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Research article

Expectation formulas for q-probability distributions: a new extension via Andrews-Askey integral

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Abstract: In this paper, we utilize the q-Chu-Vandermonde formula to derive a novel expectation formula for the q-probability distribution $W(x, y; q)$, extending previously known results. Several applications are presented, including a broader generalization of the Andrews-Askey integral. Although fractional q-calculus is not directly employed in this work, its potential for future extensions is discussed, as non-integer order derivatives and integrals could offer deeper insights into q-series and probability distributions.

Keywords: q-Chu-Vandermonde formula; the probability distribution $W(x, y; q)$; basic hypergeometric series; q-integral
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1. Introduction

Probability methods serve as valuable tools in the study of basic hypergeometric series, offering insights into their structure and properties; see, for example [1, 2]. In [3], Kadell presents a probabilistic approach to deriving Ramanujan's celebrated ${}_1\psi_1$ summation formula, offering new insights into its mathematical structure and interpretation. Furthermore, in [4–6], Wang introduces a novel probability distribution and demonstrates its versatility through various mathematical and practical applications. In recent work, Wang [7] introduced a novel discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$ represents a countably infinite sample space. The set $\mathcal{F}$ is defined as the collection of all possible subsets of Ω, forming the sigma-algebra of events. The probability measure $\mathbb{P}$ is then specified to assign probabilities to events in this space, allowing for the development of new probabilistic interpretations and results in the context of basic hypergeometric series. This construction has opened the door for further applications of the probability theory in q-calculus and hypergeometric functions, providing a fertile ground for new theoretical developments.

In this work and in the following sections, we adopt the standard q-series notation and terminology as presented in [8]. The q-shifted factorial is defined by

$$(x;q)_0 = 1, \quad (x;q)_n = \prod_{i=0}^{n-1}(1 - xq^i), \quad (x;q)_\infty = \prod_{i=0}^{\infty}(1 - xq^i).$$

To enhance clarity and ease of reference throughout our discussion, we will employ the following set of multiple notations:

$$(x_1, x_2, \cdots, x_r; q)_n = (x_1;q)_n(x_2;q)_n \cdots (x_r;q)_n,$$

where n is an integer or ∞.

The probability measure $\mathbb{P}$ is defined by

$$\mathbb{P}(\{\omega_{2k}\}) = \frac{(yq^{k+1}/x, q^{k+1}; q)_\infty q^k}{(q, yq/x, x/y; q)_\infty},$$
$$\mathbb{P}(\{\omega_{2k+1}\}) = \frac{-x(q^{k+1}, xq^{k+1}/y; q)_\infty q^k}{y(q, yq/x, x/y; q)_\infty},$$

where $k = 0, 1, 2, \cdots$ and $xy < 0$.

A random variable X has a probability distribution $W(x, y; q)$, if

$$\mathbb{P}(X(\omega_{2k}) = yq^k) = \frac{(yq^{k+1}/x, q^{k+1}; q)_\infty q^k}{(q, yq/x, x/y; q)_\infty}, \quad k = 0, 1, 2, \cdots,$$

and

$$\mathbb{P}(X(\omega_{2k+1}) = xq^k) = \frac{-x(q^{k+1}, xq^{k+1}/y; q)_\infty q^k}{y(q, yq/x, x/y; q)_\infty}, \quad k = 0, 1, 2, \cdots.$$

In fact, the distribution $W(x, y; q)$ of the random variable X serves as a natural extension of the one-dimensional distribution $W(x; q)$ as discussed in [4]. This extension incorporates additional dimensions to capture the probabilistic behavior of X in a more comprehensive manner, thus allowing for richer statistical modeling.

We denote $E(X)$ as the expectation of the random variable X defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This expectation provides a fundamental measure of the central tendency of the distribution of X, which is crucial for various applications in statistical analysis and decision-making.

Building upon earlier work, the author in [6] laid important groundwork that facilitated the subsequent findings presented in [9]. Specifically, the latter study derived an explicit expectation formula associated with the distribution $W(x, y; q)$. This derivation utilized a Maclaurin series expansion as a mathematical tool to explore the properties of $W(x, y; q)$. By expanding the distribution, the author was able to unveil significant insights into the behavior of the random variable X.

Moreover, the study in [9] highlighted various applications of this expectation formula, illustrating its utility in both theoretical and practical contexts. These applications ranged from enhancing statistical inference methods to improving modeling techniques in various fields, thereby reinforcing the relevance of the extended distribution $W(x, y; q)$ in contemporary probabilistic research.

In the subsequent sections of this paper, we will frequently use the following equations:

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty}, \tag{1.1}$$

$$(x;q)_{n-k} = \frac{(x;q)_n(-1)^k q^{\binom{k}{2}-nk}}{(q^{1-n}/x;q)_k}, \tag{1.2}$$

$$(x;q)_{n+k} = (x;q)_n(xq^n;q)_k. \tag{1.3}$$

The ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \cdots, & a_r \\ b_1, & b_2, & \cdots, & b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \cdots, a_r; q)_k}{(q, b_1, b_2, \cdots, b_s; q)_k} \left\{(-1)^k q^{\binom{k}{2}}\right\}^{1+s-r} z^k.$$

The q-integral, often referred to as the Jackson integral [10], is an extension of the classical integral in the context of q-calculus and is given by:

$$\int_0^d f(t)d_qt = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n,$$

and

$$\int_c^d f(t)d_qt = \int_0^d f(t)d_qt - \int_0^c f(t)d_qt.$$

The Andrews-Askey integral [11] represents a significant advancement in q-series theory and special functions. This remarkable integral, established by Andrews and Askey, emerges naturally from Ramanujan's ${}_1\psi_1$ summation formula and serves as a cornerstone in q-analysis. The integral's derivation elegantly demonstrates the deep interconnections between q-series, basic hypergeometric functions, and their analytic properties. Its widespread applications span across multiple domains of mathematics, from partition theory to quantum groups, highlighting the fundamental nature of this result in modern mathematical analysis. The Andrews-Askey integral [11] is given by:

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_qt = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}.$$

Building upon the Andrews-Askey integral, Al-Salam and Verma [12] established a remarkable generalization that further expanded the scope of q-integration theory. Their extension, now known as the Al-Salam–Verma q-integral, is given by:

$$\int_x^y \frac{(qt/x, qt/y, dt; q)_\infty}{(at, bt, ct; q)_\infty} d_qt = \frac{y(1-q)(q, yq/x, x/y, d/a, d/b, d/c; q)_\infty}{(ax, ay, bx, by, cx, cy; q)_\infty},$$

where $d = abcxy$, provided that no zero factors occur in the denominator of the integral. This generalization significantly broadens the applicability of the original Andrews-Askey result [11], offering a more flexible framework for handling q-integrals. The additional parameters in this extension allow for more diverse applications in q-series theory and special functions, while elegantly preserving the structural beauty of the original formula.

The theory of fractional q-derivatives and integrals, initiated by Al-Salam [13] and further developed by Agarwal [14], offers profound connections to the Andrews-Askey integral [11] and its generalizations. In particular, the fractional q-integral operators exhibit structural similarities to the Al-Salam–Verma extension [12], especially in their treatment of infinite products of the form $(at, bt, ct; q)_\infty$. These relationships become particularly evident when considering the Riemann-Liouville type q-fractional integral operator, defined for $\Re(\alpha) > 0$ as:

$$(I_q^\alpha f)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \frac{(qt/x; q)_\infty}{(t; q)_\infty} f(t)\, d_q t,$$

where $\Gamma_q(\alpha)$ denotes the q-gamma function. This formulation not only preserves the elegant structure of classical fractional calculus but also incorporates the rich theory of q-special functions. For a detailed study about this topic we may refer the readers to see [15–17].

In this paper, we present a novel extension of the Andrews-Askey integral [11] through the lens of q-probability theory. By leveraging the q-Chu-Vandermonde formula, we derive new expectation formulas for the q-probability distribution $W(x, y; q)$, which naturally lead to a broader generalization of the Andrews-Askey integral. While Al-Salam and Verma's work [12] expanded the scope of q-integration theory, our probabilistic approach reveals unexpected connections between q-series, special functions, and probability distributions. This development not only extends previous results but also establishes a framework where probability theory and q-special functions intersect, opening new avenues for investigation in both fields.

For the development of our results, we require the q-binomial coefficient (also known as the q-analog of the binomial coefficient), which is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

and the q-Chu-Vandermonde formula [8, Appendix II. (II.6)]

$${}_2\phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}; q, q\right] = \frac{a^n (c/a; q)_n}{(c; q)_n}.$$

2. A set of known results

To establish our main results concerning the q-probability distribution $W(x, y; q)$, we require several fundamental identities and transformations from the theory of q-series. The following known results play a crucial role in our development, particularly in the manipulation of q-series expressions and the derivation of our generalized integral formulas.

Proposition 2.1. *(see [7, Lemma 2.3]) If $X \sim W(x, y; q)$ and $f(x)$ is a measurable function, then*

$$E\{f(X)\} = \frac{1}{y(1-q)(q, yq/x, x/y; q)_\infty} \int_x^y (qt/x, qt/y; q)_\infty f(t) d_q t, \tag{2.1}$$

provided that the q-integral in (2.1) converges absolutely.

Proposition 2.2. *(see [7, Theorem 3.1]) If* $X \sim W(x, y; q)$, $|x| \leq 1$, *and* $|y| \leq 1$, *then*

$$E\left\{\frac{(dX;q)_\infty}{(aX,bX,cX;q)_\infty}\right\} = \frac{(dy,abxy;q)_\infty}{(ax,ay,bx,by,cy;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} ay, & by, & d/c \\ & dy, & abxy \end{matrix}; q, cx\right], \tag{2.2}$$

provided that $|a| < 1, |b| < 1, |c| < 1$.

Proposition 2.3. *(see [7, Theorem 2.5]) If* $X \sim W(x, y; q)$, *then*

$$E\left\{\frac{X^n}{(aX,bX;q)_\infty}\right\} = \frac{(abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay,by;q)_k}{(abxy;q)_k} x^k y^{n-k},$$

provided that no zero factors occur in the denominator.

Setting $n = 0$ in Proposition 2.3, the following identity is true [7, Eq (2.21)]:

$$E\left\{\frac{1}{(aX,bX;q)_\infty}\right\} = \frac{(abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty}. \tag{2.3}$$

Finally, we recall the homogeneous form of the q-shifted factorial:

$$P_0(x,y) = 1, \quad P_n(x,y) = (x-y)(x-yq)\cdots(x-yq^{n-1}) = x^n(y/x;q)_n.$$

The paper is structured as follows: In Section 2, we establish a novel expectation formula for the q-probability distribution $W(x, y; q)$, which substantially extends Wang's results [7]. This generalization leads to several significant corollaries that deepen our understanding of q-probability theory. Section 3 explores applications of our main results in two directions: First, we derive new terminating transformation formulas for q-series, and second, we present a broader generalization of the Andrews-Askey integral, demonstrating the far-reaching implications of our probabilistic approach.

3. Expectation formulas for q-probability distribution $W(x, y; q)$

In this section, we establish a novel expectation formula for the q-probability distribution $W(x, y; q)$ by employing the q-Chu-Vandermonde formula and previously established results. This development significantly extends Wang's work [7] and provides a foundation for our subsequent generalizations. The following theorem constitutes our main result.

Theorem 3.1. *Let X be a random variable with distribution* $W(x, y; q)$, $|x| \leq 1$, $|y| \leq 1$, *and* $max\{|ax|, |ay|, |bx|, |bx|, |ey|\} \leq 1$, *then*

$$\begin{aligned} &E\left\{\frac{(dX;q)_\infty P_n(aX,c)}{(aX,bX,eX;q)_\infty}\right\} \\ =&\frac{(c;q)_n(dy,abxy;q)_\infty}{(ax,ay,bx,by,ey;q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n},ax,ay;q)_k q^k}{(q,c,abxy;q)_k}\, {}_3\phi_2\left[\begin{matrix} aq^k y, & by, & d/e \\ & dy, & abq^k xy \end{matrix}; q, ex\right], \end{aligned} \tag{3.1}$$

provided that $0 < |a| < 1, |b| < 1, |e| < 1$.

Proof. We rewrite the q-Chu-Vandermonde formula

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,c;q)_k(aq^k;q)_\infty}=\frac{a^n(c/a;q)_n}{(c;q)_n(a;q)_\infty}. \tag{3.2}$$

Replacing a by ax and multiplying both sides of (3.2) by $\frac{(dx;q)_\infty}{(bx,ex;q)_\infty}$, we obtain that

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,c;q)_k}\frac{(dx;q)_\infty}{(aq^kx,bx,ex;q)_\infty}=\frac{P_n(ax,c)(dx;q)_\infty}{(c;q)_n(ax,bx,ex;q)_\infty}.$$

Again replacing x by the random variable X with the distribution $W(x,y;q)$ and then taking expectation E on both sides of the above identity, we have that

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,c;q)_k}E\left\{\frac{(dX;q)_\infty}{(aq^kX,bX,eX;q)_\infty}\right\}=\frac{1}{(c;q)_n}E\left\{\frac{P_n(aX,c)(dX;q)_\infty}{(aX,bX,eX;q)_\infty}\right\}.$$

Using the expectation formula (2.2), we obtain that

$$\begin{aligned}
&E\left\{\frac{P_n(aX,c)(dX;q)_\infty}{(aX,bX,eX;q)_\infty}\right\}\\
&=(c;q)_n\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,c;q)_k}\frac{(dy,abxyq^k;q)_\infty}{(aq^kx,aq^ky,bx,by,ey;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}aq^ky, & by, & d/e\\ & dy, & abq^kxy\end{matrix};q,ex\right]\\
&=\frac{(c;q)_n(dy,abxy;q)_\infty}{(ax,ay,bx,by,ey;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},ax,ay;q)_k q^k}{(q,c,abxy;q)_k}\,{}_3\phi_2\left[\begin{matrix}aq^ky, & by, & d/e\\ & dy, & abq^kxy\end{matrix};q,ex\right].
\end{aligned}$$

This completes the proof. □

Remark 1. *Setting $n=0$ and replacing e by c, we find that Theorem 3.1 reduces to Proposition 2.2.*

In Theorem 3.1, letting $d=0$ and $d=e$, we derive two further expectation formulas.

Corollary 3.2. *We obtain*

$$\begin{aligned}
&E\left\{\frac{P_n(aX,c)}{(aX,bX,eX;q)_\infty}\right\}\\
&=\frac{(c;q)_n(abxy;q)_\infty}{(ax,ay,bx,by,ey;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},ax,ay;q)_k q^k}{(q,c,abxy;q)_k}\,{}_2\phi_1\left[\begin{matrix}aq^ky, & by\\ & abq^kxy\end{matrix};q,ex\right].
\end{aligned}\tag{3.3}$$

Corollary 3.3. *We get*

$$E\left\{\frac{P_n(aX,c)}{(aX,bX;q)_\infty}\right\}=\frac{(c;q)_n(abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}q^{-n}, & ax, & ay\\ & c, & abxy\end{matrix};q,q\right]. \tag{3.4}$$

4. Applications and transformations

In this section, we demonstrate the utility of our expectation formula for the q-probability distribution $W(x, y; q)$ through two significant applications. First, we derive a new transformation formulas for q-series, and subsequently, we establish a broader generalization of the Andrews-Askey integral. These applications not only illustrate the power of our probabilistic approach but also reveal deeper connections between q-series theory and integral transformations. The following result demonstrates how the symmetric structure of the q-probability distribution $W(x, y; q)$ leads to elegant transformations.

Theorem 4.1. *If* $\max\{|ex|, |abxy|\} \le 1$*, then we have*

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(q^{-n},ax,ay;q)_kq^k}{(q,c,abxy;q)_k}\,{}_3\phi_2\left[\begin{matrix}aq^ky, & by, & d/e\\ & dy, & abq^kxy\end{matrix};q,ex\right]\\ =&\frac{(bx,aexy;q)_\infty}{(ex,abxy;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},ax,ay;q)_kq^k}{(q,c,aexy;q)_k}\,{}_3\phi_2\left[\begin{matrix}aq^ky, & ey, & d/b\\ & dy, & aeq^kxy\end{matrix};q,bx\right].\end{aligned}\tag{4.1}$$

Proof. Observing Eq (3.1), we note that both sides exhibit symmetry with respect to the parameters b and e. Exploiting this symmetric property by interchanging b and e on the left-hand side of (3.1), we obtain

$$\begin{aligned}&E\left\{\frac{(dX;q)_\infty P_n(aX,c)}{(aX,bX,eX;q)_\infty}\right\}\\ =&\frac{(c;q)_n(dy,abxy;q)_\infty}{(ax,ay,bx,by,ey;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},ax,ay;q)_kq^k}{(q,c,abxy;q)_k}\,{}_3\phi_2\left[\begin{matrix}aq^ky, & by, & d/e\\ & dy, & abq^kxy\end{matrix};q,ex\right]\\ =&\frac{(c;q)_n(dy,aexy;q)_\infty}{(ax,ay,ex,ey,by;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},ax,ay;q)_kq^k}{(q,c,aexy;q)_k}\,{}_3\phi_2\left[\begin{matrix}aq^ky, & ey, & d/b\\ & dy, & aeq^kxy\end{matrix};q,bx\right].\end{aligned}$$

This completes the proof. □

Setting $n = 0$ in Eq (4.1) and making suitable substitutions, we obtain Sears' well-known ${}_3\phi_2$ transformation formula [18], later examined by Liu [19] and is given by

$$\begin{aligned}&{}_3\phi_2\left[\begin{matrix}a_1, & a_2, & a_3\\ & b_1, & b_2\end{matrix};q,b_1b_2/a_1a_2a_3\right]\\ =&\frac{(b_2/a_2,b_1b_2/a_2a_3;q)_\infty}{(b_2,b_1b_2/a_1a_2a_3;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}a_1, & b_1b_2/a_1a_2a_3, & b_1/a_2\\ & b_1, & b_1b_2/a_2a_3\end{matrix};q,b_2/a_1\right].\end{aligned}$$

Thus, Eq (4.1) provides a natural generalization of Sears' transformation formula, extending its scope while preserving its fundamental structure.

Theorem 4.2. *We have that*

$$\sum_{k,m=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ m \end{bmatrix}\frac{(q/z,bx,by;q)_k(bx,by;q)_m}{(b^2xy;q)_{k+m}(bq^{1-n}y/c;q)_k(c;q)_m}(-1)^m(z/c)^k q^{\binom{m+1}{2}-nm-nk}$$
$$\times\ {}_3\phi_2\left[\begin{matrix} bq^my, & bq^ky, & q^k \\ & bq^{1-n+k}y/c, & b^2q^{k+m}xy \end{matrix};q,bq^{1-n}x/c\right]$$
$$={}_3\phi_2\left[\begin{matrix} q^{-n}, & bx, & by \\ & c, & b^2xy \end{matrix};q,z\right].$$

Proof. Recall the transformation formula in [8, Exercise 1.15. (ii)]

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & b \\ & c \end{matrix};q,z\right]=\frac{(c/b;q)_n b^n}{(c;q)_n}\,{}_3\phi_1\left[\begin{matrix} q^{-n}, & b, & q/z \\ & bq^{1-n}/c & \end{matrix};q,z/c\right]. \tag{4.2}$$

Replacing b by bx, (4.2) can be rewritten as

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k z^k}{(q,c;q)_k}\frac{1}{(bq^kx;q)_\infty}$$
$$=\frac{1}{(c;q)_n}\sum_{k=0}^{n}(-z/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},q/z;q)_k}{(q;q)_k}\frac{P_n(bx,c)(\frac{bq^{1-n+k}x}{c};q)_\infty}{(bq^kx,bq^{1-n}x/c;q)_\infty}.$$

Multiplying both sides of the identity by the factor $\frac{1}{(bx;q)_\infty}$ leads to the following equivalent form:

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k z^k}{(q,c;q)_k}\frac{1}{(bq^kx,bx;q)_\infty}$$
$$=\frac{1}{(c;q)_n}\sum_{k=0}^{n}(-z/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},q/z;q)_k}{(q;q)_k}\frac{P_n(bx,c)(\frac{bq^{1-n+k}x}{c};q)_\infty}{(bq^kx,bq^{1-n}x/c,bx;q)_\infty}.$$

Replacing x with X and applying the expectation operator E to both sides yields:

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k z^k}{(q,c;q)_k}E\left\{\frac{1}{(bq^kX,bX;q)_\infty}\right\}$$
$$=\frac{1}{(c;q)_n}\sum_{k=0}^{n}(-z/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},q/z;q)_k}{(q;q)_k}E\left\{\frac{P_n(bX,c)(\frac{bq^{1-n+k}}{c}X;q)_\infty}{(bq^kX,bq^{1-n}X/c,bX;q)_\infty}\right\}.$$

Applying Theorem 3.1 together with Eq (2.3), we obtain

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k z^k}{(q,c;q)_k}\frac{(b^2q^kxy;q)_\infty}{(bx,by,bq^kx,bq^ky;q)_\infty}$$
$$=\sum_{k=0}^{n}(-z/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},q/z;q)_k}{(q;q)_k}\frac{(bq^{1-n+k}y/c,b^2q^kxy;q)_\infty}{(bx,by,bq^kx,bq^ky,bq^{1-n}y/c;q)_\infty}$$
$$\times\sum_{m=0}^{n}\frac{(q^{-n},bx,by;q)_m q^m}{(q,c,b^2q^kxy;q)_m}\,{}_3\phi_2\left[\begin{matrix} bq^my, & bq^ky, & q^k \\ & bq^{1-n+k}y/c, & b^2q^{k+m}xy \end{matrix};q,bq^{1-n}x/c\right].$$

After algebraic simplification, we establish the identity stated in Theorem 4.2. □

Theorem 4.3. *We have that*

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(b;q)_k(ay,dy;q)_m}{(c;q)_k(adxy;q)_m}(-1)^k q^{\binom{k}{2}}(ay/q^n)^k(x/y)^m$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(1/c)^k q^{-nk}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(q,b;q)_k}{(bq^{1-n}/c;q)_k}\,{}_3\phi_2\left[\begin{matrix} q^{-k}, & ax, & ay \\ & q, & adxy \end{matrix};q,q\right]. \tag{4.3}$$

Proof. Replacing z by ax, we can rewrite (4.2) as

$$\sum_{k=0}^{n}\frac{(q^{-n},b;q)_k a^k x^k}{(q,c;q)_k} = \frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(-1/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},b;q)_k}{(q,bq^{1-n}/c;q)_k}P_k(ax,q).$$

Multiplying both sides of the above identity by $\frac{1}{(ax,dx;q)_\infty}$, we obtain that

$$\sum_{k=0}^{n}\frac{(q^{-n},b;q)_k a^k}{(q,c;q)_k}\frac{x^k}{(ax,dx;q)_\infty}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(-1/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},b;q)_k}{(q,bq^{1-n}/c;q)_k}\frac{P_k(ax,q)}{(ax,dx;q)_\infty}. \tag{4.4}$$

Letting $x = X$ and then applying the expectation E on both sides of (4.4), we derive that

$$\sum_{k=0}^{n}\frac{(q^{-n},b;q)_k a^k}{(q,c;q)_k}E\left\{\frac{X^k}{(aX,dX;q)_\infty}\right\}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(-1/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},b;q)_k}{(q,bq^{1-n}/c;q)_k}E\left\{\frac{P_k(aX,q)}{(aX,dX;q)_\infty}\right\}.$$

By Proposition 2.3 and Corollary 3.3, we obtain

$$\sum_{k=0}^{n}\frac{(q^{-n},b;q)_k a^k}{(q,c;q)_k}\frac{(adxy;q)_\infty}{(ax,ay,dx,dy;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(ay,dy;q)_m}{(adxy;q)_m}x^m y^{k-m}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(-1/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},b;q)_k}{(q,bq^{1-n}/c;q)_k}\frac{(q;q)_k(adxy;q)_\infty}{(ax,ay,dx,dy;q)_\infty}$$
$$\times\,{}_3\phi_2\left[\begin{matrix} q^{-k}, & ax, & ay \\ & q, & adxy \end{matrix};q,q\right],$$

i.e.,

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(b;q)_k}{(c;q)_k}\frac{(ay,dy;q)_m}{(adxy;q)_m}(-a)^k q^{\binom{k}{2}-nk}x^m y^{k-m}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(1/c)^k q^{-nk}\frac{(b,q;q)_k}{(bq^{1-n}/c;q)_k}$$
$$\times\,{}_3\phi_2\left[\begin{matrix} q^{-k}, & ax, & ay \\ & q, & adxy \end{matrix};q,q\right].$$

This completes the proof. □

Corollary 4.4.

$$\frac{(c;q)_n}{(c/b;q)_n b^n}\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(b;q)_k(ay;q)_m(1-dy)}{(c;q)_k(1-dyq^m)}(-1)^k q^{\binom{k}{2}-nk+m}(ay)^{k-m}$$
$$= {}_4\phi_2\left[\begin{matrix} q^{-n}, & b, & dq/a, & q \\ & bq^{1-n}/c, & dyq & \end{matrix};q,ay/c\right].$$

Proof. Setting $ax=q$ in (4.4), we have

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(b;q)_k(ay;q)_m(1-dy)}{(c;q)_k(1-dyq^m)}(-1)^k q^{\binom{k}{2}-nk+m}(ay)^{k-m}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(-1/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},b;q)_k}{(bq^{1-n}/c;q)_k}\,{}_2\phi_1\left[\begin{matrix} q^{-k}, & ay \\ & dyq \end{matrix};q,q\right]. \tag{4.5}$$

Applying the q-Chu-Vandermonde formula to the ${}_2\phi_1$ series in the right-hand side of Eq (4.5), we obtain

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(b;q)_k(ay;q)_m(1-dy)}{(c;q)_k(1-dyq^m)}(-1)^k q^{\binom{k}{2}-nk+m}(ay)^{k-m}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\sum_{k=0}^{n}(-1/c)^k q^{-\binom{k}{2}}\frac{(q^{-n},b;q)_k}{(bq^{1-n}/c;q)_k}\frac{(dq/a;q)_k(ay)^k}{(dyq;q)_k},$$

i.e.,

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}\frac{(b;q)_k(ay;q)_m(1-dy)}{(c;q)_k(1-dyq^m)}(-1)^k q^{\binom{k}{2}-nk+m}(ay)^{k-m}$$
$$=\frac{(c/b;q)_n b^n}{(c;q)_n}\,{}_4\phi_2\left[\begin{matrix} q^{-n}, & b, & dq/a, & q \\ & bq^{1-n}/c, & dyq & \end{matrix};q,ay/c\right].$$

This completes the proof. □

Setting $a=dq$ in Corollary 4.4 leads to the following identity involving the Al-Salam–Carlitz polynomials $\varphi_n^{(a)}(x|q)$.

Corollary 4.5.

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}-nk}\begin{bmatrix} n \\ k \end{bmatrix}\varphi_k^{(dy)}(q/ay|q)\frac{(b;q)_k(ay)^k}{(c;q)_k}=\frac{(c/b;q)_n b^n}{(c;q)_n},$$

where the Al-Salam–Carlitz polynomials $\varphi_n^{(a)}(x|q)$ are defined by Srivastava and Jain [20] as:

$$\varphi_n^{(a)}(x|q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a;q)_k x^k.$$

Theorem 4.6. *We have that*

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}(-1)^{k+m}q^{\binom{k}{2}+\binom{m}{2}-km+m}a^{-\frac{k}{2}}\frac{(a,aq,-qa^{1/2};q)_k}{(-a^{1/2},aq^{1+n},by;q)_k}$$
$$\times\frac{(bx,by;q)_m}{(aq,b^2a^{-1/2}xy;q)_m}\,{}_3\phi_2\left[\begin{matrix} bq^my, & ba^{-1/2}y, & q^{-(n-k)} \\ & bq^ky, & b^2a^{-1/2}q^mxy \end{matrix};q,bq^nx\right]$$
$$=\frac{(aq,ba^{-1/2}x,ba^{-1/2}y;q)_n}{(qa^{1/2},b^2a^{-1/2}xy,by;q)_n}.$$

Proof. We recall the terminating ${}_4\phi_3$ summation formula of Gasper [8, (II.14)]

$$ {}_4\phi_3\left[\begin{matrix} a, & -qa^{\frac{1}{2}}, & b, & q^{-n} \\ & -a^{\frac{1}{2}}, & aq/b, & aq^{1+n} \end{matrix};q,\frac{q^{1+n}a^{\frac{1}{2}}}{b}\right]=\frac{(aq,qa^{\frac{1}{2}}/b;q)_n}{(qa^{\frac{1}{2}},aq/b;q)_n}. \tag{4.6}$$

Upon replacing b with aq/b in Eq (4.6), we obtain

$$\sum_{k=0}^{n}\frac{(q^{-n},a,-qa^{\frac{1}{2}};q)_k(q^na^{-\frac{1}{2}})^k}{(q,-a^{\frac{1}{2}},aq^{1+n};q)_k}\frac{P_k(b,aq)}{(b;q)_k}=\frac{(aq,ba^{-\frac{1}{2}};q)_n}{(qa^{\frac{1}{2}},b;q)_n},$$

i.e.,

$$\sum_{k=0}^{n}\frac{(q^{-n},a,-qa^{\frac{1}{2}};q)_k(q^na^{-\frac{1}{2}})^k}{(q,-a^{\frac{1}{2}},aq^{1+n};q)_k}\frac{P_k(b,aq)(bq^k;q)_\infty}{(b,ba^{-\frac{1}{2}},bq^n;q)_\infty}=\frac{(aq;q)_n}{(qa^{\frac{1}{2}};q)_n}\frac{1}{(ba^{-\frac{1}{2}}q^n,b;q)_\infty}.$$

Substituting $b\to bX$ where $X\sim W(x,y;q)$, and taking the expectation E on both sides of the identity yields

$$\sum_{k=0}^{n}\frac{(q^{-n},a,-qa^{\frac{1}{2}};q)_k(q^na^{-\frac{1}{2}})^k}{(q,-a^{\frac{1}{2}},aq^{1+n};q)_k}E\left\{\frac{(bq^kX;q)_\infty P_k(bX,aq)}{(bX,ba^{-\frac{1}{2}}X,bq^nX;q)_\infty}\right\}=\frac{(aq;q)_n}{(qa^{\frac{1}{2}};q)_n}E\left\{\frac{1}{(ba^{-\frac{1}{2}}q^nX,bX;q)_\infty}\right\}.$$

Using (2.3) and (3.1), we obtain

$$\sum_{k=0}^{n}\frac{(q^{-n},a,-qa^{\frac{1}{2}};q)_k(q^na^{-\frac{1}{2}})^k}{(q,-a^{\frac{1}{2}},aq^{1+n};q)_k}\frac{(aq;q)_k(bq^ky,b^2a^{-1/2}xy;q)_\infty}{(bx,by,ba^{-1/2}x,ba^{-1/2}y,bq^ny;q)_\infty}$$
$$\times\sum_{m=0}^{k}\frac{(q^{-k},bx,by;q)_mq^m}{(q,aq,b^2a^{-1/2}xy;q)_m}\,{}_3\phi_2\left[\begin{matrix} bq^my, & ba^{-1/2}y, & q^{k-n} \\ & bq^ky, & b^2a^{-1/2}q^mxy \end{matrix};q,bq^nx\right]$$
$$=\frac{(aq,ba^{-1/2}x,ba^{-1/2}y;q)_n}{(a^{1/2}q,b^2a^{-1/2}xy;q)_n}\frac{(b^2a^{-1/2}xy;q)_\infty}{(ba^{-1/2}x,ba^{-1/2}y,bx,by;q)_\infty}.$$

After some simplifications, we obtain

$$\sum_{k=0}^{n}\frac{(q^{-n},a,aq,-qa^{\frac{1}{2}};q)_k(q^na^{-\frac{1}{2}})^k}{(q,-a^{\frac{1}{2}},aq^{1+n},by;q)_k}\sum_{m=0}^{k}\frac{(q^{-k},bx,by;q)_mq^m}{(q,aq,b^2a^{-1/2}xy;q)_m}$$
$$\times\,{}_3\phi_2\left[\begin{matrix} bq^my, & ba^{-1/2}y, & q^{-(n-k)} \\ & bq^ky, & b^2a^{-1/2}q^mxy \end{matrix};q,bq^nx\right]$$
$$=\frac{(aq,ba^{-1/2}x,ba^{-1/2}y;q)_n}{(a^{1/2}q,b^2a^{-1/2}xy,by;q)_n}.$$

This completes the proof. □

Taking $by = a^{1/2}$ in Theorem 4.6, we deduce that

Corollary 4.7.

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ m \end{bmatrix}(-1)^{k+m}q^{\binom{k}{2}+\binom{m}{2}-km+m}a^{-\frac{k}{2}}\frac{(a, aq, -qa^{1/2}; q)_k(a^{1/2}x/y, a^{1/2}; q)_m}{(-a^{1/2}, aq^{1+n}, a^{1/2}; q)_k(aq, bx; q)_m} = \delta_{n,0},$$

where

$$\delta_{n,k} = \begin{cases} 1, & n = k, \\ 0, & n \neq k \end{cases}$$

is a general Kronecker delta function.

Finally, we extend the Andrews-Askey integral in the following way:

Theorem 4.8. *If max* $\{|ax|, |ay|, |bx|, |bx|, |ey|\} \leq 1$*, then we have that*

$$\int_x^y \frac{(qt/x, qt/y, dt; q)_\infty P_n(at, c)}{(at, bt, et; q)_\infty}d_qt = y(1-q)(c;q)_n\frac{(q, yq/x, x/y, dy, abxy; q)_\infty}{(ax, ay, bx, by, ey; q)_\infty}$$
$$\sum_{k=0}^{n}\frac{(q^{-n}, ax, ay; q)_k q^k}{(q, c, abxy; q)_k}\,{}_3\phi_2\left[\begin{matrix} aq^ky, & by, & d/e \\ & dy, & abq^kxy \end{matrix}; q, ex\right].$$

Proof. Taking

$$f(X) = \frac{(dX; q)_\infty P_n(aX, c)}{(aX, bX, eX; q)_\infty}$$

in (2.1), we get

$$E\left\{\frac{(dX; q)_\infty P_n(aX, c)}{(aX, bX, eX; q)_\infty}\right\} = \frac{1}{y(1-q)(q, yq/x, x/y; q)_\infty}\int_x^y \frac{(qt/x, qt/y, dt; q)_\infty P_n(at, c)}{(at, bt, et; q)_\infty}d_qt. \tag{4.7}$$

Combining Eqs (3.1) and (4.7) completes the proof of our result. □

Corollary 4.9. *(The Al-Salam and Verma q-integral [12])*

$$\int_x^y \frac{(qt/x, qt/y, dt; q)_\infty}{(at, bt, ct; q)_\infty}d_qt = \frac{y(1-q)(q, yq/x, x/y, d/a, d/b, d/c; q)_\infty}{(ax, ay, bx, by, cx, cy; q)_\infty}, \tag{4.8}$$

where $d = abcxy$, provided that the denominator of the integral contains no zero factors.

Proof. First, taking $n = 0$ and $d = abexy$ in Theorem 4.8, we have

$$\int_x^y \frac{(qt/x, qt/y, dt; q)_\infty}{(at, bt, et; q)_\infty}d_qt = y(1-q)\frac{(q, yq/x, x/y, dy, abxy; q)_\infty}{(ax, ay, bx, by, ey; q)_\infty}\,{}_2\phi_1\left[\begin{matrix} ay, & by \\ & dy \end{matrix}; q, ex\right].$$

Using the q-Gauss sum [8, Appendix II. (II.8)]

$${}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, c/ab\right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |c/ab| < 1,$$

we get

$$
\begin{aligned}
&\int_x^y \frac{(qt/x, qt/y, dt;q)_\infty}{(at, bt, et;q)_\infty} d_q t \\
=&y(1-q)\frac{(q, yq/x, x/y, dy, abxy;q)_\infty}{(ax, ay, bx, by, ey;q)_\infty}\frac{(d/a, d/b;q)_\infty}{(dy, ex;q)_\infty}.
\end{aligned}
$$

Substituting c for e establishes Corollary 4.9. □

5. Conclusions and future directions

In this paper, we have successfully utilized the q-Chu-Vandermonde formula to derive a novel expectation formula concerning the q-probability distribution $W(x, y; q)$, significantly extending previous results in the literature. Our probabilistic approach has led to several important new results, particularly in the context of q-series transformations and integral evaluations. Most notably, we have established a broader generalization of the Andrews-Askey integral, demonstrating the power of combining probabilistic methods with classical q-series techniques.

Our work suggests several interesting directions for future research. While we have not directly applied fractional q-calculus in our current investigation, this theory offers intriguing possibilities for further extensions. The Riemann-Liouville type q-fractional integral operator could provide a natural framework for generating new families of results, particularly when applied to our probability distribution findings. Our approach to generalizing the Andrews-Askey integral suggests potential extensions to multivariate cases, which could yield rich new connections between probability theory and special functions as in [21].

Furthermore, by connecting methods from probability theory with classical q-series techniques, we have opened new paths for studying both areas. We hope this work will inspire further investigations into the rich relationship between probability theory and special functions, leading to deeper understanding of both fields.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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