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Research article

An application on edge irregular reflexive labeling for m^t-graph of cycle graph

Muhammad Amir Asif [1], **Rashad Ismail**[2], **Ayesha Razaq**[1], **Esmail Hassan Abdullatif Al-Sabri**[2], **Muhammad Haris Mateen**[1,*] **and Shahbaz Ali**[3]

[1] School of Mathematics, Minhaj University, Lahore, Pakistan
[2] Department of Mathematics, Faculty of Science and Arts, Mahayl Assir, King Khalid University, Abha, Saudi Arabia
[3] Department of Mathematics, The Islamia University of Bahawalpur, Rahim Yar Kahn Campus, Pakistan

* **Correspondence:** Email: harism.math@gmail.com.

Abstract: Graph labeling is an increasingly popular problem in graph theory. A mapping converts a collection of graph components into a set of integers known as labels. Graph labeling techniques typically label edges with positive integers, vertices with even numbers, and edge weights with consecutive numbers, known as edge irregular reflexive total labeling. This is achieved by utilizing the reflexive edge irregularity strength of the graphical structure. The edge calculates the exact values of the reflexive edge irregularity strength irregular reflexive labeling for the m^t-graph of cycle graph mC_n on $t = 1$ with $n \geq 3$ and $m \geq 4$. The maximum number of assignments assigned to each individual in a communication network, as well as providing a secure communication channel to ensure the unique identification of each employee, are potential applications for this problem.

Keywords: reflexive edge irregularity strength; m^t-graphs; edge irregular reflexive total labeling
Mathematics Subject Classification: 05C12, 05C90

1. Introduction

Graph theory is the study of the mathematical structure of networks, which have edges linked with their vertices. The Konigsberg bridge problem [1] served as the catalyst for the development of graph theory in 1735, the issue that led to the Eulerian graph's perception. A collection of elements in $V(J)$ and $E(J)$ make up a graph $J = (V, E)$. Each edge in the edge set is connected to either one or more vertices of the vertex set. The terms' size and order describe the number of edges and vertices, respectively. The graphs serve as useful models for various types of relationships and interactions in various fields such as biology, computer science, and chemistry, particularly chemical graph theory. Researchers have utilized the graphs to organize data and depict communication networks [2–4].

Graph theory uses labeling to detect unauthorized access, assigning unique labels to each communication medium. If numbers are allocated only to vertices, then named vertex labeling; if allocated only to edges, then it is named edge labeling; and if both graph elements are assigned, then it is named total labeling. Haque [5] proposed the special assignments of numbers to the edges and vertices of a graph using various total labeling techniques. A wide range of fields of science have used irregular graph labeling. For example: the labels on the vertices describe network routing problems in computers.

Chartrand et al. [6] established irregular graph labeling, with the minimum largest label of graph J being the graph strength $s(J)$. The irregular labeling is categorized into two types: irregular edge labeling and irregular vertex labeling. Ahmad et al. [7] stated that the labeling $y : V(J) \rightarrow \{1, 2, 3, \cdots, k\}$ of a simple graph is an irregular edge labeling if the weight $wt(uv) = y(u) + y(v)$ of edge $uv \in E(J)$ with $(u \neq v)$ and the edge irregularity strength is denoted as $es(J)$. Imran et al. [8] declared the labeling $y : E(J) \rightarrow \{1, 2, 3, \cdots, k\}$ of cubic graphs as irregular vertex labeling with the distinct vertex weight of v for v_i such that $wt(v) = \sum y(vv_i)$ and the vertex irregularity strength of graph J is denoted as $vs(J)$. Indriati et al. [9] established a new labeling technique, totally irregular labeling, which requires both vertex weights and edge weights to be irregular. A mapping $y : V(J) \cup E(J) \rightarrow \{1, 2, 3, \cdots, k\}$ is defined for double-star and related graphs and the total irregularity strength of graph J is the minimum k for which J has an irregular total k-labeling and is denoted by $ts(J)$. Indriati [10] introduced the vertex irregular total k-labeling $y : E(J) \cup V(J) \rightarrow \{1, 2, 3, \cdots, k\}$ of lollipop graphs with different weights of vertices such that $wt(v) = y(v) + \sum y(vv_i)$ for v_i vertices and the total vertex irregularity strength of graph J is denoted as $tvs(J)$.

Jendrol et al. [11] revealed the irregular edge total labeling $y : E(J) \cup V(J) \rightarrow \{1, 2, 3, \cdots, k\}$ of complete bipartite graphs and complete graphs with distinct edge weights $wt(uv) = y(u) + y(uv) + y(v)$ and the total edge irregularity strength of graph J is denoted as $tes(J)$. Bača et al. [12] also stated the concept of edge irregular total k-labeling for some generalized prism graphs and calculated the precise values of the total edge irregularity strength of the graphs. In [13], Bača established the generalized formula of total edge irregularity strength for any graph J with the maximum degree $\Delta(J)$ of J. Ivančo et al. [14] proposed these results and then introduced the conjecture of $tes(J)$. Ibrahim et al. [15] stated the total labels for the application of star graphs, double star graphs, and caterpillar graphs. This labeling was characterized as an irregular reflexive labeling, which affected the vertex and edge labels as well as helping to frame the problem in terms of real networks. To express a loopless vertex, the vertex label 0 was allowed in this labeling. Xin et al. [16] examined a wide range of labeling techniques, including flexible labeling for magic labeling (containing all the vertex and edge weights with the same values) and anti-magic labeling (containing all the vertex and edge weights with different values) of some simple graphs. Liang et al. [17] established a bijective mapping $y : E \rightarrow \{1, 2, \cdots, |E|\}$ for anti-magic edge labeling of the Cartesian product of graphs and prism graphs with the vertex-sum $\xi_y(v) = \sum y(vv_i)$ of graph J and where v_i vertices are neighboring vertices of v such that $\xi_y(v) \neq \xi_y(u)$, $u, v \in J$. The problem was introduced by Hartsfield and Ringel [18]. Lladó et al. [19] introduced a function $y : E \cup V \rightarrow \{1, 2, \cdots, |E| + |V|\}$ for super-magic edge covering of cycle magic graphs including wheels, subdivided wheels, and windmills with H-magic total labeling when subgraph H'

of J is isomorphic to J, such that $y(H') = \sum y(v) + \sum y(uv)$, $u, v \in J$. The results rely on a technique of partitioning sets of integers with special properties. Various properties of graphs related to labeling and indices are discussed in [20–27].

Our concern is a cycle graph which is also called a circular graph with a single cycle C_n with $n \geq 3$. Gervacio [28] stated the concept of cycle graphs in which the order and size of the graph are equal in number and each vertex has a degree of 2 that indicates the incidence of one vertex with precisely two edges. Ayache and Alameri [29] defined an intriguing graph named the m^t-graph. Let $J(V, E)$ be a simple, linked graph. The mJ graph for $t = 1$ requires taking similar m-copies of graph J, the $m(mJ) = m^2(J)$ graph for $t = 2$ requires taking similar m-copies of graph mJ, the $m(m(mJ)) = m^3(J)$ graph for $t = 3$ requires taking similar m-copies of graph $m(mJ)$, and so on using $m \geq 2$. Next, connect each copy's vertex to the matching vertex of every other copy using an edge. Following that, we obtain a new graph denoted as the m^t-graph, which is represented by $m^t(J)$. Let c be the size and d be the order of simple graph J, and then the order and size of the m^t-graph of graph J for non-negative integer t are $m^t d$ and $m^t c + \left(\frac{m(m-1)}{2}\right) dtm^{t-1}$, respectively. For cycle graph C_n with $(n \geq 3)$, the m^t-graph is denoted as $m^t(C_n)$ with $(m \geq 2)$ and for the specification of the graphical model the value of t is set as $t = 1$.

The purpose of this specification is to tackle the complexity of graphs. Furthermore, if we set $m = 2$ in the m^t-graph of cycle graph mC_n with $t = 1$, then we get the Cartesian product of the path graph with two vertices C_2 and cycle graph with n vertices C_n, i.e. $2C_n = C_2 \times C_n$, If we take $m = 3$, then we get the Cartesian product of the cycle graph with three vertices C_3 and the cycle graph with n vertices C_n, i.e., $3C_n = C_3 \times C_n$. Then the second specification for the construction of the m^t-graph of the cycle graph is $m \geq 4$ because Ke et al. [30] already stated the reflexive edge strength of the Cartesian product of two-cycle graphs. The m^t-graph of the cycle graph mC_n with $(m \geq 4)$ and $t = 1$ is formed by the following vertex set and edge set:

$$V(mC_n) = \{a_s^j : 1 \leq s \leq n; 1 \leq j \leq m\}$$

$$E(mC_n) = \{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq m\} \cup \{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq u \leq m-1; 1 \leq j \leq m-u\}$$
$$\cup \{a_1^j a_n^j : 1 \leq j \leq m\}.$$

The order and size are $|V(mC_n)| = mn$ and $|E(mC_n)| = \frac{(m^2+m)n}{2}$, respectively. The graph in Figure 1 is the generic form of the m^t-graph of the cycle graph mC_n with $t = 1$.

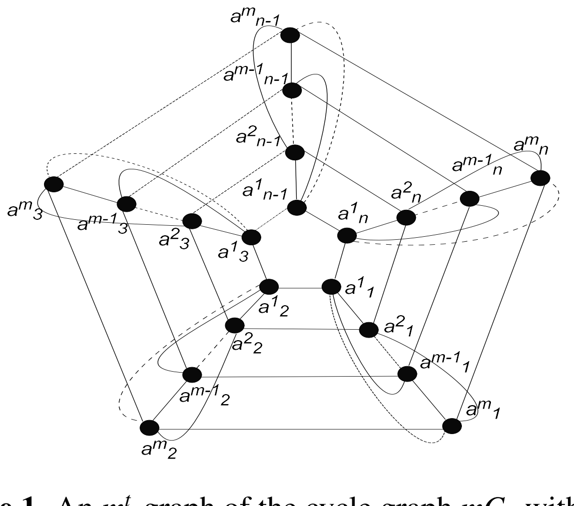

Figure 1. An m^t-graph of the cycle graph mC_n with $t = 1$.

Jendrol et al. [31] examined an edge irregular total k-coloring and a vertex irregular total k-coloring of biparite graphs and complete graphs. Ahmad et al. [32] stated the lower bounds on irregularity strength $s(J)$ will given by inequality $s(J) \geq \max_{1\leq i\leq \Delta(J)} \frac{n_i+i-1}{i}$ and the lower bounds of the total vertex irregularity strength $tvs(J)$ and the total edge irregularity strength $tes(J)$ such as:

$$tvs(J) \geq \left\lceil \frac{p+\delta(J)}{\Delta(J)+1} \right\rceil$$

$$tes(J) \geq \max\left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{|\Delta(G)|+2}{3} \right\rceil \right\}.$$

Bača et al. [13] calculated the exact values of the total edge irregularity strength of the path graph, cycle graph, star graph, wheel graph, and friendship graph, as well as also the exact values of the total vertex irregularity strength of the cycle graph, complete graph, n-sided prism graph, and star graph with n pendant vertices. Tanna et al. [33] determined a new labeling of an edge irregular reflexive total k-labeling where the labels of vertices x and y and a label of edge xy add up to the weight of edge $wt(xy)$ in graph J. The lowest value of k for which graph J has distinct edge weights is the reflexive edge strength, represented by $res(J)$, and the vertex labeling can be defined through a unique mapping y_v from the vertex set to even whole numbers as:

$$y_v : V(J) \to \{0, 2, 4, \cdots, 2k_v\}.$$

The function for edge labeling is defined through a unique mapping y_e from the edge set to non-zero whole numbers as:

$$y_e : E(J) \to \{1, 2, 3, \cdots, k_e\}.$$

In [34, 35], the weight of the edges was defined as the total of the labels of two connection vertices and the label of the selected edge. For every two different edges x_1x_2 and x_3x_4 of graph J, the edge weights are:

$$wt(x_1x_2) = y_v(x_1) + y_e(x_1x_2) + y_v(x_2)$$

$$wt(x_3x_4) = y_v(x_3) + y_e(x_3x_4) + y_v(x_4)$$

$$wt(x_1x_2) \neq wt(x_3x_4),$$

where $k = max\{k_e, 2k_v\}$ and the smallest k is the edge reflexive irregularity strength $res(J)$. In terms of $res(J)$, Baca et al. [36] proposed the following conjecture and proof for Lemma 1.

Conjecture 1. *For any graph with a maximum degree of graph J,* $\Delta(J)$ *satisfies*

$$res(J) = max\left\{\left\lceil \frac{|E(J)|}{3} + r\right\rceil \left\lfloor \frac{\Delta(J)+2}{2} \right\rfloor\right\}$$

where $r = 1$*, if* $|E(J)| \equiv 2, 3(mod\, 6)$*, and otherwise,* $r = 0$*.*

Lemma 1. *For every graph J,*

$$res(J) \geq \begin{cases} \lceil \frac{|E(J)|}{3} \rceil, & if\ |E(J)| \not\equiv 2, 3(mod\, 6), \\ \lceil \frac{|E(J)|}{3} \rceil + 1, & if\ |E(J)| \equiv 2, 3(mod\, 6), \end{cases}$$

is the lower bound of strength of the edge reflexive irregular total labeling of graph J, where $|E(J)|$ *is the size of the graph [37]. Agustin et al. [38] also determined the precise values of the ladder graph, triangular ladder graph, Cartesian product of paths and cycles, and some almost regular graph's reflexive edge irregularity strength* $res(J)$*.*

2. Main results

The purpose of this study is to examine the reflexive edge irregularity strength of m^t-graphs ($t = 1$) of the cycle graph. In Theorem 2.1, the total reflexive edge irregularity strength for the mC_n-graph with ($m \geq 4$) and ($n \geq 3$) are computed to verify the result of $res(mC_n)$.

Theorem 2.1. *Let the mC_n-graph be the m^t-graph with $t = 1$ of the cycle graph, and then for $n \geq 3$, $m \geq 4$,*

$$res(mC_n) = \begin{cases} \dfrac{(m^2+m)n}{6}, & if\ m \equiv 0,3,8,11(mod\ 12), \\ (2m-6)n + 2\left\lceil \dfrac{(m^2-11m+36)n}{12} \right\rceil, & if\ m \not\equiv 0,3,8,11(mod\ 12). \end{cases}$$

Proof. An mC_n-graph is the graph with isomorphic m-copies of the cycle graphs and each vertex in each copy of cycle graph C_n, $n \geq 3$, is attached through an edge to all the corresponding vertices of the remaining isomorphic copies of the cycle graphs containing mn vertices and $\frac{(m^2+m)n}{2}$ edges with vertex set $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq m\}$ and edge set $\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq m\} \cup \{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq m-u; 1 \leq u \leq m-1\} \cup \{a_1^j a_n^j : 1 \leq j \leq m\}$. For an mC_n-graph with $m \geq 4$, the lower bound of the reflexive edge strength in Lemma 1 can be written as:

$$res(mC_n) \geq \begin{cases} \left\lceil \dfrac{(m^2+m)n}{6} \right\rceil, & if\ |E(mC_n)| \not\equiv 2,3(\text{mod } 6), \\ \left\lceil \dfrac{(m^2+m)n}{6} \right\rceil + 1, & if\ |E(mC_n)| \equiv 2,3(\text{mod } 6). \end{cases}$$

2.1. mC_n-graph

2.1.1. Vertex labeling of the mC_n-graph

Now we define the labeling for vertices $V(mC_n) = \{a_s^j : 1 \leq s \leq n; 1 \leq j \leq m\}$ of the mC_n-graph and it is clear that vertex labeling depends only on $\{s : 1 \leq s \leq n\}$. Therefore, the vertex labeling is given by:

$$y_v(a_s^j) = \begin{cases} 0, & if\ s = 1, m \geq 4, 1 \leq j \leq m, \\ \dfrac{m^2-1}{4}, & if\ s = 2, m \equiv 1,3(\text{mod } 4), 1 \leq j \leq m, \\ \dfrac{m^2}{4}, & if\ s = 2, m \equiv 0(\text{mod } 4), 1 \leq j \leq m, \\ \dfrac{m^2}{4} - 1, & if\ s = 2, m \equiv 2(\text{mod } 4), 1 \leq j \leq m, \\ \dfrac{m^2+m-2}{2}, & if\ s = 3, m \equiv 1,2(\text{mod } 4), 1 \leq j \leq m, for\ n \geq 7. \end{cases}$$

For $(3 \leq s \leq n)$,

$$y_v(a_s^j) = \begin{cases} \dfrac{(m^2+m)s}{6}, & if\ m \equiv 0,3,8,11(\text{mod } 12), 1 \leq j \leq m, \\ (2m-6)s + 2\left\lceil \dfrac{(m^2-11m+36)s}{12} \right\rceil, & if\ m \not\equiv 0,3,8,11(\text{mod } 12), 1 \leq j \leq m. \end{cases}$$

2.1.2. Edge labeling of the mC_n-graph

We define the labeling for edge set $E(mC_n) = \{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq m\} \cup \{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq m-u; 1 \leq u \leq m-1\} \cup \{a_1^j a_n^j : 1 \leq j \leq m\}$ of the mC_n-graph. For all edges in set

$$\{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq m-u; 1 \leq u \leq m-1\},$$

we use a function $h(u)$ for labeling such that:

$$h(u) = \frac{(2m+1)u - u^2 - 2m}{2}.$$

Then the edge labeling is defined as:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + h(u), & if\ s = 1, m \geq 4, \\ j + \lceil \frac{m}{2} \rceil + h(u), & if\ m \not\equiv 2, 6, 10 (\text{mod } 12),\ s = 2, \\ j + \frac{m+4}{2} + h(u), & if\ m \equiv 2, 6, 10 (\text{mod } 12),\ s = 2, \\ j + \frac{(m^2+m)s - 3m^2 + 3m}{6} + h(u), & if\ m \equiv 0, 3, 8, 11 (\text{mod } 12)\ for\ 3 \leq s \leq n, \\ j + \frac{(m^2+m)s - 3m^2 + 3m}{6} + h(u), & if\ m \equiv 4, 7 (\text{mod } 12)\ for\ s \equiv 0 (\text{mod } 3), \\ j + \frac{(m^2+m)s - 3m^2 + 3m}{6} + h(u), & if\ m \equiv 1, 10 (\text{mod } 12)\ for\ s \equiv 0 (\text{mod } 6), \\ j + \frac{(m^2+m)s - 3m^2 + 3m}{6} + h(u), & if\ m \equiv 2, 5, 6, 9 (\text{mod } 12)\ for\ s \equiv 0 (\text{mod } 2), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 8}{6} + h(u), & if\ m \equiv 4, 7 (\text{mod } 12)\ for\ s \equiv 1 (\text{mod } 3), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 8}{6} + h(u), & if\ m \equiv 1, 10 (\text{mod } 12)\ for\ s \equiv 4 (\text{mod } 6), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 16}{6} + h(u), & if\ m \equiv 4, 7 (\text{mod } 12)\ for\ s \equiv 2 (\text{mod } 3), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 16}{6} + h(u), & if\ m \equiv 1, 10 (\text{mod } 12)\ for\ s \equiv 2 (\text{mod } 6), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 4}{6} + h(u), & if\ m \equiv 1, 10 (\text{mod } 12)\ for\ s \equiv 5 (\text{mod } 6), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 12}{6} + h(u), & if\ m \equiv 1, 10 (\text{mod } 12)\ for\ s \equiv 3 (\text{mod } 6), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 12}{6} + h(u), & if\ m \equiv 2, 5, 6, 9 (\text{mod } 12)\ for\ s \equiv 1 (\text{mod } 2), \\ j + \frac{(m^2+m)s - 3m^2 + 3m - 20}{6} + h(u), & if\ m \equiv 1, 10 (\text{mod } 12)\ for\ s \equiv 1 (\text{mod } 6). \end{cases}$$

For the edge labeling of edge set

$$\{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq m-u; 1 \leq u \leq m-1\},$$

there are two special cases in this labeling such that, for the first case, we have $s = \lceil \frac{n+1}{3} \rceil$ if s is odd and then $n \geq 3s-2$ with $m \equiv 1, 2 (\text{mod } 4)$, and if s is even then $n \geq 3s-3$ also with $m \equiv 1, 2 (\text{mod } 4)$.

Whereas, for the second case, we have $s = \lceil \frac{n}{3} \rceil$ if $n \geq 3s - 2$ and $m \equiv 0, 3 (\text{mod } 4)$. Then the edge labeling is defined as:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + 2 + h(u), & if\ m \equiv 1, 2(\text{mod } 4)\ for\ s = 3, \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m}{6} + h(u), & if\ m \equiv 0, 3, 8, 11(\text{mod } 12)\ for\ 3 \leq s \leq n, \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m}{6} + h(u), & if\ m \equiv 4, 7(\text{mod } 12)\ for\ s \equiv 0(\text{mod } 3), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m}{6} + h(u), & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 0(\text{mod } 6), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m}{6} + h(u), & if\ m \equiv 2, 5, 6, 9(\text{mod } 12)\ for\ s \equiv 0(\text{mod } 2), 4 \leq s \leq n, \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 8}{6} + h(u), & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 4(\text{mod } 6), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 8}{6} + h(u), & if\ m \equiv 4, 7(\text{mod } 12)\ for\ s \equiv 1(\text{mod } 3), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 16}{6} + h(u), & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 2(\text{mod } 6), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 16}{6} + h(u), & if\ m \equiv 4, 7(\text{mod } 12)\ for\ s \equiv 2(\text{mod } 3), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 4}{6} + h(u), & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 5(\text{mod } 6), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 20}{6} + h(u), & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 1(\text{mod } 6), \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 12}{6} + h(u), & if\ m \equiv 2, 5, 6, 9(\text{mod } 12)\ for\ s \equiv 1(\text{mod } 2), 5 \leq s \leq n, \\ j + \dfrac{(m^2+m)s - 3m^2 - 3m - 12}{6} + h(u), & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 3(\text{mod } 6). \end{cases}$$

For all edges in set

$$\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq m\},$$

we define edge labeling such as:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + \lceil \dfrac{m^2 - 2m}{4} \rceil, & if\ m \not\equiv 2, 6, 10(\text{mod } 12)\ for\ s = 1, \\ j + \dfrac{m^2 - 2m + 4}{4}, & if\ m \equiv 2, 6, 10(\text{mod } 12)\ for\ i = 1, \\ j + \lceil \dfrac{m^2 + 2m}{4} \rceil, & if\ m \not\equiv 1, 5, 9(\text{mod } 12)\ for\ s = 2, \\ j + \dfrac{m^2 + 2m - 3}{4}, & if\ m \equiv 1, 5, 9(\text{mod } 12)\ for\ s = 2, \\ j + \dfrac{(m^2+m)s - m^2 - m - 4}{6}, & if\ m \equiv 4, 7(\text{mod } 12)\ for\ s \equiv 0(\text{mod } 3), \\ j + \dfrac{(m^2+m)s - m^2 - m - 12}{6}, & if\ m \equiv 4, 7(\text{mod } 12)\ for\ s \equiv 1(\text{mod } 3), \\ j + \dfrac{(m^2+m)s - m^2 - m}{6}, & if\ m \equiv 0, 3, 8, 11(\text{mod } 12)\ for\ 3 \leq s \leq n, \\ j + \dfrac{(m^2+m)s - m^2 - m - 2}{6}, & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 5(\text{mod } 6), \\ j + \dfrac{(m^2+m)s - m^2 - m - 6}{6}, & if\ m \equiv 2, 5, 6, 9(\text{mod } 12)\ for\ 3 \leq s \leq n, \\ j + \dfrac{(m^2+m)s - m^2 - m - 6}{6}, & if\ m \equiv 1, 10(\text{mod } 12)\ for\ s \equiv 4(\text{mod } 6). \end{cases}$$

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + \frac{(m^2+m)s - m^2 - m - 8}{6}, & if\ m \equiv 4,7 (\text{mod } 12)\ for\ s \equiv 2 (\text{mod } 3), \\ j + \frac{(m^2+m)s - m^2 - m - 10}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 0,3 (\text{mod } 6), \\ j + \frac{(m^2+m)s - m^2 - m - 14}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 2 (\text{mod } 6), \\ j + \frac{(m^2+m)s - m^2 - m - 18}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 1 (\text{mod } 6). \end{cases}$$

For the edge labeling of edge set $\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq m\}$, there is a case in this labeling for which we have $s = \lceil \frac{n}{3} \rceil$ if s is odd and $m = 5$, in which case $n \geq 3s$, if s is even, $m = 5$, then $n \geq 3s - 1$, if $m = 4$ then $n \geq 3s - 1$, and if $m \geq 6$ then $n \geq 3s$. Then the edge labeling is defined as:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + \lceil \frac{m^2 - 2m}{4} \rceil, & if\ m \not\equiv 1,5,9 (\text{mod } 12)\ for\ s = 2, \\ j + \frac{m^2 - 2m - 3}{4}, & if\ m \equiv 1,5,9 (\text{mod } 12)\ for\ s = 2, \\ j + \lceil \frac{m^2 - 2m + 1}{3} \rceil, & if\ m \equiv 1,2 (\text{mod } 4)\ for\ s = 3, \\ j + \frac{(m^2+m)s - m^2 - 7m}{6}, & if\ m \equiv 0,3,8,11 (\text{mod } 12)\ for\ s \geq 3, \\ j + \frac{(m^2+m)s - m^2 - 7m - 2}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 5 (\text{mod } 6), \\ j + \frac{(m^2+m)s - m^2 - 7m - 4}{6}, & if\ m \equiv 4,7 (\text{mod } 12)\ for\ s \equiv 0 (\text{mod } 3), \\ j + \frac{(m^2+m)s - m^2 - 7m - 6}{6}, & if\ m \equiv 2,5,6,9 (\text{mod } 12)\ for\ s \geq 4, \\ j + \frac{(m^2+m)s - m^2 - 7m - 6}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 4 (\text{mod } 6), \\ j + \frac{(m^2+m)s - m^2 - 7m - 8}{6}, & if\ m \equiv 4,7 (\text{mod } 12)\ for\ s \equiv 2 (\text{mod } 3), \\ j + \frac{(m^2+m)s - m^2 - 7m - 10}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 0,3 (\text{mod } 6), \\ j + \frac{(m^2+m)s - m^2 - 7m - 12}{6}, & if\ m \equiv 4,7 (\text{mod } 12)\ for\ s \equiv 1 (\text{mod } 3), \\ j + \frac{(m^2+m)s - m^2 - 7m - 14}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 2 (\text{mod } 6), \\ j + \frac{(m^2+m)s - m^2 - 7m - 18}{6}, & if\ m \equiv 1,10 (\text{mod } 12)\ for\ s \equiv 1 (\text{mod } 6). \end{cases}$$

For the labeling of edges $\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq m\}$, there is another case where, if $n \geq 7$, then the edge labeling is defined as:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + \lceil \frac{m^2 - 2m + 5}{4} \rceil, & if\ m \equiv 1,2 (\text{mod } 4)\ for\ s = 2, \\ j + \lceil \frac{m^2 + m + 1}{3} \rceil, & if\ m \equiv 1,2 (\text{mod } 4)\ for\ s = 3. \end{cases}$$

For the labeling of all edges in set $\{a_1^j a_n^j : 1 \leq j \leq m\}$, which shows the connecting edges of the first vertex and last vertex in each copy of the cycle graph, the labeling is as follows:

$$y_e(a_1^j a_n^j) = \begin{cases} j, & if\ m \equiv 0,3(\text{mod } 4)\ for\ n \equiv 0(\text{mod } 3), n \geq 6, \\ j, & if\ m \equiv 1,2(\text{mod } 4)\ for\ n \equiv 0(\text{mod } 6), \\ j+2, & if\ m = 4\ for\ n \equiv 2(\text{mod } 3), \\ j+4, & if\ m = 5\ for\ n \equiv 5(\text{mod } 6), \\ j + \frac{m^2-5m-6}{6}, & if\ m \equiv 1,2(\text{mod } 4), m \neq 5\ for\ n \equiv 5(\text{mod } 6), \\ j + \lceil \frac{m^2-2m-2}{3} \rceil, & if\ m \equiv 0,3(\text{mod } 4)\ for\ n \equiv 1(\text{mod } 3), \\ j + \lceil \frac{m^2-2m-2}{3} & if\ m \equiv 1,2(\text{mod } 4)\ for\ n \equiv 4(\text{mod } 6), \\ j + \frac{m^2-5m}{6}, & if\ m \equiv 0,3(\text{mod } 4), m \geq 8\ and\ n \equiv 2(\text{mod } 3), \\ j + \frac{m^2-5m}{6}, & if\ m \equiv 2,5,6,9(\text{mod } 12)\ for\ n \equiv 2(\text{mod } 6), \\ j + \frac{m^2-5m-8}{6}, & if\ m \equiv 4,7(\text{mod } 12), m \geq 7\ for\ n \equiv 2(\text{mod } 3), \\ j + \frac{m^2-5m-8}{6}, & if\ m \equiv 1,10(\text{mod } 12)\ for\ n \equiv 2(\text{mod } 6), \\ j + \frac{m^2-m}{2}, & if\ m \equiv 0,3(\text{mod } 4)\ for\ n = 3, \\ j + \frac{m^2-m-2}{2}, & if\ m \equiv 1,2(\text{mod } 4)\ for\ n \equiv 3(\text{mod } 6), \\ j + \lceil \frac{m^2-2m-5}{3} \rceil, & if\ m \equiv 1,2(\text{mod } 4)\ for\ n \equiv 1(\text{mod } 6). \end{cases}$$

2.1.3. Edge weights of the mC_n-graph

As the weight of an edge is the sum of the edge label and the labels of two adjacent vertices, then for the result of edge labeling $\{y_e(a_s^u a_s^{j+u}) : 1 \leq s \leq n; 1 \leq j \leq m-u; 1 \leq u \leq m-1\}$ with the use of function $h(u)$, the weights of the edges are as follows:

$$h(u) = \frac{(2m+1)u - u^2 - 2m}{2}.$$

$$wt(a_s^u a_s^{j+u}) = \begin{cases} j + h(u), & if\ m \geq 4\ for\ s = 1, \\ j + \frac{m^2+m}{2} + h(u), & if\ m \geq 4\ for\ s = 2, \\ j + \frac{(m^2+m)s - m^2 + m}{2} + h(u), & if\ m \geq 4\ for\ 3 \leq s \leq n. \end{cases}$$

By using a condition for which $s = \lceil \frac{n+1}{3} \rceil$, if s is odd and $m \equiv 1,2(\text{mod } 4)$, then $n \geq 3s-2$, if s is even and $m \equiv 1,2(\text{mod } 4)$, then $n \geq 3s-3$, and if $m \equiv 0,3(\text{mod } 4)$, then $n \geq 3s-2$. The edge weights under this condition are defined as:

$$wt(a_s^u a_s^{j+u}) = \begin{cases} j + \frac{(m^2+m)s - m^2 - m}{2} + h(u), & if\ 3 \leq s \leq n. \end{cases}$$

For the edge labeling $\{y_e(a_s^j a_{s+1}^j) : 1 \le s \le n-1; 1 \le j \le m\}$, we formulate the edge weights as:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + \dfrac{m^2 - m}{2}, & if\ m \ge 4\ for\ s = 1, \\ j + \dfrac{(m^2 + m)s}{2}, & if\ m \ge 4\ for\ 2 \le s \le n. \end{cases}$$

There is a case in these weights for which we have $s = \lceil \frac{n}{3} \rceil$, and if s is odd and $m = 5$, then $n \ge 3s - 2$, if s is even and $m = 5$, then $n \ge 3s - 1$, if $m = 4$, then $n \ge 3s - 1$, and if $m \ge 6$, then $n \ge 3s$. Then the weights of the edges are defined as:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + \dfrac{(m^2 + m)s}{2} - m, & if\ 2 \le s \le n. \end{cases}$$

For the edge labeling $\{y_e(a_1^j a_n^j) : 1 \le j \le m\}$, the edge weights are calculated as:

$$wt(a_1^j a_n^j) = \begin{cases} j + m^2, & if\ m \equiv 0, 3 (\text{mod } 4)\ and\ n = 3, \\ j + \dfrac{(m^2 + m)n + 2m^2 - 4m}{6}, & if\ m \equiv 0, 3 (\text{mod } 4)\ and\ n \equiv 1 (\text{mod } 3), \\ j + \dfrac{(m^2 + m)n + 2m^2 - 4m}{6}, & if\ m \equiv 1, 2 (\text{mod } 4)\ and\ n \equiv 1, 4 (\text{mod } 6), \\ j + \dfrac{(m^2 + m)n + m^2 - 5m}{6}, & if\ m \equiv 0, 3 (\text{mod } 4)\ and\ n \equiv 2 (\text{mod } 3), \\ j + \dfrac{(m^2 + m)n + m^2 - 5m}{6}, & if\ m \equiv 1, 2 (\text{mod } 4)\ and\ n \equiv 2, 5 (\text{mod } 6), \\ j + \dfrac{(m^2 + m)n}{6}, & if\ m \equiv 0, 3 (\text{mod } 4)\ and\ n \equiv 0 (\text{mod } 3), n \ge 6, \\ j + \dfrac{(m^2 + m)n}{6}, & if\ m \equiv 1, 2 (\text{mod } 4)\ and\ n \equiv 0 (\text{mod } 6), \end{cases}$$

$$wt(a_1^j a_n^j) = j + \frac{(m^2 + m)n + 3m^2 - 3m}{6},\ if\ m \equiv 1, 2 (\text{mod } 4)\ then\ n \equiv 3 (\text{mod } 6).$$

It is easy to check that all the edge weights $wt(mC_n)$ for edge set $E(mC_n)$ of the mC_n-graph are distinct and consecutive integers.

$$wt(mC_n) = \{1, 2, 3, \cdots\cdots, \frac{(m^2 + m)n}{2}\}.$$

As the vertex labeling is defined with even whole numbers, the edge labeling is defined with natural numbers and a reflexive edge strength of the m^t-graph of the cycle graph with $t = 1$ and $m \ge 4$ is under the bounds of strength for every graph in Lemma 1. □

Next, we state additional findings on total reflexive edge irregularity strength. In Theorem 2.2, the total reflexive edge irregular labeling for the $4C_n$-graph where $(m = 4)$ and $(n \ge 3)$ verify the results of total reflexive edge irregularity strength $res(4C_n)$.

Theorem 2.2. *Let the $4C_n$-graph be a graph with 4 isomorphic copies of the cycle graph C_n, Then, for $n \ge 3$,*

$$res(4C_n) = 2n + 2\left\lceil \frac{2n}{3} \right\rceil.$$

Proof. A $4C_n$-graph is a graph with 4 isomorphic copies of the cycle graph C_n and $n \geq 3$ contains $4n$ vertices and $10n$ edges with vertex set $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq 4\}$ and edge set $\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq 4\} \cup \{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq 4-u; 1 \leq u \leq 3\} \cup \{a_1^j a_n^j : 1 \leq j \leq 4\}$. For a $4C_n$-graph, the use of the lower bound of the reflexive edge strength in Lemma 1 represents two results as follows:

$$res(4C_n) \geq \begin{cases} \left\lceil \frac{10n}{3} \right\rceil, & if\ 10n \not\equiv 2, 3 (\text{mod } 6), \\ \left\lceil \frac{10n}{3} \right\rceil + 1, & if\ 10n \equiv 2, 3 (\text{mod } 6). \end{cases}$$

2.2. $4C_n$-graph

2.2.1. Vertex labeling

For the precise value of the reflexive edge strength for cycle graph $res(4C_n)$, all the labels of vertices and edges are maximum as $2n + 2\lceil \frac{2n}{3} \rceil$. For this verification, the vertices $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq 4\}$ are used for vertex labeling as:

$$y_v(a_s^j) = \begin{cases} 0, & if\ s = 1\ for\ 1 \leq j \leq 4, \\ 4, & if\ s = 2\ for\ 1 \leq j \leq 4, \\ 2s + 2\left\lceil \frac{2s}{3} \right\rceil, & if\ 3 \leq s \leq n\ for\ 1 \leq j \leq 4. \end{cases}$$

2.2.2. Edge labeling

For the edges $\{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq 4-u; 1 \leq u \leq 3\}$, we define the edge labeling as:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + \frac{9u - u^2 - 8}{2}, & if\ s = 1, \\ j + 2 + \frac{9u - u^2 - 8}{2}, & if\ s = 2, \\ j + \frac{10s - 18}{3} + \frac{9u - u^2 - 8}{2}, & if\ s \equiv 0 (\text{mod } 3), \\ j + \frac{10s - 22}{3} + \frac{9u - u^2 - 8}{2}, & if\ s \equiv 1 (\text{mod } 3), \\ j + \frac{10s - 26}{3} + \frac{9u - u^2 - 8}{2}, & if\ s \equiv 2 (\text{mod } 3). \end{cases}$$

There is a case in edge labeling for which we have $s = \lceil \frac{n}{3} \rceil$ if $n \geq 3s - 2$. Then the labeling of edges is as follows:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + \frac{10s - 30}{3} + \frac{9u - u^2 - 8}{2}, & if\ s \equiv 0 (\text{mod } 3), \\ j + \frac{10s - 34}{3} + \frac{9u - u^2 - 8}{2}, & if\ s \equiv 1 (\text{mod } 3), \\ j + \frac{10s - 38}{3} + \frac{9u - u^2 - 8}{2}, & if\ s \equiv 2 (\text{mod } 3). \end{cases}$$

For the edges $\{a_s^j a_{s+1}^j : 1 \le s \le n-1; 1 \le j \le 4\}$, we define the edge labeling as:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j+2, & if\ s = 1, \\ j+6, & if\ s = 2, \\ j + \dfrac{10s-12}{3}, & if\ s \equiv 0 (\text{mod } 3), \\ j + \dfrac{10s-16}{3}, & if\ s \equiv 1 (\text{mod } 3), \\ j + \dfrac{10s-14}{3}, & if\ s \equiv 2 (\text{mod } 3). \end{cases}$$

There is a case in edge labeling for which we have $s = \lceil \frac{n}{3} \rceil$ if $n \ge 3s-1$. Then the labeling of edges is as follows:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j+2, & if\ s = 2, \\ j + \dfrac{10s-24}{3}, & if\ s \equiv 0 (\text{mod } 3), \\ j + \dfrac{10s-28}{3}, & if\ s \equiv 1 (\text{mod } 3), \\ j + \dfrac{10s-26}{3}, & if\ s \equiv 2 (\text{mod } 3). \end{cases}$$

For the edges $\{a_1^j a_n^j : 1 \le j \le 4\}$, we define the edge labeling as:

$$y_e(a_1^j a_n^j) = \begin{cases} j, & if\ n \equiv 0 (\text{mod } 3), n \ge 6, \\ j+2, & if\ n \equiv 1, 2 (\text{mod } 3), \\ j+6, & if\ n = 3. \end{cases}$$

In Figure 2, the edge reflexive irregular total labeling of the $4C_4$-graph with 4 complete connections in which the vertex labels in one complete connection are the same and also an even number with vertex set $V(4C_4) = \{0, 4, 10, 14\}$. The edges are labeled with natural numbers where the edge set $E(4C_4) = \{1, 2, 3, \cdots, 12\}$. The edge weights range from 1 to 40 and are represented by the blue label in this figure. Using Lemma 1, the reflexive edge strength of this graph $res(4C_4) = 14$.

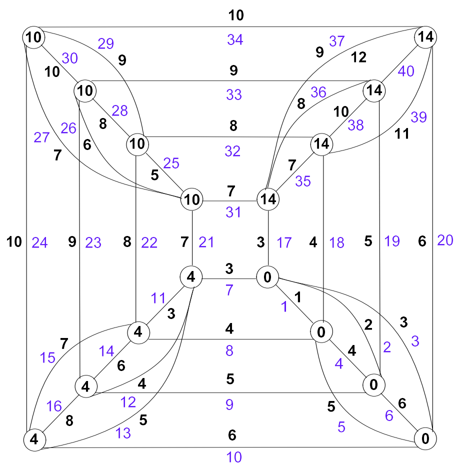

Figure 2. An edge reflexive irregular total labeling of $4C_4$.

2.2.3. Edge weights

For the edges $\{y_e(a_s^u a_s^{j+u}) : 1 \leq s \leq n; 1 \leq j \leq 4-u; 1 \leq u \leq 3\}$, the weights of the edges are:

$$wt(a_s^u a_s^{j+u}) = \begin{cases} j + \dfrac{9u - u^2 - 8}{2}, & if\ s = 1, \\ j + 10 + \dfrac{9u - u^2 - 8}{2}, & if\ s = 2, \\ j + 10s - 6 + \dfrac{9u - u^2 - 8}{2}, & if\ 3 \leq s \leq n. \end{cases}$$

There is a case in edge weights such that $s = \lceil \frac{n+1}{3} \rceil$, if $n \geq 3s - 2$. Then the weights of the edges are:

$$wt(a_s^u a_s^{j+u}) = \begin{cases} j + 10s - 10 + \dfrac{9u - u^2 - 8}{2}, & if\ s \geq 3. \end{cases}$$

For the edges $\{y_e(a_s^j a_{s+1}^j) : 1 \leq s \leq n-1; 1 \leq j \leq 4\}$, the weights of the edges are:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + 6, & if\ s = 1, \\ j + 10s, & if\ s \geq 2. \end{cases}$$

There is a case in edge weights such that $s = \lceil \frac{n}{3} \rceil$ if $n \geq 3s - 1$. Then the weights of the edges are:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + 10s - 4, & if\ s \geq 2. \end{cases}$$

For the edges $\{y_e(a_1^j a_n^j) : 1 \leq j \leq 4\}$, the weights of the edges are:

$$wt(a_1^j a_n^j) = \begin{cases} j + 16, & if\ n = 3, \\ j + \dfrac{10n + 8}{3}, & if\ n \equiv 1 (\mathrm{mod}\ 3), \\ j + \dfrac{10n - 2}{3}, & if\ n \equiv 2 (\mathrm{mod}\ 3), \\ j + \dfrac{10n}{3}, & if\ n \equiv 0 (\mathrm{mod}\ 3), n \geq 6. \end{cases}$$

It is simple to check edge weights using the fact that the edge labeling and vertex labeling are distinct and consecutive with a common difference equal to 1. In this theorem, the minimum weight is 1, maximum weight is $10n$, and $10n$ is also the size of the $4C_n$-graph. □

In Theorem 2.3, the total reflexive edge irregular labeling for the $5C_n$-graph where $m = 5$ and $n \geq 3$ corroborate the outcome of the total reflexive edge irregularity strength $res(5C_n)$.

Theorem 2.3. *Let the $5C_n$-graph be a graph with 5 isomorphic copies of the cycle graph C_n and then for $n \geq 3$,*

$$res(5C_n) = 4n + 2\left\lceil \frac{n}{2} \right\rceil.$$

Proof. An $5C_n$-graph is a graph with 5 isomersen copies of the cycle graph C_n and $n \geq 3$ containing $5n$ vertices and $15n$ edges with vertex set $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq 5\}$ and edge set $\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq 5\} \cup \{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq 5-u; 1 \leq u \leq 4\} \cup \{a_1^j a_n^j : 1 \leq j \leq 5\}$. For the $5C_n$-graph, the bounds of the reflexive edge strength in Lemma 1 represent two possible cases as follows:

$$res(5C_n) \geq \begin{cases} \lceil 5n \rceil, & if\ 15n \not\equiv 2, 3 (\text{mod } 6), \\ \lceil 5n \rceil + 1, & if\ 15n \equiv 2, 3 (\text{mod } 6). \end{cases}$$

2.3. $5C_n$-grah

2.3.1. Vertex labeling

For the exact value of the reflexive edge strength for cycle graph $res(5C_n)$, all the assignments of vertices and edges both are maximum as $4n + 2\lceil \frac{n}{2} \rceil$. For this confirmation, the vertices $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq 5\}$ are used for vertex labeling as:

$$y_v(a_s^j) = \begin{cases} 0, & if\ s = 1\ for\ 1 \leq j \leq 5, \\ 6, & if\ s = 2\ for\ 1 \leq j \leq 5, \\ 14, & if\ s = 3\ for\ 1 \leq j \leq 5, n \geq 7, \\ 4s + 2\left\lceil \dfrac{s}{2} \right\rceil, & if\ 3 \leq s \leq n\ for\ 1 \leq j \leq 5. \end{cases}$$

2.3.2. Edge labeling

For the labeling of edge set $\{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq 5 - u; 1 \leq u \leq 4\}$, we have:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + \dfrac{11u - u^2 - 10}{2}, & if\ s = 1, \\ j + 3 + \dfrac{11u - u^2 - 10}{2}, & if\ s = 2, \\ j + 5s - 10 + \dfrac{11u - u^2 - 10}{2}, & if\ 4 \leq s \leq n,\ s \equiv 0 (\text{mod } 2), \\ j + 5s - 12 + \dfrac{11u - u^2 - 10}{2}, & if\ s \equiv 1 (\text{mod } 2). \end{cases}$$

There is a case for $s = \left\lceil \frac{n+1}{3} \right\rceil$, where we have, $n \geq 3s - 2$ if s is odd and $n \geq 3s - 3$, if s is even. Then the edge labeling is as follows:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + 2 + \dfrac{11u - u^2 - 10}{2}, & if\ s = 3, \\ j + 5s - 17 + \dfrac{11u - u^2 - 10}{2}, & if\ 5 \leq s \leq n,\ s \equiv 1 (\text{mod } 2), \\ j + 5s - 15 + \dfrac{11u - u^2 - 10}{2}, & if\ s \equiv 0 (\text{mod } 2). \end{cases}$$

For the labeling of edges $\{a_s^j a_{s+1}^j : 1 \leq s \leq n - 1; 1 \leq j \leq 5\}$, we have:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + 4, & if\ s = 1, \\ j + 8, & if\ s = 2, \\ j + 5, & if\ s = 2\ for\ n \geq 7, \\ j + 11, & if\ s = 3\ for\ n \geq 7, \\ j + 5s - 6, & if\ 3 \leq s \leq n. \end{cases}$$

There is an instance for $s = \left\lceil \frac{n}{3} \right\rceil$, where we have, $n \geq 3s$, if s is odd and $n \geq 3s - 1$, if s is even. Then edge labeling is as follows:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + 3, & if\ s = 2, \\ j + 6, & if\ s = 3, \\ j + 5s - 11, & if\ s \geq 4. \end{cases}$$

For the edge labeling for edge set $\{a_1^j a_n^j : 1 \leq j \leq 5\}$, we have:

$$y_e(a_1^j a_n^j) = \begin{cases} j + 9, & if\ n \equiv 3(\text{mod } 6), \\ j + 5, & if\ n \equiv 4(\text{mod } 6), \\ j + 4, & if\ n \equiv 1, 5(\text{mod } 6), \\ j, & if\ n \equiv 0, 2(\text{mod } 6). \end{cases}$$

2.3.3. Edge weights

The edge weights for edge labels

$$\{y_e(a_s^u a_s^{j+u}) : 1 \leq s \leq n; 1 \leq j \leq 5 - u; 1 \leq u \leq 4\}$$

are as follows:

$$wt(a_s^u a_s^{j+u}) = \begin{cases} j + \dfrac{11u - u^2 - 10}{2}, & if\ s = 1, \\ j + 15 + \dfrac{11u - u^2 - 10}{2}, & if\ s = 2, \\ j + 15s - 10 + \dfrac{11u - u^2 - 10}{2}, & if\ s \geq 3. \end{cases}$$

There is an instance for $s = \left\lceil \frac{n+1}{3} \right\rceil$, where we have, $n \geq 3s - 2$, if s is odd and $n \geq 3s - 3$, if s is even. Then edge weights are as follows:

$$wt(a_s^u a_s^{j+u}) = j + 15s - 15 + \frac{11u - u^2 - 10}{2}, \quad if\ s \geq 3.$$

The edge weights for edge labels

$$\{y_e(a_s^j a_{s+1}^j) : 1 \leq s \leq n - 1; 1 \leq j \leq 5\}$$

are:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + 10, & if\ s = 1, \\ j + 15s, & if\ 2 \leq s \leq n. \end{cases}$$

In Figure 3, the edge reflexive irregular total labeling of the $5C_7$-graph with 7 complete connections in which the vertex labels in one complete connection are the same and also an even number with vertex set $V(5C_7) = \{0, 6, 14, 20, 26, 30, 36\}$. The edges are labeled with natural numbers where the edge set $E(5C_7) = \{1, 2, 3, \cdots, 33\}$. The edge weights range from 1 to 105. Using Lemma 1, the reflexive edge strength of this graph $res(5C_7) = 36$.

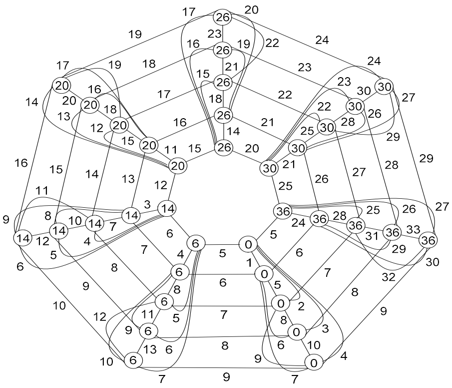

Figure 3. Edge reflexive irregular total labeling of $5C_7$.

There is a scenario for $y_e(a_s^j a_{s+1}^j)$ with $s = \lceil \frac{n}{3} \rceil$, where we have, $n \geq 3s$ if s is odd and $n \geq 3s - 1$, if s is even. Then edge weights are as follows:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + 15s - 5, & if\ s \geq 2. \end{cases}$$

The edge weights of edge labels $y_e(a_1^j a_n^j) : 1 \leq j \leq 5\}$ are:

$$wt(a_1^j a_n^j) = \begin{cases} j + 5n + 10, & if\ n \equiv 3(\text{mod } 6), \\ j + 5n + 5, & if\ n \equiv 1, 4(\text{mod } 6), \\ j + 5n, & if\ n \equiv 0, 2, 5(\text{mod } 6). \end{cases}$$

This total labeling makes it easy to verify that the edge weights are consecutive, different, and have a common difference of 1. According to this theorem, the weight ranges from 1 to $15n$, where $15n$ is also the graph's size. □

In Theorem 2.3, the total reflexive edge irregular labeling for the $8C_n$-graph where $m = 8$ and $n \geq 3$ supports the consequence of total reflexive edge irregularity strength $res(8C_n)$.

Theorem 2.4. *Let the $8C_n$-graph be a graph with 8 identical copies of the cycle graph C_n and then, for $n \geq 3$,*

$$res(8C_n) = 12n.$$

Proof. An $8C_n$-graph is a graph with 8 identical copies of the cycle graph C_n and $n \geq 3$ containing $8n$ vertices and $36n$ edges with vertex set $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq 8\}$ and edge set $\{a_s^j a_{s+1}^j : 1 \leq s \leq n-1; 1 \leq j \leq 8\} \cup \{a_s^u a_s^{j+u} : 1 \leq s \leq n; 1 \leq j \leq 8-u; 1 \leq u \leq 7\} \cup \{a_1^j a_n^j : 1 \leq j \leq 8\}$. For the $8C_n$-graph, the exact bounds of the reflexive edge strength in Lemma 1 represent two possible cases, which can be written as:

$$res(8C_n) \geq \begin{cases} \lceil 12n \rceil, & if\ 36n \not\equiv 2, 3(\text{mod } 6), \\ \lceil 12n \rceil + 1, & if\ 36n \equiv 2, 3(\text{mod } 6). \end{cases}$$

2.4. $8C_n$-graph

2.4.1. Vertex labeling

For the exact value of the reflexive edge strength for cycle graph $res(8C_n)$, all the assignments of vertices and edges are maximum as $12n$. For this confirmation, the vertices $\{a_s^j : 1 \leq s \leq n; 1 \leq j \leq 8\}$ are used for vertex labeling as:

$$y_v(a_i^8) = \begin{cases} 0, & if\ s = 1, \\ 16, & if\ s = 2, \\ 12s, & if\ s \geq 3. \end{cases}$$

2.4.2. Edge labeling

For $1 \leq s \leq n, 1 \leq j \leq 8 - u$, and $1 \leq u \leq 7$, the edge labeling is defined as:

$$y_e(a_s^u a_s^{j+u}) = \begin{cases} j + \dfrac{17u - u^2 - 16}{2}, & if\ s = 1, \\ j + 4 + \dfrac{17u - u^2 - 16}{2}, & if\ s = 2, \\ j + 12s - 28 + \dfrac{17u - u^2 - 16}{2}, & if\ s \geq 3, \\ j + 12s - 36 + \dfrac{17u - u^2 - 16}{2}, & if\ s = \lceil \dfrac{n+1}{3} \rceil\ for\ n \geq 3s - 2. \end{cases}$$

For $\{1 \leq s \leq n - 1; 1 \leq j \leq 8\}$, we define the following edge labeling:

$$y_e(a_s^j a_{s+1}^j) = \begin{cases} j + 12, & if\ s = 1, \\ j + 12, & if\ s = 2\ for\ n \geq 6,\ s = \lceil \dfrac{n}{3} \rceil, \\ j + 20, & if\ s = 2, \\ j + 12s - 12, & if\ s \geq 3, \\ j + 12i - 20, & if\ s \geq 3\ for\ n \geq 3s,\ s = \lceil \dfrac{n}{3} \rceil. \end{cases}$$

For $\{1 \leq j \leq 8\}$, we define the following edge labeling:

$$y_e(a_1^j a_n^j) = \begin{cases} j + 28, & if\ n = 3, \\ j + 16, & if\ n \equiv 1 (\mathrm{mod}\ 3), \\ j + 4, & if\ n \equiv 2 (\mathrm{mod}\ 3), \\ j, & if\ n \equiv 0 (\mathrm{mod}\ 3), n \geq 6. \end{cases}$$

In Figure 4, the edge reflexive irregular total labeling of the $8C_3$-graph with 3 complete connections in which the vertex labels in one complete connection are the same and also an even number with vertex set $V(8C_3) = \{0, 16, 36\}$. The edges are labeled with natural numbers where the edge set $E(8C_3) = \{1, 2, 3, \cdots, 36\}$. The edge weights range from 1 to 108. Using Lemma 1, the reflexive edge strength of this graph $res(8C_3) = 36$.

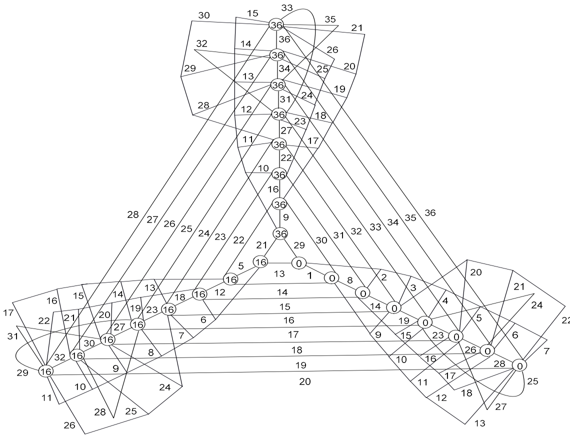

Figure 4. Edge reflexive irregular labeling of $8C_3$.

2.4.3. Edge weights

For $1 \leq s \leq n, 1 \leq j \leq 8-u$, and $1 \leq u \leq 7$, we define the following edge weights:

$$wt(a_s^u a_s^{j+u}) = \begin{cases} j + \dfrac{17u - u^2 - 16}{2}, & if\ s = 1, \\ j + 36 + \dfrac{17u - u^2 - 16}{2}, & if\ s = 2, \\ j + 36s - 28 + \dfrac{17u - u^2 - 16}{2}, & if\ s \geq 3, \\ j + 36s - 36 + \dfrac{17u - u^2 - 16}{2}, & if\ s = \lceil \dfrac{n+1}{3} \rceil, n \geq 3s - 2\ for\ s \geq 3. \end{cases}$$

For $\{1 \leq s \leq n-1; 1 \leq j \leq 8\}$, we define the following edge weights:

$$wt(a_s^j a_{s+1}^j) = \begin{cases} j + 28, & if\ s = 1, \\ j + 36s, & for\ s \geq 2, \\ j + 36s - 8, & if\ n \geq 3s,\ s = \lceil \dfrac{n}{3} \rceil\ for\ s \geq 2. \end{cases}$$

For $\{1 \leq j \leq 8\}$, we define the following edge weights:

$$wt(a_1^j a_n^j) = \begin{cases} j + 64, & if\ n = 3, \\ j + 12n + 16, & if\ n \equiv 1 (\text{mod } 3), \\ j + 12n + 4, & if\ n \equiv 2 (\text{mod } 3), \\ j + 12n, & if\ n \equiv 0 (\text{mod } 3), n \geq 6. \end{cases}$$

It is simple to confirm that the edge weights are distinct and consecutive. This theorem states that the weights vary from 1 to $36n$, where $36n$ is also the size of the graph. □

3. Application

In this paper, the graphical model is made up of certain copies of a complete graph, and edges between copies are used to form some cycles. Consider the problem illustrated by the communication between three company buildings. Each of the three buildings is represented as a complete graph, where individuals within a building are authorized with some designation and represented as vertices. These responsible vertices can receive messages from any vertex in their respective building and can communicate with each other through the direct connection as an edge. For example, if each building includes four working people such as a Manager, Engineer, Analyst, and Worker who need to communicate with each other in the same building, then the message travels from a unique vertex to any other vertex through edges and ensures a clear and traceable communication route. For inter-building communication, edges connect vertices with matching designations across different buildings, ensuring that only individuals with the same role can communicate between buildings. For graph labeling, individual vertices are labeled to represent the level of authority. The inter-building edges are also labeled to represent the number of assignments or interactions possible between individuals of the same designation. Each individual is assigned a unique label according to their building and designation (e.g., M_i, E_i, A_i, W_i for a Manager, Engineer, Analyst, and Worker representative within a company, where $i \geq 2$).

Within a building, the edges connecting these vertices are labeled with the number of assignments or interactions between individuals, facilitating seamless internal communication and collaboration. This communication model ensures efficient internal communication while maintaining controlled, designation-specific communication between buildings. The labeling scheme in this paper ensures the unique identification of each employee and assigns maximum assignments according to the given lower bound in Lemma 1, facilitating distinct communication paths within buildings and intellectual value between two special people in two different buildings through bridge edges. The graph labeling provides unique identification for each individual and indicates the communication load, helping to manage and secure communication while ensuring that role-specific information is exchanged appropriately between buildings. An illustration of a communication model between three different companies collaborating for any specific purpose or task in shown in Figure 5.

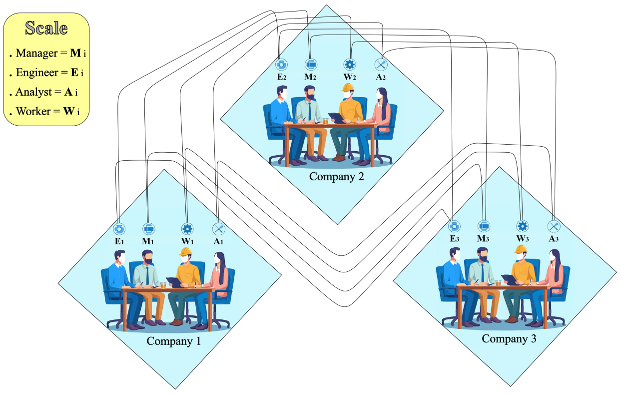

Figure 5. An illustration of the communication model between three companies.

4. Conclusions

In this paper, we have calculated the graph labels, edge weights, and reflexive edge irregularity strength of the graph. The mapping y_v is the vertex label function and y_e is the edge label function for the edge irregular reflexive total labeling for anti-magic edge weights of an mC_n-graph for cycle graph

C_n with $m \geq 4$ and $n \geq 3$. Applications of this labeling are dispensed through a secure communication channel to ensure the unique identification of each employee in a company and assign maximum assignments according to the communication load that helps to manage the number of companies. The open problem to find exact bounds of the reflexive edge irregularity strength for an m^t-graph on $t = 1$ for the star graph S_n with $m \geq 2$ and $n \geq 3$ is $res(mS_n)$, and for the m^t-graph on $t = 1$ for the fan graph F_n with $m \geq 2$ and $n \geq 3$ is $res(mF_n)$.

Author contributions

Muhammad Amir Asif: Supervision, Conceptualization, Writing-original draft; Rashad Ismail: Supervision, Conceptualization; Ayesha Razaq: Writing-original draft; Esmail Hassan Abdullatif Al-Sabri: Validation; Muhammad Haris Mateen: Supervision, Conceptualization; Shahbaz Ali: Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state that they do not have a conflict of interest in relation to the publication of this research article.

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