, $d(a_2) = 3$ and $a_1a_2 \notin E(G)$. If $s_3'(B) = 0$, we have $d(a_1) + d(a_3) \leq 11 - s_a$ and $\sum_{j=1}^{2} d(b_j) \leq 9 - s_b$. Thus $w(B) \leq \frac{1}{2} \sum_{x\in B} d(x) + \frac{1}{2}(s_a + s_b) = 17 < \frac{5}{2} \cdot 7$. If $s_3'(B) \neq 0$, assume that $d(b_1) = 3$ and $d_{\mathcal{B}}(b_1) = 3$. It is easy to see that $d(a_1) + d(a_3) \leq 11 - s_a$ and $d(b_2) \leq 5 - s_b$. We have $w(B) \leq \frac{1}{2} \sum_{x\in B} d(x) + \frac{1}{2}(s_a + s_b) + 1_{s_3'(B)} = \frac{5}{2} \cdot 7$.

By the symmetry of a_2 and a_3, we assume $s_a = 0$. If $d_{\mathcal{B}}(b_1) = 2$, then $d(b_1) = 3$. Furthermore, if $d(b_2) = 5$, we have $b_1b_2 \in E(G)$. Then either a_2 or a_3 has degree at most 4. Hence, $\sum_{i=1}^{3} d(a_i) + \sum_{j=1}^{2} d(b_j) \leq 22$. It follows that $w(B) \leq \frac{33}{2} + \frac{1}{2} < \frac{5}{2} \cdot 7$.

If $d_{\mathcal{B}}(b_1) = 3$, then $d(b_2) \leq 4$ and either a_2 or a_3 has degree at most 4. Hence $\sum_{i=1}^{3} d(a_i)+\sum_{j=1}^{2} d(b_j) \leq 21$. We have $w(B) \leq \frac{1}{2} \sum_{x\in B} d(x) + \frac{1}{2}s_b + 1_{s_3'(B)} = \frac{5}{2} \cdot 7$

Case 4.2. For any $v \in N(u)$, we have $d(v) \leq 4$.

If some vertex $v \in N(u)$ has degree 4, then v is not shared. Otherwise, a 6-4-6 path is found. Let p, q be the number of vertices of degree 3 or 4 in $N(u)$, respectively. Obviously, $p + q = 6$. It is obtained that $w(B) \leq \frac{1}{2}(6 + 3p + 4q) + \frac{p}{2} + 1_{s_3'} = 3 + 2(p + q) + 1_{s_3'} = 15 + 1_{s_3'} < \frac{5}{2} \cdot 7$.

In summary, $w(B)$ satisfies the upper bound.

4. Proof of Lemma 2.2 and Theorem 1.1

In this section, we first prove Lemma 2.2, and then provide the proof of upper bound in Theorem 1.1.

Proof. By Lemma 2.1, there exists a star-block cover $G = G_1 + G_2$ with a base $\mathcal{B}$ of G_1, where each star-block satisfies the corresponding upper bound. For each vertex of degree at least 6, it must be contained in some star-block. Recall that the degree we discuss here is the degree of this vertex in G. Then the maximum degree of vertices in G_2 is at most 5. We have

$$\begin{aligned} e(G) = w_0(G) &= w_0(G_1) + w_0(G_2) \\ &\leq w_0(G_1) + \frac{5}{2}v(G_2). \end{aligned}$$

Next we prove that $w_0(G_1) < \frac{5}{2}v(G_1)$.

Given this star-block base $\mathcal{B}$, let

- $p_1 := \left|\{v \in V(G_1) : d_{\mathcal{B}}(v) = 2 \ and \ d(v) = 3\}\right|$,
- $p_2 := \left|\{v \in V(G_1) : d_{\mathcal{B}}(v) = 3\}\right|$,
- $p_3 := \left|\{v \in V(G_1) : d_{\mathcal{B}}(v) = 2 \ and \ d(v) = 4\}\right|$.

If there is a vertex v with $d_{\mathcal{B}}(v) = 3$, the degree of vertex v will be accumulated three times in the summation of the expression $\sum_{B\in\mathcal{B}} w_0(B)$. If there is a vertex v with $d_{\mathcal{B}}(v) = 2$, the degree of vertex v will be accumulated two times.

Thus, it can be obtained that

$$\sum_{B\in\mathcal{B}} w_0(B) = w_0(G_1) + \frac{3}{2}p_1 + 3p_2 + 2p_3. \tag{4.1}$$

Now let $q_0 = |\mathcal{B}_0|$, $q_1 = |\mathcal{B}_1|$, $q_2 = |\mathcal{B}_2|$. If $q_2 \neq 0$, then $q_2 \geq 3$. By the definition of $w(B)$, we have

$$\begin{aligned}\sum_{B\in\mathcal{B}} w_0(B) &= \sum_{B\in\mathcal{B}} (w(B) - \frac{s_3}{2} - \frac{s_4}{4} - \mathbf{1}_{s_3'}) \\ &= \sum_{B\in\mathcal{B}} w(B) - \sum_{B\in\mathcal{B}} (\frac{s_3}{2} + \frac{s_4}{4} + \mathbf{1}_{s_3'}).\end{aligned}$$

Based on the relation of shared vertices and the corresponding star-blocks, we also obtain

$$\sum_{B\in\mathcal{B}} (\frac{s_3}{2} + \frac{s_4}{4} + \mathbf{1}_{s_3'}) = p_1 + \frac{3}{2}p_2 + \frac{1}{2}p_3 + q_2.$$

By Lemma 2.1, it is concluded that if $q_0 + q_1 > 0$, then

$$\begin{aligned}\sum_{B\in\mathcal{B}} w_0(B) &= \sum_{B\in\mathcal{B}_0\cup\mathcal{B}_1} w(B) + \sum_{B\in\mathcal{B}_2} w(B) - (p_1 + \frac{3}{2}p_2 + \frac{1}{2}p_3 + q_2) \\ &< \frac{5}{2}\sum_{B\in\mathcal{B}} v(B) - (p_1 + \frac{3}{2}p_2 + \frac{1}{2}p_3 + q_2) \\ &= \frac{5}{2}(v(G_1) + p_1 + 2p_2 + p_3) - (p_1 + \frac{3}{2}p_2 + \frac{1}{2}p_3 + q_2) \\ &= \frac{5}{2}v(G_1) + \frac{3}{2}p_1 + \frac{7}{2}p_2 + 2p_3 - q_2.\end{aligned}$$

Combining Eq (4.1), we have

$$w_0(G_1) < \frac{5}{2}v(G_1) + (\frac{1}{2}p_2 - q_2).$$

Note that if $q_2 = 0$, then $p_2 = 0$. Now, we show that $p_2 \leq 2q_2 - 4$ when $q_2 \neq 0$. Given the star-block base $\mathcal{B}$, we construct a bipartite graph $G^* = (X, Y)$ such that $|X| = p_2$ and $|Y| = q_2$, where a vertex x in X represents a vertex v in G_1 with $d_{\mathcal{B}}(v) = 3$ and a vertex y in Y represents a star-block in $\mathcal{B}_2$. Additionally, for $x \in X$ and $y \in Y$, xy is an edge if and only if x is contained in the star-block y. Since G is a planar graph, this auxiliary bipartite graph G^* is a planar graph too. We have $e(G^*) \leq 2v(G^*) - 4$. Note that $d_{G^*}(x) = 3$ for each vertex x in X. It follows that $3p_2 \leq 2(p_2+q_2)-4$. This means $p_2 \leq 2q_2-4$.

Thus, we have

$$w_0(G_1) < \frac{5}{2}v(G_1) - 2 \cdot \mathbf{1}_{q_2} \leq \frac{5}{2}v(G_1),$$

where $\mathbf{1}_{q_2}$ is the characteristic function of q_2.

If $q_0 + q_1 = 0$, then $q_2 > 0$. It is obtained

$$\begin{aligned}\sum_{B\in\mathcal{B}} w_0(B) &= \sum_{B\in\mathcal{B}_2} w(B) - (p_1 + \frac{3}{2}p_2 + \frac{1}{2}p_3 + q_2)\\ &\leq \frac{5}{2}\sum_{B\in\mathcal{B}_2} v(B) - (p_1 + \frac{3}{2}p_2 + \frac{1}{2}p_3 + q_2)\\ &= \frac{5}{2}v(G_1) + \frac{3}{2}p_1 + \frac{7}{2}p_2 + 2p_3 - q_2.\end{aligned}$$

Similarly,

$$\begin{aligned}w_0(G_1) &\leq \frac{5}{2}v(G_1) + (\frac{1}{2}p_2 - q_2)\\ &\leq \frac{5}{2}v(G_1) - 2\\ &< \frac{5}{2}v(G_1).\end{aligned}$$

In summary, $w_0(G_1) < \frac{5}{2}v(G_1)$. Therefore we have

$$\begin{aligned}e(G) &= w_0(G_1) + w_0(G_2)\\ &\leq w_0(G_1) + \frac{5}{2}v(G_2)\\ &\leq \frac{5}{2}v(G_1) + \frac{5}{2}v(G_2)\\ &= \frac{5}{2}v(G).\end{aligned}$$

The proof is completed. □

We next give the proof of the main result and construct the extremal graphs.

Proof. Let G be an $S_{3,4}$-free connected planar graph. We give two operations as follows: (α). Delete the 3-3 edge; (β). Delete the vertex of degree at most 2. Repeat the operations until it can no longer go on. The final graph is denoted by G'.

Here, G' is either an empty graph or an $S_{3,4}$-free planar graph satisfying the condition of Lemma 2.2. Without loss of generality, we assume that G' is connected. Otherwise, we can discuss each connected component, respectively. Hence $e(G') = 0$ or $e(G') \leq \frac{5}{2}v(G')$.

Therefore, it is easy to see that $e(G) \leq \frac{5}{2}v(G)$. □

Now we shall complete it by demonstrating that this bound is tight for infinitely many integers n.

A detailed examination of the proof of Theorem 1.1 and Lemma 2.2 reveals that equality is achieved when G_1 is an empty graph and $G = G_2$ is a 5-regular planar graph.

When $n = 12$, there is only one 5-regular planar graph, as shown in Figure 7. It is easy to see that the graph is $S_{3,4}$-free. When $n \equiv 0 \pmod{12}$, there exists an n-vertex planar graph with $12|n$ connected components, where each component is a 5-regular planar graph on 12 vertices.

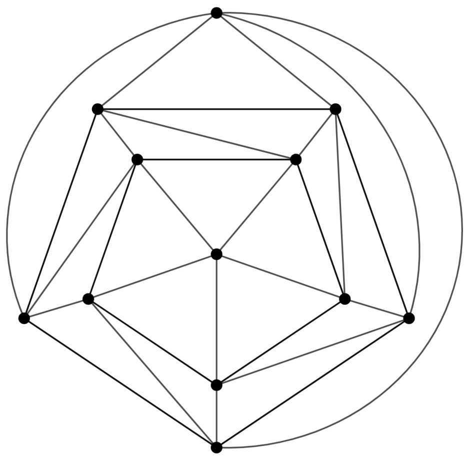

Figure 7. The extremal planar graph on 12 vertices.

5. Conclusions

In this paper, we study the planar Turán number for double stars, specifically focusing on $S_{3,4}$. Our primary contribution is the determination of the exact value of $ex_{\mathcal{P}}(n, S_{3,4})$. Moreover, we construct the extremal graphs that attain this bound.

Author contributions

Xin Xu: Conceptualization, Investigation, Writing – review and editing; Xu Zhang, Jiawei Shao: Investigation, Writing – original draft, Writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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