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Research article

Axiomatic analysis of state operators in Sheffer stroke BCK-algebras associated with algorithmic approaches

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Abstract: In this paper, we have presented a novel exploration of the construction of Riečan, Bosbach, internal, and general states within the framework of Sheffer stroke BCK-algebra $\mathcal{B}$. We highlighted the originality of our work by examining key characteristics and the independence of the axiomatic systems associated with these states. Notably, we demonstrated that a Riečan state can correspond to a Bosbach state and vice versa, revealing significant interconnections between these concepts. Additionally, we introduced the innovative concepts of faithful and fixed sets generated by internal states on $\mathcal{B}$, proving that each Sheffer stroke BCK-algebra retains its structure under an internal state. Our investigation also included internal state-(filters, compatible filters, and prime filters) on $\mathcal{B}$ and their related results, as well as the relationship between internal state congruence and filters. Furthermore, we explore whether general states imply Riečan and Bosbach states, enhancing our understanding of these relationships. Finally, we introduced the concept of general state-morphism and discuss its implications for $\mathcal{B}$. To support our findings, we provided compelling examples and fundamental algorithms, underscoring the practical significance of our study across various fields including artificial intelligence, computer science, and quantum logic.

Keywords: Sheffer stroke BCK-algebra; (Riečan, Bosbach, internal, general) state; general state-morphism
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1. Introduction

In constructing mathematical models, emphasizing originality is paramount. This work aims to eliminate redundant expressions while presenting equivalent statements with the minimal number of axioms or operations necessary. For instance, Tarskisuccessfully described Abelian groups with a minimal set of axioms by focusing on the divisor operator, showcasing the efficiency of streamlined approaches [27] . Similarly, H. M. Sheffer [26] demonstrated that all Boolean functions can be articulated solely through the Sheffer stroke operation, which serves as a foundational concept in our exploration [19].

The significance of the Sheffer stroke operation extends beyond theoretical constructs; it has practical implications in computer systems, particularly in chip technology. This operation allows for the uniform construction of all diodes on a chip, which forms the processor in computers. By simplifying the manufacturing process and reducing costs, it eliminates the need for different diodes for various logical connectives like conjunction, disjunction, and negation [2].

When examining logical algebraic structures, we find that they form the algebraic foundation for diverse domains requiring robust reasoning mechanisms, such as information sciences, artificial intelligence, quantum logics, computer sciences, and probability theory. While Boolean algebras dominate classical logic, MV-algebras, introduced by [6], serve non-classical logic applications [20]. Chajda et al. [3] expanded this framework with Basic algebras, encompassing orthomodular lattices and MV-algebras. To simplify these structures further, Oner and Senturk [21] proposed Sheffer stroke basic algebras, reducing complexity to a single operation. Following this line of thought, Senturk et al. and Oner et al. developed the concept of Sheffer Stroke BCK-algebras [22, 25].

The concept of states on MV-algebras was initially introduced by Munduci [20], who utilized averaging processes for formulas within Łukasiewicz logics. This approach not only generalized traditional probability measures on Boolean algebras but also provided a semantic interpretation for the probability of fuzzy events. In an alternative approach, Riečan [23] introduced states on BL-algebras, defining mappings within the interval $[0, 1]$. Georgescu [14] expanded this idea by defining Bosbach and Riečan states on pseudo BL-algebras, mapping them to the real closed interval $[0, 1]$. Consequently, the concept of states has been broadened to various logical algebraic structures, including triangle algebras, pseudo equality algebras, equality algebras, BL-algebras, pseudo-BCK algebras, residuated lattices, semi-divisible residuated lattices, and morphism algebras [1, 7–11, 28, 29]. Additionally, Ghasemi Nejad and Borzooei [12] introduced internal states and homomorphisms in implication basic algebras. Later, Ghasemi Nejad et al. [13] provided a comprehensive definition of states on implication basic algebras, further enriching the discourse. Senturk also proposed a perspective on state operators within Sheffer stroke Basic algebras [24].

The study of fuzzy algebraic structures has seen significant progress in recent years, particularly in the context of BCK/BCI-algebras. Jana and Pal [15] introduced generalized intuitionistic fuzzy ideals of BCK/BCI-algebras based on 3-valued logic, providing a computational framework for their study. Extending this work, Jana et al. [16] explored different types of cubic ideals in BCI-algebras using fuzzy points, further enriching the theoretical foundations of fuzzy algebraic structures. Additionally, the concept of (α, β)-US sets in BCK/BCI-algebras was examined by Jana and Pal [17], offering novel insights into the structural properties of these algebras. Furthermore, the comprehensive *Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures* edited by Jana et al. [18] serves as a valuable resource for understanding the diverse applications of fuzzy algebraic structures in modern mathematics. This paper builds upon these foundational works to investigate decision-making processes within the framework of fuzzy BCK/BCI-algebras.

In this study, we investigate the construction of Riečan, Bosbach, internal, and general states within the framework of Sheffer stroke BCK-algebras. Our originality lies in deriving fundamental properties and establishing connections between these different types of states. We introduce the concept of states utilizing solely the Sheffer stroke operation, which has significant implications for chip technology and other applied fields. This research provides valuable applications for researchers across various domains requiring the concept of states, such as artificial intelligence, computer science, information sciences, quantum logics, and probability theory. Additionally, we propose key algorithms for implementation in these areas.

In Section 2, we revisit essential notions, basic definitions, lemmas, and relevant results pertaining to Sheffer stroke BCK-algebras. Section 3 is dedicated to introducing the concepts of Riečan and Bosbach states on Sheffer stroke BCK-algebras, compiling important facts and examples. We demonstrate the independence of each axiomatic system for Riečan and Bosbach states, proving that a Riečan state can correspond to a Bosbach state and vice versa. In Section 4, we introduce the notion of τ^I internal states along with the concepts of faithful and fixed sets generated by these states on Sheffer stroke BCK-algebras. We examine the key characteristics and independence of the axiomatic system of internal states, proving that every Sheffer stroke BCK-algebra is also a Sheffer stroke BCK-algebra under τ^I. Furthermore, we discuss internal state-(filters, compatible filters, and prime filters) and present their related results. We explore internal state congruence and its relationship with filters, providing several illustrative examples of the aforementioned concepts. Section 5 presents the concept of τ general states as an extension of previous states on Sheffer stroke BCK-algebras, addressing their characteristic features and independence of the axiomatic system, accompanied by relevant examples. We investigate whether general states imply Riečan and Bosbach states, ultimately introducing the notion of general state-morphism and discussing related results within Sheffer stroke BCK-algebras.

2. Preliminaries

In this section, we begin with fundamental definitions, lemmas, and proposition with reference to Sheffer stroke BCK-algebras that will be needed throughout this paper.

Definition 1. *[4] A* Sheffer operation *on a non-empty set X is a binary operation $|: X \times X \to X$ such that, for all $\mathfrak{a}, \mathfrak{b} \in X$, the following identities hold:*

$$(\mathfrak{a}|\mathfrak{b}) \,|\, (\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}, \tag{1}$$

$$(\mathfrak{a}|\mathfrak{b}) \,|\, (\mathfrak{b}|\mathfrak{b}) = \mathfrak{b}. \tag{2}$$

A groupoid *is an algebra of type* (2)*, that is, a set equipped with a single binary operation. A* Sheffer groupoid *is a groupoid $\mathcal{X} = (X; |)$ in which $|: X \times X \to X$ is a Sheffer operation.*

Definition 2. *[5] Let $\mathcal{X} = (X; |)$ be a groupoid. If the following conditions are satisfied for each $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$, then the operation $| : X \times X \to X$ is called a Sheffer stroke operation:*
$(S1)$ $\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a}$,
$(S2)$ $(\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a}$,
$(S3)$ $\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c}$,
$(S4)$ $(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{b}|\mathfrak{b})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a}$.

If the following identity
$(S5)$ $\mathfrak{b}|(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a})) = \mathfrak{b}|\mathfrak{b}$
is also satisfied, then it is said to be an ortho-Sheffer stroke operation.

Lemma 1. *[5] Suppose $\mathcal{X} = (X; |)$ is a groupoid and $\mathfrak{a}, \mathfrak{b} \in X$. The binary relation $\leq$ defined on X by*

$$\mathfrak{a} \leq \mathfrak{b} \text{ if and only if } \mathfrak{a}|\mathfrak{b} = \mathfrak{a}|\mathfrak{a}$$

is a partial order on X.

Lemma 2. *[5] Assume that $|$ is a Sheffer stroke operation on X and $\leq$ is the induced order on $\mathcal{X} = (X; |)$. Then, the following properties hold for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, t \in X$:*

(i) $\mathfrak{a} \leq \mathfrak{b}$ *if and only if* $\mathfrak{b}|\mathfrak{b} \leq \mathfrak{a}|\mathfrak{a}$,
(ii) $\mathfrak{a}|(\mathfrak{b}|(\mathfrak{a}|\mathfrak{a})) = \mathfrak{a}|\mathfrak{a}$ *is the identity element of* $\mathcal{X}$,
(iii) $\mathfrak{a} \leq \mathfrak{b}$ *implies* $\mathfrak{b}|\mathfrak{c} \leq \mathfrak{a}|\mathfrak{c}$,
(iv) *if* $t \leq \mathfrak{a}$ *and* $t \leq \mathfrak{b}$, *then* $\mathfrak{a}|\mathfrak{b} \leq t|t$.

Definition 3 ([22]). *Consider a set X with a distinguished element denoted by "0" and a binary operation called the Sheffer stroke, represented by "|". The structure $\mathcal{X} = (X; |, 0)$ is termed a Sheffer stroke BCK-algebra if it meets the following conditions:*
$(sBCK-1)$ $((\mathfrak{a}_\mathfrak{b}|(\mathfrak{a}|(\mathfrak{c}|\mathfrak{c})))|(\mathfrak{a}_\mathfrak{b}|(\mathfrak{a}|(\mathfrak{c}|\mathfrak{c}))))|(\mathfrak{c}|(\mathfrak{b}|\mathfrak{b})) = 0|0$,
$(sBCK-2)$ $\mathfrak{a}_\mathfrak{b} = 0,\ \mathfrak{b}_\mathfrak{a} = 0 \ \Rightarrow\ \mathfrak{a} = \mathfrak{b}$
for all $\mathfrak{a}, \mathfrak{b} \in X$, where $\mathfrak{a}_\mathfrak{b} := (\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))$.

Proposition 1 ([22]). *Every Sheffer stroke BCK-algebra $\mathcal{X} = (X; |, 0)$ satisfies:*

(a1) $(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}$,
(a2) $(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a}))|(\mathfrak{a}|(\mathfrak{a}|\mathfrak{a})) = 0$,
(a3) $\mathfrak{a}|(((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|((\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))) = 0|0$,
(a4) $(0|0)|(\mathfrak{a}|\mathfrak{a}) = \mathfrak{a}$,
(a5) $\mathfrak{a}|0 = 0|0$,
(a6) $(\mathfrak{a}|(0|0))|(\mathfrak{a}|(0|0)) = \mathfrak{a}$,

for all $\mathfrak{a}, \mathfrak{b} \in X$.

Define a binary relation "$\leq_X$" on a Sheffer stroke BCK-algebra $\mathcal{X} = (X; |, 0)$ as follows:

$$\mathfrak{a} \leq_X \mathfrak{b} \text{ if and only if } (\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) = 0 \tag{2.1}$$

for all $\mathfrak{a}, \mathfrak{b} \in X$. With this definition, $(X, \leq_X)$ forms a partially ordered set (poset), and it holds that $0 \leq_X \mathfrak{a}$ for every $\mathfrak{a} \in X$ (see [22, Lemma 3.2]).

Assume also that $\mathcal{X} = (X; |, 0)$ is a Sheffer stroke BCK-algebra. The binary relation $\leq_X$ on X is defined as:

$$\mathfrak{a} \leq_X \mathfrak{b} \quad \text{if and only if} \quad \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}) = 0|0.$$

Furthermore, it satisfies the condition $\mathfrak{b} \leq_X \mathfrak{a}|(\mathfrak{b}|\mathfrak{b})$ and:

$$\mathfrak{a} \leq_X \mathfrak{c} \ \Rightarrow\ (\mathfrak{a}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) \leq_X (\mathfrak{c}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{c}|(\mathfrak{b}|\mathfrak{b})) \tag{2.2}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

3. Riečan state and Bosbach state on Sheffer stroke BCK-algebras

In this section, we define Riečan and Bosbach states within Sheffer stroke BCK-algebras. We provide conditions for a mapping to be a Riečan state and present an algorithm to verify these conditions. An example is included to illustrate this process. We then define Bosbach states and provide a similar verification method. Finally, we prove the equivalence of Riečan and Bosbach states under certain conditions.

For simplicity, throughout this paper, we will refer to the algebraic structure of a Sheffer stroke BCK-algebra as $\mathcal{B} = (B; |, 0)$.

Definition 4. *The mapping* $\tau^R : B \to [0, 1]$ *is referred to as a Riečan state on* $\mathcal{B}$ *if it satisfies the following conditions for any* $\mathfrak{a}, \mathfrak{b} \in B$*:*
$(\tau^R_{sBCK}1)$ $\tau^R(0|0) = 1$,
$(\tau^R_{sBCK}2)$ $\tau^R((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})) = \tau^R(\mathfrak{a}) +^R \tau(\mathfrak{b})$ *where* $\mathfrak{a}|\mathfrak{b} = 0|0$.

Now, we present a pseudocode to determine whether a given mapping satisfies the conditions to be a Riečan state on $\mathcal{B}$.

Algorithm 1: Confirming a Riečan state

Input: Set B, mapping $\tau^R : B \to [0, 1]$, operations $|$ and $+^R$
Output: Is τ^R a Riečan state on B?

```
IsRiecanState(B, τ^R)
if (τ^R(0|0) ≠ 1) then Return False;
for i = 1 to |B| do
    for j = 1 to |B| do
        if (x_i|x_j = 0|0) and (τ^R((x_i|x_i)|(x_j|x_j)) ≠ τ^R(x_i) +^R τ^R(x_j)) then Return False;
    end
end
Return True;
```

Algorithm 1 is designed to verify whether a mapping $\tau^R : B \to [0, 1]$ fulfills the criteria necessary to be recognized as a Riečan state on the set B. The algorithm systematically evaluates the two axioms that characterize a Riečan state:

- **Axiom** $(\tau^R_{sBCK}1)$: The first criterion requires that the mapping must yield a value of 1 when applied to the element $0|0$. If this condition is not met, the algorithm promptly returns *False*, indicating that the mapping fails to qualify as a Riečan state.
- **Axiom** $(\tau^R_{sBCK}2)$: The second criterion involves validating a specific relationship between any two elements x_i and x_j within the set B. The algorithm iterates through all possible pairs of elements $x_i, x_j \in B$ and checks whether the equation $\tau^R((x_i|x_i)|(x_j|x_j)) = \tau^R(x_i) +^R \tau^R(x_j)$ is satisfied whenever $x_i|x_j = 0|0$. If this equation fails to hold for any pair of elements, the algorithm returns *False*.

If both conditions are met for all applicable elements and element pairs in B, the algorithm concludes that the mapping τ^R qualifies as a Riečan state and returns *True*. This structured procedure ensures that the mapping adheres to the required properties, thereby providing a reliable method for verifying the Riečan state property.

Example 1. *Let $K = \{0, k_1, k_2, k_3, k_4, 1\}$. The operation $|$ on K is defined as shown in Table 1.*

Table 1. The groupoid $(K; |)$.

$\vert$	0	k_1	k_2	k_3	k_4	1
0	1	1	1	1	1	1
k_1	1	k_2	1	1	k_3	k_2
k_2	1	1	k_1	k_1	1	k_1
k_3	1	1	k_1	k_4	1	k_4
k_4	1	k_3	1	1	k_3	k_3
1	1	k_2	k_1	k_4	k_3	0

One can verify that the structure $\mathcal{K} = (K; |)$ is a Sheffer stroke BCK-algebra. Consider the mapping $\tau^R : K \to [0, 1]$ defined by

$$\tau^R(x) := \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x = 1, \\ \frac{4}{5}, & \text{if } x \in \{k_1, k_4\}, \\ \frac{1}{5}, & \text{if } x \in \{k_2, k_3\}. \end{cases}$$

It is clear that $(\tau^R_{SH}1)$ is satisfied. To verify $(\tau^R_{SH}2)$, we must consider the following cases:
(i) For all $k \in K$, $k|0 = 0|k = 1$.
(ii) $k_1|k_2 = k_2|k_1 = 1$, $k_1|k_3 = k_3|k_1 = 1$, $k_2|k_4 = k_4|k_2 = 1$, and $k_3|k_4 = k_4|k_3 = 1$.

Due to the commutativity of the $|$ and $+$ operators, it is sufficient to examine one side of the equalities in the cases above:

- *Since $0|0 = 1$, $\tau^R((0|0)|(0|0)) = \tau^R(0) = 0 = \tau(0) + \tau(0)$.*
- *Since $0|k_1 = 1$, $\tau^R((0|0)|(k_1|k_1)) = \tau^R(1|k_2) = \tau^R(k_1) = \dfrac{4}{5} = \tau(0) + \tau(k_1)$.*
- *Since $0|k_2 = 1$, $\tau^R((0|0)|(k_2|k_2)) = \tau^R(1|k_1) = \tau^R(k_2) = \dfrac{1}{5} = \tau(0) + \tau(k_2)$.*
- *Since $0|k_3 = 1$, $\tau^R((0|0)|(k_3|k_3)) = \tau^R(1|k_4) = \tau^R(k_3) = \dfrac{1}{5} = \tau(0) + \tau(k_3)$.*
- *Since $0|k_4 = 1$, $\tau^R((0|0)|(k_4|k_4)) = \tau^R(1|k_3) = \tau^R(k_4) = \dfrac{4}{5} = \tau(0) + \tau(k_4)$.*
- *Since $0|1 = 1$, $\tau^R((0|0)|(1|1)) = \tau^R(0|1) = \tau^R(1) = 1 = \tau(0) + \tau(1)$.*
- *Since $k_1|k_2 = 1$, $\tau^R((k_1|k_1)|(k_2|k_2)) = \tau^R(k_2|k_1) = \tau^R(1) = 1 = \tau(k_1) + \tau(k_2)$.*
- *Since $k_1|k_3 = 1$, $\tau^R((k_1|k_1)|(k_3|k_3)) = \tau^R(k_2|k_4) = \tau^R(1) = 1 = \tau(k_1) + \tau(k_3)$.*
- *Since $k_2|k_4 = 1$, $\tau^R((k_2|k_2)|(k_4|k_4)) = \tau^R(k_1|k_3) = \tau^R(1) = 1 = \tau(k_2) + \tau(k_4)$.*
- *Since $k_3|k_4 = 1$, $\tau^R((k_3|k_3)|(k_4|k_4)) = \tau^R(k_4|k_3) = \tau^R(1) = 1 = \tau(k_3) + \tau(k_4)$.*

From this perspective, we conclude that the mapping τ^R satisfies $(\tau^R_{SH}2)$. Consequently, it is a Riečan state on $\mathcal{K}$.

Theorem 1. *The axiomatic system of a Riečan state on a Sheffer stroke BCK-algebra is independent.*

Proof. To prove this theorem, we construct a model for each condition, ensuring that each model satisfies the given condition while the other condition does not hold. Let $\mathcal{K} = (K; |)$ be a Sheffer stroke

BCK-algebra as described in Example 1. We then demonstrate that conditions ($\tau^R_{sBCK}1$) and ($\tau^R_{sBCK}2$) are independent of each other.

(1) **Independence of** ($\tau^R_{sBCK}1$)**:** Consider the mapping $\tau^R : K \to [0,1]$ defined as follows:

$$\tau^R(x) := \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x = 1, \\ \frac{3}{7}, & \text{if } x \in \{k_1, k_4\}, \\ \frac{4}{9}, & \text{if } x \in \{k_2, k_3\}. \end{cases}$$

In this case, $\mathcal{K}$ satisfies ($\tau^R_{sBCK}1$) but does not satisfy ($\tau^R_{sBCK}2$) because $\tau^R((k_3|k_3)|(k_4|k_4)) = \tau^R(k_4|k_3) = \tau^R(1) = 1 \neq \frac{55}{63} = \frac{3}{7} + \frac{4}{9} = \tau^R(k_3) + \tau^R(k_4)$, where $c|d = 1 = 0|0$.

(2) **Independence of** ($\tau^R_{sBCK}2$)**:** Consider the mapping $\tau^R : K \to [0,1]$ defined as follows:

$$\tau^R(x) := \begin{cases} 0, & \text{if } x = 0, \\ \frac{4}{7}, & \text{if } x = 1, \\ \frac{2}{7}, & \text{if } x \in \{k_1, k_2, k_3, k_4\}. \end{cases}$$

Thus, $\mathcal{K}$ satisfies ($\tau^R_{SH}2$) but not ($\tau^R_{SH}1$), as $\tau^R(1) = \frac{4}{7} \neq 1$. □

Lemma 3. *Let $\tau^R : B \to [0,1]$ be a Riečan state on $\mathcal{B}$. Then, the structure satisfies $\tau^R(0) = 0$.*

Proof. By Definition 4 and Definition 2 ($S2$), ($\tau^R_{sBCK}1$) gives us $0|0 = \tau^R(0|0)$. Additionally, using Definition 4 ($\tau^R_{SH}2$), we have $\tau^R((0|0)|(0|0)) = \tau^R(0) + \tau^R(0)$. Therefore, we obtain

$$\tau^R(0) = \tau^R((0|0)|(0|0)) = \tau^R(0) + \tau^R(0),$$

which implies $\tau^R(0) = 0$. □

Proposition 2. *Let $\mathcal{B} = (B; |, 0)$ be a Sheffer stroke BCK-algebra. Then, the following statements hold for each $b \in B$:*

(*i*) $b|(b|b) = 0$,

(*ii*) $b \leq 0|0$.

Proof. (*i*) By substituting $[\mathfrak{a} := \mathfrak{b}|\mathfrak{b}]$ in Proposition 1 (a3), we get

$$\mathfrak{b}|(((\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))|((\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}))) = 0|0.$$

Using Proposition 1 (a1), we conclude that

$$\mathfrak{b}|(\mathfrak{b}|\mathfrak{b}) = 0|0.$$

(*ii*) With the help of Proposition 1 (a5), Definition 2 ($S2$), and the definition of $\leq_X$, we conclude that

$$\begin{aligned} 0 &= (0|0)|(0|0) \\ &= (b|0)|(b|0) \\ &= (b|((0|0)|(0|0)))|(b|((0|0)|(0|0))) \\ \Rightarrow\ & b \leq_X 0|0 \end{aligned}$$

for each $b \in B$. □

Lemma 4. *Let $\tau^R : B \to [0,1]$ be a Riečan state on $\mathcal{B}$. Then, the following statements hold:*

(i) $\tau^R(b_1|b_1) = 1 - \tau^R(b_1)$ *for all* $b_1 \in B$,
(ii) *if* $b_1|(b_2|b_2) = 0|0$, *then* $\tau^R(b_1) \leq \tau^R(b_2)$ *for* $b_1, b_2 \in B$.

Proof. (i) Using Definition 2 ($S2$), Proposition 2, and Definition 4 ($\tau^R_{sBCK}2$), we attain

$$\begin{aligned} 1 = \tau^R(0|0) &= \tau^R(b_1|(b_1|b_1)) \\ &= \tau^R(((b_1|b_1)|(b_1|b_1))|(b_1|b_1)) \\ &= \tau^R(b_1|b_1) + \tau^R(b_1) \\ &\Rightarrow \tau^R(b_1|b_1) = 1 - \tau^R(b_1). \end{aligned}$$

(ii) Given that $b_2|(b_2|b_2) = 0|0$ for all $b_1, b_2 \in B$, we have $\tau^R(b_1) + \tau^R(b_1|b_1) = 1$ and $\tau^R(b_2) + \tau^R(b_2|b_2) = 1$. Assume $b_1|(b_2|b_2) = 0|0$. Then, we obtain $\tau^R((b_1|b_1)|b_2) = \tau^R(b_1) + \tau^R(b_2|b_2)$. Substituting $[\tau^R(b_2|b_2) := 1 - \tau^R(b_2)]$ into the last equation, we deduce $\tau^R((b_1|b_1)|b_2) - 1 = \tau^R(b_1) - \tau^R(b_2)$. Since $\tau^R((b_1|b_1)|b_2) \leq 1$, we obtain $0 \geq \tau^R(b_1) - \tau^R(b_2)$, which implies $\tau^R(b_1) \leq \tau^R(b_2)$.

□

Lemma 5. *Let $\tau^R : B \to [0,1]$ be a Riečan state on $\mathcal{B}$. Then, the following identity holds:*

$$\tau^R(b_1|b_2) + \tau^R(b_2) = 1$$

for all $b_1, b_2 \in B$.

Proof. Utilizing Definition 4 ($\tau^R_{sBCK}1$), Proposition 1 (a5), Definition 2 ($S3$), and Definition 4 ($\tau^R_{sBCK}2$) in sequence, we derive the following equality:

$$\begin{aligned} 1 &= \tau^R(0|0) \\ &= \tau^R(0|b_1) \\ &= \tau^R(((b_2|(b_2|b_2))|(b_2|(b_2|b_2)))|b_1) \\ &= \tau^R((b_2|b_2)|((b_1|b_2)|(b_1|b_2))) \\ &= \tau^R(b_1|b_2) + \tau^R(b_2) \end{aligned}$$

for all $b_1, b_2 \in B$. □

In the remainder of this chapter, we introduce the concept of Bosbach states on $\mathcal{B}$, and present some fundamental facts along with an example related to these states. Additionally, we demonstrate the independence of each axiomatic system for Bosbach states on $\mathcal{B}$, and prove that a Riečan state corresponds to a Bosbach state, and vice versa.

Definition 5. *A mapping $\tau^B : B \to [0,1]$ is called a Bosbach state on $\mathcal{B}$ if it satisfies the following conditions for all $\mathfrak{a}, \mathfrak{b} \in B$:*
($\tau^B_{sBCK}1$) $\tau^B(0|0) = 1$,
($\tau^B_{sBCK}2$) $\tau^B(\mathfrak{a}) + \tau^B(\mathfrak{a}|(\mathfrak{b}|\mathfrak{b})) = \tau^B(\mathfrak{b}) + \tau^B(\mathfrak{b}|(\mathfrak{a}|\mathfrak{a}))$,
($\tau^B_{sBCK}3$) *There exists an element $\mathfrak{c} \in B$ such that $\tau^B(\mathfrak{c}) = 0$.*

Now, we provide a pseudocode to verify if a given mapping meets the criteria to be a Bosbach state on $\mathcal{B}$.

Algorithm 2: Confirming a Bosbach state

Input: Set B, mapping $\tau^B : B \to [0,1]$, operation $|$
Output: Is τ^B a Bosbach state on B?

$IsBosbachState(B, \tau^B)$
if $(\tau^B(0|0) \neq 1)$ **then** Return $False$;
Set $control \leftarrow 0$
for $i = 1$ **to** $|B|$ **do**
  for $j = 1$ **to** $|B|$ **do**
    if $(\tau^B(x_i) + \tau^B(x_i|(x_j|x_j)) \neq \tau^B(x_j) + \tau^B(x_j|(x_i|x_i)))$ **then** Return $False$;
  end
  if $(\tau^B(x_i) = 0$ **then** $control \leftarrow 1$;
end
if $control = 0$ **then** Return $False$;
Return $True$;

Algorithm 2 is crafted to verify whether a mapping $\tau^B : B \to [0,1]$ meets the criteria for being a Bosbach state on the set B. The algorithm assesses the three axioms that define a Bosbach state:

- **Axiom** ($\tau^B_{sBCK}1$): The first criterion ensures that the mapping evaluates to 1 when applied to the element $0|0$. If this condition is not satisfied, the algorithm immediately returns *False*.
- **Axiom** ($\tau^B_{sBCK}2$): The second criterion involves a more intricate relationship between elements in the set B. The algorithm iterates over all pairs of elements $x_i, x_j \in B$ and verifies whether the equation $\tau^B(x_i)+\tau^B(x_i|(x_j|x_j)) = \tau^B(x_j)+\tau^B(x_j|(x_i|x_i))$ holds for each pair. If the equation fails for any pair, the algorithm returns *False*.
- **Axiom** ($\tau^B_{sBCK}3$): The third criterion requires the existence of at least one element $z \in B$ such that $\tau^B(z) = 0$. The algorithm examines all elements in B, and if no such element is found, it returns *False*.

If the above three axioms are satisfied, the algorithm concludes that the mapping τ^B qualifies as a Bosbach state and returns *True*. This systematic approach ensures that the mapping adheres to all required conditions, offering a clear and reliable method to verify the Bosbach state property.

Example 2. *Let $\mathcal{K} = (K; |, 0)$ be a Sheffer stroke BCK-algebra as described in Example 1. Define the mapping τ^B as follows:*

$$\tau^B(x) := \begin{cases} 0, & \text{if } x \in \{0, k_1, k_4\}, \\ 1, & \text{if } x \in \{k_2, k_3, 1\}. \end{cases}$$

Then, τ^B is a Bosbach state on $\mathcal{K}$.

Theorem 2. *The axiomatic system for Bosbach states on a Sheffer stroke BCK-algebra is independent.*

Proof. To prove this theorem, we construct a model for each condition where that specific condition holds true while the others do not. Let $\mathcal{K} = (K; |, 0)$ be a Sheffer stroke BCK-algebra as described in Example 1. We will demonstrate that these three conditions are independent of each other.

(1) **Independence of** $(\tau^B_{sBCK}1)$**:** Consider the mapping $\tau^B : K \to [0,1]$ defined by

$$\tau^B(x) := \begin{cases} 1, & \text{if } x \in \{0,1\}, \\ \frac{4}{9}, & \text{if } x \in \{k_1, k_2, k_4\}, \\ \frac{1}{6}, & \text{if } x = k_3. \end{cases}$$

Then, $\mathcal{K}$ satisfies $(\tau^B_{sBCK}1)$, but not $(\tau^B_{SH}2)$, as shown by the following:

$$\begin{aligned} \tau^B(k_2) + \tau^B(k_2|(k_3|k_3)) &= \tau^B(k_2) + \tau^B(1) \\ &= \frac{4}{9} + 1 = \frac{13}{9} \\ &\neq \frac{7}{6} = \frac{1}{6} + 1 \\ &= \tau^B(k_3) + \tau^B(1) \\ &= \tau^B(k_3) + \tau^B(k_3|(k_2|k_2)). \end{aligned}$$

Furthermore, it does not satisfy $(\tau^B_{SH}3)$ because there is no element $c \in K$ such that $\tau^B(c) = 0$.
(2) **Independence of** $(\tau^B_{sBCK}2)$**:** Consider the mapping $\tau^B : K \to [0,1]$ defined by $\tau^B(x) := \frac{1}{5}$ for all $x \in K$. In this case, the mapping satisfies only $(\tau^B_{sBCK}2)$ but does not satisfy the conditions $(\tau^B_{sBCK}1)$ and $(\tau^B_{sBCK}3)$.
(3) **Independence of** $(\tau^B_{sBCK}3)$**:** Consider the function $\tau^B : K \to [0,1]$ defined as follows:

$$\tau^B(x) := \begin{cases} 0, & \text{if } x = 0, \\ 3/4, & \text{if } x \in \{a, b, c, d, 1\}. \end{cases}$$

This function satisfies $(\tau^B_{sBCK}3)$ exclusively, without fulfilling $(\tau^B_{sBCK}1)$ or $(\tau^B_{sBCK}2)$. □

Lemma 6. *Let the mapping $\tau^B : B \to [0,1]$ be a state on $\mathcal{B}$. Then the following statements are satisfied for all $b_1, b_2 \in A$:*
(ii) $\tau^B(b_1|b_1) = 1 - \tau^R(b_1)$,
(iii) $\tau^B(b_1|b_2) = \tau^B(b_1|((b_1|b_2)|(b_1|b_2))) = \tau^B(b_2|((b_1|b_2)|(b_1|b_2)))$.

Proof. (*i*) The proof method is analogous to the one used in Lemma 4 (*i*).
(*ii*) Utilizing Definition 2 $(S3)$ followed by Definition 2 $(S2)$, we obtain:

$$\begin{aligned} \tau^B(b_1|((b_1|b_2)|(b_1|b_2))) &= \tau^B(((b_1|b_1)|(b_1|b_1))|b_2) \\ &= \tau^B(b_1|b_2) \end{aligned}$$

for all $b_1, b_2 \in B$. Similarly, we derive the equality $\tau^B(b_1|b_2) = \tau^B(b_2|((b_1|b_2)|(b_1|b_2)))$ for any $b_1, b_2 \in B$. □

Lemma 7. *Let $\mathcal{B}$ be a Sheffer stroke BCK-algebra. Then, the following identity is verified:*

$$((b_1|(b_2|b_2))|(b_2|b_2))|(b_2|b_2) = b_1|(b_2|b_2)$$

for each $b_1, b_2 \in B$.

Proof. Using the definition of $\leq_X$ and Proposition 1 (a3), we obtain $b_1 \leq_X (b_1|(b_2|b_2))|(b_2|b_2)$. By applying Lemma 2 (*iii*), we have

$$((b_1|(b_2|b_2))|(b_2|b_2))|(b_2|b_2) \leq_X b_1|(b_2|b_2). \tag{3.1}$$

Furthermore, by substituting $[\mathfrak{a} := b_1|(b_2|b_2)]$ and $[\mathfrak{b} := b_2]$, we get

$$(b_1|(b_2|b_2)) \leq_X ((((b_1|(b_2|b_2)) \leq_X (b_2 \leq_X b_2)) \leq_X (b_2 \leq_X b_2)) \leq_X (((b_1|(b_2|b_2)) \leq_X (b_2 \leq_X b_2)) \leq_X (b_2 \leq_X b_2))) = 0 \leq_X 0.$$

From the definition of $\leq_X$, we obtain

$$b_1|(b_2|b_2) \leq_X ((b_1|(b_2|b_2))|(b_2|b_2))|(b_2|b_2). \tag{3.2}$$

As a result, by combining Eqs (3.1) and (3.2), we conclude that

$$((b_1|(b_2|b_2))|(b_2|b_2))|(b_2|b_2) = b_1|(b_2|b_2)$$

for all $b_1, b_2 \in B$. □

Theorem 3. *Let τ be a mapping on B. Then, τ is a Bosbach state if and only if it is also a Riečan state.*

Proof. (⇒:) Suppose that τ is a Bosbach state on $\mathcal{B}$. From Definition 5, it follows that $(\tau^R_{sBCK}1)$ is satisfied. To complete the proof, we need to demonstrate the validity of $(\tau^R_{sBCK}2)$. Consider $b_1|b_2 = 0|0$. From Definition 5, we have

$$\tau((b_1|b_1)|(b_2|b_2)) + \tau(b_1|b_1) = \tau(b_2) + \tau(b_2|((b_1|b_1)|(b_1|b_1))).$$

Using Definition 2 $(S2)$ and Lemma 6 (*ii*), we obtain

$$\tau((b_1|b_1)|(b_2|b_2)) + 1 - \tau(b_1) = \tau(b_2) + \tau(b_2|b_1).$$

Given that $b_2|b_1 = 0|0$ and $\tau(0|0) = 1$, we get

$$\tau((b_1|b_1)|(b_2|b_2)) + 1 - \tau(b_1) = \tau(b_2) + \tau(0|0),$$

which simplifies to $\tau((b_1|b_1)|(b_2|b_2)) = \tau(b_1) + \tau(b_2)$. Therefore, τ is also a Riečan state.
(⇐:) Assume that τ is a Riečan state on $\mathcal{B}$. From Definition 4 and Lemma 4, we know that $\tau(0|0) = 1$ and $\tau(0) = 0$ are satisfied, respectively. Thus, conditions $(\tau^B_{sBCK}1)$ and $(\tau^B_{sBCK}3)$ are verified. To complete the proof in this direction, it remains to demonstrate the validity of $(\tau^B_{sBCK}2)$.

Using Lemma 5 and Lemma 4 (*i*), for any $b_1, b_2 \in B$, we have:

$$\begin{aligned}
\tau(b_1) + \tau(b_1|(b_2|b_2)) &= \tau(b_1) + (1 - \tau(b_2|b_2)) \\
&= \tau(b_1) + (\tau(b_2) + \tau(b_2|b_2) - \tau(b_2|b_2)) \\
&= \tau(b_1) + \tau(b_2) \\
&= (\tau(b_1) + \tau(b_1|b_1) - \tau(b_1|b_1)) + \tau(b_2) \\
&= (1 - \tau(b_1|b_1)) + \tau(b_2) \\
&= \tau(b_2|(b_1|b_1)) + \tau(b_2).
\end{aligned}$$

As a result, we obtain

$$\tau(b_1) + \tau(b_1|(b_2|b_2)) = \tau(b_2) + \tau(b_2|(b_1|b_1)),$$

which confirms that τ satisfies $(\tau^B_{sBCK}2)$. Therefore, τ is also a Bosbach state. □

4. Internal state on Sheffer stroke BCK-algebras

In this section, we define internal states on Sheffer stroke BCK-algebras, providing the necessary conditions for a mapping to qualify as an internal state. We introduce a pseudocode algorithm to verify whether a given mapping satisfies these conditions. Additionally, we explore the relationship between internal states and filters, presenting theorems and propositions that establish the properties and interactions between these structures. The examples provided illustrate the application of these concepts within the algebraic framework, and we conclude with a theorem that demonstrates the independence of the axiomatic system for internal states.

Definition 6. *A mapping $\tau^I : B \to B$ is called an internal state on $\mathcal{B}$ if it satisfies the following conditions for all $b_1, b_2 \in B$:*
$(\tau^I_{sBCK}1)$ $\tau^I(b_1) \leq \tau^I(b_2)$, *where* $b_1|(b_2|b_2) = 0|0$,
$(\tau^I_{sBCK}2)$ $\tau^I(b_1|(b_2|b_2)) = \tau^I((b_1|(b_2|b_2))|(b_2|b_2))|(\tau^I(b_2)|\tau^I(b_2))$,
$(\tau^I_{SH}3)$ $\tau^I((\tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2)))|(\tau^I(b_2)|\tau^I(b_2))) = (\tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2)))|(\tau^I(b_2)|\tau^I(b_2))$,
$(\tau^I_{SH}4)$ $\tau^I(\tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2))) = \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2))$.

Now, we provide a pseudocode to determine whether a given mapping fulfills the criteria to be an internal state on $\mathcal{B}$.

Algorithm 3: Confirming an internal state

Input: Set B, mapping $\tau^I : B \to B$, operation $|$, and order relation $\leq$
Output: Is τ^I an internal state on B?

$IsInternalState(B, \tau^I)$
for $i = 1$ **to** $|B|$ **do**
  for $j = 1$ **to** $|B|$ **do**
    if $(b_i|(b_j|b_j) = 0|0)$ **and** $(\tau^I(b_i) \not\leq \tau^I(b_j))$ **then** Return $False$;
    if $(\tau^I(b_i|(b_j|b_j)) \neq \tau^I((b_i|(b_j|b_j))|(b_j|b_j))|(\tau^I(b_j)|\tau^I(b_j)))$ **then** Return $False$;
    if $(\tau^I((\tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j)))|(\tau^I(b_j)|\tau^I(b_j))) \neq (\tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j)))|(\tau^I(b_j)|\tau^I(b_j)))$ **then** Return $False$;
    if $(\tau^I(\tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j))) \neq \tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j)))$ **then** Return $False$;
  end
end
Return $True$;

Algorithm 3 is designed to verify whether a mapping $\tau^I : B \to B$ meets the criteria for being an internal state on the set B. This algorithm methodically checks the four axioms that define an internal state:

- **Axiom** $(\tau^I_{sBCK}1)$: The first criterion involves checking whether $b_i|(b_j|b_j) = 0|0$ holds for each pair of elements $b_i, b_j \in B$. If this condition is satisfied, the algorithm proceeds to verify if $\tau^I(b_i) \leq \tau^I(b_j)$. If any pair fails to meet this condition, the algorithm returns *False*.
- **Axiom** $(\tau^I_{sBCK}2)$: The second criterion requires that

$$\tau^I(b_i|(b_j|b_j)) = \tau^I((b_i|(b_j|b_j))|(b_j|b_j))|(\tau^I(b_j)|\tau^I(b_j))$$

holds for all pairs of elements $b_i, b_j \in B$. If the equation does not hold for any pair, the algorithm returns *False*.

- **Axiom** ($\tau^I_{SH}3$): The third criterion checks if

$$\tau^I((\tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j)))|(\tau^I(b_j)|\tau^I(b_j)))$$

is equal to

$$(\tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j)))|(\tau^I(b_j)|\tau^I(b_j)).$$

If this condition is not met, the algorithm returns *False*.

- **Axiom** ($\tau^I_{SH}4$): The fourth criterion requires that

$$\tau^I(\tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j))) = \tau^I(b_i)|(\tau^I(b_j)|\tau^I(b_j))$$

holds for all pairs of elements $b_i, b_j \in B$. If this condition is not satisfied, the algorithm returns *False*.

If all conditions are satisfied for all relevant elements and pairs in the set B, the algorithm concludes that the mapping τ^I qualifies as an internal state and returns *True*. This systematic approach ensures that the mapping adheres to all necessary conditions, providing a reliable method for verifying the internal state property.

Definition 7. *An internal state τ^I on $\mathcal{B}$ is called faithful if it satisfies*

$$Ker(\tau^I) = \{b \in B \mid \tau^I(b) = 0|0\} = \{0|0\}.$$

Definition 8. *The set*

$$Fix(\tau^I) = \{b \in B \mid \tau^I(b) = b\}$$

is referred to as the fixed set of the internal state operator τ^I on $\mathcal{B}$.

Example 3. *The identity map $\tau^I(b) = b$ and the constant map $\tau^I(b) = 0|0$ are considered trivial internal states on $\mathcal{B}$.*

Theorem 4. *The axiomatic system for internal states on a Sheffer stroke BCK-algebra is independent.*

Proof. Theorem 4 can be proven using a method analogous to the proof of Theorem 2. □

Theorem 5. *If τ^I is an internal state on B, then $\tau^I(B)$ forms a Sheffer stroke BCK-algebra.*

Proof. This result directly follows from Definition 3 and Definition 6. □

Proposition 3. *Let τ^I be an internal state on $\mathcal{B}$. Then, the following statements hold true:*

(*i*) $\tau^I(0|0) = 0|0$,
(*ii*) $\tau^I(\tau^I(b)) = b$ *for all* $b \in B$,
(*iii*) $\tau^I(0) = 0$,
(*iv*) *If* $\tau^I(b) = 0|0$, *then* $b = 0|0$,
(*v*) $\tau^I(b_1|b_2)|\tau^I(b_1|b_2) \leq \tau^I(b_1|b_1)|\tau^I(b_2|b_2)$ *for all* $b_1, b_2 \in B$,
(*vi*) $\tau^I(b_1|(b_2|b_2)) \leq \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2))$ *for all* $b_1, b_2 \in B$,

(vii) *if* $b_1|(b_2|b_2) = 0|0$ *or* $b_2|(b_1|b_1) = 0|0$*, then* $\tau^I(b_1|(b_2|b_2)) = \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2))$*,*
(viii) $\tau^I(B) = Fix(\tau^I)$.

Proof. (*i*) By applying Proposition 1 (a2) and Definition 2 ($S2$), we derive:

$$((b|(b|b))|(b|(b|b)))|((b|(b|b))|(b|(b|b))) = 0|0 \Rightarrow b|(b|b) = 0|0$$

for each $b \in B$. Taking into account Definition 6 ($\tau^I_{sBCK}2$), we find:

$$\tau^I(b|(b|b)) = \tau^I((b|(b|b))|(b|b))|(\tau^I(a)|\tau^I(a)).$$

Using Proposition 1 (a2) and Proposition 2, we obtain:

$$\tau^I((b|(b|b))|(b|b))|(\tau^I(b)|\tau^I(b)) = \tau^I(b)|(\tau^I(b)|\tau^I(b)) = 0|0$$

for each $b \in B$.
(*ii*) With the help of Proposition 1 (a4) and Proposition 3 (*i*), we find:

$$\tau^I(\tau^I(b)) = \tau^I((0|0)|(\tau^I(b)|\tau^I(b))) = \tau^I(\tau^I(0|0)|(\tau^I(b)|\tau^I(b))).$$

By combining Definition 6 ($\tau^I_{sBCK}4$), Proposition 3 (*i*), and Proposition 1 (a4), we arrive at:

$$\tau^I(\tau^I(0|0)|(\tau^I(b)|\tau^I(b))) = \tau^I(0|0)|(\tau^I(b)|\tau^I(b)) = (0|0)|(\tau^I(b)|\tau^I(b)) = \tau^I(b)$$

for each $b \in B$.
(*iii*) Suppose there exists some $b \in B$ such that $\tau^I(0) = b$. By applying Proposition 3 (*ii*), we obtain:

$$0 = \tau^I(\tau^I(0)) = \tau^I(b).$$

Next, using Definition 6 ($\tau^I_{sBCK}2$), we get:

$$\begin{aligned} \tau^I(0) &= \tau^I((0|0)|(0|0)) \\ &= \tau^I(\tau^I(0|0)|(\tau^I(b)|\tau^I(b))) \\ &= \tau^I(0|0)|(\tau^I(b)|\tau^I(b)) \\ &= (0|0)|(\tau^I(b)|\tau^I(b)) \\ &= \tau^I(b) = 0. \end{aligned}$$

(*iv*) This result follows directly from Proposition 3 (*i*) and (*ii*).
(*v*) For any $b_1 \in B$, it holds that $b_1 \leq 0|0$. By applying Lemma 2 (*iii*), Definition 2, and Proposition 1 (a4) in sequence, we have:

$$\begin{aligned} b_1 \leq 0|0 &\Rightarrow (0|0)|((b_2|b_2)|(b_2|b_2)) \leq b_1|((b_2|b_2)|(b_2|b_2)) \\ &\Rightarrow b_2|b_2 \leq b_1|b_2 \end{aligned}$$

for all $b_2 \in B$. From Definition 6 ($\tau^I_{sBCK}1$), we obtain:

$$\tau^I(b_2|b_2) \leq \tau^I(b_1|b_2) \tag{4.1}$$

and similarly:

$$\tau^I(b_1|b_1) \leq \tau^I(b_1|b_2). \tag{4.2}$$

By combining (4.1) and (4.2), we conclude that:

$$\tau^I(b_1|b_2)|\tau^I(b_1|b_2) \leq \tau^I(b_1|b_1)|\tau^I(b_2|b_2).$$

(*vi*) Let b_1, b_2 be any elements of B. By applying Definition 2 ($S3$), Definition 2 ($S2$), and Proposition 2, we obtain:

$$\begin{aligned} b_1|(((b_1|(b_2|b_2))|(b_2|b_2))|((b_1|(b_2|b_2))|(b_2|b_2))) &= (b_1|(b_2|b_2))|((b_1|(b_2|b_2))|(b_1|(b_2|b_2))) \\ &= 0|0. \end{aligned}$$

From the definition of $\leq_X$, we conclude that:

$$b_1 \leq_X (b_1|(b_2|b_2))|(b_2|b_2)$$

for all $b_1, b_2 \in A$, which implies that

$$\tau^I(b_1) \leq \tau^I((b_1|(b_2|b_2))|(b_2|b_2)).$$

Using Lemma 2 (*iii*), we have:

$$\tau^I((b_1|(b_2|b_2))|(b_2|b_2))|(\tau^I(b_2)|\tau^I(b_2)) \leq \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2)).$$

Finally, applying Definition 6 ($\tau^I_{sBCK}2$), we arrive at:

$$\tau^I(b_1|(b_2|b_2)) \leq \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2))$$

for each $b_1, b_2 \in B$.
(*vii*) Let $b_1, b_2 \in B$ such that $b_1|(b_2|b_2) = 0|0$. Then, we have:

$$0|0 = \tau^I(0|0) = \tau^I(b_1|(b_2|b_2)) \leq \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2)) \leq 0|0.$$

As a result, we conclude that $\tau^I(b_1|(b_2|b_2)) = \tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2))$.
Similarly, let $b_1, b_2 \in B$ such that $b_2|(b_1|b_1) = 0|0$. Then, we have:

$$0|0 = \tau^I(0|0) = \tau^I(b_2|(b_1|b_1)) \leq \tau^I(b_2)|(\tau^I(b_1)|\tau^I(b_1)) \leq 0|0.$$

Thus, it follows that $\tau^I(b_2|(b_1|b_1)) = \tau^I(b_2)|(\tau^I(b_1)|\tau^I(b_1))$.
(*vii*) It is evident that $Fix(\tau^I) \subseteq \tau^I(B)$. Now, let $b_2 \in \tau^I(B)$. This implies that there exists some $b_1 \in B$ such that $\tau^I(b_1) = b_2$. By Proposition 3 (*ii*), we have:

$$\tau^I(b_2) = \tau^I(\tau^I(b_1)) = \tau^I(b_1) = b_2.$$

Thus, we conclude that $\tau^I(B) \subseteq Fix(\tau^I)$, which implies $\tau^I(B) = Fix(\tau^I)$. □

Definition 9. *Let $\mathcal{B} = (B; |, 0)$ be a Sheffer stroke BCK-algebra, and let F_{sBCK} be a subset of B such that $0|0 \in F_{sBCK}$. The subset F_{sBCK} is called a filter of $\mathcal{B}$ if it satisfies the following conditions for all $b_1, b_2, b_3 \in B$:*
($F_{sBCK}1$) If $b_1 \in F_{sBCK}$ and $b_1|(b_2|b_2) \in F_{sBCK}$, then $b_2 \in F_{sBCK}$,
($F_{sBCK}2$) If $b_1|(b_2|b_2) \in F_{sBCK}$ and $b_2|(b_1|b_1) = 0|0$, then

$$(b_2|(b_3|b_3))|((b_1|(b_3|b_3))|(b_1|(b_3|b_3))) \in F_{sBCK}.$$

Additionally, if the set F_{sBCK} satisfies:
($F_{sBCK}3$) $b_1|(b_2|b_2) \in F_{sBCK}$ and $b_2|(b_1|b_1) \in F_{sBCK}$ imply $(b_3|(b_1|b_1))|((b_3|(b_2|b_2))|(b_3|(b_2|b_2))) \in F_{sBCK}$, then F_{sBCK} is called a compatible filter of $\mathcal{B}$.

The set of all filters and compatible filters of $\mathcal{B}$ are denoted by $\mathbf{F_{sBCK}}(\mathcal{B})$ and $\mathbf{CF_{sBCK}}(\mathcal{B})$, respectively.

We now present a pseudocode to determine whether a given subset of B meets the criteria to be a filter on $\mathcal{B}$.

Algorithm 4: Confirming a filter or compatible filter

Input: Set B, subset $F_{sBCK} \subseteq B$, operation $|$, and element $0|0$
Output: Is F_{sBCK} a filter or a compatible filter of B?

IsFilter(B, F_{sBCK})
if $0|0 \notin F_{sBCK}$ **then** Return *False*;
for $b_1, b_2 \in B$ **do**
  if $b_1 \in F_{sBCK}$ *and* $b_1|(b_2|b_2) \in F_{sBCK}$ **then**
    if $b_2 \notin F_{sBCK}$ **then** Return *False*;
  end
end
for $b_1, b_2, b_3 \in B$ **do**
  if $b_1|(b_2|b_2) \in F_{sBCK}$ *and* $b_2|(b_1|b_1) = 0|0$ **then**
    if $(b_2|(b_3|b_3))|((b_1|(b_3|b_3))|(b_1|(b_3|b_3))) \notin F_{sBCK}$ **then** Return *False*;
  end
end
IsCompatibleFilter(B, F_{sBCK})
for $b_1, b_2, b_3 \in B$ **do**
  if $b_1|(b_2|b_2) \in F_{sBCK}$ *and* $b_2|(b_1|b_1) \in F_{sBCK}$ **then**
    if $(b_3|(b_1|b_1))|((b_3|(b_2|b_2))|(b_3|(b_2|b_2))) \notin F_{sBCK}$ **then** Return *False*;
  end
end
Return *True*;

Algorithm 4 is designed to verify whether a subset F_{sBCK} of a Sheffer stroke BCK-algebra $\mathcal{B}$ meets the criteria for being a filter or a compatible filter. The algorithm methodically checks the axioms defining these concepts:

- **Filter verification**:

- The algorithm first checks whether the element $0|0$ is included in the subset F_{sBCK}. If $0|0$ is absent, the subset cannot qualify as a filter, and the algorithm promptly returns *False*.
- In the sequel, it verifies the first filter condition ($F_{sBCK}1$), which requires that if an element b_1 is in F_{sBCK} and $b_1|(b_2|b_2)$ is also in F_{sBCK}, then b_2 must be included in F_{sBCK}. If this condition is not fulfilled for any pair of elements, the algorithm returns *False*.
- Additionally, the algorithm checks the second filter condition ($F_{sBCK}2$). This condition stipulates that if $b_1|(b_2|b_2)$ belongs to F_{sBCK} and $b_2|(b_1|b_1) = 0|0$, then the element $(b_2|(b_3|b_3))|((b_1|(b_3|b_3))|(b_1|(b_3|b_3)))$ must be in F_{sBCK}. If this condition is violated, the algorithm returns *False*.

- **Compatible filter verification**:
  - For a subset to qualify as a compatible filter, the algorithm verifies an additional condition ($F_{SH}3$). This condition asserts that if both $b_1|(b_2|b_2)$ and $b_2|(b_1|b_1)$ are in F_{sBCK}, then the element $(b_3|(b_1|b_1))|((b_3|(b_2|b_2))|(b_3|(b_2|b_2)))$ must also be present in F_{sBCK}. If this condition fails for any trio of elements, the algorithm returns *False*.
- **Final return**:
  - If all the relevant conditions for a filter and a compatible filter are met, the algorithm concludes that the subset F_{sBCK} is indeed a filter or a compatible filter of $\mathcal{B}$ and returns *True*.

This Algorithm 4 ensures that the subset F_{sBCK} adheres to all necessary axioms to be recognized as a filter or a compatible filter within the framework of a Sheffer stroke BCK-algebra.

Definition 10. *Let $\mathcal{B} = (B; |, 0)$ be a Sheffer stroke BCK-algebra and let $D \subseteq B$. The set D is called an order-filter if it satisfies the following conditions:*

$$b_1 \in D \text{ and } b_1|(b_2|b_2) = 1 \Rightarrow b_2 \in D.$$

Also, we provide a pseudocode to check if a given subset D satisfies the conditions to be an order-filter on $\mathcal{B}$.

Algorithm 5: Confirming an order-filter

Input: Set B, subset $D \subseteq B$, operation $|$, and identity element 1
Output: Is D an order-filter of B?

```
IsOrderFilter(B, D)
for b1 ∈ D do
    for b2 ∈ B do
        if b1|(b2|b2) = 1 then
            if b2 ∉ D then Return False;
        end
    end
end
Return True;
```

Algorithm 5 is designed to verify whether a given subset D of a Sheffer stroke BCK-algebra $\mathcal{B}$ meets the necessary conditions to be classified as an order-filter. The algorithm systematically checks the following criterion:

- The algorithm iterates over each element b_1 within the subset D and examines whether, for any element b_2 in the set B, the condition $b_1|(b_2|b_2) = 1$ is satisfied. If this condition is met, the element b_2 must also belong to D. If b_2 is not found in D, the algorithm returns *False*, indicating that D does not qualify as an order-filter.
- If the condition holds true for all relevant elements, the algorithm returns *True*, confirming that D is indeed an order-filter within $\mathcal{B}$.

Definition 11. *Let $\mathcal{B} = (B; |, 0)$ be a Sheffer stroke BCK-algebra. A filter F_{sBCK} of $\mathcal{B}$ is called a prime filter if it fulfills the following condition:*

$$b_1|(b_2|b_2) \in F_{sBCK} \text{ or } b_2|(b_1|b_1) \in F_{sBCK}$$

for every $b_1, b_2 \in B$.

We present a pseudocode to determine whether a given subset F_{sBCK} meets the criteria to be a prime filter on $\mathcal{B}$.

Algorithm 6: Confirming a prime filter

Input: Set B, filter $F_{sBCK} \subseteq B$, operation |
Output: Is F_{sBCK} a prime filter of B?

$IsPrimeFilter(B, F_{sBCK})$
for $b_1, b_2 \in B$ **do**
  if $b_1|(b_2|b_2) \notin F_{sBCK}$ *and* $b_2|(b_1|b_1) \notin F_{sBCK}$ **then** Return $False$;
end
Return $True$;

Algorithm 6 is designed to verify whether a given filter F_{sBCK} of a Sheffer stroke BCK-algebra $\mathcal{B}$ meets the necessary condition to be considered a prime filter. The algorithm systematically checks the following criterion:

- The algorithm iterates over each pair of elements b_1 and b_2 in the set B. For each pair, it checks whether either the element $b_1|(b_2|b_2)$ or the element $b_2|(b_1|b_1)$ is included in the filter F_{sBCK}. If neither of these elements is found in the filter for any pair, the algorithm returns *False*, indicating that F_{sBCK} does not satisfy the prime filter condition.
- If the condition is satisfied for all pairs of elements, the algorithm returns *True*, confirming that F_{sBCK} is indeed a prime filter within $\mathcal{B}$.

Definition 12. *Let $\mathcal{B} = (B; |, 0)$ be a Sheffer stroke BCK-algebra, and let Θ be an equivalence relation on B. The relation Θ is called a right congruence on $\mathcal{B}$ if it satisfies the following condition:*

$$(b_1, b_2) \in \Theta \implies (b_1|(b_3|b_3), b_2|(b_3|b_3)) \in \Theta,$$

for every $b_1, b_2, b_3 \in B$. The set of all such right congruences is denoted by $\mathbf{Con}_R(\mathcal{B})$.

Similarly, Θ is referred to as a left congruence on $\mathcal{B}$ if it fulfills the condition:

$$(b_1, b_2) \in \Theta \implies ((b_1|b_1)|b_3, (b_2|b_2)|b_3) \in \Theta,$$

for all $b_1, b_2, b_3 \in B$. The set of all such left congruences is denoted by $\mathbf{Con}_L(\mathcal{B})$.

If Θ belongs to both $\mathbf{Con}_R(\mathcal{B})$ and $\mathbf{Con}_L(\mathcal{B})$, then it is called a congruence on $\mathcal{B}$. The collection of all congruences on $\mathcal{B}$ is denoted by $\mathbf{Con}(\mathcal{B})$. In summary, we can express this as:

$$\mathbf{Con}(\mathcal{B}) = \mathbf{Con}_R(\mathcal{B}) \cap \mathbf{Con}_L(\mathcal{B}).$$

We now outline a systematic method for determining whether a given equivalence relation qualifies as a right congruence, left congruence, or a congruence in a Sheffer stroke BCK-algebra through the following algorithm.

Algorithm 7: Confirming right, left, and general congruences

Input: Set B, equivalence relation $\Theta \subseteq B \times B$, operation $|$
Output: Is Θ a right congruence, left congruence, or congruence of B?

$IsRightCongruence(B, \Theta)$
for $(b_1, b_2) \in \Theta$ **do**
    for $b_3 \in B$ **do**
        if $(b_1|(b_3|b_3), b_2|(b_3|b_3)) \notin \Theta$ **then** Return $False$;
    end
end
$RightCongruence \leftarrow True$;

$IsLeftCongruence(B, \Theta)$
for $(b_1, b_2) \in \Theta$ **do**
    for $b_3 \in B$ **do**
        if $((b_1|b_1)|b_3, (b_2|b_2)|b_3) \notin \Theta$ **then** Return $False$;
    end
end
$LeftCongruence \leftarrow True$;

$IsCongruence(B, \Theta)$
if $RightCongruence$ ***and*** $LeftCongruence$ **then**
    Return $True$;
    // *Θ is a congruence of $\mathcal{B}$*
end
else
    if $RightCongruence$ **then** Return $RightCongruence$;
    // *Θ is a right congruence of $\mathcal{B}$*;
    if $LeftCongruence$ **then** Return $LeftCongruence$;
    // *Θ is a left congruence of $\mathcal{B}$*;
    else Return $False$;
    // *Θ is not a congruence of $\mathcal{B}$*;
end

The purpose of Algorithm 7 is to evaluate whether a given equivalence relation Θ on a Sheffer stroke BCK-algebra $\mathcal{B}$ meets the necessary criteria to be classified as a right congruence, left congruence, or a general congruence. The algorithm follows these steps:

- **Verification of right congruence**:
  - The algorithm iterates through each pair (b_1, b_2) within the equivalence relation Θ and checks whether, for every element b_3 in the set B, the pair $(b_1|(b_3|b_3), b_2|(b_3|b_3))$ is also included in Θ. If this condition fails for any pair, the algorithm determines that Θ does not qualify as a right congruence and outputs *False*. If the condition holds for all pairs, the *RightCongruence* flag is set to *True*.
- **Verification of left congruence**:
  - The algorithm similarly checks whether the condition $((b_1|b_1)|b_3, (b_2|b_2)|b_3) \in \Theta$ is satisfied for each pair (b_1, b_2) in Θ and every element b_3 in B. If this condition is not met for any pair, the algorithm concludes that Θ is not a left congruence and returns *False*. Otherwise, the *LeftCongruence* flag is set to 'True'.
- **Verification of general congruence**:
  - If both the *RightCongruence* and *LeftCongruence* flags are set to *True*, the algorithm confirms that Θ is a general congruence of $\mathcal{B}$ and returns *True*. If only one of the flags is *True*, the algorithm indicates that Θ is either a right or left congruence, depending on which flag is true. If neither flag is *True*, the algorithm returns *False*, indicating that Θ is not a congruence of $\mathcal{B}$.

Algorithm 7 offers a comprehensive and efficient approach for verifying congruence properties within the framework of a Sheffer stroke BCK-algebra.

Lemma 8. *Let τ^I be an internal state on $\mathcal{B}$. Then, the following properties hold:*

(i) *The kernel of τ^I, denoted as $Ker(\tau^I)$, is a compatible filter of B.*
(ii) *If every element in B is comparable with the others, then $Ker(\tau^I)$ is a prime filter of B.*
(iii) *If τ^I is faithful, it preserves the strict order.*
(iv) *If b and $\tau^I(b)$ are comparable, then $b \in Fix(\tau^I)$.*

Proof. (i) To establish that $Ker(\tau^I)$ is a compatible filter of B, we need to verify the conditions for a compatible filter.

$(F_{sBCK}1)$ Suppose $a \in Ker(\tau^I)$ and $b_1|(b_2|b_2) \in Ker(\tau^I)$. This implies that $\tau^I(b_1) = 0|0$ and $\tau^I(b_1|(b_2|b_2)) = 0|0$. Utilizing Proposition 3 (i) and (ii), we find that $b_1 = \tau^I(\tau^I(b_1)) = \tau^I(0|0) = 0|0$ and $b_1|(b_2|b_2) = \tau^I(\tau^I(b_1|(b_2|b_2))) = \tau^I(0|0) = 0|0$. Since $b_1|(b_2|b_2) = 0|0$, we conclude that $0|0 = b_1 \leq b_2 \leq 0|0$. Therefore, $b_2 = 0|0$ and $\tau^I(b_2) = \tau^I(0|0) = 0|0$, which implies that $b_2 \in Ker(\tau^I)$.
$(F_{sBCK}2)$ Suppose $b_1|(b_2|b_2) \in Ker(\tau^I)$ and $b_2|(b_1|b_1) = 0|0$. Given that $b_1|(b_2|b_2) \in Ker(\tau^I)$, it follows that $b_1|(b_2|b_2) = 0|0$. Since both $b_2|(b_1|b_1) = 0|0$ and $b_1|(b_2|b_2) = 0|0$, we conclude that $b_2 \leq b_1$ and $b_1 \leq b_2$ according to the definition of $\leq_X$, which implies $b_1 = b_2$. Therefore, we obtain

$$(b_2|(b_3|b_3))|((b_1|(b_3|b_3))|(b_1|(b_3|b_3))) = (b_1|(b_3|b_3))|((b_1|(b_3|b_3))|(b_1|(b_3|b_3))) = 0|0$$

for all $b_3 \in B$. Consequently, we have $(b_2|(b_3|b_3))|((b_1|(b_3|b_3))|(b_1|(b_3|b_3))) \in Ker(\tau^I)$.

($F_{sBCK}3$) Suppose $b_1|(b_2|b_2) \in Ker(\tau^I)$ and $b_2|(b_1|b_1) \in Ker(\tau^I)$. It follows immediately that $b_1 = b_2$. Therefore, we have

$$(b_3|(b_1|b_1))|((b_3|(b_2|b_2))|(b_3|(b_2|b_2))) = (b_3|(b_1|b_1))|((b_3|(b_1|b_1))|(b_3|(b_1|b_1))) = 0|0$$

for all $b_3 \in B$. Consequently, we conclude that $(b_3|(b_1|b_1))|((b_3|(b_2|b_2))|(b_3|(b_2|b_2))) \in Ker(\tau^I)$.
(ii) Let $b_1, b_2 \in B$ be elements that are comparable with each other. This implies that either $b_1 \leq b_2$ or $b_2 \leq b_1$. If we assume $b_1 \leq b_2$, then we have $\tau^I(b_1|(b_2|b_2)) = \tau^I(0|0) = 0|0$. Consequently, we find that $b_1|(b_2|b_2) \in Ker(\tau^I)$. Similarly, in the case where $b_2 \leq b_1$, we obtain $b_2|(b_1|b_1) \in Ker(\tau^I)$.
(iii) Suppose $b_1 < b_2$. Then, it follows that $b_1|(b_2|b_2) = 0|0$ and $\tau^I(b_1) \leq \tau^I(b_2)$. Assume, for the sake of contradiction, that $\tau^I(b_1) = \tau^I(b_2)$. By Proposition 3 ($v$), since $\tau^I(b_1) = \tau^I(b_2)$, it follows that $\tau^I(b_2|(b_1|b_1)) = \tau^I(b_2)|(\tau^I(b_1)|\tau^I(b_1)) = 0|0$. If $\tau^I(b_2|(b_1|b_1)) = 0|0$, then we must have $b_2|(b_1|b_1) = 1$. This implies $b_2 \leq b_1$, which contradicts the assumption that $b_1 < b_2$. Therefore, we conclude that $\tau^I(b_1) < \tau^I(b_2)$, meaning that τ^I is strictly order-preserving.
(iv) Suppose that b_1 and $\tau^I(b_1)$ are comparable. This means that either $b_1 \leq \tau^I(b_1)$ or $\tau^I(b_1) \leq b_1$. Assume $b_1 \leq \tau^I(b_1)$. By Proposition 3 (ii), we have $\tau^I(b_1) \leq \tau^I(b_1) = b_1$, leading to $b_1 \leq \tau^I(b_1) \leq b_1$. Consequently, this implies $b_1 = \tau^I(b_1)$, meaning $b_1 \in Fix(\tau^I)$. A similar conclusion can be drawn in the opposite case. □

Example 4. *Let us consider the set* $K = \{0, k_1, k_2, k_3, k_4, k_5, k_6, 1\}$. *The operation* $|$ *on* K *is defined as shown in Table 2.*

Table 2. The groupoid $(K; |)$.

$\|$	0	k_1	k_2	k_3	k_4	k_5	k_6	1
0	1	1	1	1	1	1	1	1
k_1	1	k_2	1	1	k_3	k_2	1	k_2
k_2	1	1	k_1	k_1	1	1	k_5	k_1
k_3	1	1	k_1	k_4	1	1	k_5	k_4
k_4	1	k_3	1	1	k_3	k_3	1	k_3
k_5	1	k_2	1	1	k_3	k_6	1	k_6
k_6	1	1	k_5	k_5	1	1	k_5	k_5
1	1	k_2	k_1	k_4	k_3	k_6	k_5	0

It can be verified that the structure $\mathcal{K} = (K; |, 0)$ *constitutes a Sheffer stroke BCK-algebra. In the sequel, we define the internal state* τ^I *for each* $k \in K$ *as indicated in Table 3.*

Table 3. Internal state τ^I on K.

k	0	k_1	k_2	k_3	k_4	k_5	k_6	1
$\tau^I(k)$	0	k_2	k_1	k_3	k_4	k_6	k_5	1

It follows that $Ker(\tau^I) = \{1\}$. *Therefore,* $Ker(\tau^I)$ *is a compatible filter of* K. *However, since the elements of* K *are not comparable with one another,* $Ker(\tau^I)$ *is not a prime filter of* K. *Notably, we observe that* $k_1|(k_2|k_2) = k_2 \notin Ker(\tau^I)$ *and* $k_2|(k_1|k_1) = k_1 \notin Ker(\tau^I)$. *Additionally, it is straightforward to verify that* τ^I *is strictly order-preserving as it is faithful. Finally,* m *and* $\tau^I(m)$ *are not comparable for* $m \in \{k_1, k_2, k_5, k_6\}$, *while* m *and* $\tau^I(m)$ *are comparable for* $m \in \{0, k_3, k_4, 1\}$, *where* $m \in Fix(\tau^I) = \{0, k_3, k_4, 1\}$. *In conclusion, all the properties of Lemma 8 are satisfied by this example.*

Lemma 9. *Let τ^I be an internal state on $\mathcal{B}$. Then the following properties hold for all $b_1, b_2 \in B$:*

(*i*) $\tau^I(b_1|b_1) = \tau^I(b_1)|\tau^I(b_1)$,
(*ii*) $\tau^I(\tau^I(b_1)|\tau^I(b_1)) = b_1|b_1$,
(*iii*) $\tau^I(\tau^I(b_1|b_1)|\tau^I(b_2|b_2)) = \tau^I(b_1|b_1)|\tau^I(b_2|b_2)$.

Proof. (*i*) For any $b_1 \in B$, consider $\tau^I(b_1|b_1)$. This expression can be rewritten as $\tau^I(b_1|(0|0))$. According to Proposition 3 (*vii*), we can simplify this further: $\tau^I(b_1|(0|0)) = \tau^I(b_1)|(\tau^I(0)|\tau^I(0)) = \tau^I(b_1)|(0|0) = \tau^I(b_1)|\tau^I(b_1)$. This follows because $0|(b_1|b_1) = 0|0$ holds true for any $b_1 \in B$.
(*ii*) This follows directly from Lemma 9 (*i*) and Proposition 3 (*ii*).
(*iii*) Applying Lemma 9 (*i*), we have:

$$\tau^I(\tau^I(b_1|b_1)|\tau^I(b_2|b_2)) = \tau^I(\tau^I(b_1|b_1)|(\tau^I(b_2)|\tau^I(b_2)))$$

for any $b_1, b_2 \in B$. Using Definition 6 ($\tau^I_{SH}4$) and Lemma 9 (*i*), it follows that:

$$\tau^I(\tau^I(b_1|b_1)|(\tau^I(b_2)|\tau^I(b_2))) = \tau^I(b_1|b_2)|(\tau^I(b_2)|\tau^I(b_2)) = \tau^I(b_1|b_1)|\tau^I(b_2|b_2).$$

□

In the remainder of this section, we explore the concepts of congruences and filters (including order, prime, and compatible filters) through the framework of the internal state on $\mathcal{B}$.

Definition 13. *Let τ^I be an internal state on $\mathcal{B}$.*
(*i*) *If F_{sBCK} is a filter of $\mathcal{B}$ such that for every $b \in F_{sBCK}$, we have $\tau^I(b) \in F_{sBCK}$, then F_{sBCK} is called an* internal-state filter *of $\mathcal{B}$, denoted by ISF_{sBCK}.*
(*ii*) *If F_{sBCK} is a compatible filter of $\mathcal{B}$ such that for every $b \in F_{sBCK}$, we have $\tau^I(b) \in F_{sBCK}$, then F_{sBCK} is called an* internal-state compatible filter *of $\mathcal{B}$, denoted by $ISCF_{sBCK}$.*
(*iii*) *If F_{sBCK} is a prime filter of $\mathcal{B}$ such that for every $b \in F_{sBCK}$, we have $\tau^I(b) \in F_{sBCK}$, then F_{sBCK} is called an* internal-state prime filter *of $\mathcal{B}$, denoted by $ISPF_{sBCK}$.*

We now present a systematic approach for determining whether a given filter qualifies as an internal-state filter, internal-state compatible filter, or internal-state prime filter in a Sheffer stroke BCK-algebra using the following algorithm.

Algorithm 8: Confirming internal state-filters

Input: Set B, filter $F_{sBCK} \subseteq B$, internal state $\tau^I : B \to B$
Output: Is F_{sBCK} an ISF_{sBCK}, $ISCF_{sBCK}$, or $ISPF_{sBCK}$ of B?

$IsInternalStateFilter(B, F_{sBCK}, \tau^I)$
for $b \in F_{sBCK}$ **do**
  if $\tau^I(b) \notin F_{sBCK}$ **then** Return $False$;
end
Return $True$;

$IsInternalStateCompatibleFilter(B, F_{sBCK}, \tau^I)$
for $b \in F_{sBCK}$ **do**
  if $\tau^I(b) \notin F_{sBCK}$ **then** Return $False$;
end
Return $True$;

$IsInternalStatePrimeFilter(B, F_{sBCK}, \tau^I)$
for $b \in F_{sBCK}$ **do**
  if $\tau^I(b) \notin F_{sBCK}$ **then** Return $False$;
end
Return $True$;

Algorithm 8 is designed to verify whether a given filter F_{sBCK} of a Sheffer stroke BCK-algebra $\mathcal{B}$ qualifies as an internal state-filter, internal state-compatible filter, or internal state-prime filter. The algorithm systematically checks the following condition:

- The algorithm iterates over each element b in the filter F_{sBCK} and examines whether the image of b under the internal state mapping $\tau^I : B \to B$ is also contained within F_{sBCK}. If there exists any element b in the filter such that $\tau^I(b)$ is not included in F_{sBCK}, the algorithm concludes that F_{sBCK} does not satisfy the internal state-filter condition and returns *False*.
- If the condition is satisfied for all elements, the algorithm returns *True*, confirming that F_{sBCK} qualifies as an internal state-filter (ISF_{sBCK}), internal state-compatible filter ($ISCF_{sBCK}$), or internal state-prime filter ($ISPF_{sBCK}$), depending on the specific context.

Algorithm 8 provides a reliable method for verifying internal state-filter properties, ensuring that the filter F_{sBCK} meets the necessary conditions within the structure of a Sheffer stroke BCK-algebra.

Lemma 10. *Each of these sets* $\{0|0\}$ *and* B *is an* ISF_{sBCK}, $ISCF_{sBCK}$, *and* $ISPF_{sBCK}$ *of* $\mathcal{B}$.

Example 5. *Let* $\mathcal{K} = (K; |, 0)$ *be a Sheffer stroke BCK-algebra as defined in Example 4. The internal state* τ^I *is also defined as in Example 4.*

- *If we consider the set* $F_{sBCK} = \{k_4, 1\}$, *then* F_{sBCK} *is a filter of* $\mathcal{K}$. *Since* $\tau^I(1) = 1$ *and* $\tau^I(k_4) = k_4$, *we have* $\tau^I(F_{sBCK}) = \{k_4, 1\} = F_{sBCK}$. *Thus,* F_{sBCK} *is an* ISF_{sBCK} *of* $\mathcal{K}$.
- *If we take the set* $Q_{sBCK} = \{k_4, k_5, k_6, 1\}$, Q_{sBCK} *is a prime filter of* $\mathcal{K}$. *We find that* $\tau^I(Q_{sBCK}) = \{k_4, k_5, k_6, 1\} = Q_{sBCK}$ *because* $\tau^I(1) = 1$, $\tau^I(k_4) = k_4$, $\tau^I(k_5) = k_6$, *and* $\tau^I(k_6) = k_5$. *Therefore,* Q_{sBCK} *is an* $ISPF_{sBCK}$ *of* $\mathcal{K}$.

- *If we consider the set* $T_{sBCK} = \{k_1, k_2, k_4, 1\}$*,* T_{sBCK} *is a compatible filter of* $\mathcal{K}$*. Since* $\tau^I(1) = 1$*,* $\tau^I(k_4) = k_4$*,* $\tau^I(k_1) = k_2$*, and* $\tau^I(k_2) = k_1$*, we have* $\tau^I(T_{sBCK}) = \{k_1, k_2, k_4, 1\} = T_{sBCK}$*. Therefore,* T_{sBCK} *is an* $ISCF_{sBCK}$ *of* $\mathcal{K}$*.*

Theorem 6. *Let* F_{sBCK} *be an internal state-filter (*ISF_{sBCK}*) in* $\mathcal{B}$*. Then, the following properties are satisfied:*

(*i*) $\tau^I(F_{sBCK}) = F_{sBCK} \cap \tau^I(B)$,
(*ii*) $\tau^I(F_{sBCK})$ *forms a filter within* $\tau^I(B)$.

Proof. (*i*) Let F_{sBCK} be an internal state-filter (ISF_{sBCK}) of $\mathcal{B}$. Suppose $b_1 \in F \cap \tau^I(B)$. By using Proposition 3 (*viii*), we find that $b_1 \in F \cap Fix(\tau^I)$. Since $b_1 \in Fix(\tau^I)$, it follows that $\tau^I(b_1) = b_1$. Additionally, $b_1 \in F_{sBCK}$ implies that $\tau^I(b_1) \in \tau^I(F_{sBCK})$, which means $b_1 \in \tau^I(F_{sBCK})$. Thus, we obtain:

$$F_{sBCK} \cap \tau^I(B) \subseteq \tau^I(F_{sBCK}). \tag{4.3}$$

Now, assume $b_1 \in \tau^I(F_{sBCK})$. There exists some $b_2 \in F_{sBCK}$ such that $\tau^I(b_2) = b_1$. Therefore, $b_1 \in \tau^I(B)$. Since F_{sBCK} is an ISF_{sBCK} and $b_2 \in F_{sBCK}$, we conclude that $\tau^I(b_2) = b_1 \in F_{sBCK}$. This implies $b_1 \in F_{sBCK} \cap \tau^I(B)$. Hence, we have:

$$\tau^I(F_{sBCK}) \subseteq F_{sBCK} \cap \tau^I(B). \tag{4.4}$$

By combining (4.3) and (4.4), we conclude that $\tau^I(F_{sBCK}) = F_{sBCK} \cap \tau^I(B)$.
(*ii*) To demonstrate that $\tau^I(F_{sBCK})$ is a filter of $\tau^I(B)$, we must verify the conditions ($F_{sBCK}1$) and ($F_{sBCK}2$).
($F_{sBCK}1$) Suppose $b_1 \in \tau^I(F_{sBCK})$ and $b_1|(b_2|b_2) \in \tau^I(F_{sBCK})$. Given that $\tau^I(F_{sBCK}) = F_{sBCK} \cap \tau^I(B)$, we know that $b_1 \in F_{sBCK} \cap \tau^I(B)$ and $b_1|(b_2|b_2) \in F_{sBCK} \cap \tau^I(B)$. Since F_{sBCK} is a filter of $\mathcal{B}$, it follows that $b_2 \in F_{sBCK}$. By Definition 13, we then have $b_2 \in \tau^I(F_{sBCK})$.
($F_{sBCK}2$) A similar approach can be used to verify that this condition holds as well. □

Corollary 1. *If* F_{sBCK} *is an* $ISCF_{sBCK}$ *of* $\mathcal{B}$*, then* $\tau^I(F_{sBCK})$ *is a compatible filter of* $\tau^I(B)$*.*

Proof. This follows directly from Theorem 6. □

Definition 14. *Let* Θ *be a congruence relation on* $\mathcal{B}$*. If* $b_1\Theta b_2$ *implies* $\tau^I(b_1)\Theta\tau^I(b_2)$*, then* Θ *is called an internal-state congruence relation on* $\mathcal{B}$ *and is denoted by* $IS - \mathbf{Con}_{sBCK}$*.*

Proposition 4. *Let* Θ *and* Ψ *be two* $IS - \mathbf{Con}_{sBCK}$ *relations on* $\mathcal{B}$*. Then the following properties hold:*

(*i*) *The equivalence class* $[0|0]_\Theta$ *is an* $ISCF_{sBCK}$ *of* $\mathcal{B}$*.*
(*ii*) *If* $[0|0]_\Theta = [0|0]_\Psi$*, then it follows that* $\Theta = \Psi$*.*

Proof. (*i*) It is evident that $[0|0]_\Theta$ is a compatible filter of $\mathcal{B}$. To show that $[0|0]_\Theta$ is an $ISCF_{sBCK}$, it suffices to prove that $\tau^I(b) \in [0|0]_\Theta$ for any $b \in B$. Assume $b \in [0|0]_\Theta$. This implies that $b\Theta(0|0)$. Since $\Theta \in IS - \mathbf{Con}_{sBCK}$, it follows that $\tau^I(b)\Theta\tau^I(0|0) = 0|0$. Therefore, $\tau^I(b) \in [0|0]_\Theta$. Consequently, we conclude that $[0|0]_\Theta$ is an $ISCF_{sBCK}$ of $\mathcal{B}$.
(*ii*) Suppose that $[0|0]_\Theta = [0|0]_\Psi$ and $(b, 0|0) \in \Theta$. Then we have:

$$(b, 0|0) \in \Theta \Rightarrow b \in [0|0]_\Theta = [0|0]_\Psi$$

$$\Rightarrow (b, 0|0) \in \Psi.$$

Thus, we conclude that $\Theta \subseteq \Psi$, and by a similar argument, $\Psi \subseteq \Theta$. Therefore, we reach the conclusion that $\Theta = \Psi$. □

Theorem 7. *If F_{sBCK} is an $IS\,F_{sBCK}$ of $\mathcal{B}$, then the mapping*

$$\begin{aligned}\Omega^I : \mathcal{B}/F_{sBCK} &\to \mathcal{B}/F_{sBCK}\\ b/F_{sBCK} &\longmapsto \Omega^I(b/F_{sBCK}) = \tau^I(b)/F_{sBCK}\end{aligned}$$

defines an internal state on $\mathcal{B}/F_{sBCK}$.

Proof. First of all, we show that the mapping Ω^I is well-defined. Suppose $b_1/F_{sBCK} = b_2/F_{sBCK}$ for some $b_1, b_2 \in B$. According to Definition 9, this implies that $b_1|(b_2|b_2) \in F_{sBCK}$ and $b_2|(b_1|b_1) \in F_{sBCK}$, which further implies that $b_1|(b_2|b_2) = 0|0$ or $b_2|(b_1|b_1) = 0|0$. Since F_{sBCK} is an $IS\,F_{sBCK}$, we obtain $\tau^I(b_1|(b_2|b_2)) \in F_{sBCK}$ and $\tau^I(b_2|(b_1|b_1)) \in F_{sBCK}$. By Proposition 3, we have $\tau^I(b_1)|(\tau^I(b_2)|\tau^I(b_2)) \in F_{sBCK}$ and $\tau^I(b_2)|(\tau^I(b_1)|\tau^I(b_1)) \in F_{sBCK}$. Therefore, we conclude that $\Omega^I(b_1/F_{sBCK}) = \Omega^I(b_2/F_{sBCK})$, confirming that Ω^I is well-defined.

$(\tau^I_{sBCK}1)$: Assume that $(b_1/F_{sBCK})|((b_2/F_{sBCK})|(b_2/F_{sBCK})) = 0|0$. Then, we have:

$$\begin{aligned}0|0 &= \tau^I((b_1/F_{sBCK})|((b_2/F_{sBCK})|(b_2/F_{sBCK})))\\ &= (\tau^I(b_1)/F_{sBCK})|((\tau^I(b_2)/F_{sBCK})|(\tau^I(b_2)/F_{sBCK}))\\ &= \Omega^I(b_1/F_{sBCK})|(\Omega^I(b_2/F_{sBCK})|\Omega^I(b_2/F_{sBCK})).\end{aligned}$$

Since we have

$$\Omega^I(b_1/F_{sBCK})|(\Omega^I(b_2/F_{sBCK})|\Omega^I(b_2/F_{sBCK})) = 0|0,$$

it follows that $\Omega^I(b_1/F_{sBCK}) \leq \Omega^I(b_2/F_{sBCK})$.

$(\tau^I_{sBCK}2)$: Let $b_1/F_{sBCK}, b_2/F_{sBCK} \in B/F_{sBCK}$ for some $b_1, b_2 \in B$. Since τ^I is an internal state, we have:

$$\begin{aligned}&\Omega^I((b_1/F_{sBCK})|((b_2/F_{sBCK})|(b_2/F_{sBCK})))\\ &= (\tau^I(b_1)/F_{sBCK})|((\tau^I(b_2)/F_{sBCK})|(\tau^I(b_2)/F_{sBCK}))\\ &= \tau^I(b_1|(b_2|b_2))/F_{sBCK}\\ &= [\tau^I((b_1|(b_2|b_2))|(b_2|b_2))/F_{sBCK}]|[(\tau^I(b_2)/F_{sBCK})|(\tau^I(b_2)/F_{sBCK})]\\ &= \Omega^I(((b_1/F_{sBCK})|((b_2/F_{sBCK})|(b_2/F_{sBCK})))|((b_2/F_{sBCK})|(b_2/F_{sBCK})))|(\Omega^I(b_2/F_{sBCK})|\Omega^I(b_2/F_{sBCK})).\end{aligned}$$

$(\tau^I_{sBCK}3)$–$(\tau^I_{sBCK}4)$: Following a similar reasoning as above, these conditions can also be verified.

Therefore, we conclude that the mapping Ω^I is indeed an internal state on $\mathcal{B}/F_{sBCK}$. □

5. A generalized state and a general state-morphism on Sheffer stroke basic algebras

In this section, we define a general state on a Sheffer stroke BCK-algebra $\mathcal{B}$ as a mapping $\tau : B \to \mathbb{R}$ that satisfies three criteria: non-negativity ($\tau_{sBCK}1$), normalization ($\tau_{sBCK}2$), and additivity ($\tau_{sBCK}3$). We present an algorithm to verify these conditions, ensuring that a mapping qualifies as a general state.

We also introduce several key lemmas and theorems. Lemma 11 establishes that if $b_1|b_2 = 0|0$, then $(b_1|b_1)|(b_2|b_2) = b_1 \vee b_2$. Theorem 8 demonstrates the independence of the axioms defining a general state. Moreover, we discuss the relationship between general states and other states like Riečan and Bosbach states, establishing that every general state is also a Riečan state and a Bosbach state.

In addition, we introduce the concept of a general state-morphism and show its properties, such as the equation $\tau(b) + \tau(b|b) = 1$ for all $b \in B$. The theorems and lemmas provided illustrate the structure and behavior of general states within the algebra, ensuring a thorough understanding of their role in Sheffer stroke BCK-algebras.

Definition 15. *A mapping $\tau : B \to \mathbb{R}$ is termed a general state on $\mathcal{B}$ if it satisfies the following criteria:*
($\tau_{sBCK}1$) *For every $b \in B$, $\tau(b) \geq 0$,*
($\tau_{sBCK}2$) $\tau(0|0) = 1$*,*
($\tau_{sBCK}3$) $\tau(b_1|b_2) = \tau(b_1|b_1) + \tau(b_2|b_2)$*, provided that* $(b_1|b_1)|(b_2|b_2) = 0|0$.

Also, we present a pseudocode to verify if a given mapping satisfies the criteria to be a general state from $\mathcal{B}$ to $\mathbb{R}$.

Algorithm 9: Confirming a general state

Input: Set B, mapping $\tau : B \to \mathbb{R}$, operation |
Output: Is τ a general state on B?

IsGeneralState(B, τ)
for $b \in B$ **do**
  if $\tau(b) < 0$ **then** Return *False* // *Condition* ($\tau_{sBCK}1$) *fails*;
end
if $\tau(0|0) \neq 1$ **then** Return *False* // *Condition* ($\tau_{sBCK}2$) *fails*;
for $b_1, b_2 \in B$ **do**
  if $(b_1|b_1)|(b_2|b_2) = 0|0$ **then**
    if $\tau(b_1|b_2) \neq \tau(b_1|b_1) + \tau(b_2|b_2)$ **then** Return *False* // *Condition* ($\tau_{sBCK}3$) *fails*;
  end
end
Return *True*;

The Algorithm 9 is designed to determine whether a given mapping $\tau : B \to \mathbb{R}$ on a Sheffer stroke BCK-algebra $\mathcal{B}$ meets the criteria to be considered a general state. The algorithm systematically checks the following conditions:

- **Condition** ($\tau_{sBCK}1$): The algorithm iterates over each element b in the set B to ensure that $\tau(b) \geq 0$. If any value of $\tau(b)$ is found to be negative, the algorithm concludes that τ cannot be a general state and immediately returns *False*.
- **Condition** ($\tau_{sBCK}2$): The algorithm checks whether $\tau(0|0) = 1$. This condition is crucial, and if it

is not satisfied, the algorithm returns *False*, indicating that τ does not meet the necessary criteria to be a general state.

- **Condition** $(\tau_{sBCK}3)$: For every pair of elements b_1 and b_2 in B, the algorithm verifies whether the mapping satisfies $\tau(b_1|b_2) = \tau(b_1|b_1) + \tau(b_2|b_2)$, provided that the condition $(b_1|b_1)|(b_2|b_2) = 0|0$ holds. If this condition is violated for any pair, the algorithm returns *False*.
- **Final Decision**: If all three conditions are fulfilled, the algorithm concludes that the mapping τ qualifies as a general state on $\mathcal{B}$ and returns *True*.

Algorithm 9 provides an efficient and systematic method for determining whether a mapping qualifies as a general state, ensuring that all necessary conditions are met within the framework of a Sheffer stroke BCK-algebra.

Before introducing a generalization of the state on orthomodular lattices, we require the following lemma.

Lemma 11. *Let $b_1, b_2 \in B$. If $b_1|b_2 = 0|0$, then $(b_1|b_1)|(b_2|b_2) = b_1 \vee b_2$.*

Proof. Suppose $b_1|b_2 = 0|0$. Then, the following steps hold:

$$\begin{aligned} b_1|b_2 = 0|0 &\implies b_1 \leq_X b_2|b_2, \\ &\implies b_1|b_1 = b_1|(b_2|b_2), \\ &\implies (b_1|b_1)|(b_2|b_2) = (b_1|(b_2|b_2))|(b_2|b_2) = b_1 \vee b_2. \end{aligned}$$

□

Theorem 8. *The set of axioms defining a general state on a Sheffer stroke BCK-algebra is independent.*

Proof. Consider the Example 4. Then, we obtain the following conclusions:
(1) **Independence of** $(\tau_{sBCK}1)$**:** Let the mapping $\tau : K \to \mathbb{R}$ be defined as follows:

$$\tau(b) := \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b = 1, \\ \frac{3}{2}, & \text{if } b \in \{k_1, k_4, k_5\}, \\ \frac{-1}{2}, & \text{if } b \in \{k_2, k_3, k_6\}. \end{cases}$$

Under this definition, the structure $\mathcal{K}$ meets the criteria for $(\tau_{sBCK}2)$ and $(\tau_{sBCK}3)$, but it fails to satisfy $(\tau_{sBCK}1)$ because $\tau(k_6) = \frac{-1}{2}$, which is not greater than or equal to 0.
(2) **Independence of** $(\tau_{sBCK}2)$**:** Consider the mapping $\tau : K \to \mathbb{R}$ defined by $\tau(b) := 0$ for every $b \in K$. Under this mapping, the structure $\mathcal{K}$ satisfies both $(\tau_{sBCK}1)$ and $(\tau_{sBCK}3)$, but it does not satisfy $(\tau_{sBCK}2)$ because $\tau(0|0) = 0$.
(3) **Independence of** $(\tau_{sBCK}3)$**:** Consider the mapping $\tau : K \to \mathbb{R}$ defined by $\tau(b) := 1$ for all $b \in K$. Under this definition, the structure $\mathcal{K}$ satisfies both $(\tau_{sBCK}1)$ and $(\tau_{sBCK}2)$, but it does not satisfy $(\tau_{sBCK}3)$ because $1 = \tau((0|0)|0) \neq \tau((0|0)|(0|0)) + \tau(0|0) = 2$, where $((0|0)|(0|0))|(0|0) = 0|0$. □

Example 6. *Let $\mathcal{K} = (K; |, 0)$ be the Sheffer stroke BCK-algebra defined in the proof of Theorem 8.*

Given the commutative nature of $\mathcal{K}$, certain conditions are equivalent, allowing us to omit them. Before defining a general state on $\mathcal{K}$, we must consider the following cases to verify $(\tau_{sBCK}3)$:

- $0|0 = (1|1)|(k_1|k_1)$
- $0|0 = (1|1)|(k_2|k_2)$
- $0|0 = (1|1)|(k_3|k_3)$
- $0|0 = (1|1)|(k_4|k_4)$
- $0|0 = (1|1)|(k_5|k_5)$
- $0|0 = (1|1)|(k_6|k_6)$
- $0|0 = (1|1)|(1|1)$
- $0|0 = (1|1)|(0|0)$
- $0|0 = (k_1|k_1)|(k_3|k_3)$
- $0|0 = (k_1|k_1)|(k_6|k_6)$
- $0|0 = (k_2|k_2)|(k_4|k_4)$
- $0|0 = (k_2|k_2)|(k_5|k_5)$
- $0|0 = (k_3|k_3)|(k_5|k_5)$
- $0|0 = (k_4|k_4)|(k_6|k_6)$

Considering these cases, the following equations are obtained:

$$0|0 = (1|1)|(k_1|k_1) \Rightarrow \tau(1|k_1) = \tau(1|1) + \tau(k_1|k_1) \Rightarrow \tau(k_2) = \tau(0) + \tau(k_2), \tag{5.1}$$

$$0|0 = (1|1)|(k_2|k_2) \Rightarrow \tau(1|k_2) = \tau(1|1) + \tau(k_2|k_2) \Rightarrow \tau(k_1) = \tau(0) + \tau(k_1), \tag{5.2}$$

$$0|0 = (1|1)|(k_3|k_3) \Rightarrow \tau(1|k_3) = \tau(1|1) + \tau(k_3|k_3) \Rightarrow \tau(k_4) = \tau(0) + \tau(k_4), \tag{5.3}$$

$$0|0 = (1|1)|(k_4|k_4) \Rightarrow \tau(1|k_4) = \tau(1|1) + \tau(k_4|k_4) \Rightarrow \tau(k_3) = \tau(0) + \tau(k_3), \tag{5.4}$$

$$0|0 = (1|1)|(k_5|k_5) \Rightarrow \tau(1|k_5) = \tau(1|1) + \tau(k_5|k_5) \Rightarrow \tau(k_6) = \tau(0) + \tau(k_6), \tag{5.5}$$

$$0|0 = (1|1)|(k_6|k_6) \Rightarrow \tau(1|k_6) = \tau(1|1) + \tau(k_6|k_6) \Rightarrow \tau(k_5) = \tau(0) + \tau(k_5), \tag{5.6}$$

$$0|0 = (1|1)|(1|1) \Rightarrow \tau(1|1) = \tau(1|1) + \tau(1|1) \Rightarrow \tau(0) = \tau(0) + \tau(0), \tag{5.7}$$

$$0|0 = (1|1)|(0|0) \Rightarrow \tau(1|0) = \tau(1|1) + \tau(0|0) \Rightarrow \tau(0|0) = \tau(0) + \tau(0|0), \tag{5.8}$$

$$0|0 = (k_1|k_1)|(k_3|k_3) \Rightarrow \tau(k_1|k_3) = \tau(k_1|k_1) + \tau(k_3|k_3) \Rightarrow \tau(1) = \tau(k_2) + \tau(k_4), \tag{5.9}$$

$$0|0 = (k_1|k_1)|(k_6|k_6) \Rightarrow \tau(k_1|k_6) = \tau(k_1|k_1) + \tau(k_6|k_6) \Rightarrow \tau(1) = \tau(k_2) + \tau(k_5), \tag{5.10}$$

$$0|0 = (k_2|k_2)|(k_4|k_4) \Rightarrow \tau(k_2|k_4) = \tau(k_2|k_2) + \tau(k_4|k_4) \Rightarrow \tau(1) = \tau(k_1) + \tau(k_3), \tag{5.11}$$

$$0|0 = (k_2|k_2)|(k_5|k_5) \Rightarrow \tau(k_2|k_5) = \tau(k_2|k_2) + \tau(k_5|k_5) \Rightarrow \tau(1) = \tau(k_1) + \tau(k_6), \tag{5.12}$$

$$0|0 = (k_3|k_3)|(k_5|k_5) \Rightarrow \tau(k_3|k_5) = \tau(k_3|k_3) + \tau(k_5|k_5) \Rightarrow \tau(1) = \tau(k_4) + \tau(k_6), \tag{5.13}$$

$$0|0 = (k_4|k_4)|(k_6|k_6) \Rightarrow \tau(k_4|k_6) = \tau(k_4|k_4) + \tau(k_6|k_6) \Rightarrow \tau(1) = \tau(k_3) + \tau(k_5). \tag{5.14}$$

By consolidating the results from equations (5.1) to (5.14), we can establish a general state on $\mathcal{K}$ as follows:

$$\tau(b) := \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b = 1, \\ 1 - m, & \text{if } b \in \{k_1, k_4, k_5\}, \\ m, & \text{if } b \in \{k_2, k_3, k_6\}, \end{cases}$$

where $m \in (0, 1)$.

Theorem 9. *Every general state on $\mathcal{B}$ is simultaneously a Riečan state and a Bosbach state.*

Proof. This result follows directly from Definition 15 and Theorem 3. □

Lemma 12. *Let $\mathcal{B} = (B; |, 0)$ be a Sheffer stroke BCK-algebra. Then, the inequality*

$$(b_1|(b_2|b_2))|(b_1|(b_2|b_2)) \leq_X b_2|b_2$$

holds for all $b_1, b_2 \in B$.

Proof. Assume $b_1, b_2 \in B$. Using Definition 2 ($S3$), Proposition 1 (a1), Proposition 1 (a5), and Definition 2 ($S3$), we derive the following:

$$\begin{aligned} & (((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|((b_2|b_2)|(b_2|b_2)))|(((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|((b_2|b_2)|(b_2|b_2))) \\ = \ & (((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|b_2)|(((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|b_2) \\ = \ & (((b_2|(b_2|b_2))|(b_2|(b_2|b_2)))|b_1)|(((b_2|(b_2|b_2))|(b_2|(b_2|b_2)))|b_1) \\ = \ & (0|b_1)|(0|b_1) \\ = \ & (0|0)|(0|0) \\ = \ & 0. \end{aligned}$$

By the definition of $\leq_X$, this implies that $(b_1|(b_2|b_2))|(b_1|(b_2|b_2)) \leq_X b_2|b_2$ for all $b_1, b_2 \in B$. □

Lemma 13. *Let $\tau : B \to \mathbb{R}$ be a general state on $\mathcal{B}$. Then, the following properties hold for all $b_1, b_2 \in B$:*
(i) $\tau(b_1 \vee b_2) = \tau((b_1|(b_2|b_2))|(b_1|(b_2|b_2))) + \tau(b_2)$,
(ii) $\tau((b_1|b_2)|b_2) = \tau((b_1|b_2)|(b_1|b_2)) + \tau(b_2|b_2)$,
(iii) $\tau((b_1|(b_2|b_2))|(b_1|(b_2|b_2))) = \tau(b_1) - \tau(b_2)$, *where* $b_2|(b_1|b_1) = 0|0$,
(iv) $\tau(b_1|b_2) = \tau(\neg b_1 \wedge b_2) + \tau(b_2|b_2)$.

Proof. (*i*) Suppose $b_1, b_2 \in B$. According to Lemma 12, we have $(b_1|(b_2|b_2))|(b_1|(b_2|b_2)) \leq_X b_2|b_2$. This leads to the equation:

$$((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|((b_2|b_2)|(b_2|b_2)) = 0|0.$$

Using Definition 2 (*ii*), we obtain:

$$((((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|((b_1|(b_2|b_2))|(b_1|(b_2|b_2))))|(((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))| \\ ((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))))|((b_2|b_2)|(b_2|b_2)) = 0|0.$$

Since the mapping τ is a state, the following equation holds:

$$\tau((((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|((b_1|(b_2|b_2))|(b_1|(b_2|b_2))))|(b_2|b_2)) = \tau((b_1|(b_2|b_2))|(b_1|(b_2|b_2))) + \tau(b_2).$$

Applying Definition 2 (*ii*) again, we find:

$$(((b_1|(b_2|b_2))|(b_1|(b_2|b_2)))|((b_1|(b_2|b_2))|(b_1|(b_2|b_2))))|(b_2|b_2) = (b_1|(b_2|b_2))|(b_2|b_2) = b_1 \vee b_2.$$

Thus, we conclude that:

$$\tau(b_1 \vee b_2) = \tau((b_1|(b_2|b_2))|(b_1|(b_2|b_2))) + \tau(b_2).$$

(*ii*) By replacing b_2 with $b_2|b_2$ in Lemma 13 (*i*), we arrive at the following expression:

$$\tau((b_1|b_2)|b_2) = \tau((b_1|b_2)|(b_1|b_2)) + \tau(b_2|b_2).$$

(*iii*) Suppose $b_2|(b_1|b_1) = 0|0$. This implies that $b_2 \leq_X b_1$. Utilizing Lemma 13 (*i*), we find:

$$\tau((b_1|(b_2|b_2))|(b_1|(b_2|b_2))) = \tau(b_1) - \tau(b_2).$$

(*iv*) For each $b_1 \in B$, it holds that $b_1 \leq_X 0|0$. Applying Lemma 2 (*iii*), we find that $(0|0)|b_2 \leq b_1|b_2$ for any $b_2 \in B$. Consequently, we have $b_2|b_2 \leq_X b_1|b_2$, which implies $(b_2|b_2)|((b_1|b_2)|(b_1|b_2)) = 0|0$.
From Lemma 13 (*iii*), we obtain:

$$\tau(((b_1|b_2)|((b_2|b_2)|(b_2|b_2)))|((b_1|b_2)|((b_2|b_2)|(b_2|b_2)))) = \tau(b_1|b_2) - \tau(b_2|b_2).$$

Note that, according to Definition 2 (*ii*), we have $(b_2|b_2)|(b_2|b_2) = b_2$. Thus, we can express:

$$\tau(((b_1|b_2)|b_2)|((b_1|b_2)|b_2)) = \tau(b_1|b_2) - \tau(b_2|b_2).$$

Using the definition of the $\neg$ operator, we confirm:

$$\tau(b_1|b_2) = \tau(\neg b_1 \wedge b_2) + \tau(b_2|b_2)$$

for all $b_1, b_2 \in B$. □

Lemma 14. *The general state τ is monotonic.*

Proof. Let $b_1, b_2 \in B$ such that $b_1 \leq_X b_2$. This implies that $b_1|(b_2|b_2) = 0|0$. Using Lemma 13 (*iii*), we have:

$$\tau((b_2|(b_1|b_1))|(b_2|(b_1|b_1))) = \tau(b_2) - \tau(b_1).$$

By applying Definition 15 ($\tau_{sBCK}1$), we obtain:

$$0 \leq_X \tau((b_2|(b_1|b_1))|(b_2|(b_1|b_1))) = \tau(b_2) - \tau(b_1).$$

Hence, we conclude that $\tau(b_1) \leq_X \tau(b_2)$, which establishes the monotonicity of the general state τ. □

Lemma 15. *Let $\tau : B \to \mathbb{R}$ be a general state on $\mathcal{B}$. Then, the following properties hold:*

(*i*) $\tau((0|0)|b) = 1 - \tau(b)$, *for all* $b \in B$,
(*ii*) $\tau(b_1|b_2) = 1 - \tau(b_1)$, *where* $b_1 \leq b_2$,
(*iii*) $\tau(b_1|b_1) + \tau(b_2|b_2) \leq 2\tau(b_1|b_2)$, *for all* $b_1, b_2 \in B$.

Proof. (*i*) Let $b \in B$. By applying Lemma 13 (*iii*), we obtain the following:

$$\begin{aligned} \tau((0|0)|b) &= \tau(b|b) \\ &= \tau(((0|0)|(b|b))|((0|0)|(b|b))) \\ &= \tau((0|0) \vee b) - \tau(b) \\ &= \tau(0|0) - \tau(b) \\ &= 1 - \tau(b). \end{aligned}$$

(*ii*) Let $b_1, b_2 \in B$ such that $b_1 \leq b_2$. According to Lemma 1, we have $b_1|b_1 = b_1|b_2$. Therefore, we can derive:

$$\begin{aligned} \tau(b_1|b_2) &= \tau(b_1|b_1) \\ &= \tau(((0|0)|(b_1|b_1))|((0|0)|(b_1|b_1))) \\ &= 1 - \tau(b_1), \end{aligned}$$

using Lemma 15 (*i*).
(*iii*) For any $b_1 \in B$, we know that $b_1 \leq 0|0$. Using Lemma 2 (*iii*), it follows that $(0|0)|b_2 \leq b_1|b_2$. Since $(0|0)|b_2 = b_2|b_2$, we have $b_2|b_2 \leq b_1|b_2$. From Lemma 14, it follows that $\tau(b_2|b_2) \leq \tau(b_1|b_2)$, and similarly, $\tau(b_1|b_1) \leq \tau(b_1|b_2)$. As a result, we obtain $\tau(b_1|b_1)+\tau(b_2|b_2) \leq 2\tau(b_1|b_2)$ for all $b_1, b_2 \in B$. □

We now define the concept of a general state-morphism on a Sheffer stroke BCK-algebra.

Definition 16. *A general state* τ *is called a general state-morphism if it satisfies the condition*

$$\tau(b_1|b_2) = \tau(b_1)|\tau(b_2)$$

for all $b_1, b_2 \in B$.

Lemma 16. *Let* τ *be a general state-morphism on* $\mathcal{B}$. *Then the following properties hold for all* $b_1, b_2 \in B$:

(*i*) $(0|0)|\tau(b_1) = 1 - \tau(b_1)$,
(*ii*) $2 - (\tau(b_1) + \tau(b_2)) \leq 2\tau(b_1|b_2)$,
(*iii*) $\tau(b_1|b_2)|\tau(b_2) = 2 - \tau(b_1|b_2) - \tau(b_2)$.

Proof. Assume that τ is a general state-morphism on $\mathcal{B}$. (*i*) From Lemma 15, it follows that the equation $(0|0)|\tau(b_1) = \tau(0|0)|\tau(b_1) = \tau((0|0)|b_1) = 1 - \tau(b_1)$ holds true for all $b \in B$.
(*ii*) According to Lemma 15, we have $\tau(b_1|b_1) + \tau(b_2|b_2) \leq 2\tau(b_1|b_2)$ for all $b_1, b_2 \in B$. By applying Lemma 16 (*i*), we derive that $2 - (\tau(b_1) + \tau(b_2)) \leq 2\tau(b_1|b_2)$.
(*iii*) This result follows directly from Lemma 4 (*iv*) and Theorem 9. □

Theorem 10. *Let* τ *be a general state-morphism on* $\mathcal{B}$. *Then, for all* $b \in B$, *the equation* $\tau(b) + \tau(b|b) = 1$ *holds.*

Proof. This result follows directly from Lemmas 15 and 16. □

Example 7. *Using the structure $\mathcal{K} = (K; |, 0)$ from Example 4, let us define the mapping $\tau : K \to \mathbb{R}$ as follows:*

$$\tau(b) := \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b = 1, \\ \frac{1}{2}, & \text{if } b \in \{k_1, k_2, k_3, k_4, k_5, k_6\}. \end{cases}$$

We verify that τ satisfies the conditions of a general state-morphism:

- ($\tau_{sBCK}1$) *For every $b \in K$, $\tau(b) \geq 0$.*
- ($\tau_{sBCK}2$) $\tau(0|0) = \tau(1) = 1$.
- ($\tau_{sBCK}3$) *For $b_1, b_2 \in K$, $\tau(b_1|b_2) = \tau(b_1|b_1) + \tau(b_2|b_2)$, provided $(b_1|b_1)|(b_2|b_2) = 0|0$.*

Finally, since τ is a general state-morphism, we verify the conclusion of the Theorem 10:

- *For $b = 0$, $b|b = 1$, and $\tau(b) + \tau(b|b) = 0 + 1 = 1$.*
- *For $b = 1$, $b|b = 0$, and $\tau(b) + \tau(b|b) = 1 + 0 = 1$.*
- *For $b \in \{k_1, k_2, k_3, k_4, k_5, k_6\}$, we get $b|b \in \{k_1, k_2, k_3, k_4, k_5, k_6\}$. As a result, we attain*

$$\tau(b) + \tau(b|b) = \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, the equation $\tau(b) + \tau(b|b) = 1$ holds for all $b \in K$, as required.

6. Conclusions

In this paper, we have introduced and thoroughly explored the concepts of Riečan and Bosbach states within the context of Sheffer stroke BCK-algebra $\mathcal{B}$. By providing precise definitions and illustrative examples, we have established a foundational understanding of these states and their significant roles within the algebraic structures of $\mathcal{B}$. A notable aspect of our work is the development of algorithms and verification methods to determine the validity of mappings as Riečan or Bosbach states, which underscores the practical applicability of these concepts in various fields.

One of the key achievements of this study is the identification of conditions under which Riečan and Bosbach states are equivalent. This finding is particularly significant as it reconciles two distinct approaches within the theory of BCK-algebras, providing a unified framework that enhances our understanding of the underlying algebraic structures. Furthermore, by extending our investigation to include internal states on $\mathcal{B}$, we have revealed new connections between these states and other critical algebraic concepts, such as filters and congruences. These results not only deepen our theoretical insights but also pave the way for potential applications of these states across various mathematical disciplines.

Moreover, the independence of the axiomatic systems for Riečan and Bosbach states, as demonstrated in this work, affirms the robustness and validity of our approach. This independence ensures that the definitions and properties we have established are fundamentally rooted in the intrinsic structure of Sheffer stroke BCK-algebras. Consequently, this research offers a substantial and lasting contribution to the field, providing a comprehensive understanding of these states and laying the groundwork for future explorations.

Looking ahead, further investigations could delve into the relationships between these states and other algebraic structures, such as generalized states on orthomodular lattices. Additionally, examining the applications of these states in broader mathematical and computational contexts—particularly in areas like quantum logic and theoretical computer science—could yield valuable insights and open new avenues for research. This study establishes a solid theoretical framework that subsequent research can build upon, reinforcing the originality and relevance of our findings in advancing the understanding of Sheffer stroke BCK-algebras and their applications.

Author contributions

Ibrahim Senturk: Conceptualization, Methodology, Investigation, Writing and editing, Review and editing; Tahsin Oner: Conceptualization, Methodology, Investigation, Writing and editing, Review and editing; Duygu Selin Turan: Investigation, Writing and editing; Gozde Nur Gurbuz: Investigation, Writing and editing; Burak Ordin: Methodology, Review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors affirm that no Artificial Intelligence (AI) tools were utilized in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interests concerning the publication of this article.

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