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Research article

Interpolative best proximity point results via γ-contraction with applications

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Abstract: In this paper, we introduce a ρ-interpolative Kannan and Ćirić-Reich-Rus type fuzzy proximal contraction using a γ-contraction. We prove some best proximity theorems of this new approximation using the concept of ρ-proximal admissibility in complete fuzzy metric spaces. We generalize some previous studies and present fixed point results of the best proximity theorems in complete fuzzy metric spaces. Also, we extend some best proximity results to the partially ordered fuzzy metric spaces.

Keywords: fuzzy metric space; ordered fuzzy metric space; interpolative proximal fuzzy contraction
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

Fixed point theory is an important field of study that analyzes the points that are equal to the appearance of functions that provide various different contractions. The Banach contraction principle, which is the most fundamental subject of fixed point theory, is one of the fundamental theorems of fixed point theory and has a wide application in many analyses [3]. Although Banach proved the Banach fixed point theorem, known as this basic principle, in metric spaces, which is an important field of study in mathematics, it has actually been the subject of research in other areas other than metric spaces. This subject, which attracts a lot of attention from researchers, has been studied in other applied fields as well as mathematics, and important results have been obtained. Also widely used in mathematics, this theory was applied to demonstrate the uniqueness of solutions of linear or nonlinear differential and integral equations.

Due to its very useful applications, this principle has been modified many times in different versions, generalized, updated and used in other spaces, and the results obtained have been demonstrated in practice. Although Banach contractions are continuous mappings, Kannan generalized this principle to prove some fixed point theorems for mappings that do not require continuity [18]. In addition, Chatterjea [6], Reich [23], and Ćirić [8] obtained new and more general results by producing different contractions. Later, researchers further generalized these different contractions and presented different versions in a metric space. Karapınar introduced the notion of an interpolative Kannan and Ćirić-Reich-Rus type contraction in a complete metric space [16,17]. Although the Banach contraction principle has been proven in metric spaces, many authors have generalized this issue in various spaces such as fuzzy metric spaces.

Fixed point theory shows the existence of a point x that satisfies the equality $Tx = x$ of the mapping T defined on a non-empty X. However, for the non-void sets A and E of X, the mapping defined as $T: A \to \mathrm{E}$ may not have such a fixed point x. Indeed, best proximity point theorems explore the existence of such optimal approximate solutions, known as best proximity points, of the equation $Tx = x$ when there is no solution. It tries to determine an approximate solution x that will minimize the distance between x and Tx. If the intersection of sets A and E is different from the empty one, the best proximity point is a fixed point. The concept of best proximity theory has been studied in metric spaces by many researchers. Eldred and Veeramani [10] presented on non-self contractions for the existence of a best proximity point. Basha [4] proved the best proximity point theorem for proximal contractions, which generalizes Banach contraction. Hussain et al. [14] introduced best proximity point theorems of Suzuki α-ψ-proximal contraction. Aydi et al. [2] proved some fixed point results for ω-interpolative Ćirić-Reich-Rus-type contraction mappings. Later, Saleem et al. [26] extended some best proximity results ω-interpolative Ćirić-Reich-Rus-type contraction to partial ordered metric spaces and graphical metric spaces.

Zadeh introduced the notion of fuzzy set [31]. Fuzzy metric spaces have different concepts. Kramosil and Michalek [19] introduced the concept of fuzzy metric space using continuous t-norms, which generalize the concept of probabilistic metric space to the fuzzy case. Afterward, Grabiec [12] defined the completeness of the fuzzy metric space. Moreover, George and Veeramani [11] modified the concept of fuzzy metric spaces and obtained a Hausdorff topology for such fuzzy metric spaces. Recently, Gregori et al. [13] applied fuzzy metrics to the color image process and used the concept of fuzzy metrics to filter noisy images and solve some engineering problems of special interest. Fixed point theory has been studied by many authors in fuzzy metric spaces. In a way, the concept of the best proximity point theory, which covers the concept of fixed point theory, has an important role in fuzzy metric spaces. The concept of best proximity theory has been studied in different type of fuzzy metric spaces by many researchers, and important results have been obtained. Vetro and Salimi [30] proved the existence and uniqueness of the best proximity points by using different contractive conditions in non-Archimedean fuzzy metric space. Hussain [15] initiated some new classes of proximal contraction mappings in a non-Archimedean fuzzy metric space. Choudhury et al. [7] and Abbas et al. [1] presented some best proximity points of proximal contractions in complete partially ordered non-Archimedean fuzzy metric space. Latif et al. [20,21] and Rakić et al. [22] obtained some notable results regarding the best proximity theorems in different types of fuzzy metric spaces. Saha et al. [25] presented a fuzzy extension of the proximity point problem which is by its nature a problem of global optimization in fuzzy metric space.

In this paper, we introduce ρ-interpolative Ćirić-Reich-Rus type fuzzy proximal contractions.

We prove some best proximity theorems of ρ-interpolative Ćirić-Reich-Rus type fuzzy proximal contraction in complete fuzzy metric spaces. We support our main theorems with a few examples. As a result, we establish ρ-interpolative Kannan-type fuzzy proximal contractions. We prove the fixed point results of the best proximity theorems in complete fuzzy metric spaces. Also, we extend some best proximity results to the partially ordered fuzzy metric spaces.

Definition 1.1. [27] *A binary operation* $*: [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangular norm (in short, continuous* t*-norm) if it satisfies the following conditions:*

(i) $*$ *is commutative and associative;*

(ii) $*$ *is continuous;*

(iii) $*(\bar{a},1) = \bar{a}$ *for every* $\bar{a} \in [0,1]$;

(iv) $*(\bar{a},\bar{e}) \leq *(\bar{u},\bar{o})$ *whenever* $\bar{a} \leq \bar{u}$, $\bar{e} \leq \bar{o}$ *and* $\bar{a},\bar{e},\bar{u},\bar{o} \in [0,1]$.

Definition 1.2. [11] *A fuzzy metric space is an ordered triple* $(X, F, *)$ *such that* X *is a nonempty set,* $*$ *is a continuous t-norm, and* F *is a fuzzy set on* $X^2 \times (0,\infty)$, *satisfying the following conditions, for all* $\bar{a},\bar{e},\bar{u} \in X$ *and* $s, \hat{\jmath} > 0$:

(v) $F(\bar{a},\bar{e},\hat{\jmath}) > 0$;

(i) $F(\bar{a},\bar{e},\hat{\jmath}) = 1$ *iff* $\bar{a} = \bar{e}$;

(ii) $F(\bar{a},\bar{e},\hat{\jmath}) = F(\bar{e},\bar{a},\hat{\jmath})$;

(iii) $F(\bar{a},\bar{u},\hat{\jmath}+s) \geq F(\bar{a},\bar{e},\hat{\jmath}) * F(\bar{e},\bar{u},s)$;

(iv) $F(\bar{a},\bar{e},\cdot): (0,\infty) \to [0,1]$ *is continuous.*

Definition 1.3. [11] *Let* $(X, F, *)$ *be a fuzzy metric space. Then,*

(i) *A sequence* $\{\bar{a}_n\}$ *in* X *is said to converge to* $\bar{a}$ *in* X, *denoted by* $\bar{a}_n \to \bar{a}$, *if* $\lim_{n\to+\infty} F(\bar{a}_n,\bar{a},\hat{\jmath}) = 1$ *for all* $\hat{\jmath} > 0$, *i.e., for each* $r \in (0,1)$ *and* $\hat{\jmath} > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $F(\bar{a}_n,\bar{a},\hat{\jmath}) > 1 - r$ *for all* $n \geq n_0$.

(ii) *A sequence* $\{\bar{a}_n\}$ *is a Cauchy sequence if for all* $r \in (0,1)$ *and* $\hat{\jmath} > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $F(\bar{a}_n,\bar{a}_m,\hat{\jmath}) \geq 1 - r$ *for all* $m > n \geq n_0$.

(iii) *The fuzzy metric space* $(X, F, *)$ *is called complete if every Cauchy sequence is convergent.*

Remark 1.1. [5,12] *Let* $(X, F, *)$ *be a fuzzy metric space. Then,*

(i) *The limit of the convergent sequence in* X *is unique.*

(ii) *The mapping* $F(\bar{a},\bar{e},\cdot)$ *is non-decreasing on* $(0,\infty)$ *for all* $\bar{a},\bar{e} \in X$.

(iii) F *is a continuous mapping on* $X^2 \times (0,\infty)$.

Definition 1.4. [28] *Let* $\gamma: [0,1) \to \mathbb{R}$ *be a strictly increasing, continuous mapping, and for each sequence* $\{\bar{a}_n\}_{n\in\mathbb{N}}$ *of positive numbers* $\lim_{n\to+\infty} \bar{a}_n = 1$ *if and only if* $\lim_{n\to+\infty} \gamma(\bar{a}_n) = +\infty$. *Let* Γ *be the family of all* γ *functions.*

Example 1.1. *Let* $\gamma \in \Gamma$. *The different types of the mapping* $\gamma(t)$ *are the following:*

$$a)\ \frac{1}{1-t},\quad b)\ \frac{1}{1-t} + t,\quad c)\ \frac{1}{1-t^2},\quad d)\ \frac{1}{\sqrt{1-t}},$$

for all $t \in [0,1)$.

Definition 1.5. [9] *Let* $(X, F, *)$ *be an FMS and* $K: X \to X$ *be a given mapping. We say that* K *is a triangular* ρ*-admissible mapping if there exists a function* $\rho: X \times X \times (0,\infty) \to (0,\infty)$ *such that*

(i) $\rho(\bar{a},\bar{e},\hat{\jmath}) \leq 1$ *implies* $\rho(K\bar{a},K\bar{e},\hat{\jmath}) \leq 1$ *for all* $\bar{a},\bar{e} \in X$ *and for all* $\hat{\jmath} > 0$;

(ii) $\rho(\bar{a},\bar{e},\hat{\jmath}) \leq 1$ *and* $\rho(\bar{e},\bar{u},\hat{\jmath}) \leq 1$ *imply* $\rho(\bar{a},\bar{u},\hat{\jmath}) \leq 1$ *for all* $\bar{a},\bar{e},\bar{u} \in X$ *and for all* $\hat{\jmath} > 0$.

Lemma 1.1. [9] *Let* $(X, F, *)$ *be an FMS and* K *be a triangular* ρ*-admissible mapping. Assume that there exists* $\bar{a}_0 \in X$ *such that* $\rho(\bar{a}_0, K\bar{a}_0, \jmath) \leq 1$. *Define a sequence* $\{\bar{a}_n\}$ *by* $\bar{a}_{n+1} = K\bar{a}_n$ *for all* $n \in \mathbb{N}$. *Then*

$$\rho(\bar{a}_m, \bar{a}_n, \jmath) \leq 1 \; for \; all \; m, n \in \mathbb{N} \; with \; m < n.$$

Now, before presenting the best proximity point results in fuzzy metric spaces, which is the main concept of our study, it is necessary to emphasize some expressions that should be used:

Definition 1.6. [24] *Let* $A_0(\jmath)$ *and* $E_0(\jmath)$ *be two nonempty subsets of a fuzzy metric space* $(X, F, *)$. *We will use the following notations:*

$$F(A, E, \jmath) = sup\{F(\bar{a}, \bar{e}, \jmath): \bar{a} \in A, \bar{e} \in E\},$$

$$A_0(\jmath) = \{\bar{a} \in A: F(\bar{a}, \bar{e}, \jmath) = F(A, E, \jmath) \; for \; some \; \bar{e} \in E \; and \; for \; all \; \jmath > 0\},$$

$$E_0(\jmath) = \{\bar{e} \in E: F(\bar{a}, \bar{e}, \jmath) = F(A, E, \jmath) \; for \; some \; \bar{a} \in A \; and \; for \; all \; \jmath > 0\}.$$

Definition 1.7. [29] *Let* (A, E) *be a pair of nonempty subsets of* X *with* $A_0 \neq \emptyset$. *Then the pair* (A, E) *is said to have the fuzzy weak P-property if and only if*

$$\begin{cases} F(\bar{a}_1, \bar{e}_1, \jmath) = F(A, E, \jmath) \\ F(\bar{a}_2, \bar{e}_2, \jmath) = F(A, E, \jmath) \end{cases} \Rightarrow F(\bar{a}_1, \bar{a}_2, \jmath) \geq F(\bar{e}_1, \bar{e}_2, \jmath),$$

where $\bar{a}_1, \bar{a}_2 \in A_0$ *and* $\bar{e}_1, \bar{e}_2 \in E_0$.

2. Main results

In this section, we define the concept of the ρ-proximal admissibility, ρ-interpolative Ćirić-Reich-Rus-type γ-fuzzy proximal and Kannan-type γ-fuzzy proximal contractions, and related the best proximity point theorems.

Definition 2.1. *Let* $K: A \to E$ *and* $\rho: A \times A \times (0, \infty) \to (0, \infty)$, *then* K *is known as* ρ*-proximal admissible if*

$$\left.\begin{array}{l} \rho(\bar{a}_1, \bar{a}_2, \jmath) \leq 1 \\ F(\bar{e}_1, K\bar{a}_1, \jmath) = F(A, E, \jmath) \\ F(\bar{e}_2, K\bar{a}_2, \jmath) = F(A, E, \jmath) \end{array}\right\} \Rightarrow \rho(\bar{e}_1, \bar{e}_2, \jmath) \leq 1,$$

for all $\bar{a}_1, \bar{a}_2, \bar{e}_1, \bar{e}_2 \in A$.

Remark 2.1. *If* K *is a self-mapping, then every* ρ*-proximal admissible becomes* ρ*-admissible mapping.*

Definition 2.2. *Let* A *and* E *be two nonempty, closed subsets of* $(X, F, *)$ *FMS. A mapping* $K: A \to E$ *is said to be* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction if there exist* $\gamma \in \Gamma$, $\rho: A \times A \times (0, +\infty) \to \mathbb{R}^+$, *positive real numbers* α, β *satisfying* $\alpha + \beta < 1$ *and* $\delta \in (0,1)$ *such that*

$$\rho(\bar{a}, \bar{e}, \jmath)\gamma(F(K\bar{a}, K\bar{e}, \jmath)) \geq \gamma(F(\bar{a}, \bar{e}, \jmath)^{\alpha} F(\bar{a}, K\bar{a}, \jmath)^{\beta} F(\bar{e}, K\bar{e}, \jmath)^{(1-\alpha-\beta)}) + \delta, \quad (1)$$

for all $\bar{a}, \bar{e} \in A \backslash B\_\{est\}(K)$ *with* $\rho(\bar{a}, \bar{e}, \jmath) \leq 1$, $F(K\bar{a}, K\bar{e}, \jmath) < 1$ *and for all* $\jmath > 0$, *where* $B\_\{est\}(K): \{\bar{a} \in A: F(\bar{a}, K\bar{a}, \jmath) = F(A, E, \jmath)\}$.

Example 2.1. *Let* $X = \mathbb{R} \times \mathbb{R}$ *be endowed with a standard fuzzy metric* $F(\bar{a}, \bar{e}, \jmath) = exp(-\frac{d(\bar{a}, \bar{e})}{\jmath})$

for all $\hat{\jmath} > 0$ *such that* $d(\bar{a}, \bar{e}) = |\bar{a}_1 - \bar{e}_1| + |\bar{a}_2 - \bar{e}_2|$ *for all* $\bar{a} = (\bar{a}_1, \bar{a}_2)$ *and* $\bar{e} = (\bar{e}_1, \bar{e}_2) \in X$. *Clearly,* $(X, F, *)$ *is a complete FMS where* $*$ *is a product* t*-norm. Define* A *and* E *be two nonempty subsets of* X *given as*

$$A = \{(0,0), (0,1), (0,2), (0,3)\}$$

and

$$E = \{(1,0), (1,1), (1,2), (1,3)\}.$$

So that, $d(A, E) = 1$ *and* $F(\bar{a}, \bar{e}, \hat{\jmath}) = exp(-\frac{1}{\hat{\jmath}})$ *for all* $\bar{a}, \bar{e} \in A$ *and* $\hat{\jmath} > 0$. *Obviously,* A, E *are nonempty closed subsets of* X. *It is clear that* $A_0 = A$ *and* $E_0 = E$. *Define a mapping* $K: A \to E$ *as*

$$K(\bar{a}) = \begin{cases} (1,1) & , \quad if \ \bar{a} \in \{(0,0), (0,1)\} \\ (1,0) & , \quad if \ \bar{a} \in \{(0,2), (0,3)\} \end{cases}.$$

Clearly $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$. *Also suppose that* $\rho: A \times A \times (0, +\infty) \to \mathbb{R}^+$ *is given by*

$$\rho(\bar{a}, \bar{e}, \hat{\jmath}) = \frac{1 - e^{-\frac{1}{\hat{\jmath}}}}{1 - e^{-\frac{2}{\hat{\jmath}}}} \ for \ all \ \bar{a}, \bar{e} \in A \ and \ \hat{\jmath} > 0.$$

Let $\gamma: [0,1) \to \mathbb{R}$ *be defined by* $\gamma(t) = \frac{1}{1-t}$. *Now, we will show that* K *is* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction for all* $\hat{\jmath} > 0$. *Let* $\alpha = 0{,}2$ *and* $\beta = 0{,}2$. *For all* $\bar{u}, \bar{o} \in A$, *we have*

$$F(K\bar{u}, K\bar{o}, \hat{\jmath}) = exp(-\frac{1}{\hat{\jmath}}). \tag{2}$$

Case 1. *If* $\bar{u} = (0,0)$, $\bar{o} = (0,2)$, *then, we have*

$$F(\bar{u}, \bar{o}, \hat{\jmath})^{\alpha} F(\bar{u}, K\bar{u}, \hat{\jmath})^{\beta} F(\bar{o}, K\bar{o}, \hat{\jmath})^{(1-\alpha-\beta)} = exp(-\frac{2{,}6}{\hat{\jmath}}). \tag{3}$$

Using (2), (3), and from the inequality (1), we obtain

$$\begin{aligned} \rho(\bar{u}, \bar{o}, \hat{\jmath})\gamma(F(K\bar{u}, K\bar{o}, \hat{\jmath})) &= \left(\frac{1-e^{-\frac{1}{\hat{\jmath}}}}{1-e^{-\frac{2}{\hat{\jmath}}}}\right)\left(\frac{1}{1-e^{-\frac{1}{\hat{\jmath}}}}\right) = \frac{1}{1-e^{-\frac{2}{\hat{\jmath}}}} \\ &> \gamma(F(\bar{u}, \bar{o}, \hat{\jmath})^{\alpha} F(\bar{u}, K\bar{u}, \hat{\jmath})^{\beta} F(\bar{o}, K\bar{o}, \hat{\jmath})^{(1-\alpha-\beta)}) \\ &= \gamma\left(exp(-\frac{2{,}6}{\hat{\jmath}})\right) = \frac{1}{1-e^{-\frac{2{,}6}{\hat{\jmath}}}}, \end{aligned}$$

for all $\hat{\jmath} > 0$.

Case 2. *If* $\bar{u} = (0,0)$, $\bar{o} = (0,3)$, *then, we have*

$$F(\bar{u}, \bar{o}, \hat{\jmath})^{\alpha} F(\bar{u}, K\bar{u}, \hat{\jmath})^{\beta} F(\bar{o}, K\bar{o}, \hat{\jmath})^{(1-\alpha-\beta)} = exp(-\frac{3{,}4}{\hat{\jmath}}). \tag{4}$$

Using (2), (4), and from the inequality (1), we obtain

$$\rho(\bar{u},\bar{o},\hat{\jmath})\gamma(F(K\bar{u},K\bar{o},\hat{\jmath})) = \left(\frac{1-e^{-\frac{1}{\hat{\jmath}}}}{1-e^{-\frac{2}{\hat{\jmath}}}}\right)\left(\frac{1}{1-e^{-\frac{1}{\hat{\jmath}}}}\right) = \frac{1}{1-e^{-\frac{2}{\hat{\jmath}}}}$$

$$> \gamma(F(\bar{u},\bar{o},\hat{\jmath})^{\alpha}F(\bar{u},K\bar{u},\hat{\jmath})^{\beta}F(\bar{o},K\bar{o},\hat{\jmath})^{(1-\alpha-\beta)})$$

$$= \gamma\left(exp(-\frac{3,4}{\hat{\jmath}})\right) = \frac{1}{1-e^{-\frac{3,4}{\hat{\jmath}}}},$$

for all $\hat{\jmath} > 0$.

Case 3. *If* $\bar{u} = (0,1)$, $\bar{o} = (0,2)$, *then, we have*

$$F(\bar{u},\bar{o},\hat{\jmath})^{\alpha}F(\bar{u},K\bar{u},\hat{\jmath})^{\beta}F(\bar{o},K\bar{o},\hat{\jmath})^{(1-\alpha-\beta)} = exp\left(-\frac{2,2}{\hat{\jmath}}\right). \tag{5}$$

Using (2), (5), and from the inequality (1), we obtain

$$\rho(\bar{u},\bar{o},\hat{\jmath})\gamma(F(K\bar{u},K\bar{o},\hat{\jmath})) = \left(\frac{1-e^{-\frac{1}{\hat{\jmath}}}}{1-e^{-\frac{2}{\hat{\jmath}}}}\right)\left(\frac{1}{1-e^{-\frac{1}{\hat{\jmath}}}}\right) = \frac{1}{1-e^{-\frac{2}{\hat{\jmath}}}}$$

$$> \gamma(F(\bar{u},\bar{o},\hat{\jmath})^{\alpha}F(\bar{u},K\bar{u},\hat{\jmath})^{\beta}F(\bar{o},K\bar{o},\hat{\jmath})^{(1-\alpha-\beta)})$$

$$= \gamma\left(exp\left(-\frac{2,2}{\hat{\jmath}}\right)\right) = \frac{1}{1-e^{-\frac{2,2}{\hat{\jmath}}}},$$

for all $\hat{\jmath} > 0$.

Case 4. *If* $\bar{u} = (0,1)$, $\bar{o} = (0,3)$, *then, we have*

$$F(\bar{u},\bar{o},\hat{\jmath})^{\alpha}F(\bar{u},K\bar{u},\hat{\jmath})^{\beta}F(\bar{o},K\bar{o},\hat{\jmath})^{(1-\alpha-\beta)} = exp(-\frac{2,8}{\hat{\jmath}}). \tag{6}$$

Using (2), (6), and from the inequality (1), we obtain

$$\rho(\bar{u},\bar{o},\hat{\jmath})\gamma(F(K\bar{u},K\bar{o},\hat{\jmath})) = \left(\frac{1-e^{-\frac{1}{\hat{\jmath}}}}{1-e^{-\frac{2}{\hat{\jmath}}}}\right)\left(\frac{1}{1-e^{-\frac{1}{\hat{\jmath}}}}\right) = \frac{1}{1-e^{-\frac{2}{\hat{\jmath}}}}$$

$$> \gamma(F(\bar{u},\bar{o},\hat{\jmath})^{\alpha}F(\bar{u},K\bar{u},\hat{\jmath})^{\beta}F(\bar{o},K\bar{o},\hat{\jmath})^{(1-\alpha-\beta)})$$

$$= \gamma\left(exp\left(-\frac{2,8}{\hat{\jmath}}\right)\right) = \frac{1}{1-e^{-\frac{2,8}{\hat{\jmath}}}}, \tag{7}$$

for all $\hat{\jmath} > 0$. *There can be at least one* $\delta \in (0,1)$ *that satisfies the inequality (1) for all cases. Therefore,* K *is a* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction.*

Theorem 2.1. L*et* $(X,F,*)$ *be a complete FMS and* (A,E) *be a pair of closed subsets of* X *such that* $A_0(\hat{\jmath})$ *is nonempty. Let* $K: A \to E$ *be a continuous mapping, satisfying*

(i) $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$ *and* (A,E) *abide by the fuzzy weak P-property.*

(ii) There exist $\bar{a}_0, \bar{a}_1 \in A$ *such that* $\rho(\bar{a}_1, K\bar{a}_0, \hat{\jmath}) \leq 1$ *and* $F(\bar{a}_1, K\bar{a}_0, \hat{\jmath}) = F(A,E,\hat{\jmath})$.

(iii) K *is* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction.*

Then, K *has a unique best proximity point in* A.

Proof. Let $\bar{a}_0 \in A_o(\hat{\jmath})$. Since $K(A_o(\hat{\jmath})) \subseteq E_o(\hat{\jmath})$, there is an element $\bar{a}_1$ in $A_0(\hat{\jmath})$ such that

$$\rho(\bar{a}_0,\bar{a}_1,\hat{\jmath}) \leq 1 \; and \; F(\bar{a}_1,K\bar{a}_0,\hat{\jmath}) = F(A,E,\hat{\jmath}). \tag{8}$$

Since $K(A_o(\hat{\jmath})) \subseteq E_o(\hat{\jmath})$, there is an element $\bar{a}_2$ in $A_0(\hat{\jmath})$ such that

$$F(\bar{a}_2, K\bar{a}_1, \hat{\jmath}) = F(A, E, \hat{\jmath}). \tag{9}$$

Then, from (8) and (9), and using the definition of ρ-proximal admissibility, we have

$$\rho(\bar{a}_0, \bar{a}_1, \hat{\jmath}) \leq 1,$$
$$F(\bar{a}_1, K\bar{a}_0, \hat{\jmath}) = F(A, E, \hat{\jmath}),$$
$$F(\bar{a}_2, K\bar{a}_1, \hat{\jmath}) = F(A, E, \hat{\jmath}),$$

such that $\rho(\bar{a}_1, \bar{a}_2, \hat{\jmath}) \leq 1$. Thus,

$$\rho(\bar{a}_1, \bar{a}_2, \hat{\jmath}) \leq 1 \text{ and } F(\bar{a}_2, K\bar{a}_1, \hat{\jmath}) = F(A, E, \hat{\jmath}).$$

Since $K(A_o(\hat{\jmath})) \subseteq E_o(\hat{\jmath})$, there is an element $\bar{a}_3$ in $A_0(\hat{\jmath})$ such that

$$F(\bar{a}_3, K\bar{a}_2, \hat{\jmath}) = F(A, E, \hat{\jmath}).$$

Since K is ρ-proximal admissible, we conclude that $\rho(\bar{a}_2, \bar{a}_3, \hat{\jmath}) \leq 1$. Thus we obtain

$$\rho(\bar{a}_2, \bar{a}_3, \hat{\jmath}) \leq 1 \text{ and } F(\bar{a}_3, K\bar{a}_2, \hat{\jmath}) = F(A, E, \hat{\jmath}).$$

On similar steps, we construct a sequence $\{\bar{a}_n\}$ in $A_o(\hat{\jmath})$ such that

$$\rho(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath}) \leq 1 \text{ and } F(\bar{a}_{n+1}, K\bar{a}_n, \hat{\jmath}) = F(A, E, \hat{\jmath}), \tag{10}$$

for all $n \geq 0$. If for some $n_{0,}$ we have $\bar{a}_{n_0} = \bar{a}_{n_0+1}$, then $\bar{a}_{n_0}$ is a best proximity point of K. Assume that, $\bar{a}_n \neq \bar{a}_{n+1}$ for all $n \geq 0$. Then, by using (1) and from (10), we obtain

$$\begin{aligned}
&\gamma(F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath})) \\
&\geq \gamma(F(\bar{a}_n, \bar{a}_{n-1}, \hat{\jmath})^{\alpha} F(\bar{a}_n, K\bar{a}_n, \hat{\jmath})^{\beta} F(\bar{a}_{n-1}, K\bar{a}_{n-1}, \hat{\jmath})^{(1-\alpha-\beta)}) + \delta \\
&= \gamma(F(\bar{a}_n, \bar{a}_{n-1}, \hat{\jmath})^{\alpha} F(\bar{a}_n, \bar{a}_{n+1}, \hat{\jmath})^{\beta} F(\bar{a}_{n-1}, \bar{a}_n, \hat{\jmath})^{(1-\alpha-\beta)}) + \delta \\
&= \gamma(F(\bar{a}_n, \bar{a}_{n-1}, \hat{\jmath})^{(1-\beta)} F(\bar{a}_n, \bar{a}_{n+1}, \hat{\jmath})^{\beta}).
\end{aligned} \tag{11}$$

From the condition (i), since (A, E) satisfies the fuzzy weak P-property, we deduce that $F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath}) \geq F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath})$. Thus from (11), we have

$$\gamma(F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath})) \geq \gamma(F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath})) \geq \gamma(F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath})^{(1-\beta)} F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath})^{\beta}) + \delta.$$

Since γ is a strictly increasing and $\delta \in (0,1)$ arbitrary, we have

$$F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath}) > F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath})^{(1-\beta)} F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath})^{\beta},$$

implies

$$F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath}) > F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath}).$$

Consequently, we obtain

$$\gamma(F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath})) \geq \gamma(F(K\bar{a}_n, K\bar{a}_{n-1}, \hat{\jmath}) + \delta,$$

that is

$$\gamma(F(\bar{a}_{n+1}, \bar{a}_n, \hat{\jmath})) \geq \gamma(F(\bar{a}_{n-1}, \bar{a}_n, \hat{\jmath})) + \delta. \tag{12}$$

Repeating this process, we obtain

$$\gamma(F(\bar{a}_{n+1},\bar{a}_n,\hat{j})) \geq \gamma(F(\bar{a}_{n-1},\bar{a}_n,\hat{j})) + \delta \geq \gamma(F(\bar{a}_0,\bar{a}_1,\hat{j})) + n\delta, \tag{13}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$, from (13), we obtain

$$\lim_{n\to+\infty} \gamma(F(\bar{a}_{n+1},\bar{a}_n,\hat{j})) = +\infty.$$

Then, from the property of γ, we have

$$\lim_{n\to+\infty} F(\bar{a}_{n+1},\bar{a}_n,\hat{j}) = 1. \tag{14}$$

Now, we want to show that $\{\bar{a}_n\}$ is a Cauchy sequence. Suppose to the contrary, we assume that $\{\bar{a}_n\}$ is not a Cauchy sequence. Then, there are $\varepsilon \in (0,1)$ and $\hat{j}_0 > 0$ such that for all $k \in \mathbb{N}$, there exist $n_k, m_k \in \mathbb{N}$ with $m_k > n_k \geq k$ and

$$F(\bar{a}_{m_k},\bar{a}_{n_k},\hat{j}_0) \leq 1-\varepsilon. \tag{15}$$

From Remark 1.1 (ii), we have

$$F(\bar{a}_{m_k},\bar{a}_{n_k},\tfrac{\hat{j}_0}{2}) \leq 1-\varepsilon. \tag{16}$$

Assume that n_k is the least integer exceeding n_k satisfying the inequality (16). Then, we have

$$F(\bar{a}_{m_k-1},\bar{a}_{n_k},\tfrac{\hat{j}_0}{2}) > 1-\varepsilon. \tag{17}$$

Using the inequality (1) with $\bar{a} = \bar{a}_{m_k-1}$, $\bar{e} = \bar{a}_{n_k-1}$ and $\hat{j} = \hat{j}_0$, we have

$$\gamma(F(\bar{a}_{m_k},\bar{a}_{n_k},\hat{j}_0)) > \gamma(F(\bar{a}_{m_k-1},\bar{a}_{n_{k-1}},\hat{j}_0)).$$

As γ is nondecreasing, we have

$$F(\bar{a}_{m_k},\bar{a}_{n_k},\hat{j}_0) > F(\bar{a}_{m_k-1},\bar{a}_{n_{k-1}},\hat{j}_0). \tag{18}$$

Now, using (15), (17), and (18), we have

$$\begin{aligned} 1-\varepsilon &\geq F(\bar{a}_{m_k},\bar{a}_{n_k},\hat{j}_0) \\ &> F(\bar{a}_{m_k-1},\bar{a}_{n_{k-1}}\hat{j}_0) \\ &\geq F(\bar{a}_{m_k-1},\bar{a}_{n_k},\tfrac{\hat{j}_0}{2}) * F(\bar{a}_{n_k},\bar{a}_{n_k-1},\tfrac{\hat{j}_0}{2}) \\ &> (1-\varepsilon) * F\left(\bar{a}_{n_k},\bar{a}_{n_k-1},\tfrac{\hat{j}_0}{2}\right), \end{aligned} \tag{19}$$

by taking $k \to +\infty$ in (19) and (14), we obtain

$$\lim_{k\to+\infty} F(\bar{a}_{m_k}, \bar{a}_{n_k}, \hat{j}_0) = 1 - \varepsilon. \tag{20}$$

Using inequality (1) with $\bar{a} = \bar{a}_{m_k-1}$, $\bar{e} = \bar{a}_{n_k-1}$ and so $\hat{j} = \hat{j}_0$, we have

$$\begin{aligned}
&\gamma\left(F\left(\bar{a}_{m_k}, \bar{a}_{n_k}, \hat{j}_0\right)\right)\\
&\geq \gamma(F(\bar{a}_{m_k-1}, \bar{a}_{n_k-1}, \hat{j}_0)^{\alpha} F(\bar{a}_{m_k-1}, K\bar{a}_{m_k-1}, \hat{j}_0)^{\beta} F(\bar{a}_{n_k-1}, K\bar{a}_{n_k-1}, \hat{j}_0)^{(1-\alpha-\beta)}) + \delta\\
&\geq \gamma(F(\bar{a}_{m_k-1}, \bar{a}_{n_k-1}, \hat{j}_0)^{\alpha} F(\bar{a}_{m_k-1}, K\bar{a}_{m_k-1}, \hat{j}_0)^{\beta} F(\bar{a}_{n_k-1}, K\bar{a}_{n_k-1}, \hat{j}_0)^{(1-\alpha-\beta)}) + \delta\\
&= \gamma(F(\bar{a}_{m_k-1}, \bar{a}_{n_k-1}, \hat{j}_0)^{\alpha} F(\bar{a}_{m_k-1}, \bar{a}_{m_k}, \hat{j}_0)^{\beta} F(\bar{a}_{n_k-1}, \bar{a}_{n_k}, \hat{j}_0)^{(1-\alpha-\beta)}) + \delta. \qquad (21)
\end{aligned}$$

Taking the limit as $k \to +\infty$ in (21), applying (1), from (14), (20), and continuity of γ, we obtain

$$\gamma((1-\varepsilon)) \geq \gamma((1-\varepsilon)^{\alpha}(1-\varepsilon)^{\beta}(1-\varepsilon)^{(1-\alpha-\beta)}) + \delta.$$

Then, we have

$$\gamma((1-\varepsilon)) \geq \gamma((1-\varepsilon)) + \delta,$$

a contradiction. Thus, $\{\bar{a}_n\}$ is a Cauchy sequence in X. Since A is a closed subset of the complete fuzzy metric space $(X, F, *)$, then there exists $\bar{u} \in A$ so that

$$\lim_{n\to+\infty} F(\bar{a}_n, \bar{u}, \hat{j}) = 1. \tag{22}$$

Since K is continuous,

$$\lim_{n\to+\infty} F(K\bar{a}_n, K\bar{u}, \hat{j}) = 1. \tag{23}$$

Combining (10), (22), and (23), we have

$$F(A, E, \hat{j}) = \lim_{n\to+\infty} F(\bar{a}_{n+1}, K\bar{a}_n, \hat{j}) = F(\bar{u}, K\bar{u}, \hat{j}).$$

This proves that $\bar{u}$ is a best proximity point of K.

Now, to prove the uniqueness of the best proximity point of mapping K, suppose that $\bar{e} \in A_0$ is another best proximity point (different from $\bar{u}$) of the mapping K such that

$$\rho(\bar{u}, \bar{e}, \hat{j}) = 1,$$

$$F(\bar{u}, K\bar{u}, \hat{j}) = F(A, E, \hat{j}),$$

$$F(\bar{e}, K\bar{e}, \hat{j}) = F(A, E, \hat{j}).$$

Since the pair of subsets (A, E) satisfies fuzzy weak P-property, then we have $F(\bar{u}, \bar{e}, \hat{j}) \geq F(K\bar{u}, K\bar{e}, \hat{j})$. From the inequality (1), we have

$$\gamma(F(\bar{u},\bar{e},\hat{\jmath})) \geq \gamma(F(K\bar{u},K\bar{e},\hat{\jmath})) \geq \rho(\bar{u},\bar{e},\hat{\jmath})\gamma(F(K\bar{u},K\bar{e},\hat{\jmath}))$$
$$\geq \gamma(F(\bar{u},\bar{e},j)^{\alpha}F(\bar{a},K\bar{a},j)^{\beta}F(\bar{e},K\bar{e},j)^{(1-\alpha-\beta)}) + \delta$$
$$= \gamma(F(\bar{u},\bar{e},\hat{\jmath})^{\alpha}) + \delta. \quad (24)$$

From the property of γ and (24), we have

$$F(\bar{u},\bar{e},\hat{\jmath}) > F(\bar{u},\bar{e},\hat{\jmath})^{\alpha} > F(\bar{u},\bar{e},\hat{\jmath}),$$

a contradiction. Therefore, the best proximity point of the mapping K is unique.

Example 2.2. *Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a standard fuzzy metric $F(\bar{a},\bar{e},\hat{\jmath}) = \left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{d(\bar{a},\bar{e})}$ for all $\hat{\jmath} > 0$ such that $d(\bar{a},\bar{e}) = |\bar{a}_1 - \bar{e}_1| + |\bar{a}_2 - \bar{e}_2|$ for all $\bar{a} = (\bar{a}_1,\bar{a}_2)$ and $\bar{e} = (\bar{e}_1,\bar{e}_2) \in X$. Clearly, $(X,F,*)$ is a complete FMS where $*$ is a product t-norm. Define A and E be two nonempty subsets of X given as*

$$A = \{(0,n): n \in \mathbb{R}^+ \cup \{0\}\}$$

and

$$E = \{(1,n): n \in \mathbb{R}^+ \cup \{0\}\}.$$

So that, $d(A,E) = 1$ and $F(\bar{a},\bar{e},\hat{\jmath}) = \frac{\hat{\jmath}}{\hat{\jmath}+1}$ for all $\bar{a},\bar{e} \in A$ and $\hat{\jmath} > 0$. Obviously, A, E are nonempty closed subsets of X. Also, the pair (A,E) admits the fuzzy weak P-property. It is clear that $A_0 = A$ and $E_0 = E$. Define a mapping $K: A \to E$ as $K(\bar{a}) = (1,\frac{n}{8})$ for all $\bar{a} \in A$. Clearly $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$. Clearly $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$. Also suppose that $\rho: A \times A \times (0,+\infty) \to \mathbb{R}^+$ is given by

$$\rho(\bar{a},\bar{e},\hat{\jmath}) = \frac{1 - \left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{d(\bar{a},\bar{e})}{4}}}{1 - \left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{d(\bar{a},\bar{e})}{3}}} \; for\ all\ \bar{a},\bar{e} \in A\ and\ \hat{\jmath} > 0.$$

Let $\gamma: [0,1) \to \mathbb{R}$ be defined by $\gamma(t) = \frac{1}{1-t^2}$. Now, we will show that K is ρ-interpolative Ćirić-Reich-Rus-type fuzzy proximal contraction for all $\hat{\jmath} > 0$. Let $\alpha = \frac{5}{10}$ and $\beta = \frac{3}{10}$.

If $\bar{a} = (0,n)$, $\bar{e} = (0,m)$ for all $n,m \in \mathbb{R}^+ \cup \{0\}$, then, we have

$$d(K\bar{a},K\bar{e}) = d((1,\tfrac{n}{8}),(1,\tfrac{m}{8})) = \frac{|n-m|}{8},$$

$$d(\bar{a},\bar{e}) = d((0,n),(0,m)) = |n-m|,$$

$$d(\bar{a},K\bar{a}) = d((0,n),(1,\tfrac{n}{8})) = 1 + \frac{7n}{8},$$

$$d(\bar{e},K\bar{e}) = d((0,m),(1,\tfrac{m}{8})) = 1+\tfrac{7m}{8}.$$

According to the equations above, we have

$$\frac{|n-m|}{8} < |n-m| + (1+\tfrac{7n}{8}) + (1+\tfrac{7m}{8})$$

and for $\alpha = \frac{5}{10}$, $\beta = \frac{3}{10}$, *we have*

$$\frac{|n-m|}{8} < \frac{5}{10}|n-m| + \frac{3}{10}(1+\tfrac{7n}{8}) + \frac{2}{10}(1+\tfrac{7m}{8}).$$

Then, we have

$$\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{|n-m|}{8}} > \left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{5}{10}|n-m|}\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{3}{10}(1+\frac{7n}{8})}\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{2}{10}(1+\frac{7m}{8})}$$

is equivalent to

$$\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{d(K\bar{a},K\bar{e})} > \left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\alpha d(\bar{a},\bar{e})}\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\beta d(\bar{a},K\bar{a})}\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{(1-\alpha-\beta)d(\bar{e},K\bar{e})},$$

for all $\hat{\jmath} > 0$. *Then, we obtain*

$$F(K\bar{a},K\bar{e},\hat{\jmath}) > F(\bar{a},\bar{e},\hat{\jmath})^{\alpha}F(\bar{a},K\bar{a},\hat{\jmath})^{\beta}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha-\beta)}.$$

Since γ *is nondecreasing, we have*

$$\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) > \gamma(F(\bar{a},\bar{e},\hat{\jmath})^{\alpha}F(\bar{a},K\bar{a},\hat{\jmath})^{\beta}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha-\beta)}),$$

and also for $\rho(\bar{a},\bar{e},\hat{\jmath}) = \frac{1-\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{d(\bar{a},\bar{e})}{4}}}{1-\left(\frac{\hat{\jmath}}{\hat{\jmath}+1}\right)^{\frac{d(\bar{a},\bar{e})}{3}}}$, *we have*

$$\rho(\bar{a},\bar{e},\hat{\jmath})\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) > \gamma(F(\bar{a},\bar{e},\hat{\jmath})^{\alpha}F(\bar{a},K\bar{a},\hat{\jmath})^{\beta}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha-\beta)}),$$

for all $\hat{\jmath} > 0$. *There can be at least one* $\delta \in (0,1)$ *that satisfies the inequality (1) for all cases. Therefore,* K *is a* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction. Since all the conditions of Theorem 2.1. are satisfied,* $(0,0)$ *is a best proximity point of* K.

Corollary 2.1. *Let* $(X,F,*)$ *be a complete FMS and* (A,E) *be a pair of closed subsets of* X *such that* $A_0(\hat{\jmath})$ *is nonempty. Let* $K: A \to E$ *be a continuous mapping, satisfying*

(i) $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$ *and* (A,E) *abide by the fuzzy weak P-property.*

(ii) *There exist* $\gamma \in \Gamma$, $\delta \in (0,1)$, *and positive real numbers* α,β *with* $\alpha+\beta<1$ *such that*

$$\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},\bar{e},\hat{\jmath})^{\alpha}F(\bar{a},K\bar{a},\hat{\jmath})^{\beta}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha-\beta)}) + \delta,$$

for all $\bar{a},\bar{e} \in A\backslash B\_\{est\}(K)$ *with* $F(K\bar{a},K\bar{e},\hat{\jmath}) < 1$ *and for all* $\hat{\jmath} > 0$. *Then,* K *has a unique best proximity point in* A.

Proof. By considering $\rho(\bar{a},\bar{e},\hat{\jmath}) = 1$ and arguing similarly as in the proof of Theorem 2.1., we have the required proof.

Definition 2.2. *Let* A *and* E *be two nonempty, closed subsets of* $(X,F,*)$ *FMS. A mapping* $K: A \to E$ *is said to be* ρ*-interpolative Kannan-type* γ*-fuzzy proximal contraction if there exist* $\gamma \in \Gamma$, $\rho: A \times A \times (0,+\infty) \to \mathbb{R}^+$ *and positive real numbers* $\alpha, \delta \in (0,1)$ *such that*

$$\rho(\bar{a},\bar{e},\hat{\jmath})\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},K\bar{a},\hat{\jmath})^{\alpha}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha)}) + \delta,$$

for all $\bar{a},\bar{e} \in A\backslash B\_\{est\}(K)$ *with* $\rho(\bar{a},\bar{e},\hat{\jmath}) \leq 1$, $F(K\bar{a},K\bar{e},j) < 1$ *and for all* $\hat{\jmath} > 0$, *where* $B\_\{est\}(K): \{\bar{a} \in A: F(\bar{a},K\bar{a},\hat{\jmath}) = F(A,E,\hat{\jmath})\}$.

Theorem 2.2. *Let* $(X,F,*)$ *be a complete FMS and* (A,E) *be a pair of closed subsets of* X *such that* $A_0(\hat{\jmath})$ *is nonempty. Let* $K: A \to E$ *be a continuous mapping, satisfying*

(i) $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$ *and* (A,E) *abide by the fuzzy weak P-property.*
(ii) There exist $\bar{a}_0, \bar{a}_1 \in A$ *such that* $\rho(\bar{a}_1,K\bar{a}_0,\hat{\jmath}) \leq 1$ *and* $F(\bar{a}_1,K\bar{a}_0,\hat{\jmath}) = F(A,E,\hat{\jmath})$.
(iii) K *is* ρ*-interpolative Kannan-type* γ*-fuzzy proximal contraction.*

Then, K *has a unique best proximity point in* A.

Proof. The proof of Theorem 2.2. can be shown in steps similar to the proof of Theorem 2.1.

Corollary 2.2. *Let* $(X,F,*)$ *be a complete FMS and* (A,E) *be a pair of closed subsets of* X *such that* $A_0(\hat{\jmath})$ *is nonempty. Let* $K: A \to E$ *be a continuous mapping, satisfying*

(i) $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$ *and* (A,E) *abide by the fuzzy weak P-property.*
(ii) There exist $\gamma \in \Gamma$ *and positive real numbers* $\alpha,\delta \in (0,1)$ *such that*

$$\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},K\bar{a},\hat{\jmath})^{\alpha}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha)}) + \delta, \qquad (25)$$

for all $\bar{a},\bar{e} \in A\backslash B\_\{est\}(K)$ *with* $F(K\bar{a},K\bar{e},\hat{\jmath}) < 1$ *and for all* $\hat{\jmath} > 0$. *Then,* K *has a unique best proximity point in* A.

Proof. By considering $\rho(\bar{a},\bar{e},\hat{\jmath}) = 1$ and arguing similarly as in the proof of Theorem 2.2, we have the required proof.

Example 2.3. *Let* $X = \mathbb{R} \times \mathbb{R}$ *be endowed with a standard fuzzy metric* $F(\bar{a},\bar{e},\hat{\jmath}) = exp(-\frac{d(\bar{a},\bar{e})}{\hat{\jmath}})$ *for all* $\hat{\jmath} > 0$ *such that* $d(\bar{a},\bar{e}) = |\bar{a}_1 - \bar{e}_1| + |\bar{a}_2 - \bar{e}_2|$ *for all* $\bar{a} = (\bar{a}_1,\bar{a}_2)$ *and* $\bar{e} = (\bar{e}_1,\bar{e}_2) \in X$. *Clearly,* $(X,F,*)$ *is a complete FMS where* $*$ *is a product* t*-norm. Define* A *and* E *be two nonempty subsets of* X *given as*

$$A = \{(0,x): x \in \mathbb{R}^+ \cup \{0\}\}$$

and

$$E = \{(2,x): x \in \mathbb{R}^+ \cup \{0\}\}.$$

So that, $d(A,E) = 2$ *and* $F(\bar{a},\bar{e},\hat{\jmath}) = exp(-\frac{2}{\hat{\jmath}})$ *for all* $\bar{a},\bar{e} \in A$ *and* $\hat{\jmath} > 0$. *Obviously,* A, E *are nonempty closed subsets of* X. *Also, the pair* (A,E) *admits the fuzzy weak P-property. It is clear that* $A_0 = A$ *and* $E_0 = E$. *Define a mapping* $K: A \to E$ *as* $K(\bar{a}) = (2,\frac{x}{4})$. *Clearly* $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$. *Let* $\gamma: [0,1) \to \mathbb{R}$ *be defined by* $\gamma(t) = \frac{1}{\sqrt{1-t}}$. *Now, we will show that* K *is* ρ*-interpolative Kannan-type fuzzy proximal contraction. Let* $\alpha = 0{,}5$.

If $\bar{a} = (0,x)$, $\bar{e} = (0,y)$ *for all* $x,y \in \mathbb{R}^+ \cup \{0\}$, *then, we have*

$$d(K\bar{a},K\bar{e}) = d((2,\tfrac{x}{4}),(2,\tfrac{y}{4})) = \tfrac{|x-y|}{4},$$

$$d(\bar{a},K\bar{a}) = d((0,x),(2,\tfrac{x}{4})) = 2+\tfrac{3x}{4},$$

$$d(\bar{e},K\bar{e}) = d((0,y),(2,\tfrac{y}{4})) = 2+\tfrac{3y}{4}.$$

According to the equations above, we have

$$\frac{|x-y|}{4} < (2+\frac{3x}{4}) + (2+\frac{3y}{4})$$

is equivalent to

$$d(K\bar{a},K\bar{e}) < d(\bar{a},K\bar{a}) + d(\bar{e},K\bar{e}).$$

Then, we have

$$d(K\bar{a},K\bar{e}) < \alpha d(\bar{a},K\bar{a}) + (1-\alpha) d(\bar{e},K\bar{e})$$

and so

$$exp(-\tfrac{d(K\bar{a},K\bar{e})}{\hat{j}}) > exp(-\tfrac{\alpha d(K\bar{a},K\bar{e})}{\hat{j}}) exp(-\tfrac{(1-\alpha) d(K\bar{a},K\bar{e})}{\hat{j}}),$$

for all $\hat{j} > 0$. *That is equivalent to*

$$F(K\bar{a},K\bar{e},\hat{j}) > F(\bar{a},K\bar{a},\hat{j})^{\alpha} F(\bar{e},K\bar{e},\hat{j})^{(1-\alpha)}.$$

Since the γ *is nondecreasing, we have*

$$\gamma(F(K\bar{a},K\bar{e},\hat{j})) > \gamma(F(\bar{a},K\bar{a},\hat{j})^{\alpha} F(\bar{e},K\bar{e},\hat{j})^{(1-\alpha)}),$$

for all $\hat{j} > 0$. *There can be at least one* $\delta \in (0,1)$ *that satisfies the inequality (25) for all cases. Since all the conditions of Corollary 2.2. are satisfied,* $(0,0)$ *is a best proximity point of* K.

3. Some results

In this part, we have the best proximity point theorems for ordered ρ-interpolative Ćirić-Reich-Rus-type and Kannan-type γ-fuzzy proximal contractions on a fuzzy metric space endowed with a partial ordering/graph

$$H = \{\bar{a},\bar{e} \in A \text{ such that } \bar{a} \preceq \bar{e} \text{ or } \bar{e} \preceq \bar{a}\}$$

and

$$\rho: A \times A \times (0,\infty) \to (0,\infty), \qquad \text{where } \rho(\bar{a},\bar{e},\hat{j}) = \begin{cases} 1 & , \quad \bar{a},\bar{e} \in H, \\ 0 & , \quad otherwise. \end{cases}$$

Definition 3.1. *[1] Let* $(X,F,*,\preceq)$ *be a partially ordered FMS and* (A,E) *be a pair of nonempty subsets of* X. *A mapping* $K: A \to E$ *is called proximal fuzzy order preserving, if*

$$\begin{cases} \bar{e}_1 \preceq \bar{e}_2, \\ F(\bar{a}_1, K\bar{e}_1, \hat{\jmath}) = F(A, E, \hat{\jmath}) \\ F(\bar{a}_2, K\bar{e}_2, \hat{\jmath}) = F(A, E, \hat{\jmath}), \end{cases} \Rightarrow \bar{a}_1 \preceq \bar{a}_2,$$

for all $\bar{a}_1, \bar{a}_2, \bar{e}_1, \bar{e}_2 \in A$.

Definition 3.2. *Let* $(X, F, *, \preceq)$ *be a partially ordered FMS. A mapping* $K: A \to E$ *is said to be an ordered interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction if there exist* $\gamma \in \Gamma$, ρ *positive real numbers* α, β *satisfying* $\alpha + \beta < 1$ *and* $\delta \in (0,1)$ *such that*

$$\gamma(F(K\bar{a}, K\bar{e}, \hat{\jmath})) \geq \gamma(F(\bar{a}, \bar{e}, \hat{\jmath})^{\alpha} F(\bar{a}, K\bar{a}, \hat{\jmath})^{\beta} F(\bar{e}, K\bar{e}, \hat{\jmath})^{(1-\alpha-\beta)}) + \delta,$$

for all $\bar{a}, \bar{e} \in A \backslash B\_\{est\}(K)$ *with* $F(K\bar{a}, K\bar{e}, \hat{\jmath}) < 1$, $(\bar{a}, \bar{e}) \in H$ *and for all* $\hat{\jmath} > 0$.

The following result is a direct consequence of Theorem 2.1.

Theorem 3.1. *Let* $(X, F, *, \preceq)$ *be a partially ordered FMS and* (A, E) *be a pair of closed subsets of* X *such that* $A_0(\hat{\jmath})$ *is nonempty. Let* $K: A \to E$ *be a continuous mapping satisfying*

(i) $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$ *and* (A, E) *abide by the fuzzy weak P-property.*
(ii) K is proximal fuzzy order preserving.
(iii) There exist $\bar{a}_0, \bar{a}_1 \in A$ *such that* $\rho(\bar{a}_1, K\bar{a}_0, \hat{\jmath}) \leq 1$ *and* $(\bar{a}_0, \bar{a}_1) \in H$.
(iv) K *is ordered interpolative Ćirić-Reich-Rus-type* γ*-fuzzy proximal contraction.*

Then, K *has a best proximity point in* A.

Definition 3.3. *Let* $(X, F, *, \preceq)$ *be a partially ordered FMS. A mapping* $K: A \to E$ *is said to be ordered interpolative Kannan-type* γ*-fuzzy proximal contraction if there exist* $\gamma \in \Gamma$ *and positive real numbers* $\alpha, \delta \in (0,1)$ *such that*

$$\gamma(F(K\bar{a}, K\bar{e}, \hat{\jmath})) \geq \gamma(F(\bar{a}, \bar{e}, \hat{\jmath})^{\alpha} F(\bar{e}, K\bar{e}, \hat{\jmath})^{(1-\alpha)}) + \delta,$$

for all $\bar{a}, \bar{e} \in A \backslash B\_\{est\}(K)$ *with* $F(K\bar{a}, K\bar{e}, \hat{\jmath}) < 1$, $(\bar{a}, \bar{e}) \in H$ *and for all* $\hat{\jmath} > 0$.

The following result is a direct consequence of Theorem 2.2.

Theorem 3.2. *Let* $(X, F, *, \preceq)$ *be a partially ordered FMS and* (A, E) *be pair of closed subsets of* X *such that* $A_0(\hat{\jmath})$ *is nonempty. Let* $K: A \to E$ *be a continuous mapping, satisfying*

(i) $K(A_0(\hat{\jmath})) \subseteq E_0(\hat{\jmath})$ *and* (A, E) *abide by the fuzzy weak P-property.*
(ii) K *is proximal fuzzy order preserving.*
(iii) There exist $\bar{a}_0, \bar{a}_1 \in A$ *such that* $\rho(\bar{a}_1, K\bar{a}_0, \hat{\jmath}) \leq 1$ *and* $(\bar{a}_0, \bar{a}_1) \in H$.
(iv) K *is ordered interpolative Kannan-type* γ*-fuzzy proximal contraction.*

Then, K *has a best proximity point in* A.

4. Applications to the fixed point theory

In this section we prove related results to the fixed point theory for ρ-interpolative Ćirić-Reich-Rus-Type and Kannan-type γ-fuzzy contractions. If $A = E = X$, then the following contractions can be defined. Since $F(\bar{a}, K\bar{a}, \hat{\jmath}) = F(A, E, \hat{\jmath}) = 1$ for self-mappings, meaning $\bar{a} = K\bar{a}$, whereby the best proximity point reduces to the fixed point. In this context, ρ-interpolative Ćirić-Reich-Rus-type and Kannan-type γ-fuzzy contractions also reduce to the fixed point problem.

Definition 4.1. *Let* $(X, F, *)$ *be an FMS. The mapping* $K: X \to X$ *is said to be* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy contraction if there exist* $\gamma \in \Gamma$, $\rho: X \times X \times (0, +\infty) \to \mathbb{R}^+$, *positive real numbers* α, β *satisfying* $\alpha + \beta < 1$ *and* $\delta \in (0,1)$ *such that*

$$\rho(\bar{a},\bar{e},\hat{\jmath})\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},\bar{e},\hat{\jmath})^{\alpha}F(\bar{a},K\bar{a},\hat{\jmath})^{\beta}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha-\beta)})+\delta,$$

for all $\bar{a},\bar{e}\in X\backslash Fix(K)$ *with* $F(K\bar{a},K\bar{e},\hat{\jmath})<1$ *and for all* $\hat{\jmath}>0$.

The following result is a consequence of Theorem 2.1.

Theorem 4.1. *Let* $(X,F,*)$ *be a complete FMS and* $K: X\to X$ *be continuous* ρ*-interpolative Ćirić-Reich-Rus-type* γ*-fuzzy contraction. If there exists* $\bar{a}_0\in K$ *such that* $\rho(\bar{a}_0,K\bar{a}_0,\hat{\jmath})\leq 1$, *then* K *has a fixed point in* X.

By considering $\rho(\bar{a},\bar{e},\hat{\jmath})=1$ in Theorem 4.1, we state the following.

Corollary 4.1. *Let* $(X,F,*)$ *be a complete FMS,* $K: X\to X$ *be mapping,* $\gamma\in\Gamma$ *and for positive real numbers* γ,β *with* $\gamma+\beta<1$ *and* $\delta\in(0,1)$ *such that*

$$\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},\bar{e},\hat{\jmath})^{\alpha}F(\bar{a},K\bar{a},\hat{\jmath})^{\beta}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha-\beta)})+\delta,$$

for all $\bar{a},\bar{e}\in X\backslash Fix(K)$ *with* $F(K\bar{a},K\bar{e},\hat{\jmath})<1$ *and for all* $\hat{\jmath}>0$. *Then* K *has a fixed point in* X.

Definition 4.2. *Let* $(X,F,*)$ *be an FMS. The mapping* $K: X\to X$ *is said to be* ρ*-interpolative Kannan-type* γ*-fuzzy contraction if there exist* $\gamma\in\Gamma$, $\rho: X\times X\times(0,+\infty)\to\mathbb{R}^{+}$ *and positive real numbers* $\alpha,\gamma\in(0,1)$ *such that*

$$\rho(\bar{a},\bar{e},\hat{\jmath})\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},K\bar{a},\hat{\jmath})^{\alpha}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha)})+\delta,$$

for all $\bar{a},\bar{e}\in X\backslash Fix(K)$ *with* $F(K\bar{a},K\bar{e},\hat{\jmath})<1$ *and for all* $\hat{\jmath}>0$.

The following result is a consequence of Theorem 2.2.

Theorem 4.2. *Let* $(X,F,*)$ *be a complete FMS and* $K: X\to X$ *be continuous* ρ*-interpolative Kannan-type* γ*-fuzzy contraction. If there exists* $\bar{a}_0\in X$ *such that* $\rho(\bar{a}_0,K\bar{a}_0,\hat{\jmath})\leq 1$, *then* K *has a fixed point in* X.

By considering $\rho(\bar{a},\bar{e},\hat{\jmath})=1$ in Theorem 4.2., we state the following.

Corollary 4.2. *Let* $(X,F,*)$ *be a complete FMS,* $K: X\to X$ *be mapping,* $\gamma\in\Gamma$ *and for positive real numbers* $\alpha,\gamma\in(0,1)$ *such that*

$$\gamma(F(K\bar{a},K\bar{e},\hat{\jmath})) \geq \gamma(F(\bar{a},K\bar{a},\hat{\jmath})^{\alpha}F(\bar{e},K\bar{e},\hat{\jmath})^{(1-\alpha)})+\delta,$$

for all $\bar{a},\bar{e}\in X\backslash Fix(K)$ *with* $F(K\bar{a},K\bar{e},\hat{\jmath})<1$ *and for all* $\hat{\jmath}>0$. *Then* K *has a fixed point in* X.

5. Conclusions

The article contains definitions and theorems that reveal the existence of the best proximity point for ρ-interpolative Ćirić-Reich-Rus-type γ-fuzzy proximal contraction. The article presents the a new definition of proximal contraction by using of the γ-function, followed by an example and our main theorem. Later, an example where the best proximity point is obtained is given to support the results. The results of our main theorem as an application in fixed point theory are proved. Some applications of our main theorem under appropriate and necessary conditions are presented, along with a partial ordering defined on the fuzzy metric.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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