AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 1488–1500.
DOI: 10.3934/math.2025069
Received: 01 October 2024
Revised: 16 December 2024
Accepted: 15 January 2025
Published: 22 January 2025

Research article

Monodromic singularities without curves of zero angular speed

Isaac A. García and Jaume Giné*

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, Lleida 25001, Spain

* **Correspondence:** Email: jaume.gine@udl.cat.

Abstract: We consider planar analytic vector fields $\mathcal{X}$ having a monodromic singular point with Poincaré map Π. We use the fact that there always exists a real analytic invariant curve $F = 0$ of $\mathcal{X}$ in a neighborhood of that singularity. We find some relations between Π and F that can be used to determine new conditions that guarantee the analyticity of Π at the singularity. In the special case that F becomes an inverse integrating factor of $\mathcal{X}$, we rediscover formulas obtained previously by other methods. Applications to the center-focus problem and also to vector fields with degenerate infinity are given.

Keywords: center; periodic orbits; Poincaré map; invariant curve
Mathematics Subject Classification: 34Cxx, 37G15, 37G10

1. Introduction and main results

We focus in this work on families of planar real analytic vector fields $\mathcal{X} = P(x, y; \lambda)\partial_x + Q(x, y; \lambda)\partial_y$ defined in a neighborhood of a monodromic singularity that can be placed at the origin without loss of generality. Here $\lambda \in \mathbb{R}^p$ denotes the finite number of parameters in the family. We recall that a monodromic singular point of $\mathcal{X}$ is a singularity of $\mathcal{X}$ such that the associated flow rotates around it, and therefore a Poincaré map Π is well defined in a sufficiently small transversal section with an end point at the singularity. Centers and foci are examples of monodromic singularities. We will restrict the family to the monodromic parameter space $\Lambda \subset \mathbb{R}^p$ defined as the parameter subset for which the origin is a monodromic singularity of $\mathcal{X}$. The common characterization of Λ is via the blow-up procedure developed by Dumortier in [1]; see also Arnold [2]. It is worth emphasizing that Algaba and co-authors present in [3, 4] an algorithmic procedure to determine the parameter restrictions that defines Λ in terms of the Newton diagram $\mathbf{N}(\mathcal{X})$ of the vector field $\mathcal{X}$; see also [5].

In [6], an explicit analytic first-order ordinary differential equation was obtained for the Poincaré map Π associated with a monodromic singularity (without local zero angular speed curves) of $\mathcal{X}$ under the assumption of the existence of a Laurent inverse integrating factor V of $\mathcal{X}$. Later on, in [7], an example of focus without Puiseux (a generalization of Laurent) inverse integrating factor is presente. This example shows that the differential equation obtained in [6] for Π is not universal.

Initially, our first main objective in this work was to find an explicit analytic ordinary differential equation $f(\rho, \Pi, \Pi', \ldots, \Pi^k) = 0$ for the Poincaré map $\Pi(\rho)$ associated to a monodromic singularity of $\mathcal{X}$ where f only depends on objects that do not include the flow of $\mathcal{X}$. If this objective were achieved, an alternative proof of the fact that Π admits an asymptotic Dulac series can be constructed via Bruno's theory [8, 9] of asymptotic solutions of analytic differential equations at singularities. In this work we take the former universal object to be the analytic invariant curve $F(x, y) = 0$ through the monodromic singularity that there always exists, as it was proved in [10]. Using this idea, we find a first order ordinary differential equation for Π only depending on F; see Theorem 1.1 in Section 1.1, although we have been able to prove its analyticity only under some assumptions; see, for example, Theorem 1.6. This equation generalizes the one obtained in [6] in the sense that both coincides in the particular case that F becomes an inverse integrating factor of $\mathcal{X}$, as we show in Section 2.

1.1. The relations between Π *and* F

We denote by $\mathbf{N}(\mathcal{X})$ the Newton diagram of $\mathcal{X}$ that is composed of edges joining both positive semi-axis; see [11] for a detailed account of this construction and also [3]. Each edge of $\mathbf{N}(\mathcal{X})$ has endpoints in $\mathbb{N}^2$ and slope $-p/q$ with $(p, q) \in \mathbb{N}^2$ coprimes. From now on we denote by $W(\mathbf{N}(\mathcal{X})) \subset \mathbb{N}^2$ the set of all weights (p, q). We will use the *weighted polar* blow-up $(x, y) \mapsto (\rho, \varphi)$ given by

$$x = \rho^p \cos\varphi, \quad y = \rho^q \sin\varphi, \tag{1.1}$$

with Jacobian

$$J(\varphi, \rho) = \rho^{p+q-1}(p\cos^2\varphi + q\sin^2\varphi).$$

In these coordinates, the system associated to $\mathcal{X}$ is written in the form $\dot{\rho} = R(\varphi, \rho)$, $\dot{\varphi} = \Theta(\varphi, \rho)$ with $R(\varphi, 0) = 0$ and $\Theta(\varphi, \rho) = G_r(\varphi) + O(\rho)$. Therefore, the equations of the orbits of $\mathcal{X}$ near the origin are governed by the equation

$$\frac{d\rho}{d\varphi} = \mathcal{F}(\varphi, \rho) := \frac{R(\varphi, \rho)}{\Theta(\varphi, \rho)} \tag{1.2}$$

well defined in $C \backslash \Theta^{-1}(0)$ being the cylinder $C = \left\{(\theta, \rho) \in \mathbb{S}^1 \times \mathbb{R} : 0 \leq \rho \ll 1\right\}$ with $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. the local set of *zero angular speed* is defined as $\Theta^{-1}(0) = \{(\varphi, \rho) \in C : \Theta(\varphi, \rho) = 0, \rho \geq 0\}$, and the set of *characteristic directions* $\Omega_{pq} = \{\varphi^* \in \mathbb{S}^1 : \Theta(\varphi^*, 0) = G_r(\varphi^*) = 0\}$. Notice that the set $\{\rho = 0\}$ is invariant for the flow of (1.2), and it becomes either a periodic orbit or a polycycle according to whether $\Omega_{pq} = \emptyset$ or $\Omega_{pq} \neq \emptyset$, respectively.

We also define the (p, q)-*critical parameters* as the elements of the subset $\Lambda_{pq} \subset \Lambda$ of the monodromic parameter space corresponding to vector fields with local curves of zero angular speed, that is,

$$\Lambda_{pq} = \{\lambda \in \Lambda : \Theta^{-1}(0) \backslash \{\rho = 0\} \neq \emptyset\}.$$

We emphasize that $\Lambda_{pq} = \emptyset$ when $\Omega_{pq} = \emptyset$ but the converse is not true.

Given $(p, q) \in W(\mathbf{N}(\mathcal{X}))$, we take the (p, q)-quasihomogeneous expansion

$$\mathcal{X} = \sum_{j \geq r} \mathcal{X}_j, \tag{1.3}$$

where $r \geq 1$ and $\mathcal{X}_j$ are (p, q)-quasihomogeneous vector fields of degree j, and $\mathcal{X}_r$ is called the leading vector field of $\mathcal{X}$.

Let $F(x, y) = 0$ be a real invariant analytic curve of $\mathcal{X}$ with $F(0, 0) = 0$, and denote by $K(x, y)$ the *cofactor* of F, that is, $\mathcal{X}(F) = KF$ holds. We recall that F always exists, as it was proved in [10] using the separatrix theorem of Camacho-Sad. In weighted polar coordinates this equation is transformed into

$$\hat{\mathcal{X}}(\hat{F}) = \hat{K}\hat{F}, \tag{1.4}$$

where $\hat{\mathcal{X}} = \partial_\varphi + \mathcal{F}(\varphi, \rho)\partial_\rho$, $\hat{F}(\varphi, \rho) = F(\rho^p \cos\varphi, \rho^q \sin\varphi)$ and $\hat{K}$ is the cofactor of the invariant curve $\hat{F} = 0$ of $\hat{\mathcal{X}}$ whose expression is

$$\hat{K}(\varphi, \rho) = \frac{K(\rho^p \cos\varphi, \rho^q \sin\varphi)}{\rho^r \Theta(\varphi, \rho)}. \tag{1.5}$$

Let $\Phi(\varphi; \rho_0)$ be the flow of $\hat{\mathcal{X}}$ with initial condition $\Phi(0; \rho_0) = \rho_0$. In [10], it is proved that

$$I_{pq}(\rho_0) = PV \int_0^{2\pi} \hat{K}(\varphi, \Phi(\varphi; \rho_0))\, d\varphi, \tag{1.6}$$

exists for any initial condition $\rho_0 > 0$ sufficiently small. Working in $\Lambda \backslash \Lambda_{pq}$ we may also define the integral

$$\zeta_{pq}(\rho_0) = \int_0^{2\pi} \frac{\partial \mathcal{F}}{\partial \rho}(\varphi, \Phi(\varphi; \rho_0))\, d\varphi, \tag{1.7}$$

and the difference

$$\alpha_{pq}(\rho_0) = \zeta_{pq}(\rho_0) - I_{pq}(\rho_0).$$

A fundamental result in this work is the following one.

Theorem 1.1. *The equation*

$$\hat{F}(0, \Pi(\rho_0)) = \hat{F}(0, \rho_0)\, \exp\left(I_{pq}(\rho_0)\right) \tag{1.8}$$

holds. Moreover, in $\Lambda \backslash \Lambda_{pq}$, the relations

$$\zeta_{pq}(\rho_0) = \log\left(\Pi'(\rho_0)\right), \tag{1.9}$$

and

$$\hat{F}(0, \Pi(\rho_0)) = \hat{F}(0, \rho_0)\, \exp(-\alpha_{pq}(\rho_0))\, \Pi'(\rho_0) \tag{1.10}$$

are satisfied.

Remark 1.2. We believe that the fundamental equation (1.10) holds in the whole monodromic space Λ and not only in $\Lambda \backslash \Lambda_{pq}$, but we have no proof.

Remark 1.3. If $\mathcal{X}$ has $\ell > 1$ invariant analytic curves $F_i = 0$ with cofactors K_i for $i = 1, \ldots, \ell$, then it also has the ℓ-parameter invariant analytic curve $F = \prod_i F_i^{m_i} = 0$ with arbitrary multiplicities $m_i \in \mathbb{N}$ and cofactor $K = \sum_i m_i K_i$. Sometimes we may find m_i that simplifies the expression of K. As an example, the case when F becomes an inverse integrating factor of $\mathcal{X}$, hence $K = \mathrm{div}(\mathcal{X})$, was studied in [6].

Il'Yashenko in [12] proves that Π has a Dulac asymptotic expansion possessing a linear leading term. More specifically, one has $\Pi(\rho_0) = \eta_1\rho_0 + o(\rho_0)$. The computation of the first (generalized) Poincaré-Lyapunov quantity η_1 is quite involved and needs the use of cumbersome blow-ups computations; see for example [13]. Some sufficient conditions that guarantee the computation of η_1 are stated below. We will user the following expansions. The functions Θ and R appearing in (1.2) have Taylor expansions at $\rho = 0$ given by $\Theta(\varphi,\rho) = G_r(\varphi) + O(\rho)$ and $R(\varphi,\rho) = F_r(\varphi)\rho + O(\rho^2)$. Moreover $F(x,y) = F_s(x,y) + \cdots$ and $K(x,y) = K_{\bar{r}}(x,y) + \cdots$ are the (p,q)-quasihomogeneous expansions of the cofactor K associated to the analytic invariant curve $F = 0$. Here F_s and $K_{\bar{r}}$ are the leading (p,q)-quasihomogeneous polynomials of weighted degree s and $\bar{r}$, respectively, and the dots denote higher (p,q)-quasihomogeneous terms.

Proposition 1.4. *Assume that both I_{pq} and ζ_{pq} can be extended with continuity to the origin. Then*

$$\log(\eta_1^s) = I_{pq}(0) = s\,\zeta_{pq}(0),$$

that is,

$$\log(\eta_1^s) = PV\int_0^{2\pi}\frac{K_{\bar{r}}(\cos\varphi,\sin\varphi)}{G_r(\varphi)}d\varphi = sPV\int_0^{2\pi}\frac{F_r(\varphi)}{G_r(\varphi)}d\varphi,$$

provided that both principal values exist.

Remark 1.5. It is worth emphasizing that, in general, I_{pq} cannot be extended by continuity at $\rho_0 = 0$. In [10], it is shown that expression (3.4) of η_1 is wrong in several examples.

Clearly, if $\Phi(\varphi;\rho_0)$ is analytic at $\rho_0 = 0$ for all $\varphi \in \mathbb{S}^1$ then Π is analytic at $\rho_0 = 0$. A sufficient condition for that to happen is that $\Omega_{pq} = \emptyset$. Instead, a weaker condition to ensure the analyticity of Π is the following one.

Theorem 1.6. *If I_{pq} is analytic at the origin, then Π is too.*

1.2. Center-focus problem

The classical center-focus problem has been studied for decades. In [14, 15], some particular degenerate systems with a monodromic singularity were studied. In [16, 17], it is proved that some degenerate systems with a monodromic singularity are limit of differential systems with monodromic linear part. In [18–20] some sufficient conditions to have a center at a completely degenerate critical point are given. In [21], the relation between the reversivility and the center problem is studied. The textbook [22] is a good summary about the relations between the the center and cyclicity problems. In [23], a geometrical criteria to determine the existence of a center for certain differential systems is given. In [24], the Hopf-bifurcation formulas for some differential systems are established. Final, in [25], the authors solve the center problem for monodromic sigularities with characteristic directions and with inverse integrating factor and [26] the linear term of all the monodromic families known is obtained. However, no general characterization was known until the work [10], where it is proved the following theorem.

Theorem 1.7 ([10]). *Let X be a family of analytic planar vector fields having a monodromic singular point at the origin and K the cofactor associated to a real analytic invariant curve through the origin. Then $I_{pq}(\rho_0)$ exists for any initial condition $\rho_0 > 0$ that is sufficiently small, and the origin is a center if and only if $I_{pq}(\rho_0) \equiv 0$.*

We show other necessary center condition in $\Lambda\backslash\Lambda_{pq}$.

Proposition 1.8. *If the origin is a center of $\mathcal{X}$ and we restrict to the parameter space $\Lambda\backslash\Lambda_{pq}$ then $\alpha_{pq}(\rho_0) = \zeta_{pq}(\rho_0) \equiv 0$.*

Notice that from Proposition 1.8 and relation $\alpha_{pq}(\rho_0) = \zeta_{pq}(\rho_0) - I_{pq}(\rho_0)$, we obtain a new proof of the necessary part in Theorem 1.7.

Example 1.9. *We emphasize that there are focus with parameters in $\Lambda\backslash\Lambda_{pq} \neq \emptyset$ and $\alpha_{pq}(\rho_0) \not\equiv 0$. As an example, we consider the family*

$$\begin{aligned} \dot{x} &= \lambda_1(x^6 + 3y^2)(-y + \mu x) + \lambda_2(x^2 + y^2)(y + Ax^3), \\ \dot{y} &= \lambda_1(x^6 + 3y^2)(x + \mu y) + \lambda_2(x^2 + y^2)(-x^5 + 3Ax^2 y). \end{aligned} \tag{1.11}$$

In [27], it is proved that

$$\Lambda = \{(\lambda_1, \lambda_2, \mu, A) \in \mathbb{R}^4 : 3\lambda_1 - \lambda_2 > 0, \;\; \lambda_1 - \lambda_2 > 0\}$$

and the Poincaré map is the linear map $\Pi(\rho_0) = \eta_1\rho_0$. Moreover, $F(x, y) = (x^2 + y^2)(x^6 + 3y^2)$ is an inverse integrating factor of the full family (1.11)*. Using the weights $(p, q) = (1, 1) \in W(\mathbf{N}(\mathcal{X}))$ we have $\mathcal{X} = \mathcal{X}_2 + \cdots$, hence $r = 2$, and the forthcoming formula* (2.3) *yields $\alpha_{11}(\rho_0) = -\log(\eta_1^7)$. The value $\eta_1 = \exp(2\pi\lambda_1\mu + 2\sqrt{3}\pi A\lambda_2/3)$ is a consequence of the works [6, 27]. Notice that in this example $0 \in \Omega_{11}$.*

1.3. Monodromic degenerate infinity vector fields

We consider analytic degenerate infinity vector fields

$$\mathcal{X} = \mathcal{X}_n + A\,\mathcal{X}_E \tag{1.12}$$

with $\mathcal{X}_n = P_{p+n}(x, y)\partial_x + Q_{q+n}(x, y)\partial_y$ a (p, q)-quasihomogeneous polynomial vector field of degree n, $\mathcal{X}_E = px\partial_x + qy\partial_y$ is the radial Euler field, and $A(x, y)$ a real analytic function in $\mathbb{R}^2$ whose Taylor expansion at the origin starts with (p, q)-quasihomogeneous terms of degree $m - 1$ with $m > n + 1$. In the work [28] the particular homogeneous case $(p, q) = (1, 1)$ is analyzed, and here we generalize it.

Proposition 1.10. *Any degenerate infinity vector field* (1.12) *has the homogeneous algebraic invariant curve*

$$F(x, y) = p\,x\,Q_{q+n}(x, y) - q\,y\,P_{p+n}(x, y) = 0$$

with cofactor

$$K(x, y) = \mathrm{div}(\mathcal{X}_n) + (n + p + q)A(x, y).$$

Moreover, it also has the inverse integrating factor $V = FH^{\frac{m-n-1}{d}}$ provided that A is (p, q)-quasihomogeneous of degree $m - 1$, where H is a (p, q)-quasihomogeneous first integral of degree d of $\mathcal{X}_n$.

Proposition 1.11. *Any degenerate infinity vector field* (1.12) *with a monodromic singularity at the origin becomes a focus provided that* $\mathrm{div}(\mathcal{X}_n) + (n + p + q)A(x, y)$ *is a positive or negative defined* *function in a neighborhood of the origin. If additionally the origin is a center of $\mathcal{X}_n$ and A is (p,q)-quasihomogeneous then the Poincaré map associated with the origin is analytic and has the form $\Pi(\rho) = \rho + \eta_{m-n}\rho^{m-n}(1 + O(\rho))$ with*

$$\eta_{m-n} = \int_0^{2\pi} \frac{(p\cos^2\varphi + q\sin^2\varphi)A(\cos\varphi,\sin\varphi)}{F(\cos\varphi,\sin\varphi)\,(H(\cos\varphi,\sin\varphi))^{(m-n-1)/d}}\,d\varphi.$$

In particular, the origin is a center if and only if $\eta_{m-n} = 0$ and the cyclicity of the origin in family (1.12) *is zero.*

Remark 1.12. We note that the (p,q)-quasihomogeneous first integral H of the monodromic vector field $\mathcal{X}_n$ appearing in Proposition 1.10 only exists in case the origin be a center of $\mathcal{X}_n$. This statement can be inferred from statement (ii) of Lemma 4 in [29] because the transformed first integral to (p,q)-weighted polar coordinates becomes a Laurent first integral of $\mathcal{X}_n$. In particular, $\Pi(\rho) = \rho + o(\rho)$ in agreement with Proposition 1.11.

2. The special case: F is an inverse integrating factor of $\mathcal{X}$

In this section we restrict our attention to the monodromic subset $\Lambda\backslash\Lambda_{pq}$ so that $\Theta^{-1}(0)\backslash\{\rho = 0\} = \emptyset$. In particular Ω_{pq} can be empty or not.

Let us assume the particular case that $F(x,y)$ is an analytic inverse integrating factor of $\mathcal{X}$; that is, $\mathrm{div}(\mathcal{X}/F) \equiv 0$ holds. Then the function

$$V(\varphi,\rho) = \frac{\hat{F}(\varphi,\rho)}{\rho^r\, J(\varphi,\rho)\, \Theta(\varphi,\rho)} \tag{2.1}$$

is an inverse integrating factor of $\hat{\mathcal{X}}$ in $C\backslash\{\Theta^{-1}(0) \cup \{\rho = 0\}\}$, that is, V satisfies the equation $\hat{\mathcal{X}}(V) = \partial_\rho(\mathcal{F})\,V$. Applying the differential operator $\hat{\mathcal{X}}$ on both sides of (2.1) and taking into account (1.4), we obtain the relation between $\hat{K}$ and $\partial_\rho\mathcal{F}$ given by

$$\partial_\rho\mathcal{F} = \hat{K} - \hat{\mathcal{X}}(\log|\rho^r\, J\,\Theta|). \tag{2.2}$$

On the other hand, using (2.2) and recalling that $\alpha_{pq}(\rho_0) = \zeta_{pq}(\rho_0) - I_{pq}(\rho_0)$, we obtain

$$\begin{aligned}
\alpha_{pq}(\rho_0) &= -\int_0^{2\pi} \hat{\mathcal{X}}(\log|\rho^r\, J\,\Theta|) \circ (\varphi, \Phi(\varphi;\rho_0))\,d\varphi \\
&= -\int_0^{2\pi} \frac{d}{d\varphi}\Big(\log|\rho^r\, J\,\Theta| \circ (\varphi, \Phi(\varphi;\rho_0))\Big)\,d\varphi, \\
&= -\log\left|\frac{\Pi^r(\rho_0)\, J(0,\Pi(\rho_0))\,\Theta(0,\Pi(\rho_0))}{\rho^r\, J(0,\rho_0)\,\Theta(0,\rho_0)}\right|,
\end{aligned} \tag{2.3}$$

where we have used the 2π-periodicity of J and Θ in the variable φ. Using this expression of α_{pq} and taking into account (2.1), Eq (1.10) in $\Lambda\backslash\Lambda_{pq}$ is written in the simpler form $V(0,\Pi(\rho_0)) = V(0,\rho_0)\,\Pi'(\rho_0)$. In this way we rediscover the formula obtained in [6] by other methods. This formula for the special case of degenerate differential systems without characteristic directions was given in [30]; see also [31].

3. Proofs of the results

3.1. Proof of Theorem 1.1

Proof. To prove the first statement, we evaluate (1.4) along the flow $\Phi(\varphi;\rho_0)$, and we obtain

$$\frac{d}{d\varphi}\hat{F}(\varphi,\Phi(\varphi;\rho_0)) \quad = \quad \hat{K}(\varphi,\Phi(\varphi;\rho_0))\,\hat{F}(\varphi,\Phi(\varphi;\rho_0)),$$

hence

$$PV\int_0^{2\pi}\frac{\frac{d}{d\varphi}\hat{F}(\varphi,\Phi(\varphi;\rho_0))}{\hat{F}(\varphi,\Phi(\varphi;\rho_0))}\,d\varphi = PV\int_0^{2\pi}\hat{K}(\varphi,\Phi(\varphi;\rho_0))\,d\varphi, \tag{3.1}$$

where the last principal value exist for any initial condition $\rho_0 > 0$ sufficiently small as it was proved in [10]. Therefore, (3.1) takes the form

$$\hat{F}(2\pi,\Phi(2\pi;\rho_0)) = \hat{F}(0,\rho_0)\,\exp\left(I_{pq}(\rho_0)\right).$$

Using the 2π-periodicity of $\hat{F}$ in the variable φ and the definition of Π, the former equation becomes (1.8).

To prove the second part, we use the definition of $\Phi(\varphi;\rho_0)$, that is,

$$\frac{\partial\Phi}{\partial\varphi}(\varphi;\rho_0) = \mathcal{F}(\varphi,\Phi(\varphi;\rho_0)), \quad \Phi(0;\rho_0) = \rho_0 > 0. \tag{3.2}$$

Working in $\Lambda\backslash\Lambda_{pq}$ we know that $\mathcal{F}$ is analytic in $C\backslash\{\rho = 0\}$; hence, Φ is also analytic there, and differentiating both expressions in (3.2) with respect to ρ_0 yields

$$\frac{\partial}{\partial\varphi}\left(\frac{\partial\Phi}{\partial\rho_0}(\varphi;\rho_0)\right) = \frac{\partial\mathcal{F}}{\partial\rho}(\varphi,\Phi(\varphi;\rho_0))\,\frac{\partial\Phi}{\partial\rho_0}(\varphi;\rho_0), \quad \frac{\partial\Phi}{\partial\rho_0}(0;\rho_0) = 1. \tag{3.3}$$

Clearly $(\varphi,\Phi(\varphi;\rho_0)) \in C\backslash\{\rho = 0\}$ for all $\varphi \in [0,2\pi]$, and therefore the function $\partial_\rho\mathcal{F}(\varphi,\Phi(\varphi;\rho_0))$ is continuous in $\mathbb{S}^1 \times \{0 < \rho_0 \ll 1\}$. Thus, we may integrate the first equality in (3.3), yielding

$$\int_0^{2\pi}\frac{\partial\mathcal{F}}{\partial\rho}(\varphi,\Phi(\varphi;\rho_0))\,d\varphi = \int_0^{2\pi}\frac{\frac{\partial}{\partial\varphi}\left(\frac{\partial\Phi}{\partial\rho_0}(\varphi;\rho_0)\right)}{\frac{\partial\Phi}{\partial\rho_0}(\varphi;\rho_0)}\,d\varphi.$$

So we obtain

$$\int_0^{2\pi}\frac{\partial\mathcal{F}}{\partial\rho}(\varphi,\Phi(\varphi;\rho_0))\,d\varphi = \left[\log\left(\frac{\partial\Phi}{\partial\rho_0}(\varphi;\rho_0)\right)\right]_{\varphi=0}^{\varphi=2\pi} = \log\left(\frac{\partial\Phi}{\partial\rho_0}(2\pi;\rho_0)\right)$$

that can be written as Eq (1.9). Finally, Eq (1.8) is rewritten as the fundamental ordinary differential equation (1.10). □

3.2. *Proof of Proposition 1.4*

Proof. Using the definition (1.5) of $\hat{K}$ together with the (p,q)-quasihomogeneous expansion $K(x,y) = K_{\bar{r}}(x,y) + \cdots$ and the Taylor expansion $\Theta(\varphi,\rho) = G_r(\varphi) + O(\rho)$ with $\bar{r} \geq r$ we obtain

$$\hat{K}(\varphi,\rho) = \rho^{\bar{r}-r}\frac{K_{\bar{r}}(\cos\varphi,\sin\varphi)}{G_r(\varphi)} + O(\rho^{\bar{r}-r+1}).$$

From [10], we know that it is proved that the principal value $I_{pq}(\rho_0)$ defined in (1.6) exists for any $\rho_0 > 0$ and small. If I_{pq} can be extended with continuity to the origin, then, using that $\Phi(\varphi;0) = 0$, we would have

$$I_{pq}(0) = \begin{cases} 0 & \text{if } \ \bar{r} > r, \\ PV\displaystyle\int_0^{2\pi}\frac{K_{\bar{r}}(\cos\varphi,\sin\varphi)}{G_r(\varphi)}d\varphi & \text{if } \ \bar{r} = r, \end{cases}$$

in case that this principal value exists. Moreover, from the (p,q)-quasihomogeneous expansion $F(x,y) = F_s(x,y) + \cdots$, we could use Eq (1.8) to express

$$I_{pq}(\rho_0) \;=\; \log\left|\frac{\hat{F}(0,\Pi(\rho_0))}{\hat{F}(0,\rho_0)}\right| = \log\left|\frac{\Pi^s(\rho_0)F_s(\cos\varphi,\sin\varphi) + O(\Pi^{s+1}(\rho_0))}{\rho_0^s(F_s(\cos\varphi,\sin\varphi) + O(\rho_0))}\right|$$

whose extension to $\rho_0 = 0$ gives $I_{pq}(0) = \log(\eta_1^s)$, taking into account that $\Pi(\rho_0) = \eta_1\rho_0 + o(\rho_0)$. Comparing both expressions of $I_{pq}(0)$, we have that if $\bar{r} > r$, then $\eta_1 = 1$, whereas

$$\log(\eta_1^s) = PV\int_0^{2\pi}\frac{K_{\bar{r}}(\cos\varphi,\sin\varphi)}{G_r(\varphi)}d\varphi, \tag{3.4}$$

when $\bar{r} = r$ under the restriction that the former principal value exists.

On the other hand, if ζ_{pq} can be extended with continuity to the origin, then, by (1.9),

$$\log(\eta_1) = \zeta_{pq}(0) = PV\int_0^{2\pi}\frac{F_r(\varphi)}{G_r(\varphi)}d\varphi,$$

assuming this last principal value exists. Then the proposition follows. □

3.3. *Proof of Theorem 1.6*

Proof. We define

$$f(\rho_0,\Pi) = \hat{F}(0,\Pi) - \hat{F}(0,\rho_0)\exp\left(I_{pq}(\rho_0)\right).$$

The analyticity of I_{pq} at the origin implies that (1.8) can be written as $f(\rho_0,\Pi) = 0$, where f is an analytic function in a neighborhood of $(\rho_0,\Pi) = (0,0)$. Therefore, the Poincaré map $\Pi(\rho_0) = \eta_1\rho_0 + o(\rho_0)$ must be a branch of f at the origin and, consequently, admits the convergent Puiseux expansion

$$\Pi(\rho_0) = \sum_{i\geq 0}\eta_{i+1}\,\rho_0^{1+\frac{i}{n}} \tag{3.5}$$

with some index $n \in \mathbb{N}^*$. Now we are going to compute n.

By Proposition 1.4 we know that $I_{pq}(0) = \log(\eta_1^s)$. Then $\exp(I_{pq}(\rho_0)) = \eta_1^s + O(\rho_0)$ with $\eta_1 > 0$. Using that $\hat{F}(0,\rho_0) = \rho_0^s(F_s(1,0) + O(\rho_0))$, we obtain that $f(\rho_0,\Pi) = F_s(1,0)(\Pi^s - \eta_1^s\rho_0^s) + \cdots$ from where we deduce that the Newton diagram of f only contains the edge L joining the endpoints $(s, 0)$ and $(0, s)$ provided that $F_s(1, 0) \neq 0$. We can take $F_s(1, 0) \neq 0$ because, without loss of generality, we take $0 \notin \Omega_{pq}$ after a rotation (if necessary), and moreover we claim that the eventual real roots of $F_s(\cos\varphi, \sin\varphi)$ must belong to Ω_{pq}. To prove the claim, we observe that

$$\hat{F}(\varphi, \rho) = F(\rho^p \cos\varphi, \rho^q \sin\varphi) = \rho^s [F_s(\cos\varphi, \sin\varphi) + o(\rho)],$$

hence $\hat{F}(\varphi, \rho)/\rho^s = 0$ is an invariant curve of the differential equation (1.2). Clearly that invariant curve can only intersect the monodromic polycycle $\rho = 0$ at its singularities $(\varphi, \rho) = (\varphi^*, 0)$ with $\varphi^* \in \Omega_{pq}$.

Once we know that $F_s(1, 0) \neq 0$ so that the edge L has slope -1, then we compute

$$f(\rho_0, \rho_0\mu) = \rho_0^s((\mu^s - \eta_1^s)F_s(1, 0) + O(\rho_0)).$$

From this expression we deduce that the determining polynomial $\mathcal{P}(\mu)$ associated to L is $\mathcal{P}(\mu) = F_s(1, 0)(\mu^s - \eta_1^s)$. Therefore $\mu = \eta_1$ is a simple root of $\mathcal{P}$, and it follows that $n = 1$ in the Puiseux expansion (3.5); this is a classical result that is proved, for instance, in [32]. Therefore, Π admits a convergent power series expansion at the origin finishing the proof. □

3.4. Proof of Proposition 1.8

Proof. Let the origin be a center. Then Eq (1.10) must have the solution $\Pi(\rho_0) = \rho_0$ and therefore $\hat{F}(0, \rho_0) = \hat{F}(0, \rho_0) \exp(-\alpha_{pq}(\rho_0))$ holds. We claim that $\hat{F}(0, \rho_0) \not\equiv 0$ so it must occur $\alpha_{pq}(\rho_0) \equiv 0$ proving the proposition.

To prove the claim, we use the (p, q)-quasihomogeneous expansion $F(x, y) = F_s(x, y) + \cdots$ with $F_s(x, y) \not\equiv 0$ a (p, q)-quasihomogeneous polynomial of degree s and the dots are (p, q)-quasihomogeneous terms of higher degree. Notice that $s \geq 1$ because $(x, y) = (0, 0)$ is an isolated real zero of F. Now we consider the expression

$$\hat{F}(\varphi, \rho) = F(\rho^p \cos\varphi, \rho^q \sin\varphi) = \rho^s [F_s(\cos\varphi, \sin\varphi) + O(\rho)], \tag{3.6}$$

and we observe that $\hat{G}(\varphi, \rho) = \hat{F}(\varphi, \rho)/\rho^s = 0$ is an invariant curve $\hat{\mathcal{X}}$. Thus either

$$\hat{G}^{-1}(0) \cap \{\rho = 0\} = \emptyset$$

or

$$\hat{G}^{-1}(0) \cap \{\rho = 0\} \subset \{(\varphi, \rho) = (\varphi^*, 0) : \varphi^* \in \Omega_{pq}\}$$

by uniqueness of solutions of $\hat{\mathcal{X}}$. But $\hat{G}^{-1}(0) \backslash \{\rho = 0\} = \emptyset$ by the monodromy of the polycycle $\{\rho = 0\}$. Therefore $\hat{F}(\varphi, \rho)$ has an isolated zero at $\rho = 0$ for any $\varphi \in \mathbb{S}^1$, and, in particular, the claim follows.

The second part, that $\zeta_{pq}(\rho_0) \equiv 0$, is a trivial consequence of the relation (1.9). □

3.5. Proof of Proposition 1.10

Proof. The first part is straightforward since $\mathcal{X}(F) = KF$ holds because $\mathcal{X}_n$ is a (p, q)-quasihomogeneous polynomial vector field of degree n; hence its components P_{p+n} and Q_{q+n} satisfy Euler relations

$$\mathcal{X}_E(P_{p+n}) = (p + n)P_{n+p}, \quad \mathcal{X}_E(Q_{q+n}) = (q + n)Q_{q+n}. \tag{3.7}$$

The second part is also straightforward since $\mathcal{X}(V) = \mathrm{div}(\mathcal{X})V$ holds, taking into account Euler relations (3.7) and $\mathcal{X}_E(A) = (m - 1)A$, together with the fact that system $\mathcal{X}_n(H) = 0$ and $\mathcal{X}_E(H) = dH$ can be solved as $\partial_x H = dQ_{q+n}H/F$ and $\partial_y H = -dP_{p+n}H/F$. □

3.6. Proof of Proposition 1.11

Proof. In (p, q)-weighted polar coordinates, the angular speed of vector field (1.12) is $\dot{\varphi} = G_r(\varphi) = F(\cos\varphi, \sin\varphi)/(p\cos^2\varphi + q\sin^2\varphi)$. Therefore, a monodromic necessary condition for the origin of $\mathcal{X}$ is that F has no real factors in $\mathbb{R}[x, y]$ so that $G_r \neq 0$ in $\mathbb{S}^1$. In particular, F has an isolated singularity at the origin, $\mathcal{X} \in \mathrm{Mo}^{(p,q)}$ so Π is analytic at the origin, $\Lambda_{pq} = \emptyset$ and the origin is the unique real finite singularity of $\mathcal{X}$. We observe that $\mathcal{X}_n$ has a monodromic singular point at the origin when $\mathcal{X}$ has it because both vector fields share the same angular speed $\dot{\varphi}$. Indeed, the origin is a center of $\mathcal{X}_n$ under our hypothesis that implies the existence of the first integral H of Proposition 1.10, see Remark 1.12.

The monodromy of the origin implies that $\mathbf{N}(\mathcal{X}) = \mathbf{N}(\mathcal{X}_n)$ has only one edge and that when we only vary the parameters of $\mathcal{X}$ in the monodromic space Λ, then the Newton diagram $\mathbf{N}(\mathcal{X}|_\Lambda)$ of the restricted vector field $\mathcal{X}|_\Lambda$ is fixed. The inverse integrating factor

$$V(\varphi, \rho) = \rho^{m-n} \left(H(\cos\varphi, \sin\varphi)\right)^{(m-n-1)/d}.$$

The explicit Taylor expansion of Π at the origin is just a consequence of statement (ii) of Theorem 4 in [6]. Finally, we can use Theorem 7 in [6] to conclude that

$$\eta_{m-n} = \int_0^{2\pi} \frac{\mathcal{F}(\varphi, r)}{V(\varphi, r)}\, d\varphi = I_1 + r^{n-m+1} I_2,$$

where I_i are integrals independents of r and

$$I_1 = \int_0^{2\pi} \frac{(p\cos^2\varphi + q\sin^2\varphi)A(\cos\varphi, \sin\varphi)}{F(\cos\varphi, \sin\varphi)\left(H(\cos\varphi, \sin\varphi)\right)^{(m-n-1)/d}}\, d\varphi.$$

Since the expression of η_{m-n} must be independent of $r > 0$ and sufficiently small, by Theorem 7 in [6], we deduce that $I_2 = 0$, and the proposition follows. □

4. Conclusions

In this paper we have considered planar analytic vector fields $\mathcal{X}$ having a monodromic singular point with Poincaré map Π. Using the fact that always exists a real analytic invariant curve $F = 0$ of $\mathcal{X}$ in a neighborhood of that singularity in the paper are given the relations between Π and F that can be used to determine new conditions in order to guarantee the analyticity of Π at the singularity.

The special case when F is inverse integrating factor of $\mathcal{X}$ we rediscover the formula obtained previously in [25] by an other method. Finally some applications to the center-focus problem and also to vector fields with degenerate infinity are given.

Author contributions

All authors carried out the main results of this article, drafted the manuscript, and read and approved the final manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are partially supported by the Agencia Estatal de Investigación grant PID2020-113758GB-I00 and an AGAUR (Generalitat de Catalunya) grant number 2021SGR 01618.

The author is partially supported by a MICIN grant number PID2020-113758GB-I00 and an AGAUR grant number 2017SGR-1276.

Conflict of interest

Prof. Jaume Giné is the Guest Editor of special issue "Advances in Qualitative Theory of Differential Equations" for AIMS Mathematics. Prof. Jaume Giné was not involved in the editorial review and the decision to publish this article. The authors declare no conflicts of interest.

References

1. F. Dumortier, Singularities of vector fields on the plane, *J. Differ. Equations*, **23** (1977), 53–106. https://doi.org/10.1016/0022-0396(77)90136-X
2. V. I. Arnold, D. Embarek, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Moscow: Mir, 1980.
3. A. Algaba, C. García, M. Reyes, Characterization of a monodromic singular point of a planar vector field, *Nonlinear Anal.*, **74** (2011), 5402–5414. https://doi.org/10.1016/j.na.2011.05.023
4. A. Algaba, C. García, M. Reyes, A new algorithm for determining the monodromy of a planar differential system, *Appl. Math. Comput.*, **237** (2014), 419–429. https://doi.org/10.1016/j.amc.2014.03.129
5. I. A. García, J. Giné, M. Grau, A necessary condition in the monodromy problem for analytic differential equations on the plane, *J. Symb. Comput.*, **41** (2006) 943–958. https://doi.org/10.1016/j.jsc.2006.04.007
6. I. A. García, J. Giné, Principal Bautin ideal of monodromic singularities with inverse integrating factors, preprint Universitat de Lleida. https://doi.org/10.48550/arXiv.2412.09205
7. I. A. García, J. Giné, A. L. Rodero, Existence and non-existence of Puiseux inverse integrating factors in analytic monodromic singularities, preprint Universitat de Lleida.
8. A. D. Bruno, *Volume 57: power geometry in algebraic and differential equations*, Elsevier, 2000.
9. A. D. Bruno, Asymptotic behaviour and expansions of solutions of an ordinary differential equation, *Russ. Math. Surv.*, **59** (2004), 429. https://doi.org/10.1070/RM2004v059n03ABEH000736
10. I. A. García, J. Giné, Characterization of centers by its complex separatrices, preprint Universitat de Lleida. https://doi.org/10.48550/arXiv.2412.09197
11. A. D. Bruno, *Local methods in nonlinear differential equations*, Part I the local method of nonlinear analysis of differential equations Part II the sets of analyticity of a normalizing transformation, Berlin: Springer-Verlag, 1989.

12. Y. S. Il'yashenko, Finiteness theorems for limit cycles, *Russ. Math. Surv.*, **45** (1990), 129. https://doi.org/10.1070/RM1990v045n02ABEH002335
13. N. B. Medvedeva, The principal term of the asymptotic expansion of the monodromy transformation: calculation in blowing-up geometry, *Sib. Math. J.*, **38** (1997), 114–126. https://doi.org/10.1007/BF02674907
14. A. Gasull, J. Llibre, V. Mañosa, F. Mañosas, The focus-centre problem for a type of degenerate system, *Nonlinearity*, **13** (2000), 699. https://doi.org/10.1088/0951-7715/13/3/311
15. A. Gasull, V. Mañosa, F. Mañosas, Monodromy and stability of a class of degenerate planar critical points, *J. Differ. Equations*, **217** (2005), 363–376. https://doi.org/10.1006/jdeq.2001.4095
16. H. Giacomini, J. Giné, J. Llibre, The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems, *J. Differ. Equations*, **227** (2006), 406–426. https://doi.org/10.1016/j.jde.2006.03.012
17. H. Giacomini, J. Giné, J. Llibre, Corrigendum to: "The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems" [J. Differential Equations 227 (2006) 406–426], *J. Differ. Equations*, **232** (2007), 702. https://doi.org/10.1016/j.jde.2006.10.004
18. J. Giné, Sufficient conditions for a center at a completely degenerate critical point, *Int. J. Bifurcat. Chaos*, **12** (2002), 1659–1666. https://doi.org/10.1142/s0218127402005315
19. J. Giné, On the centers of planar analytic differential systems, *Int. J. Bifurcat. Chaos*, **17** (2007), 3061–3070. https://doi.org/10.1142/S0218127407018865
20. J. Giné, On the degenerate center problem, *Int. J. Bifurcat. Chaos*, **21** (2011), 1383–1392. https://doi.org/10.1142/S0218127411029082
21. J. Giné, S. Maza, The reversibility and the center problem, *Nonlinear Anal.*, **74** (2011), 695–704. https://doi.org/10.1016/j.na.2010.09.028
22. D. Shafer, V. Romanovski, *The center and cyclicity problems: a computational algebra approach*, Birkhäuser Boston, MA, 2009. https://doi.org/10.1007/978-0-8176-4727-8
23. A. Algaba, C. García, J. Giné, Geometric criterium in the center problem, *Mediterr. J. Math.*, **13** (2016), 2593–2611. https://doi.org/10.1007/s00009-015-0641-0
24. W. W. Farr, C. Li, I. S. Labouriau, W. F. Langford, Degenerate Hopf-bifurcation formulas and Hilbert's 16th problem, *SIAM J. Math. Anal.*, **20** (1989), 13–30. https://doi.org/10.1137/0520002
25. I. A. García, J. Giné, Center problem with characteristic directions and inverse integrating factors, *Commun. Nonlinear Sci. Numer. Simul.*, **108** (2022), 106276. https://doi.org/10.1016/j.cnsns.2022.106276
26. I. A. García, J. Giné, The linear term of the Poincaré map at singularities of planar vector fields, *J. Differ. Equations*, **396** (2024), 44–67. https://doi.org/10.1016/j.jde.2024.02.055
27. I. A. García, J. Giné, The Poincaré map of degenerate monodromic singularities with Puiseux inverse integrating factor, *Adv. Nonlinear Anal.*, **12** (2023), 20220314. https://doi.org/10.1515/anona-2022-0314
28. J. Chavarriga, I. A. García, J. Giné, On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity, *Int. J. Bifurcat. Chaos*, **11** (2001), 711–722. https://doi.org/10.1142/S0218127401002390

29. I. A. García, J. Giné, A. L. Rodero, Existence and non-existence of Puiseux inverse integrating factors in analytic monodromic singularities, *Stud. Appl. Math.*, **153** (2024), e12724. https://doi.org/10.1111/sapm.12724

30. I. A. García, H. Giacomini, M. Grau, The inverse integrating factor and the Poincaré map, *Trans. Amer. Math. Soc.*, **362** (2010), 3591–3612. https://doi.org/10.1090/S0002-9947-10-05014-2

31. I. A. García, S. Maza, A new approach to center conditions for simple analytic monodromic singularities, *J. Differ. Equations*, **248** (2010), 363–380. https://doi.org/10.1016/j.jde.2009.09.002

32. M. M. Vainberg, V. A. Trenogin, *Theory of branching of solutions of nonlinear equations*, Monographs and textbooks on pure and applied mathematics, Noordhoff International Publishing, 1974.