Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 1438–1447.
DOI:10.3934/math.2025066
Received: 28 August 2024
Revised: 26 December 2024
Accepted: 07 January 2025
Published: 22 January 2025

Research article

Gallai's path decomposition conjecture for block graphs

Xiaohong Chen and Baoyindureng Wu*

College of Mathematics and System Science, Xinjiang University, Urumqi, Xinjiang 830046, China

* **Correspondence:** Email: baoywu@163.com.

Abstract: Let G be a graph of order n. A path decomposition $\mathcal{P}$ of G is a collection of edge-disjoint paths that covers all the edges of G. Let $p(G)$ denote the minimum number of paths needed in a path decomposition of G. Gallai conjectured that if G is connected, then $p(G) \leq \lceil \frac{n}{2} \rceil$. In this paper, we prove that the above conjecture holds for all block graphs.

Keywords: path decomposition; Gallai's conjecture; block graphs; cut vertex; end block
Mathematics Subject Classification: 05C38, 05C70

1. Introduction

Graph covering and partitioning problems are among the most classical and central subjects in graph theory. They also have extensive applications in a variety of routing problems, such as robot navigation and city snow plowing planning [16]. In this paper, all graphs considered are finite, undirected, and simple. We refer to [4] for unexplained terminology and notation.

Let $G = (V(G), E(G))$ be a graph. The order $|V(G)|$ and size $|E(G)|$ are denoted by $n(G)$ and $e(G)$, respectively. The degree and the neighborhood of a vertex v are denoted by $d_G(v)$ and $N_G(v)$, respectively. A vertex is called *odd* or *even* depending on whether its degree is odd or even, respectively. A graph in which every vertex is odd or even is called an *odd graph* or an *even graph*. The number of odd vertices of G is denoted by $n_o(G)$. The *union* of simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Let X be a set of vertices of $V(G)$. We use $G - X$ to denote the graph that arises from G by deleting the set X. For any $u, v \in V(G)$, we use P_{uv} to denote the path of G with ends u and v. As usual, we use K_n to denote the complete graph of order n.

A *path decomposition* $\mathcal{P}$ of a graph G is a collection of edge-disjoint paths that covers all the edges of G. We use $p(G)$ to denote the minimum number of paths needed for a path decomposition of G. Erdős asked what is the minimum number of paths into which every connected graph can be decomposed. As a response to the question of Erdős, Gallai made the following conjecture:

Conjecture 1.1 ([19]). *For any connected graph G of order n, then $p(G) \leq \lceil \frac{n}{2} \rceil$.*

Indeed, one can see that this bound is sharp by considering a graph in which every vertex has an odd degree; then in any path decomposition of G, each vertex must be the end vertex of some path, and so at least $\frac{n}{2}$ paths are required. In 1968, Lovász [19] proved the following two theorems.

Theorem 1.2 ([19]). *Every graph on n vertices can be decomposed into at most $\frac{n}{2}$ paths and cycles.*

Theorem 1.3 ([19]). *Every odd graph on n vertices can be decomposed into $\frac{n}{2}$ paths.*

In 1980, Donald [12] showed that if G is allowed to be disconnected, then $p(G) \leq \lfloor \frac{3n}{4} \rfloor$, which was improved by Dean and Kouider [11], is as follows.

Theorem 1.4 ([11]). *For any graph G with n vertices (possibly disconnected), $p(G) \leq \frac{2n}{3}$.*

Furthermore, Conjecture 1.1 was verified for several classes of graphs. Let G_E denote the subgraph of G induced by the vertices of even degree. In 1996, Pyber [21] proved the following theorem.

Theorem 1.5 ([21]). *If G is a graph on n vertices such that G_E is a forest, then $p(G) \leq \frac{n}{2}$.*

A *block* of a graph is a maximal 2-connected subgraph of this graph. Theorem 1.5 was strengthened by Fan [13], who proved the following.

Theorem 1.6 ([13]). *If G is a graph on n vertices such that each block of G_E is a triangle-free graph of maximum degree at most 3, then $p(G) \leq \frac{n}{2}$.*

The *girth* of a graph is the length of the shortest cycle in G, denoted by $g(G)$. Harding and McGuinness [15] investigated graphs with high girth and proved the following result.

Theorem 1.7 ([15]). *For any graph G with $g(G) \geq 4$, $p(G) \leq \frac{n_o(G)}{2} + \lfloor (\frac{g(G)+1}{2g(G)}) n_e(G) \rfloor$.*

In 2022, Chu, Fan, and Zhou [10] proved the following result.

Theorem 1.8 ([10]). *For any triangle-free graph G with n vertices, $p(G) \leq \frac{3n}{5}$.*

More results regarding Conjecture 1.1 can be found in [3, 5–9, 13, 14, 17]. But in general, it is still open.

A vertex v is a *cut vertex* of G if the number of components increases in $G - v$. We call a block of G an *end block* if it exactly contains one cut vertex of G. Specifically, if G is a maximal 2-connected graph, we also call G an end block of itself. A maximal complete subgraph of G is called a *clique* of G. For two blocks A and B, we call *A adjacent to B* if A and B have a common vertex. A connected graph is called a *block graph* if each of its blocks is a clique. A *non-complete* block graph has at least two blocks. We refer to [1, 2, 18, 20, 22] for some recent results on block graphs. In this paper, we prove that Gallai's conjecture holds for block graphs.

2. Path decomposition of block graph

In this section, we discuss the path decomposition of block graphs. First, we present a path decomposition of K_n, as shown in Lemma 2.1, in which the subscripts of vertices are taken modular n.

Lemma 2.1. *Let n be a positive integer. If $V(K_n) = \{v_1, v_2, \cdots, v_n\}$, then there exists a path decomposition $\mathcal{P}(K_n)$ of K_n as follows:*

(1) if n is even, then $\mathcal{P}(K_n) = \{P_i : 1 \le i \le \frac{n}{2}\}$, where $P_i = v_i v_{i+1} v_{i+(n-1)} v_{i+2} \cdots v_{i+(\frac{n}{2}+1)} v_{i+\frac{n}{2}}$;

(2) if n is odd, then $\mathcal{P}(K_n) = \{P_{\frac{n+1}{2}}\} \cup \{P'_i : 1 \le i \le \frac{n-1}{2}\}$, where $P'_i = P_i \setminus E(P_{\frac{n+1}{2}})$, $P_i = v_i v_{i+1} v_{i+(n-1)} v_{i+2} \cdots v_{i+(\frac{n-1}{2})} v_{i+\frac{n+1}{2}}$, and

$$P_{\frac{n+1}{2}} = \begin{cases} v_n v_1 v_2 v_{n-1} v_{n-2} v_3 \cdots v_{\frac{n-3}{2}} v_{\frac{n-1}{2}} v_{\frac{n+3}{2}}, & if\ n = 1\ (mod\ 4), \\ v_n v_1 v_2 v_{n-1} v_{n-2} v_3 \cdots v_{\frac{n+5}{2}} v_{\frac{n+3}{2}} v_{\frac{n-1}{2}}, & if\ n = 3\ (mod\ 4). \end{cases}$$

Thus,

$$p(K_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n+1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

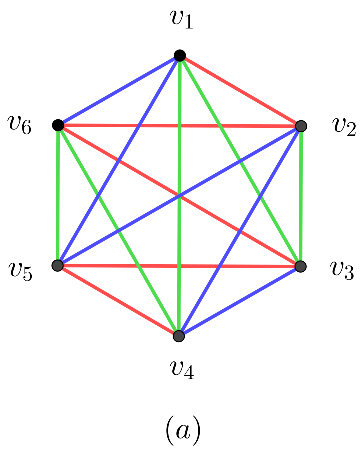
Proof. It is clear that $\mathcal{P}(K_n)$ given in the statement of the lemma is a path decomposition of K_n of cardinality $\frac{n}{2}$ (if n is even) and $\frac{n+1}{2}$ (if n is odd). Thus, $p(K_n) \le \frac{n}{2}$ if n is even, and $p(K_n) \le \frac{n+1}{2}$ if n is odd. On the other hand, since $p(K_n) \ge \frac{n(n-1)}{2}/(n-1) = \frac{n}{2}$, we have

$$p(K_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even}, \\ \frac{n+1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

□

Remark 1. For an illustration of the decomposition $\mathcal{P}(K_n)$ of K_n given in the above lemma, one may see it in Figure 1(a) and (b), which are the examples when $n = 6$ and $n = 7$, respectively. It is worth noting that for the case when n is odd, in the decomposition $\mathcal{P}(K_n)$ in the above lemma, by the transitivity of K_n, the $\frac{n+1}{2}$ vertices can be selected arbitrarily as the end vertices of exactly two paths, and the remaining $\frac{n-1}{2}$ vertices are not the end vertex of any path of $\mathcal{P}(K_n)$.

For convenience, in the remainder of this paper we will refer to the end vertices of the paths in $\mathcal{P}(K_n)$ simply as the end vertices of $\mathcal{P}(K_n)$. Before proving our main theorem, we tackle some of its special cases.



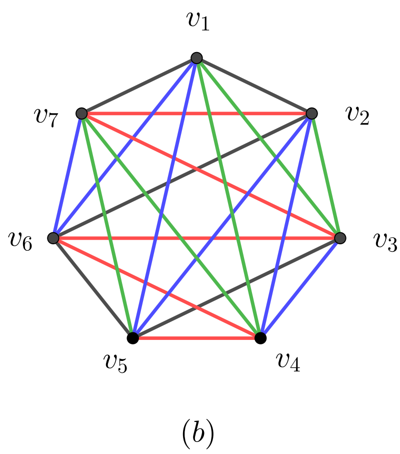


Figure 1. (a) A path decomposition $\mathcal{P}(K_6)$ of K_6 in which each vertex is the end vertex of a path of $\mathcal{P}(K_6)$. (b) A path decomposition $\mathcal{P}(K_7)$ of K_7 where each vertex of $\{v_2, v_3, v_5, v_7\}$ is the end vertex of exactly two paths of $\mathcal{P}(K_7)$.

Lemma 2.2. *Let G be a block graph in which all blocks share a common vertex. If all blocks of G have even order, then*

$$p(G) = \begin{cases} \frac{n}{2}, & \text{if } r \text{ is odd,} \\ \frac{n-1}{2}, & \text{if } r \text{ is even,} \end{cases}$$

where n is the order of G and r is the number of blocks of G.

Proof. Let $B_1, \ldots, B_r$ be all blocks of G and x be their common vertex. It is easy to check that $n = \sum_{i=1}^{r} n(B_i) - r + 1$. If r is odd, then G is odd. By Theorem 1.3, $p(G) = \frac{n}{2}$.

If r is even, then $d_G(x)$ is even, and $n_o(G) = n - 1$, and hence $p(G) \geq \frac{n_o(G)}{2} = \frac{n-1}{2}$. By Lemma 2.1, let $\mathcal{P}(B_i)$ be a path decomposition of B_i with $|\mathcal{P}(B_i)| = \frac{n(B_i)}{2}$ for each $i \in \{1, \ldots, r\}$. Furthermore, let P_i be the path in $\mathcal{P}(B_i)$ with x as its end. One can see that $\mathcal{P}(G) = \bigcup_{i=1}^{r}(\mathcal{P}(B_i) \setminus \{P_i\}) \cup \{P_i \cup P_{i+1} : i \in \{1, 3, \cdots, r-1\}\}$ is a path decomposition of G with

$$|\mathcal{P}(G)| = \sum_{i=1}^{r} |\mathcal{P}(B_i)| - \frac{r}{2} = \sum_{i=1}^{r} \frac{n(B_i)}{2} - \frac{r}{2} = \frac{\sum_{i=1}^{r} n(B_i) - r}{2} = \frac{n-1}{2}.$$

Thus, we have $p(G) \leq \frac{n-1}{2}$. This proves $p(G) = \frac{n-1}{2}$ if r is even. □

For any odd integer $t > 0$, let $\mathcal{H}_t$ be the family of graphs, each element H of which is a block graph and obtained from attaching an odd number of cliques with even order to a vertex of K_t. We use $c(H)$ to denote the number of cut vertices of H. Clearly, $1 \leq c(H) \leq t$. For an illustration, one may see an example in Figure 2 for the case when $t = 7$.

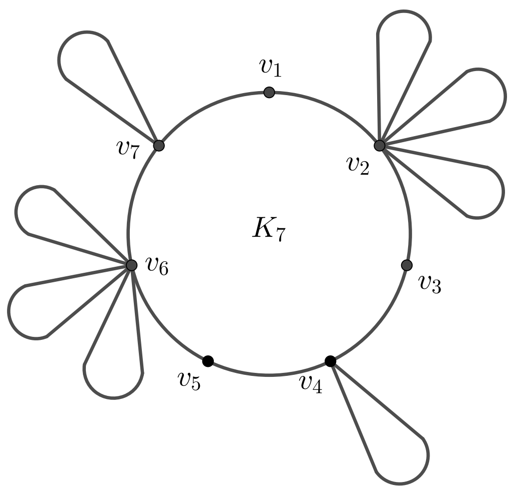


Figure 2. An example of $H \in \mathcal{H}_7$, where v_2, v_4, v_6, v_7 are the cut vertices of H_7, each of which is connected to an odd number of end blocks with order even.

Lemma 2.3. *Let $t > 0$ be an odd integer. If $H \in \mathcal{H}_t$ with n vertices, then*

$$p(H) \leq \begin{cases} \frac{n-1}{2}, & \text{if } 2 \leq c(H) \leq t-1, \\ \frac{n}{2}, & \text{if } c(H) = 1 \text{ or } c(H) = t. \end{cases}$$

Proof. Let $V(K_t) = \{v_1, v_2, \cdots, v_t\}$. For each $i \in \{1, 2, \cdots, t\}$, let G_i be a subgraph of H consisting of those blocks containing v_i but K_t. In addition, let r_i be the number of the blocks of G_i. Since r_i is odd, G_i is odd. By Lemma 2.2, G_i has a path decomposition $\mathcal{P}(G_i)$ with $|\mathcal{P}(G_i)| = \frac{n(G_i)}{2}$. Let $Q_{v_i} \in \mathcal{P}(G_i)$ be the path with v_i as its end vertex.

Let $\mathcal{P}(K_t)$ be a path decomposition of K_t as described in Lemma 2.1. By Remark 1, there are $\frac{t+1}{2}$ vertices (these vertices can be selected arbitrarily) that are the end vertices of $\mathcal{P}(K_t)$ and the remaining $\frac{t-1}{2}$ vertices are not the end vertices of $\mathcal{P}(K_t)$. We use a to denote the number of the cut vertices of H_t that are end vertices of $\mathcal{P}(K_t)$ and b to denote the number of the cut vertices of H_t that are not end vertices of $\mathcal{P}(K_t)$.

Clearly, if $c(H) = 1$ or $c(H) = t$, then $a = b+1$. Now we assume that $2 \leq c(H) \leq t-1$. If $c(H) \leq \frac{t+1}{2}$, we choose the all cut vertices of H as the end vertices of $\mathcal{P}(K_t)$. If $c(H) > \frac{t+1}{2}$, we first choose $\frac{t+1}{2}$ cut vertices of H as end vertices of $\mathcal{P}(K_t)$, then the number of remaining cut vertices is at most $\frac{t-3}{2}$. It is easy to obtain that $a \geq b + 2$, since $\frac{t+1}{2} - \frac{t-3}{2} = 2$.

For $v_i \in V(K_t)$, if v_i is an end vertex of $\mathcal{P}(K_t)$, then there is a path $P_{v_i} \in \mathcal{P}(K_t)$ with v_i as its end. Clearly, $P_{v_i} \cup Q_{v_i}$ is a path of H. Let $V_1 = \{v_i : v_i$ is a cut vertex of H and is an end vertex of $\mathcal{P}(K_t)\}$ and $V_2 = \{v_i : v_i$ is a cut vertex of H and is not an end vertex of $\mathcal{P}(K_t)\}$. One can see that

$$\mathcal{P}(H) = (\mathcal{P}(K_t) \setminus \{P_{v_i} : v_i \in V_1\}) \cup (\bigcup_{v_i \in V_1}(\mathcal{P}(G_i) \setminus \{Q_{v_i}\})) \cup (\bigcup_{v_i \in V_2}\mathcal{P}(G_i)) \cup \{P_{v_i} \cup Q_{v_i} : v_i \in V_1\}$$

is a path decomposition of H with

$$\begin{aligned} |\mathcal{P}(H)| &= (|\mathcal{P}(K_t)| - a) + (\sum_{v_i \in V_1} \frac{n(G_i)}{2} - a) + \sum_{v_i \in V_2} \frac{n(G_i)}{2} + a \\ &= \frac{t+1}{2} + \sum_{v_i \in V_1} \frac{n(G_i)}{2} + \sum_{v_i \in V_2} \frac{n(G_i)}{2} - a. \end{aligned}$$

On the other hand,

$$n = t + \sum_{v_i \in V_1} n(G_i) + \sum_{v_i \in V_2} n(G_i) - (a+b).$$

If $a \geq b + 2$, then $|\mathcal{P}(H)| \leq \frac{n-1}{2}$, and thus $p(H) \leq \frac{n-1}{2}$. If $a = b + 1$, then $|\mathcal{P}(H)| \leq \frac{n}{2}$, and thus $p(H) \leq \frac{n}{2}$. □

Now we will prove our main result.

Theorem 2.4. *If G is a non-complete block graph of order n, then $p(G) \leq \frac{n}{2}$.*

Proof. Let $B_1, \ldots, B_k$ be all blocks of G. For each $i \in \{1, 2, \cdots, k\}$, we use n_i to denote the order of B_i. Let $\mathcal{P}(B_i)$ be a path decomposition of B_i as described in Lemma 2.1. To show $p(G) \leq \frac{n}{2}$, it is enough to show that G has a path decomposition $\mathcal{P}(G)$ with $|\mathcal{P}(G)| \leq \frac{n}{2}$.

The proof is by induction on k. Since G is a non-complete block graph, we have $k \geq 2$. If $k = 2$, then $n = n_1 + n_2 - 1$. Let x be the common vertex of B_1 and B_2. By Lemma 2.1, for each $i \in \{1, 2\}$, B_i has a path decomposition $\mathcal{P}(B_i)$ in which x can be chosen as an end vertex of $\mathcal{P}(B_i)$. We consider the following three cases.

Case 1. n_1 and n_2 are even.

By Lemma 2.1, we have $|\mathcal{P}(B_i)| = \frac{n_i}{2}$ for each $i \in \{1,2\}$. Let P_x and Q_x be the two paths of $\mathcal{P}(B_1)$ and $\mathcal{P}(B_2)$ with x as their ends, respectively. Clearly, $P_x \cup Q_x$ is a path of G. Thus, $(\mathcal{P}(B_1) \setminus \{P_x\}) \cup (\mathcal{P}(B_2) \setminus \{Q_x\}) \cup \{P_x \cup Q_x\}$ is a path decomposition of G with cardinality

$$|\mathcal{P}(B_1)| + |\mathcal{P}(B_2)| - 1 = \frac{n_1}{2} + \frac{n_2}{2} - 1 = \frac{n-1}{2}.$$

Case 2. n_1 and n_2 have distinct parity.

Without loss of generality, assume that n_1 is odd and n_2 is even. By Lemma 2.1, $\mathcal{P}(B_1)$ is a path decomposition of B_1 with $|\mathcal{P}(B_1)| = \frac{n_1+1}{2}$ in which x is the end of a path P_x. Also, $\mathcal{P}(B_2)$ is a path decomposition of B_2 with $|\mathcal{P}(B_2)| = \frac{n_2}{2}$ in which x is the end of a path Q_x. Let $P_x \cup Q_x$ be a path of G. Thus, $(\mathcal{P}(B_1) \setminus \{P_x\}) \cup (\mathcal{P}(B_2) \setminus \{Q_x\}) \cup \{P_x \cup Q_x\}$ is a path decomposition of G with cardinality

$$|\mathcal{P}(B_1)| + |\mathcal{P}(B_2)| - 1 = \frac{n_1+1}{2} + \frac{n_2}{2} - 1 = \frac{n}{2}.$$

Case 3. n_1 and n_2 are odd.

By Lemma 2.1, we have $|\mathcal{P}(B_i)| = \frac{n_i+1}{2}$ for each $i \in \{1,2\}$. Let P_x^1 and P_x^2 be the two paths of $\mathcal{P}(B_1)$ with x as their ends, and Q_x^1 and Q_x^2 be the two paths of $\mathcal{P}(B_2)$ with x as their ends. Let $P_x^1 \cup Q_x^1$ and $P_x^2 \cup Q_x^2$ be two paths of G. Thus, $(\mathcal{P}(B_1) \setminus \{P_x^1, P_x^2\}) \cup (\mathcal{P}(B_2) \setminus \{Q_x^1, Q_x^2\}) \cup \{P_x^1 \cup Q_x^1, P_x^2 \cup Q_x^2\}$ is a path decomposition of G with cardinality

$$|\mathcal{P}(B_1)| + |\mathcal{P}(B_2)| - 2 = \frac{n_1+1}{2} + \frac{n_2+1}{2} - 2 = \frac{n-1}{2}.$$

Thus, $p(G) \leq \frac{n}{2}$ for $k = 2$.

Now we consider the case when $k \geq 3$. Assuming that this result is true for any block graph with the number of blocks less than k. We consider the following two cases.

Case 1. G has an end block with odd order.

Let B be a block of G with odd order, and $x \in V(B)$ be a cut vertex of G. By Lemma 2.1, $\mathcal{P}(B)$ is a path decomposition of B with $|\mathcal{P}(B)| = \frac{n(B)+1}{2}$ in which x can be chosen as the end vertex of $\mathcal{P}(B)$. Suppose Q_x^1 and Q_x^2 are two paths of $\mathcal{P}(B)$ with x as their end vertices. Let $G' = G - (V(B) \setminus \{x\})$. Clearly, $n(G') = n - n(B) + 1$. By the induction hypothesis, G' has a path decomposition $\mathcal{P}(G')$ with $|\mathcal{P}(G')| \leq \frac{n-n(B)+1}{2}$. Now we consider the following subcases.

Subcase 1.1. $d_{G'}(x)$ is odd.

Since an odd vertex serves as an end vertex of at least one path in $\mathcal{P}(G')$, $P_x \in \mathcal{P}(G')$ denotes a path with x as its end. Let $Q_x^1 \cup P_x$ be a path of G. Clearly, $(\mathcal{P}(G') \setminus \{P_x\}) \cup (\mathcal{P}(B) \setminus \{Q_x^1\}) \cup \{Q_x^1 \cup P_x\}$ is a path decomposition of G with cardinality

$$|\mathcal{P}(G')| + |\mathcal{P}(B)| - 1 \leq \frac{n - n(B) + 1}{2} + \frac{n(B)+1}{2} - 1 = \frac{n}{2}.$$

Subcase 1.2. $d_{G'}(x)$ is even.

Since $d_{G'}(x)$ is even, there are at least two paths in $\mathcal{P}(G')$ with x as their ends, or there are no such paths at all.

First assume that there are two paths P_x^1 and P_x^2 in $\mathcal{P}(G')$ with x as their ends. Let $P_x^1 \cup Q_x^1$ and $P_x^2 \cup Q_x^2$ be two paths of G. Obviously, $(\mathcal{P}(G') \setminus \{P_x^1, P_x^2\}) \cup (\mathcal{P}(B) \setminus \{Q_x^1, Q_x^2\}) \cup \{P_x^1 \cup Q_x^1, P_x^2 \cup Q_x^2\}$ is a path decomposition of G with cardinality

$$|\mathcal{P}(G')| + |\mathcal{P}(B)| - 2 \leq \frac{n - n(B) + 1}{2} + \frac{n(B) + 1}{2} - 2 = \frac{n-2}{2}.$$

Now assume that there is no path in $\mathcal{P}(G')$ with x as its end. Assuming that $P_{uv} \in \mathcal{P}(G')$ containing x. Let P_{ux} and P_{xv} be subpaths of P_{uv}. Clearly, $P_{ux} \cup Q_x^1$ and $P_{xv} \cup Q_x^2$ be two paths of G. Hence $(\mathcal{P}(G') \setminus \{P_{uv}\}) \cup (\mathcal{P}(B) \setminus \{Q_x^1, Q_x^2\}) \cup \{P_{ux} \cup Q_x^1, P_{xv} \cup Q_x^2\}$ is a path decomposition with cardinality

$$|\mathcal{P}(G')| + |\mathcal{P}(B)| - 1 \leq \frac{n - n(B) + 1}{2} + \frac{n(B) + 1}{2} - 1 = \frac{n}{2}.$$

Case 2. All end blocks in G have even order.

If all blocks of G are end blocks, then they share the same vertex. By Lemma 2.2, $p(G) \leq \frac{n}{2}$. So, next assume that not all blocks of G, are end blocks. Take an end block B of G. Suppose $x \in V(B)$ is the cut vertex of G and the end blocks containing x are $B^1, B^2, \cdots, B^r$ where $B^1 = B$. Let $H = \bigcup_{i=1}^{r} B^i$ and $G'' = G - (V(H) \setminus \{x\})$ (see Figure 3). Trivially, $n(G'') = n - n(H) + 1$. By the induction hypothesis, G'' has a path decomposition $\mathcal{P}(G'')$ with $|\mathcal{P}(G'')| \leq \frac{n-n(H)+1}{2}$.

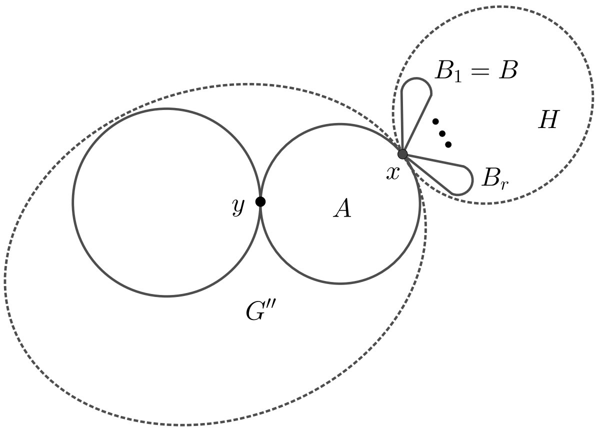


Figure 3. The subgraph H of G and $G'' = G - (V(H) \setminus \{x\})$.

Subcase 2.1. There exists such a subgraph H with r being even.

By Lemma 2.2, H has a path decomposition $\mathcal{P}(H)$ with $|\mathcal{P}(H)| = \frac{n(H)-1}{2}$. Clearly, $\mathcal{P}(G'') \cup \mathcal{P}(H)$ is a path decomposition of G with cardinality

$$|\mathcal{P}(G'')| + |\mathcal{P}(H)| \leq \frac{n - n(H) + 1}{2} + \frac{n(H) - 1}{2} = \frac{n}{2}.$$

Subcase 2.2. For each H, r is odd.

Since r is odd, H is odd. By Theorem 1.3, H has a path decomposition $\mathcal{P}(H)$ with $|\mathcal{P}(H)| = \frac{n(H)}{2}$ in which x is the end of a path Q_x. Let G_0 be the graph obtained from G after deleting all end blocks. Assume that A is an end block of G_0 containing x. We consider the following two subcases.

Subcase 2.2.1. The order of A is even.

Since the order of A is even, $d_A(x)$ is odd. Moreover, by $d_{G''}(x) = d_A(x)$, $d_{G''}(x)$ is odd. Thus, there must exist a path P_x in $\mathcal{P}(G'')$ with x as its end vertex. Let $P_x \cup Q_x$ be a path of G. One can see that $(\mathcal{P}(G'') \setminus \{P_x\}) \cup (\mathcal{P}(H) \setminus \{Q_x\}) \cup \{P_x \cup Q_x\}$ is a path decomposition with cardinality

$$|\mathcal{P}(G'')| + |\mathcal{P}(H)| - 1 \leq \frac{n - n(H) + 1}{2} + \frac{n(H)}{2} - 1 = \frac{n-1}{2}.$$

Subcase 2.2.2. The order of A is odd.

If G_0 is a complete graph, by Lemma 2.3, $p(G) \leq \frac{n}{2}$. Now we assume that G_0 is not a complete graph. Let $y \in V(A)$ be the cut vertex of G_0. Let $G''' = G - (V(H') \setminus \{y\})$, where H' is the union of A and the end blocks of G that contain the vertices of A other than y (see Figure 4). By the induction hypothesis, G''' has a path decomposition $\mathcal{P}(G''')$ with $|\mathcal{P}(G''')| \leq \frac{n(G''')}{2}$. Now we consider the following two cases.

(1) If there is only one cut vertex x in H', by Lemma 2.3, H' has a path decomposition $\mathcal{P}(H')$ with $|\mathcal{P}(H')| \leq \frac{n(H')}{2}$, in which x and y are the end vertices of $\mathcal{P}(A)$. We denote $Q_y^1, Q_y^2 \in \mathcal{P}(H')$ as two paths with y as their ends. On the other hand, there must exist a path $P_{uv} \in \mathcal{P}(G''')$ pass through y. Let P_{uy} and P_{yv} be subpaths of P_{uv}. Clearly, $P_{uy} \cup Q_y^1$ and $P_{yv} \cup Q_y^2$ are two paths of G. Thus, $(\mathcal{P}(G''') \setminus \{P_{uv}\}) \cup (\mathcal{P}(H') \setminus \{Q_y^1, Q_y^2\}) \cup \{P_{uy} \cup Q_y^1, P_{yv} \cup Q_y^2\}$ is a path decomposition of G with cardinality

$$|\mathcal{P}(G''')| + |\mathcal{P}(H')| - 1 \leq \frac{n - n(H') + 1}{2} + \frac{n(H')}{2} - 1 = \frac{n-1}{2}.$$

(2) If there are at least two cut vertices in H', by Lemma 2.3, H' has a path decomposition $\mathcal{P}(H')$ with $|\mathcal{P}(H')| = \frac{n(H')-1}{2}$. Hence $\mathcal{P}(G''') \cup \mathcal{P}(H')$ is a path decomposition of G with cardinality

$$|\mathcal{P}(G''')| + |\mathcal{P}(H')| \leq \frac{n(G''')}{2} + \frac{n(H') - 1}{2} = \frac{n}{2}.$$

The proof is now finished. □

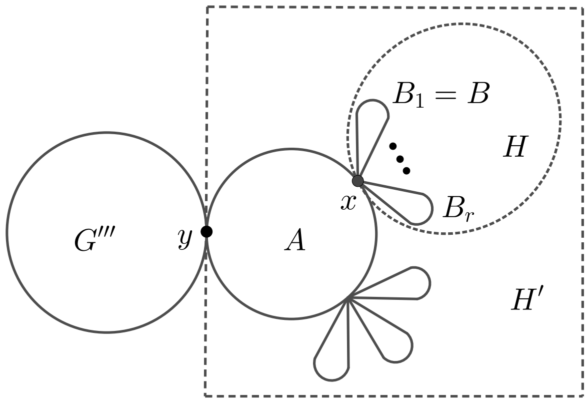


Figure 4. The subgraph H' of G and $G''' = G - (V(H') \setminus \{y\})$.

Author contributions

X.C.: Conceptualization, Funding acquisition, Methodology, Validation, Visualization, Writing-original draft; B.W.: Funding acquisition, Methodology, Project administration, Supervision, Validation, Visualization, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by NSFC (No.12061073) and the Research Innovation Program for Postgraduates of Xinjiang Uygur Autonomous Region under Grant No. XJ2023G020.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. K. J. Balodis, M. E. Kroeker, L. Mol, O. R. Oellermann, On the mean order of connected induced subgraphs of block graphs, *Australas. J. Combin.*, **76** (2020), 128–148.
2. A. Behtoei, M. Jannesari, B. Taeri, A characterization of block graphs, *Discrete Appl. Math.*, **158** (2010), 219–221. https://doi.org/10.1016/j.dam.2009.09.024
3. M. Bonamy, T. J. Perrett, Gallai's path decomposition conjecture for graphs of small maximum degree, *Discrete Math.*, **342** (2019), 1293–1299. https://doi.org/10.1016/j.disc.2019.01.005
4. J. A. Bondy, U. S. R. Murty, *Graph theory*, London: Springer, 2008.
5. F. Botler, A. Jiménez, On path decompositions of $2k$-regular graphs, *Discrete Math.*, **340** (2017), 1405–1411. https://doi.org/10.1016/j.disc.2016.09.029
6. F. Botler, A. Jiménez, M. Sambinelli, Gallai's path decomposition conjecture for triangle-free planar graphs, *Discrete Math.*, **342** (2019), 1403–1414. https://doi.org/10.1016/j.disc.2019.01.020
7. F. Botler, M. Sambinelli, Towards Gallai's path decomposition conjecture, *J. Graph Theor.*, **97** (2020), 161–184. https://doi.org/10.1002/jgt.22647
8. F. Botler, M. Sambinelli, R. S. Coelho, O. Lee, Gallai's path decomposition conjecture for graphs with treewidth at most 3, *J. Graph Theor.*, **93** (2020), 328–349. https://doi.org/10.1002/jgt.22489
9. Y. Chu, G. Fan, Q. Liu, On Gallai's conjecture for graphs with maximum degree 6, *Discrete Math.*, **344** (2021), 112212. https://doi.org/10.1016/j.disc.2020.112212
10. Y. Chu, G. Fan, C. Zhou, Path decomposition of triangle-free graphs, *Discrete Math.*, **345** (2022), 112866. https://doi.org/10.1016/j.disc.2022.112866
11. N. Dean, M. Kouider, Gallai's conjecture for disconnected graphs, *Discrete Math.*, **213** (2000), 43–54. https://doi.org/10.1016/S0012-365X(99)00167-3
12. A. Donald, An upper bound for the path number of a graph, *J. Graph Theor.*, **4** (1980), 189–201. https://doi.org/10.1002/jgt.3190040207
13. G. Fan, Path decompositions and Gallai's conjecture, *J. Comb. Theory B*, **93** (2005), 117–125. https://doi.org/10.1016/j.jctb.2004.09.008
14. X. Geng, M. Fang, D. Li, Gallai's conjecture for outerplanar graphs, *J. Interdiscip. Math.*, **18** (2015), 593–598. https://doi.org/10.1080/09720502.2014.1001570
15. P. Harding, S. McGuinness, Gallai's conjecture for graphs of girth at least four, *J. Graph Theor.*, **75** (2014), 256–274. https://doi.org/10.1002/jgt.21735

16. N. Immorlica, M. Mahdian, V. S. Mirrokni, Cycle cover with short cycles, In: *STACS 2005*, Berlin, Heidelberg: Springer, 2005, 641–653. https://doi.org/10.1007/978-3-540-31856-9_53

17. A. Jiménez, Y. Wakabayashi, On path-cycle decompositions of triangle-free graphs, *Discrete Mathematics & Theoretical Computer Science*, **19** (2017), 7. https://doi.org/10.23638/DMTCS-19-3-7

18. V. B. Le, N. N. Tuy, The square of a block graph, *Discrete Math.*, **310** (2010), 734–741. https://doi.org/10.1016/j.disc.2009.09.004

19. L. Lovász, On covering of graphs, In: *Theory of graphs*, New York: Academic Press, 1968, 231–236.

20. D. Pradhan, A. Jha, On computing a minimum secure dominating set in block graphs, *J. Comb. Optim.*, **35** (2018), 613–631. https://doi.org/10.1007/s10878-017-0197-y

21. L. Pyber, Covering the edges of a connected graph by paths, *J. Comb. Theory B*, **66** (1996), 152–159. https://doi.org/10.1006/jctb.1996.0012

22. C. Wang, L. Chen, C. Lu, *k*-Power domination in block graphs, *J. Comb. Optim.*, **31** (2016), 865–873. http://doi.org/10.1007/s10878-014-9795-0