AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 1589–1601.
DOI: 10.3934/math.2025073
Received: 24 September 2024
Revised: 02 December 2024
Accepted: 16 December 2024
Published: 23 January 2025

Research article

Nonlinear neutral differential equations of second-order: Oscillatory properties

Maged Alkilayh*

Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: m.alkilayh@qu.edu.sa.

Abstract: This paper investigates the oscillatory behavior of second-order differential equations featuring a mixed neutral term along with a p-Laplace differential operator. The analysis employs Riccati transformations and includes a comparative study with first-order equations, which facilitates the development of oscillation criteria. The findings culminate in a significant theorem that addresses the oscillation properties of these equations. In addition, two illustrative examples are presented to demonstrate the practical application of the established criteria.

Keywords: mixed neutral; differential equations; oscillation conditions; second-order
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

Oscillation theory is a vital field of mathematics dedicated to exploring the dynamics of both oscillatory and nonoscillatory systems. This theory investigates how systems consistently fluctuate around a central value or shift between various states. Such oscillatory behavior is widespread in the natural world and in technological applications, manifesting itself in various areas such as mechanical, electrical, and biological systems.

For example, in mechanical systems, oscillation theory is essential for analyzing the stability of electrical circuits, where periodic fluctuations can affect performance. Similarly, in ecological models, it provides insight into population dynamics, helping to understand how species interact and change over time. This makes oscillation theory a crucial tool for both theoretical analysis and practical applications in multiple disciplines; see [1, 2]. In this work, we focus our attention on the oscillation of the second-order nonlinear differential equation with the form

$$\left(f(s)\left|u'(s)\right|^{p-2}u'(s)\right)' + q(s)\left|w(h(s))\right|^{p-2}w(h(s)) = 0,\ s \geq s_0, \tag{1.1}$$

where the first and second terms mean the $p-$Laplace type operator $(1 < p < \infty)$ and $u(s) = w(s) + b_1(s) w(a_1(s)) + b_2(s) w(a_2(s))$. Throughout, we assume that:

(G_1) $b_1, b_2, q \in C([s_0, \infty), [0, \infty))$, $f \in C([s_0, \infty), (0, \infty))$, $\eta(s_0) = \int_{s_0}^{\infty} f^{-1/(p-1)}(\upsilon) \mathrm{d}\upsilon < \infty$.
(G_2) $a_1, a_2, h \in C([s_0, \infty), \mathbb{R}), a_1(s) \leq s, a_2(s) \geq s, h(s) \leq s, \lim_{s\to\infty} a_1(s) = \lim_{s\to\infty} a_2(s) = \lim_{s\to\infty} h(s) = \infty$ and $q(s)$ is not identically zero for large s.

By a solution of (1.1), we mean a nontrivial function $w(s) \in C^1([s_0, \infty), \mathbb{R})$, which has the properties $f(s)(u'(s))^{(p-1)} \in C^1([s_0, \infty), \mathbb{R})$, and $w(s)$ satisfies (1.1) on $[s_w, \infty)$, then u is a solution of (1.1). In our study, we focus on the solutions that satisfy $\sup\{|w(s)| : s_w \leq s < \infty\} > 0$, for every $s_w \geq s_0$. If the solution to (1.1) is neither eventually positive nor eventually negative in the end, it is said to be oscillatory. Otherwise, this solution is called non-oscillatory. Therefore, we say that this equation is oscillatory if all of its solutions are oscillatory.

The p-Laplace equations of the oscillatory behavior of differential delay equation issues have advanced significantly over the past few decades; see [3, 4]. It has proven to be quite helpful in the engineering and applied sciences fields in building a range of applied mathematical models that faithfully represent real-world events. neutral-type DDEs are involved in second-order equation delay systems issues, which were studied by some authors; see [5–7].

The theory of oscillatory conditions generally appears from a large adequate instance for use in drug organization, physics, engineering, trash hold hypothesis in biology, and airship control; see [8, 9]. The impulsive conditions play an important role in modeling phenomena, principally in relating the dynamics of the population area under discussion to unexpected modification. Exploring Laplace differential equations yields a multitude of essential utilities in mechanical systems, electrical circuits, and the regulation of chemical processes; see [10–12]. Furthermore, their utility extends to ecological systems, epidemiology, and the modeling of population dynamics, as detailed in references [13–16].

Investigations by some authors in [16–18] have yielded techniques and methodologies aimed at enhancing the oscillatory attributes of these equations. Furthermore, the work [19, 20] has expanded this inquiry to include differential equations of neutral variety. In recent years, there has also been a significant exploration of oscillation behaviors in fourth-order delay differential equations with the $p-$ Laplace type operator, as evidenced by studies such as [21–23]. Now, in some detail, Liu et al. [24] introduced good conditions concerning solutions to even-order differential equations that feature a mixed term in the canonical case.

$$\left(f(s)\left|u^{(n)}(s)\right|^{p-2} u^{(n)}(s)\right)' + r(s)\left|u^{(n)}(s)\right|^{p-2} u^{(n)}(s) + q(s)|w(h(s))|^{p-2} w(h(s)) = 0,$$

where

$$u(s) = w(s) + b(s) w(a(s)).$$

Graef et al. [7] improved some results for equations with mixed types.

$$(f(s)(w(s) + b_1(s) w(a_1(s)) + b_2(s) w(a_2(s)))')' + b(s) u'(s) + q(s) w(h(s)) = 0,$$

where

$$\int_{s_0}^{\infty} \frac{1}{f(s)} \exp\left(-\int_{s_0}^{s} \frac{b(\upsilon)}{f(\upsilon)} \mathrm{d}\upsilon\right) \mathrm{d}s = \infty.$$

The authors in [25,26] established new conditions to improve and extend some of the results for the oscillation of the equation of fourth-order with the $p-$ Laplace type operator

$$\left(f(s)|u'''(s)|^{p-2}u'''(s)\right)' + q(s)|w(h(s))|^{p-2}w(h(s)) = 0,$$

where $\int_{s_0}^{\infty} f^{-1/(p-1)}(\upsilon)\,d\upsilon = \infty$.

This manuscript aims to broaden the scope of inquiry by incorporating second-order differential equations with mixed neutral terms under condition $\eta(s_0) < \infty$, using some methods. In this context, the paper introduces innovative criteria for analyzing oscillatory solutions of equation (1.1). Three examples are shown to illustrate the results that we obtained.

2. Basic lemmas

In this section, we first introduce some important lemmas and notation.

Lemma 2.1. *[11] The equation*

$$\eta(\upsilon^*) \leq \max_{\varrho\in\mathbb{R}} \eta(\upsilon) = QL + \frac{(p-1)^{(p-1)}}{p^p} Q^p N^{-(p-1)}, N > 0,$$

holds, if $\eta(\upsilon) := Q\upsilon - N(\upsilon - L)^{p/(p-1)}$, *where* Q, N, *and* L *are real constants.*

Lemma 2.2. *[27] Let* F *and* ϖ *be constants. Then*

$$Fv - \varpi v^{p/(p-1)} \leq \frac{(p-1)^{(p-1)}}{p^p} \frac{F^p}{\varpi^{(p-1)}}, \ \varpi > 0. \tag{2.1}$$

The following notation will be used in the remaining sections of this work:

$$M(\varrho, v) = \int_{\varrho}^{v} f^{-1/(p-1)}(\xi)\,d\xi,$$

$$x_1(s) := 1 - b_1(s)\frac{M(a_1(s), \infty)}{M(s, \infty)} - b_2(s),$$

and

$$x_2(s) := 1 - b_1(s) - b_2(s)\frac{M(s_1, a_2(s))}{M(s_1, s)}.$$

3. Main results

In this section, we discuss our main results about (1.1).

Lemma 3.1. *The function* w *is identified as the ultimate positive solution to (1.1). Then, either*

$$\left(\frac{u(s)}{M(s,\infty)}\right)' \geq 0, \tag{3.1}$$

or

$$\left(\frac{u(s)}{M(s_1,s)}\right)' \leq 0, \ \textit{for all } s \geq s_1. \tag{3.2}$$

Proof. Given that $w(s)$ constitutes the final positive solution of (1.1). Obviously, for all $s \geq s_1$, $u(s) \geq w(s) > 0$ and $f(s)(u'(s))^{(p-1)}$ it is not increasing. Since

$$\left(f(s)(u'(s))^{(p-1)}\right)' = -q(s)\, w^{(p-1)}(h(s)) \leq 0,$$

therefore, u' is either eventually negative or eventually positive.

Assume first that $u' < 0$ on $[s, \infty)$. Since

$$u(s) \geq -\int_s^{\infty} f^{-1/(p-1)}(\upsilon)\, f^{1/(p-1)}(\upsilon)\, u'(\upsilon)\, \mathrm{d}\upsilon \geq -M(s, \infty)\, f^{1/(p-1)}(s)\, u'(s), \tag{3.3}$$

and so

$$\left(\frac{u(s)}{M(s, \infty)}\right)' = \frac{M(s, \infty)\, u'(s) + f^{-1/(p-1)}(s)\, u(s)}{(M(s, \infty))^2} \geq 0.$$

Assume now that $u' > 0$ on $[s_1, s)$, we obtain

$$u(s) \geq \int_{s_1}^{s} f^{-1/(p-1)}(\upsilon)\, f^{1/(p-1)}(\upsilon)\, u'(\upsilon)\, \mathrm{d}\upsilon \geq M(s_1, s)\, f^{1/(p-1)}(s)\, u'(s),$$

it follows that

$$\left(\frac{u(s)}{M(s_1, s)}\right)' = \frac{M(s_1, s)\, u'(s) - f^{-1/(p-1)}(s)\, u(s)}{M^2(s_1, s)} \leq 0.$$

Thus, the proof is complete. □

Lemma 3.2. *The function w is identified as the ultimate positive solution to (1.1) and $u'(s) < 0$. Then*

$$\left(f(s)(u'(s))^{(p-1)}\right)' \leq -\varpi M^{(p-1)}(s, \infty)\, q(s)\, x_1^{(p-1)}(h(s)). \tag{3.4}$$

Proof. Given that $w(s)$ constitutes the final positive solution of (1.1). Suppose that $u'(s) < 0$ on $[s_1, \infty)$. From Lemma 3.1, we have

$$u(a_1(s)) \leq \frac{M(a_1(s), \infty)}{M(s, \infty)} u(s),$$

based on the fact that $a_1(s) \leq s$. Therefore,

$$\begin{aligned} w(s) &= u(s) - b_1(s)\, w(a_1(s)) - b_2(s)\, w(a_2(s)) \\ &\geq u(s) - b_1(s)\, u(a_1(s)) - b_2(s)\, u(a_2(s)) \\ &\geq \left(1 - b_1(s)\frac{M(a_1(s), \infty)}{M(s, \infty)} - b_2(s)\right) u(s) \\ &= x_1(s)\, u(s). \end{aligned}$$

Hence, (1.1) becomes

$$\begin{aligned} \left(f(s)(u'(s))^{(p-1)}\right)' &\leq -q(s)\, w^{(p-1)}(h(s)) \leq -q(s)\, x_1^{(p-1)}(h(s))\, u^{(p-1)}(h(s)) \\ &\leq -u^{(p-1)}(s)\, q(s)\, x_1^{(p-1)}(h(s)). \end{aligned} \tag{3.5}$$

Since $\left(f(s)(u'(s))^{(p-1)}\right)' \leq 0$, we have

$$f(s)(u'(s))^{(p-1)} \leq f(s_1)(u'(s_1))^{(p-1)} := -\varpi < 0, \tag{3.6}$$

for all $s \geq s_1$, from (3.3) and (3.6), we have

$$u^{(p-1)}(s) \geq \varpi M^{(p-1)}(s,\infty) \text{ for all } s \geq s_1. \tag{3.7}$$

Combining (3.5) with (3.7) yields

$$\left(f(s)(u'(s))^{(p-1)}\right)' \leq -\varpi M^{(p-1)}(s,\infty)\, q(s)\, x_1^{(p-1)}(h(s)),$$

for all $s \geq s_1$. This completes the proof. □

Theorem 3.3. *Let $x_2(s) \geq x_1(s) > 0$. If*

$$\limsup_{s\to\infty} \int_{s_1}^{s} \frac{1}{f^{1/(p-1)}(\varrho)} \left(\int_{s_1}^{\varrho} M^{(p-1)}(\upsilon,\infty)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, d\upsilon\right)^{1/(p-1)} d\varrho = \infty, \tag{3.8}$$

then, (1.1) is oscillatory.

Proof. Given that $w(s)$ constitutes the final positive solution of (1.1). Then $w(a_1(s))$, $w(a_2(s))$, and $w(h(s))$ are positive. From (1.1) and $u(s) = w(s) + b_1(s)w(a_1(s)) + b_2(s)w(a_2(s))$, we see that $u(s) \geq w(s) > 0$ and $f(s)(u'(s))^{(p-1)}$ do not increase. Therefore, u' is either eventually negative or eventually positive.

Suppose first that $u'(s) > 0$ on $[s_1,\infty)$. From Lemma 3.2 and integrating (3.4) from s_1 to s, we obtain

$$\begin{aligned} f(s)(u'(s))^{(p-1)} &\leq f(s_1)(u'(s_1))^{(p-1)} - \varpi \int_{s_1}^{s} M^{(p-1)}(\upsilon,\infty)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, d\upsilon \\ &\leq -\varpi \int_{s_1}^{s} M^{(p-1)}(\upsilon,\infty)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, d\upsilon. \end{aligned}$$

Integrating the last inequality from s_1 to s, we obtain

$$u(s) \leq u(s_1) - \varpi^{1/(p-1)} \int_{s_1}^{s} \frac{1}{f^{1/(p-1)}(\varrho)} \left(\int_{s_1}^{\varrho} M^{(p-1)}(\upsilon,\infty)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, d\upsilon\right)^{1/(p-1)} d\varrho.$$

At $s \to \infty$, we arrive at a contradiction with (3.8).

Assume now that $u'(s) < 0$ on $[s_1,\infty)$. From Lemma 3.4 and integrating (3.14) from s_2 to s, we obtain

$$\begin{aligned} f(s)(u'(s))^{(p-1)} &\leq f(s_2)(u'(s_2))^{(p-1)} - \int_{s_2}^{s} u^{(p-1)}(h(\upsilon))\, q(\upsilon)\, x_2^{(p-1)}(h(\upsilon))\, d\upsilon \\ &\leq f(s_2)(u'(s_2))^{(p-1)} - u^{(p-1)}(h(s_2)) \int_{s_2}^{s} q(\upsilon)\, x_2^{(p-1)}(h(\upsilon))\, d\upsilon. \end{aligned}$$

Since $x_2(s) > x_1(s)$, we obtain

$$f(s)(u'(s))^{(p-1)} \leq f(s_2)(u'(s_2))^{(p-1)} - u^{(p-1)}(h(s_2)) \int_{s_2}^{s} q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, d\upsilon, \tag{3.9}$$

we have shown that, according to Eq (3.10), the positivity of $u'(s)$ as $s \to \infty$ is contradicted. Therefore, the proof is finished. □

Lemma 3.4. *Given that $w(s)$ constitutes the final positive solution of (1.1) and $u'(s) > 0$. Then*

$$\int_{s_1}^{s} q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon \to \infty \ \ \text{as } s \to \infty. \tag{3.10}$$

Proof. The function w is identified as the ultimate positive solution to (1.1). Suppose that $u'(s) > 0$ on $[s_1, \infty)$. From Lemma 3.1, we arrive at

$$u(a_2(s)) \le \frac{M(s_1, a_2(s))}{M(s_1, s)} u(s). \tag{3.11}$$

From the definition of u, we obtain

$$\begin{aligned} w(s) &= u(s) - b_1(s)\, w(a_1(s)) - b_2(s)\, w(a_2(s)) \\ &\ge u(s) - b_1(s)\, u(a_1(s)) - b_2(s)\, u(a_2(s)). \end{aligned} \tag{3.12}$$

Using that (3.11) and $u(a_1(s)) \le u(s)$ where $a_1(s) < s$ in (3.12), we obtain

$$\begin{aligned} w(s) &\ge u(s)\left(1 - b_1(s) - b_2(s)\frac{M(s_1, a_2(s))}{M(s_1, s)}\right) \\ &\ge x_2(s)\, u(s). \end{aligned} \tag{3.13}$$

Thus, (1.1) becomes

$$\begin{aligned} \left(f(s)\left(u'(s)\right)^{(p-1)}\right)' &\le -q(s)\, w^{(p-1)}(h(s)) \le -q(s)\, x_2^{(p-1)}(h(s))\, u^{(p-1)}(h(s)) \\ &\le -u^{(p-1)}(h(s))\, q(s)\, x_2^{(p-1)}(h(s)) \end{aligned} \tag{3.14}$$

On the other hand, it is follows from (3.8) that $\int_{s_1}^{s} M^{(p-1)}(\upsilon, \infty)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon$ must be unbounded. Further, since $M'(s, \infty) < 0$, it is easy to see that

$$\int_{s_1}^{s} q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon \to \infty \ \ \text{as } s \to \infty.$$

This completes the proof. □

Lemma 3.5. *Given that $w(s)$ constitutes the final positive solution of (1.1) and $u'(s) < 0$. If*

$$E(s) = \xi(s)\left(\frac{f(s)\left(u'(s)\right)^{(p-1)}}{u^{(p-1)}(s)} + \frac{1}{M^{(p-1)}(s, \infty)}\right) \ \ \text{on } [s_1, \infty). \tag{3.15}$$

Then

$$E'(s) \le -\xi(s)\, q(s)\, x_1^{(p-1)}(h(s)) + \left(\frac{\xi(s)}{M^{(p-1)}(s, \infty)}\right)' + \frac{f(s)}{p^p}\frac{\left(\xi'(s)\right)^p}{\left(\xi(s)\right)^{(p-1)}}, \tag{3.16}$$

where $\xi \in C^1([s_0, \infty), (0, \infty))$.

Proof. The function w is identified as the ultimate positive solution to (1.1). Suppose that $u'(s) < 0$ on $[s_1, \infty)$, we can notice that $E \geq 0$ on $[s_1, \infty)$. By computing the derivative of (3.15), we can conclude at

$$\begin{aligned} E'(s) &= \frac{\xi'(s)}{\xi(s)} E(s) + \xi(s) \frac{\left(f(s)(u'(s))^{(p-1)}\right)'}{u^{(p-1)}(s)} - (p-1)\xi(s) f(s) \left(\frac{u'(s)}{u(s)}\right)^{(p-1)+1} \\ &\quad + \frac{(p-1)\xi(s)}{f^{1/(p-1)}(s) M^p(s,\infty)} \\ &\leq \frac{\xi'(s)}{\xi(s)} E(s) + \xi(s) \frac{\left(f(s)(u'(s))^{(p-1)}\right)'}{u^{(p-1)}(s)} - \frac{(p-1)}{(\xi(s) f(s))^{1/(p-1)}} \left(E(s) - \frac{\xi(s)}{M^{(p-1)}(s,\infty)}\right)^{p/(p-1)} \\ &\quad + \frac{(p-1)\xi(s)}{f^{1/(p-1)}(s) M^p(s,\infty)}. \end{aligned} \tag{3.17}$$

Combining (3.5) and (3.17), we have

$$\begin{aligned} E'(s) &\leq -\frac{(p-1)}{(\xi(s) f(s))^{1/(p-1)}} \left(E(s) - \frac{\xi(s)}{M^{(p-1)}(s,\infty)}\right)^{p/(p-1)} - \xi(s) q(s) x_1^{(p-1)}(h(s)) \\ &\quad + \frac{(p-1)\xi(s)}{f^{1/(p-1)}(s) M^p(s,\infty)} + \frac{\xi'(s)}{\xi(s)} E(s). \end{aligned} \tag{3.18}$$

Using Lemma 2.1 with $Q := \xi'(s)/\xi(s)$, $N := (p-1)(\xi(s) f(s))^{-1/(p-1)}$, $L := \xi(s)/M^{(p-1)}(s,\infty)$ and $\upsilon := E$, we get

$$\begin{aligned} \frac{\xi'(s)}{\xi(s)} E(s) - \frac{(p-1)}{(\xi(s) f(s))^{1/(p-1)}} \left(E(s) - \frac{\xi(s)}{M^{(p-1)}(s,\infty)}\right)^{p/(p-1)} &\leq \frac{1}{p^p} f(s) \frac{(\xi'(s))^p}{(\xi(s))^{(p-1)}} \\ &\quad + \frac{\xi'(s)}{M^{(p-1)}(s,\infty)}, \end{aligned}$$

which, in view of (3.18), we have

$$\begin{aligned} E'(s) &\leq \frac{\xi'(s)}{M^{(p-1)}(s,\infty)} + \frac{1}{p^p} f(s) \frac{(\xi'(s))^p}{(\xi(s))^{(p-1)}} - \xi(s) q(s) x_1^{(p-1)}(h(s)) \\ &\quad + \frac{(p-1)\xi(s)}{f^{1/(p-1)}(s) M^p(s,\infty)} \\ &\leq -\xi(s) q(s) x_1^{(p-1)}(h(s)) + \left(\frac{\xi(s)}{M^{(p-1)}(s,\infty)}\right)' + \frac{f(s)}{p^p} \frac{(\xi'(s))^p}{(\xi(s))^{(p-1)}}. \end{aligned}$$

This completes the proof. □

Lemma 3.6. *Given that $w(s)$ constitutes the final positive solution of (1.1) and $u'(s) > 0$. If*

$$D(s) = \varsigma(s) \frac{f(s)(u'(s))^{(p-1)}}{u^{(p-1)}(h(s))}, \quad on\ [s_1, \infty), \tag{3.19}$$

then

$$D'(s) \leq \frac{\varsigma'(s)}{\varsigma(s)} D(s) - \varsigma(s) q(s) x_2^{(p-1)}(h(s)) - (p-1)\varsigma(s) f(s) h'(s) \frac{(u'(s))^p}{u^p(h(s))}, \tag{3.20}$$

where $\varsigma \in C^1([s_0, \infty), (0, \infty))$.

Proof. The function w is identified as the ultimate positive solution to (1.1). Suppose that $u'(s) > 0$ on $[s_1, \infty)$. From (3.19), we see that $D \geq 0$ on $[s_1, \infty)$. Differentiating (3.19), we arrive at

$$D'(s) = \frac{\varsigma'(s)}{\varsigma(s)} D(s) + \varsigma(s) \frac{\left(f(s)(u'(s))^{(p-1)}\right)'}{u^{(p-1)}(h(s))} - (p-1)\varsigma(s) f(s) \frac{(u'(s))^{(p-1)} u'(h(s)) h'(s)}{u^p(h(s))}. \quad (3.21)$$

Combining (3.14) and (3.21), we have

$$D'(s) \leq \frac{\varsigma'(s)}{\varsigma(s)} D(s) - \varsigma(s) q(s) x_2^{(p-1)}(h(s)) - (p-1)\varsigma(s) f(s) \frac{(u'(s))^{(p-1)} u'(h(s)) h'(s)}{u^p(h(s))}.$$

Since $\left(f(s)(u'(s))^{(p-1)}\right)' < 0$ and $h(s) \leq s$, we arrive at

$$D'(s) \leq \frac{\varsigma'(s)}{\varsigma(s)} D(s) - \varsigma(s) q(s) x_2^{(p-1)}(h(s)) - (p-1)\varsigma(s) f(s) h'(s) \frac{(u'(s))^p}{u^p(h(s))}.$$

This completes the proof. □

Now, we discuss a numerical example to highlight the significance of the conditions we obtained in Theorem 3.3.

Example 3.7. *Consider the neutral equation*

$$\left(s^2 \left(w(s) + \frac{1}{3} w\left(\frac{s}{2}\right) + \frac{1}{2} w(3s)\right)'\right)' + \frac{t_0}{s} w\left(\frac{s}{3}\right) = 0, \ s > 0, t_0 > 1. \quad (3.22)$$

Let $p = 2,\ f(s) = s^2,\ b_1(s) = 1/3,\ b_2(s) = 1/2,\ a_1(s) = s/2,\ a_2(s) = 3s, h(s) = s/3,\ q(s) = t_0/s$ *and*

$$\eta(s) = \int_{s_0}^{\infty} f^{-1/(p-1)}(\upsilon) \, d\upsilon = \frac{1}{s}.$$

Moreover, we find

$$\limsup_{s\to\infty} \int_{s_1}^{s} \frac{1}{f^{1/(p-1)}(\varrho)} \left(\int_{s_1}^{\varrho} M^{(p-1)}(\upsilon, \infty) q(\upsilon) x_1^{(p-1)}(h(\upsilon)) \, d\upsilon \right)^{1/(p-1)} d\varrho$$
$$= \limsup_{s\to\infty} \int_{s_1}^{s} \frac{1}{\upsilon^2} \left(\int_{s_1}^{\varrho} \int_{s}^{\infty} \varkappa^{-2} \left(\frac{t_0}{\upsilon} \left(1 - \frac{\int_{s/2}^{\infty} r^{-2} dr}{3 \int_{s}^{\infty} \upsilon^{-2} d\upsilon} - \frac{1}{2} \right) \right) d\varkappa d\varrho \right) d\upsilon = \infty.$$

From Theorem 3.3, the Eq (3.22) is oscillatory.

Theorem 3.8. *Assume that* $x_2(s) \geq x_1(s) > 0$. *If*

$$\limsup_{s\to\infty} M^{(p-1)}(s, \infty) \int_{s_1}^{s} q(\upsilon) x_1^{(p-1)}(h(\upsilon)) \, d\upsilon > 1, \quad (3.23)$$

then, every solutions of (1.1) is oscillatory.

Proof. The function w is identified as the ultimate positive solution to (1.1). Then there exists $s_1 \geq s_0$ such that $w(a_1(s)) > 0$, $w(a_2(s)) > 0$, and $w(h(s)) > 0$ for all $s \geq s_1$. The same way we prove Theorem 3.3, the sign of u' becomes consistently positive or negative eventually. First, let $u' < 0$. Integrating (3.5) from s_1 to s, we see that

$$\begin{aligned} f(s)\left(u'(s)\right)^{(p-1)} &\leq f(s_1)\left(u'(s_1)\right)^{(p-1)} - \int_{s_1}^{s} u^{(p-1)}(\upsilon)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon \\ &\leq -u^{(p-1)}(s) \int_{s_1}^{s} q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon. \end{aligned} \tag{3.24}$$

Using (3.3) in (3.24), we obtain

$$-f(s)\left(u'(s)\right)^{(p-1)} \geq -M^{(p-1)}(s,\infty)\, f(s)\left(u'(s)\right)^{(p-1)} \int_{s_1}^{s} q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon. \tag{3.25}$$

Divide both sides of inequality (3.25) by $-f(s)\left(u'(s)\right)^{(p-1)}$ and taking the limsup, we obtain

$$\limsup_{s\to\infty} M^{(p-1)}(s,\infty) \int_{s_1}^{s} q(\upsilon)\, x_1^{(p-1)}(h(\upsilon))\, \mathrm{d}\upsilon \leq 1,$$

we arrive at a contradiction with (3.23).

Let $u' > 0$ on $[s_1, \infty)$. From (3.23) and $M(s,\infty) < \infty$, we have that (3.10) holds. We notice that this part of the proof is exactly like the part of Theorem 3.3, so the proof is complete. □

Theorem 3.9. *If $x_2(s) > 0$, $x_1(s) > 0$, and $f' > 0$ such that*

$$\limsup_{s\to\infty} \frac{M^{(p-1)}(s,\infty)}{\xi(s)} \int_{s}^{s} \left(\xi(\upsilon)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon)) - \frac{f(\upsilon)}{p^p} \frac{\left(\xi'(\upsilon)\right)^p}{\left(\xi(\upsilon)\right)^{(p-1)}} \right) \mathrm{d}\upsilon > 1, \tag{3.26}$$

and

$$\limsup_{s\to\infty} \int_{s}^{s} \left(\varsigma(\upsilon)\, q(\upsilon)\, x_2^{(p-1)}(h(\upsilon)) - \frac{1}{p^p} \frac{f(\upsilon)\left(\varsigma'(\upsilon)\right)^p}{\varsigma^{(p-1)}(\upsilon)\left(h'(\upsilon)\right)^{(p-1)}} \right) \mathrm{d}\upsilon = \infty, \tag{3.27}$$

where the functions $\varsigma, \xi \in C^1([s_0,\infty),(0,\infty))$ and $s \in [s_0,\infty)$. Then, (1.1) is oscillatory.

Proof. Given that $w(s)$ constitutes the final positive solution of (1.1) on $[s_0,\infty)$. Then $w(a_1(s)) > 0$, $w(a_2(s)) > 0$, and $w(h(s)) > 0$ for all $s \geq s_1$. From Theorem 3.3, it follows that u' eventually takes on a consistent sign, either remaining negative or remaining positive, beyond a certain point. First, assuming that $u' < 0$ on $[s_1,\infty)$. By following the approach used in the proof of Theorem 3.3, we can obtain that u is a solution of the inequality (3.5). From Lemma 3.5 and due to (3.3). Integrating (3.16) from s_2 to s, we arrive at

$$\begin{aligned} \int_{s_2}^{s} \left(\xi(\upsilon)\, q(\upsilon)\, x_1^{(p-1)}(h(\upsilon)) - \frac{f(\upsilon)}{p^p} \frac{\left(\xi'(\upsilon)\right)^p}{\left(\xi(\upsilon)\right)^{(p-1)}} \right) \mathrm{d}\upsilon &\leq \left(\frac{\xi(s)}{M^{(p-1)}(s,\infty)} - E(s) \right)\Bigg|_{s_2}^{s} \\ &\leq -\left(\xi(s) \frac{f(s)\left(u'(s)\right)^{(p-1)}}{u^{(p-1)}(s)} \right)\Bigg|_{s_2}^{s}. \end{aligned} \tag{3.28}$$

From (3.3), we have

$$-\frac{f^{1/(p-1)}(s)\,u'(s)}{u(s)} \leq \frac{1}{M(s,\infty)},$$

which, in view of (3.28), implies

$$\frac{M^{(p-1)}(s,\infty)}{\xi(s)}\int_{s_2}^{s}\left(\xi(\upsilon)\,q(\upsilon)\,x_1^{(p-1)}(h(\upsilon)) - \frac{f(\upsilon)}{p^p}\frac{(\xi'(\upsilon))^p}{(\xi(\upsilon))^{(p-1)}}\right)\mathrm{d}\upsilon \leq 1.$$

Applying the limit superior to both sides, we are led to a contradiction with (3.26).

Now, let $u'(s) > 0$ on $[s_1,\infty)$. From Lemma 3.6 and from (3.19) and (3.20), we have

$$D'(s) \leq \frac{\varsigma'(s)}{\varsigma(s)}D(s) - \varsigma(s)\,q(s)\,x_2^{(p-1)}(h(s)) - \frac{(p-1)\,h'(s)}{\varsigma^{1/(p-1)}(s)\,f^{1/(p-1)}(s)}D^{p/(p-1)}(s).$$

Using Lemma 2.1, with $F = \varsigma'(s)/\varsigma(s)$, $\varpi = (p-1)\,h'(s)/\left(\varsigma^{1/(p-1)}(s)\,f^{1/(p-1)}(s)\right)$ and $v = D$, we have

$$D'(s) \leq -\varsigma(s)\,q(s)\,x_2^{(p-1)}(h(s)) + \frac{1}{p^p}\frac{f(s)\,(\varsigma'(s))^p}{\varsigma^{(p-1)}(s)\,(h'(s))^{(p-1)}}. \tag{3.29}$$

Integrating (3.29) from s_2 to s, we arrive at

$$\int_{s_2}^{s}\left(\varsigma(\upsilon)\,q(\upsilon)\,x_2^{(p-1)}(h(\upsilon)) - \frac{1}{p^p}\frac{f(\upsilon)\,(\varsigma'(\upsilon))^p}{\varsigma^{(p-1)}(\upsilon)\,(h'(\upsilon))^{(p-1)}}\right)\mathrm{d}\upsilon \leq D(s_2).$$

We are taking lim sup on both sides of this inequality; we find a contradiction with (3.27), and therefore the proof is finished. □

Now, we include the following numerical example to highlight the importance of the criteria we obtained in Theorem 3.9.

Example 3.10. *Let the equation*

$$\left(s^{2(p-1)}\left[\left(w(s) + \frac{1}{2}w\left(\frac{s}{3}\right) + \frac{1}{3}w(\sigma s)\right)'\right]^{(p-1)}\right)' + \xi s w\left(\frac{s}{2}\right) = 0, \tag{3.30}$$

where $\xi > 1$, $s \geq 1$ and $\sigma > 1$. Let $p = 2$, $f(s) = s^2$, $b_1(s) = 1/3$, $b_2(s) = 1/3$, $a_1(s) = s/3$, $a_2(s) = \sigma s$, $h(s) = s/2$, and $q(s) = \xi s$. So, we find

$$\eta(s) = \int_{s_0}^{\infty} f^{-1/(p-1)}(\upsilon)\,\mathrm{d}\upsilon = s^{-1}.$$

If we set $\varsigma(\upsilon) = 1$, we obtain

$$\limsup_{s\to\infty}\int_{s}^{s}\left(\varsigma(\upsilon)\,q(\upsilon)\,x_2^{(p-1)}(h(\upsilon)) - \frac{1}{p^p}\frac{f(\upsilon)\,(\varsigma'(\upsilon))^p}{\varsigma^{(p-1)}(\upsilon)\,(h'(\upsilon))^{(p-1)}}\right)\mathrm{d}\upsilon = \infty, p < 2.$$

By Theorem 3.9, every solution of (3.30) is oscillatory.

4. Conclusions

In this paper, we investigate the oscillatory criteria of a non-canonical second-order differential equation featuring mixed neutral terms, driven by a p-Laplace differential operator. Our approach begins with transforming the equation into a canonical form, which facilitates the application of the Riccati technique to derive new oscillatory properties. The results we present not only extend but also simplify existing criteria found in the literature.

To demonstrate the practical applicability of our findings, we include several illustrative examples that highlight the effectiveness of our main results.

Looking to the future, our aim is to explore the oscillatory characteristics inherent in third-order differential equations with neutral terms, as well as to investigate fractional-order equations governed by a p-Laplace differential operator. We are already making progress in the study of these specific types of equations, which promise to enhance our understanding of oscillatory behavior in various contexts.

Use of Generative-AI tools declaration

The author declares that the Artificial Intelligence (AI) tools were not used in the creation of this article.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

The author declares that there are no conflicts of interest.

References

1. L.Erbe, Q. Kong, B. G. Zhong, *Oscillation Theory for Functional Differential Equations*, New York: Marcel Dekker, 1995.
2. F. Masood, O. Moaaz, S. S. Santra, U. Fernandez-Gamiz, H. A. El-Metwally, Y. Marib, Oscillation theorems for fourth-order quasi-linear delay differential equations, *AIMS Mathematics*, **8** (2023), 16291–16307. http://doi.org/10.3934/math.2023834
3. T. Li, Y. V. Rogovchenko, Oscillation criteria for even-order neutral differential equations, *Appl. Math. Lett.*, **61** (2019), 35–41. http://doi.org/10.1016/j.aml.2016.04.012
4. T. Li, M. T. Şenel, C. Zhang, Oscillation of solutions to second-order half-linear differential equations with neutral terms, *Electron. J. Differ. Equ.*, **2013** (2013), 1–7.
5. T. Li, Y. V. Rogovchenko, On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations, *Appl. Math. Lett.*, **105** (2020), 106293. http://doi.org/10.1016/j.aml.2020.106293

6. A. Saad, B. Omar, Y. Mehmet, Some important criteria for oscillation of non-linear differential equations with middle term, *Mathematics*, **9** (4), 346. http://doi.org/10.3390/math9040346

7. J. R. Graef, O. Ozdemir, A. Kaymaz, E. Tunc, Oscillation of damped second-order linear mixed neutral differential equations, *Monatsh. Math.*, **194** (2021), 85–104. http://doi.org/10.1007/s00605-020-01469-6

8. A. Maged, R. Lothar, Some numerical aspects of Arnoldi-Tikhonov regularization, *Appl. Numer. Math.*, **185** (2023), 503–515. http://doi.org/10.1016/j.apnum.2022.12.009

9. A. Maged, R. Lothar, Q. Ye, A method for computing a few eigenpairs of large generalized eigenvalue problems, *Appl. Numer. Math.*, **183** (2023), 108–117. http://doi.org/10.1016/j.apnum.2022.08.018

10. O. Bazighifan, P. Kumam, Oscillation Theorems for Advanced Differential Equations with p-Laplacian Like Operators, *Mathematics*, **8** (2020), 821. doilinkhttps://doi.org/10.3390/math8050821

11. T. Li, Comparison theorems for second-order neutral differential equations of mixed type, *EJDE*, **2010** (2010), 167.

12. T. Li, B. Baculíková, J. Džurina, Oscillation results for second-order neutral differential equations of mixed type, *Tatra Mt. Math. Publ.*, **48** (2011), 101–116. http://doi.org/10.2478/v10127-011-0010-8

13. O. Moaaz, Ch. Park, A. Muhib, O. Bazighifan, Oscillation criteria for a class of even-order neutral delay differential equations, *J. Appl. Math. Comput.*, **63** (2020), 607–617. http://doi.org/10.1007/s12190-020-01331-w

14. O. Bazighifan, An approach for studying asymptotic properties of solutions of neutral differential equations, *Symmetry*, **12** (2020), 555. http://doi.org/10.3390/sym12040555

15. R. Arul, V. S. Shobha, Oscillation of second order nonlinear neutral differential equations with mixed neutral term, *J. Appl. Math. Phys.*, **3** (2015), 1080–1089. http://doi.org/10.4236/jamp.2015.39134

16. O. Bazighifan, Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations, *Adv. Differ. Equ.*, **2020** (2020), 201. http://doi.org/10.1186/s13662-020-02661-6

17. G. E. Chatzarakis, S. R. Grace, I. Jadlovská, T. Li, E. Tunç, Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients, *Complexity*, **2019** (2019), 5691758. http://doi.org/10.1155/2019/5691758

18. T. Li, M. T. Senel, C. Zhang, Oscillation of solutions to second-ordered half-linear differntial equations with neutearl terms, *Electron. J. Differ. Eq.*, **2013** (2013), 229.

19. Y. Qi, J. Yu, Oscillation of second order nonlinear mixed neutral differential equations with distributed deviating arguments, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 543–560. https://doi.org/10.1007/s40840-014-0035-7

20. C. Zhang, B. Baculíková, J. Džurina, T. Li, Oscillation results for second-order mixed neutral differential equations with distributed deviating arguments, *Math. Slovaca*, **66** (2016), 615–626. http://doi.org/10.1515/ms-2015-0165

21. T. Li, M. T. Senel, C. Zhang, Oscillation of solutions to second-order half-linear differential equations with neutral terms, *Electron, J. Differ. Equ.*, **2013** (2013), 229.

22. E. Thandapani, S. Selvarangam, M. Vijaya, R. Rama, Oscillation results for second order nonlinear differential equation with delay and advanced arguments, *Kyungpook Math. J.*, **56** (2016), 137–146. http://doi.org/10.5666/KMJ.2016.56.1.137

23. E. Thandapani, R. Rama, Comparison and oscillation theorems for second order nonlinear neutral differential equations of mixed type, *Serdica Math. J.*, **39** (2013), 1–16.

24. L. Shouhua, Z. Quanxin, Y. Yuanhong, Oscillation of even-order half-linear functional differential equations with damping, *Comput. Math. Appl.*, **61** (2011), 2191–2196. https://doi.org/10.1016/j.camwa.2010.09.011

25. O. Bazighifan, A. Thabet, Improved Approach for Studying Oscillatory Properties of Fourth-Order Advanced Differential Equations with p-Laplacian Like Operator, *Mathematics*, **8** (2020), 656. http://doi.org/10.3390/math8050656

26. L. Tongxing, B. Blanka, D. Jozef, Z. Chenghui, Oscillation of fourth order neutral differential equations with p-Laplacian like operators, *Bound. Value Probl.*, **56** (2014), 41–58. http://doi.org/10.1186/1687-2770-2014-56

27. R. P. Agarwal, C. Zhang, T. Li, Some remarks on oscillation of second order neutral differential equations, *Appl. Math. Comput.*, **274** (2016), 178–181. http://doi.org/10.1016/j.amc.2015.10.089