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Research article

Comparative study of tuberculosis infection by using general fractional derivative

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Abstract: The current study examined the general fractional derivative of the fractional order in the context of an incomplete treatment of tuberculosis (TB). Utilizing the fixed-point technique and nonlinear analysis, we arrived at certain theoretical conclusions on the existence and stability of the solution. The well-known Laplace transform method was used to calculate the arithmetic output of the model under consideration. This approach depends upon an elementary principle of fractional calculus. For each case of general fractional derivative, numerical simulation was also provided, each one linked with a particular fractional order in (0,1).

Keywords: TB; tuberculosis disease; generalized derivative; fractional operator; fixed point condition; Laplace transform
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1. Introduction

This research examines the general fractional derivative mathematical model of TB [1]. Since TB is one of the leading causes of death in the modern world, research on the subject is critically needed [2, 3]. Numerous scientists and researchers have utilized experimental evidence to show that TB is a communicable epidemic often brought on by bacteria in human lungs. According to a WHO report, nearly 10 million individuals worldwide contracted the disease in 2017 and 1.5 million died from it. Researchers worry that a global epidemic of diabetes illnesses could put a greater number of people at risk. Several mathematical models have been developed to study the dynamics of infectious illness transmission and the development of practical mitigation techniques. Waaler et al. introduced the TB disease infection model for the first time in disease epidemiology [4]. Yang et al. developed a model to investigate the various scenarios of TB transmission with insufficient therapy [5]. To study the outcomes of TB's worldwide stability and the impact of heterogeneity on TB dispersion, several scholars proposed mathematical models for the disease. Zhang et al. looked into a dynamic problem of infectious classes with hospitalized and non-hospitalized TB sickness [6]. A mathematical model of TB was taken into consideration by Robert in order to examine the effects of deterioration and reinfection. In addition, Egonmwan et al. proposed a model for TB illness to assess the impact of analysis and therapy for contagious individuals [7].

In this study, a mathematical model of TB [8, 9] is analyzed using a unique approach of fractional calculus. One should know the importance of fractional calculus and mathematical modeling in real life [10]. Mathematical modeling is proving to be a very useful tool in modern mathematical studies [11]. Primary approaches to transform real-world problems into mathematical terms, allowing us to apply the findings back to the real world to achieve predictive goals. Every company aims to meet its objectives or forecast the future. However, there are instances when the fractional calculus describes models more accurately than traditional methods [12,13]. Nowadays, many technical, engineering and physical phenomena are modeled using fractional calculus [14]. Fractional calculus has a rich history spanning around 300 years.

Fractional calculus is an obvious next step after classical calculus [15]. Applications of fractional calculus are increasing, as it provides better mathematical representations of everyday occurrences [16]. Any natural or physical phenomena whose outline is useful in the understanding of the issue can be modeled using a simulation model [17]. Fractional calculus has influenced a wide range of fields, including biology, engineering, fluid, control theory, image processing, visco-elasticity, astronomy, and electricity [18,19]. Fractional calculus is a fascinating branch of mathematics, handling integrals and derivatives of arbitrary order [20, 21]. Fractional integrals and derivatives of order $\xi > 0$ have a range of interpretations, in contrast to conventional definitions of derivatives and integrals [22]. The theory of singular kernels in fractional calculus was greatly influenced by several researchers, including Samko, Riemann, Caputo, Kilbas, and others [23, 24]. Researchers such as Miller-Ross, Atangana-Baleanu, Wiman, Yang and others investigated integrals and derivatives with kernels without singularity [25, 26].

The general fractional derivative is believed to be the most effective way to explain the models of complex processes [27]. In fractional calculus, the behavior of non-smooth functions is described by the Caputo derivative and general fractional derivative, two distinct forms of fractional derivative. The Caputo derivative considers a function's initial conditions and is used to represent processes where initial conditions are unknown. On the other hand, a more modern operator, the general derivative, offers more freedom in modeling non-smooth systems. Since the function is supposed to be of order ξ, which can be broken further into different operators, the conditions of a function are not taken into consideration.

Some derivatives better fit specific real phenomena. The main objective of this study is to provide a systematic application of the general fractional derivative to analyze the same system using different kernels (singular and non-singular) at the same time.

The novelty of this work lies in the application of new approach to the existing problem of TB infection with and without treatment [28, 29]. We have also validate the results of the system using fixed-point theorem and show the uniqueness of the solution. This approach will certainly model different applications of the general fractional derivative. The remaining sections of the paper are structured as follows: The pre-requisites are defined in Section 2. The TB model is briefly discussed in

Section 3, while Section 4 ensures the existence of the results. Section 5 deals with uniqueness of the solution. In Section 6, a solution of the TB model is found using the general fractional derivative with the help of Laplace transform. Graphical results are shown in Section 7, while the paper is concluded in Section 8.

2. Pre-requisites

This section provides some background on recently established general fractional operators. The Caputo and Riemann-Liouville derivatives of fractional exponent are provided by

$$ {}_0^C\mathrm{D}_t^\gamma f(t) = \int_0^t \dot{f}(s)\nabla_\iota(t-s)ds, \tag{2.1}$$

$$ {}_0\mathrm{D}_t^\gamma f(t) = \frac{d}{dt}\int_0^t f(s)\nabla_\iota(t-s)ds, \tag{2.2}$$

where $\gamma \in (0,1)$ is the exponent of derivative, $f : [0,+\infty) \to \mathbb{R}$ is a continuous function with $\dot{f} \in L^1_{\text{loc}}(0,+\infty), 0 \leqslant t \leqslant T < +\infty$, ∇_ι is known as kernel. The operator is made to adhere to the linear condition,

$$ {}_0^C\mathrm{D}_t^\gamma(jf(t)+kg(t)) = j{}_0^C\mathrm{D}_t^\gamma f(t) + k{}_0^C\mathrm{D}_t^\gamma g(t), \tag{2.3}$$

$$ {}_0\mathrm{D}_t^\gamma(jf(t)+kg(t)) = j{}_0\mathrm{D}_t^\gamma f(t) + k{}_0\mathrm{D}_t^\gamma g(t). \tag{2.4}$$

It is evident that for any $t > 0$, as long as certain requirements of $\nabla_\iota(t)$ are met, a completely function of monotone type $\mathfrak{I}_\iota(t)$ occurs,

$$ \nabla_\iota(t) * \mathfrak{I}_\iota(t) = \int_0^\infty \nabla_\iota(s)\mathfrak{I}_\iota(t-s)ds = 1, \tag{2.5}$$

further, for $f \in L^1_{\text{loc}}(0,+\infty)$, we can rewrite the above

$$ {}_0\mathrm{D}_t^{-\gamma}\left[{}_0^C\mathrm{D}_t^\gamma f(t)\right] = f(t) - f(0), \tag{2.6}$$

where ${}_0\mathrm{D}_t^{-\gamma}$ denotes the general Riemann-Liouville integral of fractional order, given as

$$ {}_0\mathrm{D}_t^{-\gamma} f(t) = \int_0^t f(s)\mathfrak{I}_\iota(t-s)ds. \tag{2.7}$$

The right-side of Caputo and Riemann-Liouville fractional derivatives are

$$ {}_t^C\mathrm{D}_T^\gamma f(t) = \int_t^T \dot{f}(s)\nabla_r(s-t)ds, \tag{2.8}$$

$$ {}_t\mathrm{D}_T^\gamma f(t) = \frac{d}{dt}\int_t^T f(s)\nabla_r(s-t)ds, \tag{2.9}$$

and

$$ {}_t\mathrm{D}_T^{-\gamma} f(t) = \int_t^T f(s)\mathfrak{I}_r(s-t)ds. \tag{2.10}$$

Based on the findings above, the integration by component formula is therefore satisfied by the above-mentioned fractional-order operators such as

$$\int_0^T f(s)_0 D_s^\gamma g(s)ds = \int_0^T g(s)_s {}^C D_s^\gamma f(s)ds, \tag{2.11}$$

$$\int_0^T f(s)_0 {}^C D_s^\gamma g(s)ds = \int_0^T g(s)_s D_s^\gamma f(s)ds. \tag{2.12}$$

By incorporating the various kernels into the various general operator definitions, we may derive 3 specific cases of general operator. In the first case, the kernel is $\nabla_\iota(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)}$; so, we have power function $\Im_\iota(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)}$ reforming the integral operator's associated kernel (2.7).

Further, take kernel $\nabla_\iota(t) = \frac{B(\gamma)}{1-\gamma} \mathrm{E}_\gamma \left(\frac{-\gamma}{1-\gamma} t^\gamma \right)$ where E_γ and $\mathrm{B}(\gamma)$ are Mittag-Leffler and normalization functions. We also have

$$\Im_\iota(t) = \frac{1-\gamma}{B(\gamma)} \delta(t) + \frac{\gamma}{\mathrm{B}(\gamma)\Gamma(\gamma)} t^{\gamma-1}. \tag{2.13}$$

So, Eqs (2.1) and (2.2) may be used to get the derivatives of AB-Caputo and AB-Riemann-Liouville. The AB type integral is,

$${}_0\mathrm{D}_t^{-\gamma} f(t) = \frac{1-\gamma}{\mathrm{B}(\gamma)} f(t) + \frac{\gamma}{\mathrm{B}(\gamma)\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)ds. \tag{2.14}$$

Now, at last, the CF derivative is found by taking kernel $\nabla_\iota(t) = \frac{B(\gamma)}{1-\gamma} \exp\left(\frac{-\gamma}{1-\gamma} t \right)$.

2.1. Laplace transform

Laplace transformation is a key transformation in mathematics. It converts the system into a algebraic system, which is easily solvable. The symbol $L\{f(t)\}$ represents the Laplace transform of $f(t)$ and is explained below:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \quad s > 0. \tag{2.15}$$

2.1.1. Laplace transform of Riemann-Liouville fractional differential operator

Laplace transform of Riemann-Liouville's fractional derivative [10] is explained as follows:

$$L\left\{ {}^C D^\gamma g(t) \right\} = [sL(g(t)) - g(0)] s^{(\gamma-1)}. \tag{2.16}$$

2.1.2. Laplace transform of Caputo-Fabrizio (CF) fractional differential operator

The Laplace transformation of CF [12] operator is:

$$L\left\{ {}^{CF} D^\gamma g(t) \right\} = \frac{1}{2} . \frac{B(\gamma)(2-\gamma)}{1-\gamma} . \frac{sL\{g(t)\} - g(0)}{s + \frac{\gamma}{1-\gamma}}. \tag{2.17}$$

2.1.3. Laplace transform of Atangana-Baleanu fractional derivative

If $f(t)$ is any function and ${}^{ABC}D^{\gamma}f(t)$ is the Atangana-Baleanu fractional differential operator [25] of $f(t)$, then the Laplace transform of this operator is given as follows:

$$L\{{}^{ABC}D^{\gamma}f(t)\} = \frac{B(\gamma)}{1-\gamma}\frac{p^{\gamma}L\{f(t)\} - p^{\gamma-1}f(0)}{p^{\gamma}+\frac{\gamma}{1-\gamma}}. \tag{2.18}$$

3. TB mathematical model

By mathematical modelling, it is possible to identify the endemics' unique transmission patterns and get insight into how infection impacts a new population. The incomplete-treatment TB model was developed by Ihsan et al [30]. Here, the total population is split in five subclasses: susceptible class S, exposed class E, infectious class without treatment I, infectious class with treatment T, and recovered class R. The proposed model is given below:

$$\left.\begin{aligned}
\tfrac{dS}{dt} &= \Delta - \xi S(I+\eta T) - \phi S,\\
\tfrac{dE}{dt} &= \xi S(I+\eta T) - (\phi+\lambda)E + (1-\rho)\delta T,\\
\tfrac{dI}{dt} &= \lambda E - (\phi+\sigma_1+\alpha)I,\\
\tfrac{dT}{dt} &= \alpha I - (\phi+\sigma_2+\delta)T,\\
\tfrac{dR}{dt} &= \rho\delta T - \phi R,
\end{aligned}\right\} \tag{3.1}$$

where Δ denote the rate of conscription, ξ represents the rate of effective contact, η represents the decrease in infection and ϕ is the rate of natural death. λ denotes the rate of transfer from E to I, ρ represents the capable treatment, σ_1 shows the rate of infection without treatment, and σ_2 shows the rate of infection with treatment. α shows the rate of the infected population and δ is rate of leaving population T.

4. Existence of the result

Here, we establish the existence of the result for the defined model.

4.1. For the Caputo-Fabrizio derivative

We use the fixed-point theorem to define the existence of the result. Converting the model into integral equations

$$S(t) - S(0) = {}_0^{CF}I_t^{\gamma}[\Delta - \xi S(I+\eta T) - \phi S], \tag{4.1}$$

$$E(t) - E(0) = {}_0^{CF}I_t^{\gamma}[\xi S(I+\eta T) - (\phi+\lambda)E + (1-\rho)\delta T], \tag{4.2}$$

$$I(t) - I(0) = {}_0^{CF}I_t^{\gamma}[\lambda E - (\phi+\sigma_1+\alpha)I], \tag{4.3}$$

$$T(t) - T(0) = {}_0^{CF}I_t^{\gamma}[\alpha I - (\phi+\sigma_2+\delta)T], \tag{4.4}$$

$$R(t) - R(0) = {}_0^{CF}I_t^{\gamma}[\rho\delta T - \phi R]. \tag{4.5}$$

Then by Nieto's definition [31], we get

$$\begin{aligned}
S(t) = S(0) &+ \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)}[\Delta - \xi S(I+\eta T) - \phi S]\\
&+ \tfrac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t [\Delta - \xi S(I+\eta T) - \phi S]ds,
\end{aligned}$$

$$E(t) = E(0) + \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)}[\xi S(I+\eta T) - (\phi+\lambda)E + (1-\rho)\delta T]$$
$$+\tfrac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t [\xi S(I+\eta T) - (\phi+\lambda)E + (1-\rho)\delta T]ds,$$

$$I(t) = I(0) + \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}[\lambda E - (\phi+\sigma_1+\alpha)I] + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t [\lambda E - (\phi+\sigma_1+\alpha)I]ds,$$

$$T(t) = T(0) + \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}[\alpha I - (\phi+\sigma_2+\delta)T] + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t [\alpha I - (\phi+\sigma_2+\delta)T]ds,$$

$$R(t) = R(0) + \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}[\rho\delta T - \phi R] + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t [\rho\delta T - \phi R]ds.$$

Further, consider the kernels to be

$$N_1(t,S) = \Delta - \xi S(I+\eta T) - \phi S,$$
$$N_2(t,E) = \xi S(I+\eta T) - (\phi+\lambda)E + (1-\rho)\delta T,$$
$$N_3(t,I) = \lambda E - (\phi+\sigma_1+\alpha)I,$$
$$N_4(t,T) = \alpha I - (\phi+\sigma_2+\delta)T,$$
$$N_5(t,R) = \rho\delta T - \phi R.$$

4.1.1. Theorem

If the kernels N_1, N_2, N_3, N_4, and N_5 satisfy the following constraints:

$$\|N_1(t,S) - N_1(t,S_1)\| \le H\,\|S_1(t) - S(t)\|, \qquad where\ \|\xi(I+\eta T)+\phi\| \le H < 1,$$
$$\|N_2(t,E) - N_2(t,E_1)\| \le H_1\,\|E_1(t) - E(t)\|, \qquad where\ \|(\phi+\lambda)\| \le H_1 < 1,$$
$$\|N_3(t,I) - N_3(t,I_1)\| \le H_2\,\|I_1(t) - I(t)\|, \qquad where\ \|(\phi+\sigma_1+\alpha)\| \le H_2 < 1,$$
$$\|N_4(t,T) - N_4(t,T_1)\| \le H_3\,\|T_1(t) - T(t)\|, \qquad where\ \|(\phi+\sigma_2+\delta)\| \le H_3 < 1,$$
$$\|N_5(t,R) - N_5(t,R_1)\| \le H_4\,\|R_1(t) - R(t)\|, \qquad where\ \|\phi\| \le H_4 < 1,$$

then system has a solution.

Proof. We shall prove for N_1 at first. Assume S and S_1 are two functions, hence

$$\begin{aligned}\|N_1(t,S) - N_1(t,S_1)\| &\le \|\Delta - \xi S(I+\eta T) - \phi S - \{\Delta - \xi S_1(I+\eta T) - \phi S_1\}\| \\ &= \|\xi(I+\eta T)(S_1 - S) + \phi(S_1 - S)\| \\ &= \|\xi(I+\eta T) + \phi\|\,\|S_1 - S\|,\end{aligned}$$

$$\|N_1(t,S)-N_1(t,S_1)\| \le H\,\|S_1(t)-S(t)\|\,,$$

where $\|\xi(I+\eta T)+\phi\| \le H < 1$.

Similarly

$$\begin{aligned}\|N_2(t,E)-N_2(t,E_1)\| &\le \|(\phi+\lambda)\,E_1-(\phi+\lambda)\,E\| \\ &\le \|(\phi+\lambda)\|\,\|E_1(t)-E(t)\|\,,\end{aligned}$$

$$\|N_2(t,E)-N_2(t,E_1)\| \le H_1\,\|E_1(t)-E(t)\|\,,$$

where $\|(\phi+\lambda)\| \le H_1 < 1$.

$$\begin{aligned}\|N_3(t,I)-N_3(t,I_1)\| &\le \|\lambda E-(\phi+\sigma_1+\alpha)I-\{\lambda E-(\phi+\sigma_1+\alpha)I\}\| \\ &\le \|(\phi+\sigma_1+\alpha)\|\,\|I_1(t)-I(t)\|\,,\end{aligned}$$

$$\|N_3(t,I)-N_3(t,I_1)\| \le H_2\,\|I_1(t)-I(t)\|\,,$$

where $\|(\phi+\sigma_1+\alpha)\| \le H_2 < 1$.

$$\begin{aligned}\|N_4(t,T)-N_4(t,T_1)\| &\le \|\alpha I-(\phi+\sigma_2+\delta)T-\{\alpha I-(\phi+\sigma_2+\delta)T_1\}\| \\ &\le \|(\phi+\sigma_2+\delta)\|\,\|T_1(t)-T(t)\|\,,\end{aligned}$$

$$\|N_4(t,T)-N_4(t,T_1)\| \le H_3\,\|T_1(t)-T(t)\|\,,$$

where $\|(\phi+\sigma_2+\delta)\| \le H_3 < 1$, and

$$\begin{aligned}\|N_5(t,R)-N_5(t,R_1)\| &\le \|\rho\delta T-\phi R-\{\rho\delta T-\phi R_1\}\| \\ &\le \|\phi\|\,\|R_1(t)-R(t)\|\,,\end{aligned}$$

$$\|N_5(t,R)-N_5(t,R_1)\| \le H_4\,\|R_1(t)-R(t)\|\,,$$

where $\|\phi\| \le H_4 < 1$.

Now, by recursive relation

$$S_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}N_1(t,S_{n-1}) + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_1(s,S_{n-1})ds, \tag{4.6}$$

$$E_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}N_2(t,E_{n-1}) + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_2(s,E_{n-1})ds, \tag{4.7}$$

$$I_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}N_3(t,I_{n-1}) + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_3(s,I_{n-1})ds, \tag{4.8}$$

$$T_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)}N_4(t,T_{n-1}) + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_4(s,T_{n-1})ds, \tag{4.9}$$

$$R_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_5(t, R_{n-1}) + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_5(s, R_{n-1}) ds. \tag{4.10}$$

Consider the difference of two successive terms is

$$\begin{aligned} U_n(t) &= S_n(t) - S_{n-1}(t) \\ &= \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_{n-1}) + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S_{n-1}) ds \\ &\quad - \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_{n-2}) - \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S_{n-2}) ds, \end{aligned}$$

$$\begin{aligned} U_n(t) &= \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_{n-1}) - \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_{n-2}) \\ &\quad + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t \{N_1(s, S_{n-1}) - N_1(s, S_{n-2})\} ds. \end{aligned}$$

Now

$$\begin{aligned} \|U_n(t)\| &= \|S_n(t) - S_{n-1}(t)\| \\ &= \left\| \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_{n-1}) - \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_{n-2}) + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t \{N_1(s, S_{n-1}) - N_1(s, S_{n-2})\} ds \right\|, \end{aligned}$$

$$\begin{aligned} \|U_n(t)\| \le \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} &\|N_1(t, S_{n-1}) - N_1(t, S_{n-2})\| \\ + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} &\left\| \int_0^t \{N_1(s, S_{n-1}) - N_1(s, S_{n-2})\} ds \right\|. \end{aligned} \tag{4.11}$$

Since N_1 satisfies the Lipschitz condition, then

$$\|U_n(t)\| \le \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} H \|S_{n-1} - S_{n-2}\| + \frac{2\gamma}{(2-\gamma)B(\gamma)} J \int_0^t \|S_{n-1} - S_{n-2}\| ds.$$

Similarly

$$\|V_n(t)\| = \|E_n(t) - E_{n-1}(t)\|,$$

or

$$\begin{aligned} \|V_n(t)\| = &\Big\| \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_2(t, E_{n-1}) + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_2(s, E_{n-1}) ds \\ &- \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_2(t, E_{n-2}) - \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_2(s, E_{n-2}) ds \Big\|, \end{aligned}$$

or

$$\begin{aligned} \|V_n(t)\| \le \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} &\|N_2(t, E_{n-1}) - N_2(t, E_{n-2})\| \\ + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} &\left\| \int_0^t \{N_2(s, E_{n-1}) - N_2(s, E_{n-2})\} ds \right\|. \end{aligned} \tag{4.12}$$

Since N_2 satisfies Lipschitz condition, then

$$\|V_n(t)\| \le \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} H_1 \|E_{n-1} - E_{n-2}\| + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_1 \int_0^t \|E_{n-1} - E_{n-2}\| ds.$$

Similarly, we can get

$$\|W_n(t)\| \le \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} H_2 \|I_{n-1} - I_{n-2}\| + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_2 \int_0^t \|I_{n-1} - I_{n-2}\| \, ds,$$

$$\|P_n(t)\| \le \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} H_3 \|T_{n-1} - T_{n-2}\| + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_3 \int_0^t \|T_{n-1} - T_{n-2}\| \, ds,$$

$$\|Q_n(t)\| \le \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} H_4 \|R_{n-1} - R_{n-2}\| + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_4 \int_0^t \|R_{n-1} - R_{n-2}\| \, ds.$$

Since we have demonstrated that the kernels meet the Lipschitz condition and that the aforementioned Eqs (4.11), (4.12), and others are bounded, the following relations may be established by applying the recursive approach to the findings obtained in Eqs (4.11), (4.12), and others:

$$\|U_n(t)\| \le \|S(0)\| + \left\{\frac{2(1-\gamma)H}{(2-\gamma)B(\gamma)}\right\}^n + \left\{\frac{2\gamma J t}{(2-\gamma)B(\gamma)}\right\}^n,$$

$$\|V_n(t)\| \le \|E(0)\| + \left\{\frac{2(1-\gamma)H_1}{(2-\gamma)B(\gamma)}\right\}^n + \left\{\frac{2\gamma J_1 t}{(2-\gamma)B(\gamma)}\right\}^n,$$

$$\|W_n(t)\| \le \|I(0)\| + \left\{\frac{2(1-\gamma)H_2}{(2-\gamma)B(\gamma)}\right\}^n + \left\{\frac{2\gamma J_2 t}{(2-\gamma)B(\gamma)}\right\}^n,$$

$$\|P_n(t)\| \le \|T(0)\| + \left\{\frac{2(1-\gamma)H_3}{(2-\gamma)B(\gamma)}\right\}^n + \left\{\frac{2\gamma J_3 t}{(2-\gamma)B(\gamma)}\right\}^n,$$

$$\|Q_n(t)\| \le \|R(0)\| + \left\{\frac{2(1-\gamma)H_4}{(2-\gamma)B(\gamma)}\right\}^n + \left\{\frac{2\gamma J_4 t}{(2-\gamma)B(\gamma)}\right\}^n.$$

Hence, the existence of results is validated, which are continuous too. So we obtain

$$\begin{aligned} S(t) &= S_n(t) + A_n(t), \\ E(t) &= E_n(t) + B_n(t), \\ I(t) &= I_n(t) + C_n(t), \\ T(t) &= T_n(t) + D_n(t), \\ R(t) &= R_n(t) + L_n(t), \end{aligned}$$

where A_n, B_n, C_n, D_n , *and* L_n are remainders of series solution. Hence,

$$S(t) - S_n(t) = \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S_n) + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S_n) ds,$$
$$S(t) - S_n(t) = \tfrac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S - A_n(t)) + \tfrac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S - A_n(s)) ds.$$

Similarly, we have

$$E(t) - E_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_2(t, E - B_n(t)) + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_2(s, E - B_n(s))ds,$$
$$I(t) - I_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_3(t, I - C_n(t)) + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_3(s, I - C_n(s))ds,$$
$$T(t) - T_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_4(t, T - D_n(t)) + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_4(s, T - D_n(s))ds,$$
$$R(t) - R_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_5(t, R - L_n(t)) + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_5(s, R - L_n(s))ds.$$

Now, it is clear that

$$S(t) - S_n(t) = \frac{2(1-\gamma)}{(2-\gamma)B(\gamma)} N_1(t, S - A_n(t)) + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S - A_n(s))ds,$$

$$S(t) - S(0) - \frac{2(1-\gamma)N_1(t,S)}{(2-\gamma)B(\gamma)} - \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S)ds$$
$$= A_n(t) + \frac{2(1-\gamma)N_1(t, S - A_n(t))}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S - A_n(s))ds.$$

Now

$$\left\| S(t) - \frac{2(1-\gamma)N_1(t,S)}{(2-\gamma)B(\gamma)} - S(0) - \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S)ds \right\|$$
$$\leq \|A_n(t)\| + \left\{ \frac{2(1-\gamma)H}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} Kt \right\} \|A_n(t)\|,$$
$$\left\| E(t) - \frac{2(1-\gamma)N_2(t,E)}{(2-\gamma)B(\gamma)} - E(0) - \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_2(s, E)ds \right\|$$
$$\leq \|B_n(t)\| + \left\{ \frac{2(1-\gamma)H_1}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_1 t \right\} \|B_n(t)\|,$$
$$\left\| I(t) - \frac{2(1-\gamma)N_3(t,I)}{(2-\gamma)B(\gamma)} - I(0) - \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_3(s, I)ds \right\|$$
$$\leq \|C_n(t)\| + \left\{ \frac{2(1-\gamma)H_2}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_2 t \right\} \|C_n(t)\|,$$
$$\left\| T(t) - \frac{2(1-\gamma)N_4(t,T)}{(2-\gamma)B(\gamma)} - T(0) - \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_4(s, T)ds \right\|$$
$$\leq \|D_n(t)\| + \left\{ \frac{2(1-\gamma)H_3}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_3 t \right\} \|D_n(t)\|,$$
$$\left\| R(t) - \frac{2(1-\gamma)N_5(t,R)}{(2-\gamma)B(\gamma)} - R(0) - \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_5(s, R)ds \right\|$$
$$\leq \|L_n(t)\| + \left\{ \frac{2(1-\gamma)H_4}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} J_4 t \right\} \|L_n(t)\|.$$

Taking $n \to \infty$ we have

$$S(t) = S(0) + \frac{2(1-\gamma)N_1(t,S)}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)} \int_0^t N_1(s, S)ds,$$

$$E(t) = E(0) + \frac{2(1-\gamma)N_2(t,E)}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_2(s,E)ds,$$

$$I(t) = I(0) + \frac{2(1-\gamma)N_3(t,I)}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_3(s,I)ds,$$

$$T(t) = T(0) + \frac{2(1-\gamma)N_4(t,T)}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_4(s,T)ds,$$

and

$$R(t) = R(0) + \frac{2(1-\gamma)N_5(t,R)}{(2-\gamma)B(\gamma)} + \frac{2\gamma}{(2-\gamma)B(\gamma)}\int_0^t N_5(s,R)ds.$$

We can assert that the system's solution exists on the basis of the aforementioned equations. In the same way, we can also show the existence of the solution for the remaining two cases when kernel changes.

5. Uniqueness of result

In this section, we prove that the results mentioned in the above section are entirely unique. For this, we suppose that there exists another set of results for the setup given by Eqs (4.1)–(4.5), say $S_1(t)$, $E_1(t)$, $I_1(t)$, $T_1(t)$, and $R_1(t)$. Then we have

$$\begin{aligned} S(t) - S_1(t) = {} & \tfrac{2(1-\gamma)}{B(\gamma)(2-\gamma)}\left[N_1(t,S) - N_1(t,S_1)\right] \\ & + \tfrac{2(\gamma)}{B(\gamma)(2-\gamma)}\int_0^t \left[N_1(s,S) - N_1(s,S_1)\right] \mathrm{ds}, \end{aligned} \tag{5.1}$$

on both sides taking the norm, we get

$$\begin{aligned} \|S - S_1\| = {} & \tfrac{2(1-\gamma)}{B(\gamma)(2-\gamma)}\left[\|N_1(t,S) - N_1(t,S_1)\|\right] \\ & + \tfrac{2(\gamma)}{B(\gamma)(2-\gamma)}\int_0^t \left[\|K_1(s,S) - K_1(s,S_1)\|\right] \mathrm{ds}, \end{aligned} \tag{5.2}$$

using Lipchitz condition, we obtain

$$\|S - S_1\| < \frac{2(1-\gamma)}{B(\gamma)(2-\gamma)}HZ + \left(\frac{2(\gamma)}{B(\gamma)(2-\gamma)}J_1Zt\right)^n, \tag{5.3}$$

which is true for all n, so

$$S = S_1, \tag{5.4}$$

similarly

$$E = E_1, I = I_1, T = T_1 \, and \, R = R_1. \tag{5.5}$$

Thus, it proves the uniqueness of the solution.

In the same way, we can also show the uniqueness of the solution for the remaining of two cases.

6. Solution of the TB model by general fractional derivative by Laplace transform

The TB model given by Ihsan et al. is:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \Delta - \xi S(I + \eta T) - \phi S,\\
\frac{dE}{dt} &= \xi S(I + \eta T) - (\phi + \lambda)E + (1 - \rho)\delta T,\\
\frac{dI}{dt} &= \lambda E - (\phi + \sigma_1 + \alpha)I,\\
\frac{dT}{dt} &= \alpha I - (\phi + \sigma_2 + \delta)T,\\
\frac{dR}{dt} &= \rho\delta T - \phi R.
\end{aligned}\right\} \tag{6.1}$$

However, we are interested in the solution by using the general operator of fractional order. So, we replace integer ordered derivatives by the general operator of order γ. Hence

$$\left.\begin{aligned}
{}^cD^\gamma S(t) &= \Delta - \xi S(I + \eta T) - \phi S,\\
{}^cD^\gamma E(t) &= \xi S(I + \eta T) - (\phi + \lambda)E + (1 - \rho)\delta T,\\
{}^cD^\gamma I(t) &= \lambda E - (\phi + \sigma_1 + \alpha)I,\\
{}^cD^\gamma T(t) &= \alpha I - (\phi + \sigma_2 + \delta)T,\\
{}^cD^\gamma R(t) &= \rho\delta T - \phi R,
\end{aligned}\right\} \tag{6.2}$$

where ${}^cD^\gamma$ represents the general operator of fractional order γ and the remaining notations are already explained in Section 3. Now applying Laplace transform to both sides of the equation of system (6.2), we obtain

$$L\{{}^cD^\gamma S(t)\} = L\{\Delta - \xi S(I + \eta T) - \phi S\}. \tag{6.3}$$

By incorporating the various kernels into the various general operator definitions, we may derive three specific cases of the general operator. In the first case, when kernel is $\nabla_\iota(t) = \frac{t^{-\xi}}{\Gamma(1-\xi)}$, we have power function $\Im_\iota(t) = \frac{t^{\xi-1}}{\Gamma(\xi)}$ transforming the integral operator's associated kernel.

In the next condition, take kernel $\nabla_\iota(t) = \frac{B(\xi)}{1-\xi}\mathrm{E}_\xi\left(\frac{-\xi}{1-\xi}t^\xi\right)$ where E_ξ and $\mathrm{M}(\xi)$ are Mittag-Leffler and normalization functions. We also have

$$\Im_\iota(t) = \frac{1-\xi}{B(\xi)}\delta(t) + \frac{\xi}{\mathrm{B}(\xi)\Gamma(\xi)}t^{\xi-1}. \tag{6.4}$$

This may be used to get the derivatives of AB-Caputo and AB-Riemann-Liouville. The AB type integral is

$${}_0\mathrm{D}_t^{-\xi} f(t) = \frac{1-\xi}{\mathrm{B}(\xi)}f(t) + \frac{\xi}{\mathrm{B}(\xi)\Gamma(\xi)}\int_0^t (t-s)^{\xi-1}f(s)ds. \tag{6.5}$$

Now, at last, the CF derivative is found by taking kernel $\nabla_\iota(t) = \frac{B(\xi)}{1-\xi}\exp\left(\frac{-\xi}{1-\xi}t\right)$. It is clear that the general fractional derivative can be further converted into three different operators, namely Riemann-Liouvilles, Atangana-Baleanu, and Caputo-Fabrizio [32–34]; also, we have already discussed their Laplace transform in section 2. Now, we derive the numerical solution of the model in each case:

6.1. Case I Caputo-Fabrizio operator

From Eq (6.3), we have

$$L\left\{{}^{CF}D^\gamma S(t)\right\} = L\{\Delta - \xi S(I + \eta T) - \phi S\}. \tag{6.6}$$

By using Eq (2.17), we obtain

$$\frac{pL\{S(t)\} - S(0)}{p + \gamma(1 - p)} = L\{\Delta - \xi S(I + \eta T) - \phi S\}, \tag{6.7}$$

or

$$pL\{S(t)\} - S(0) = \{p + \gamma(1 - p)\} L\{\Delta - \xi S(I + \eta T) - \phi S\}, \tag{6.8}$$

or

$$pL\{S(t)\} = S(0) + \{p + \gamma(1 - p)\} L\{\Delta - \xi S(I + \eta T) - \phi S\}, \tag{6.9}$$

or

$$L\{S(t)\} = \frac{S(0)}{p} + \left\{1 + \gamma\left(\frac{1}{p} - 1\right)\right\} L\{\Delta - \xi S(I + \eta T) - \phi S\}. \tag{6.10}$$

By inverse Laplace operator, we get

$$S(t) = S(0) + L^{-1}\left[\left\{1 + \gamma\left(\frac{1}{p} - 1\right)\right\} L\{\Delta - \xi S(I + \eta T) - \phi S\}\right]. \tag{6.11}$$

Using iterative technique, we obtain

$$S_{n+1}(t) = S(0) + L^{-1}\left[\left\{1 + \gamma\left(\tfrac{1}{p} - 1\right)\right\} L\{\Delta - \xi S_n(I_n + \eta T_n) - \phi S_n\}\right]. \tag{6.12}$$

In the same way, we get the remaining expressions as

$$E_{n+1}(t) = E(0) + L^{-1}\left[\tfrac{p+\gamma(1-p)}{p} L\{\xi S_n(I_n + \eta T_n) - (\phi + \lambda)E_n + (1 - \rho)\delta T_n\}\right], \tag{6.13}$$

$$I_{n+1}(t) = I(0) + L^{-1}\left[\frac{p + \gamma(1 - p)}{p} L\{\lambda E_n - (\phi + \sigma_1 + \alpha)I_n\}\right], \tag{6.14}$$

$$T_{n+1}(t) = T(0) + L^{-1}\left[\frac{p + \gamma(1 - p)}{p} L\{\alpha I_n - (\phi + \sigma_2 + \delta)T_n\}\right], \tag{6.15}$$

and

$$R_{n+1}(t) = R(0) + L^{-1}\left[\frac{p + \gamma(1 - p)}{p} L\{\rho\delta T_n - \phi R_n\}\right]. \tag{6.16}$$

6.2. Case II Riemann Liouville's operator

From Eq (6.3), we have

$$L\left\{{}^{RL}D^{\gamma} S(t)\right\} = L\{\Delta - \xi S(I + \eta T) - \phi S\}. \tag{6.17}$$

Using Eq (2.16), we have

$$p^{\gamma} L\{S(t)\} - p^{\gamma-1} S(0) = L\{\Delta - \xi S(I + \eta T) - \phi S\}, \tag{6.18}$$

or

$$p^{\gamma} L\{S(t)\} = p^{\gamma-1} S(0) + L\{\Delta - \xi S(I + \eta T) - \phi S\}, \tag{6.19}$$

or

$$L\{S(t)\} = \frac{S(0)}{p} + \frac{1}{p^{\gamma}} L\{\Delta - \xi S(I+\eta T) - \phi S\}. \tag{6.20}$$

Applying inverse Laplace transform both sides of the above equation, we get

$$S(t) = S(0) + L^{-1}\left[\frac{1}{p^{\gamma}} L\{\Delta - \xi S(I+\eta T) - \phi S\}\right]. \tag{6.21}$$

By the iterative method, we obtain

$$S_{n+1}(t) = S(0) + L^{-1}\left[\frac{1}{p^{\gamma}} L\{\Delta - \xi S_n(I_n+\eta T_n) - \phi S_n\}\right]. \tag{6.22}$$

Similarly, we can find other results

$$E_{n+1}(t) = E(0) + L^{-1}\left[\tfrac{1}{p^{\gamma}} L\{\xi S_n(I_n+\eta T_n) - (\phi+\lambda)E_n + (1-\rho)\delta T_n\}\right], \tag{6.23}$$

$$I_{n+1}(t) = I(0) + L^{-1}\left[\frac{1}{p^{\gamma}} L\{\lambda E_n - (\phi+\sigma_1+\alpha)I_n\}\right], \tag{6.24}$$

$$T_{n+1}(t) = T(0) + L^{-1}\left[\frac{1}{p^{\gamma}} L\{\alpha I_n - (\phi+\sigma_2+\delta)T_n\}\right], \tag{6.25}$$

and

$$R_{n+1}(t) = R(0) + L^{-1}\left[\frac{1}{p^{\gamma}} L\{\rho\delta T_n - \phi R_n\}\right]. \tag{6.26}$$

6.3. Case III Atangana-Baleanu operator

From Eq (6.3), we have

$$L\left\{{}^{ABC}D^{\gamma} S(t)\right\} = L\{\Delta - \xi S(I+\eta T) - \phi S\}. \tag{6.27}$$

Applying Eq (2.18), we find

$$\frac{B(\gamma)}{(1-\gamma)} . \frac{p^{\gamma} L\{S(t)\} - p^{\gamma-1} S(0)}{p^{\gamma} + \frac{\gamma}{1-\gamma}} = L\{\Delta - \xi S(I+\eta T) - \phi S\}, \tag{6.28}$$

or

$$\frac{B(\gamma)}{(1-\gamma)} . p^{\gamma} L\{S(t)\} - p^{\gamma-1} S(0) = \left(p^{\gamma} + \frac{\gamma}{1-\gamma}\right) \times L\{\Delta - \xi S(I+\eta T) - \phi S\}, \tag{6.29}$$

or

$$p^{\gamma} L\{S(t)\} - p^{\gamma-1} S(0) = \tfrac{(1-\gamma)}{B(\gamma)}\left(p^{\gamma} + \tfrac{\gamma}{1-\gamma}\right) \times L\{\Delta - \xi S(I+\eta T) - \phi S\}, \tag{6.30}$$

or

$$p^{\gamma} L\{S(t)\} = p^{\gamma-1} S(0) + \tfrac{(1-\gamma)}{B(\gamma)}\left(p^{\gamma} + \tfrac{\gamma}{1-\gamma}\right) L\{\Delta - \xi S(I+\eta T) - \phi S\}, \tag{6.31}$$

or

$$L\{S(t)\} = \frac{S(0)}{p} + \frac{(1-\gamma)}{B(\gamma)}\left(1 + \frac{\gamma p^{-\gamma}}{1-\gamma}\right) L\{\Delta - \xi S(I+\eta T) - \phi S\}. \tag{6.32}$$

By inverse Laplace transform, we get

$$S(t) = S(0) + L^{-1}\left[\frac{(1-\gamma)}{B(\gamma)}\left(1 + \frac{\gamma p^{-\gamma}}{1-\gamma}\right)L\{\Delta - \xi S(I + \eta T) - \phi S\}\right]. \tag{6.33}$$

This implies

$$S(t) = S(0) + L^{-1}\left[\frac{(1-\gamma+\gamma p^{-\gamma})}{B(\gamma)}L\{\Delta - \xi S(I + \eta T) - \phi S\}\right]. \tag{6.34}$$

By the iterative technique, we get

$$S_{n+1}(t) = S(0) + L^{-1}\left[\frac{(1-\gamma+\gamma p^{-\gamma})}{B(\gamma)}L\{\Delta - \xi S_n(I_n + \eta T_n) - \phi S_n\}\right]. \tag{6.35}$$

We can find other expressions in the same way

$$E_{n+1}(t) = E(0) + L^{-1}\left[\frac{(1-\gamma+\gamma p^{-\gamma})}{B(\gamma)}L\{\xi S_n(I_n + \eta T_n) - (\phi + \lambda)E_n + (1-\rho)\delta T_n\}\right], \tag{6.36}$$

$$I_{n+1}(t) = I(0) + L^{-1}\left[\frac{(1-\gamma+\gamma p^{-\gamma})}{B(\gamma)}L\{\lambda E_n - (\phi + \sigma_1 + \alpha)I_n\}\right], \tag{6.37}$$

$$T_{n+1}(t) = T(0) + L^{-1}\left[\frac{(1-\gamma+\gamma p^{-\gamma})}{B(\gamma)}L\{\alpha I_n - (\phi + \sigma_2 + \delta)T_n\}\right], \tag{6.38}$$

and

$$R_{n+1}(t) = R(0) + L^{-1}\left[\frac{(1-\gamma+\gamma p^{-\gamma})}{B(\gamma)}L\{\rho\delta T_n - \phi R_n\}\right]. \tag{6.39}$$

7. Numerical and graphical results

The TB infection model is numerically analyzed for the better understanding of the treatment. During the investigation, we used some primary conditions and parameter values. The details of the numeric values are explained in Table 1.

Table 1. Initial values and parameters.

S.N.	Variable	Symbol	Value
1	Primary no. of vulnerable people	S_0	7
2	Starting exposed persons	E_0	2
3	Initial no. of infections without treatment	I_0	1
4	Initial no. of infections with treatment	T_0	0
5	Initially recovered persons	R_0	0
6	Rate of conscription	Δ	0.2
7	Rate of effective contact	ξ	0.7
8	Decrease in infections	η	0.1
9	Rate of natural death	ϕ	0.1
10	Rate of transfer from E to I	λ	0.25
11	Capable treatment	ρ	0.9
12	Infected person without treatment	σ_1	0.15
13	Infected person with treatment	σ_2	0.05
14	Rate for infected population	α	0.2
15	Rate of leaving population T	δ	0.1

We can plot the graphs using the values of the parameters listed in Table 1 and the mathematical results found in the previous section. Figures 1–15 have been plotted for γ= 0.5, 0.7, 0.9, and 1.

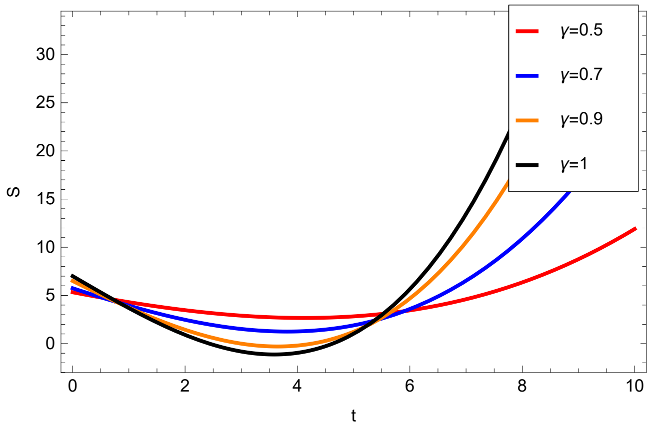

Figure 1. Graph of Susceptible with respect to t, for γ= 0.5, 0.7, 0.9, and 1 in the Caputo-Fabrizio case.

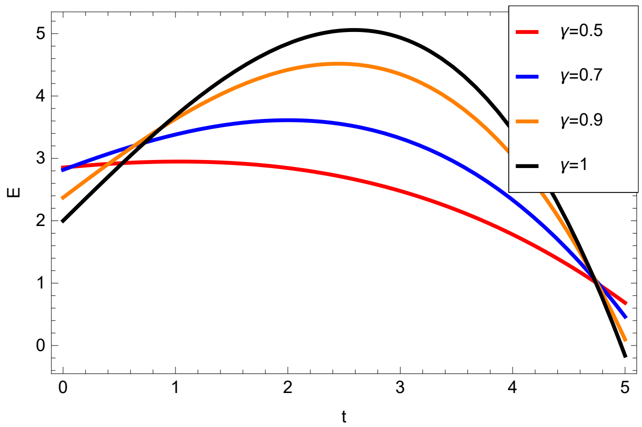

Figure 2. Exposed w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Caputo-Fabrizio case.

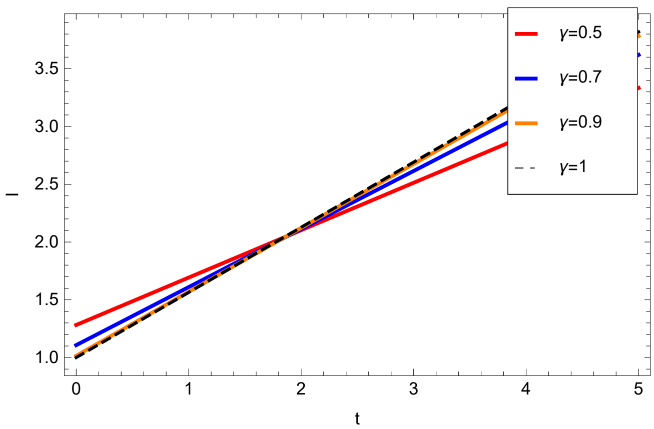

Figure 3. Infected without treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Caputo-Fabrizio case.

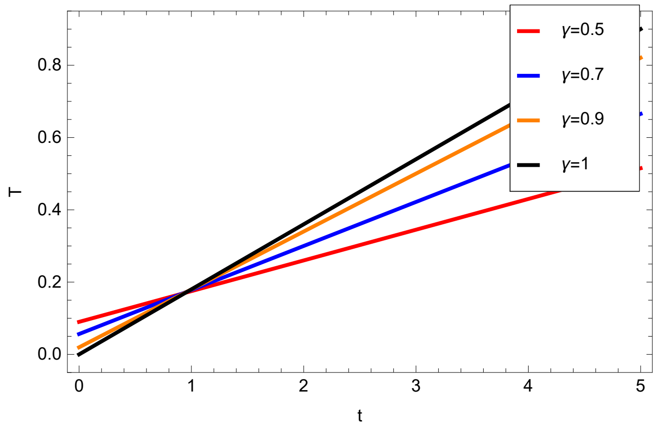

Figure 4. Infected with treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Caputo-Fabrizio case.

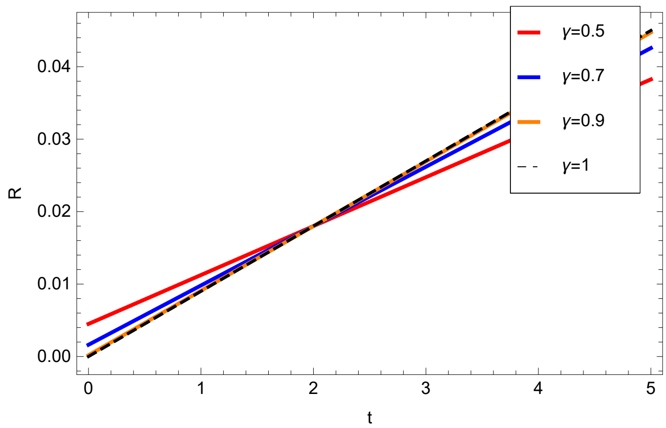

Figure 5. Recovered with treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Caputo-Fabrizio case.

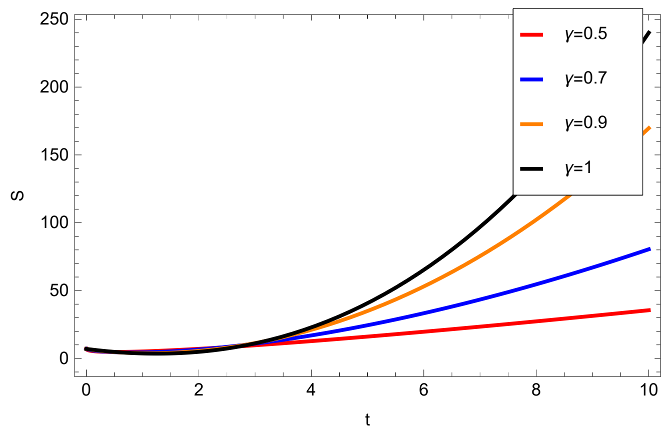

Figure 6. Susceptible w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Riemann-Liouville's case.

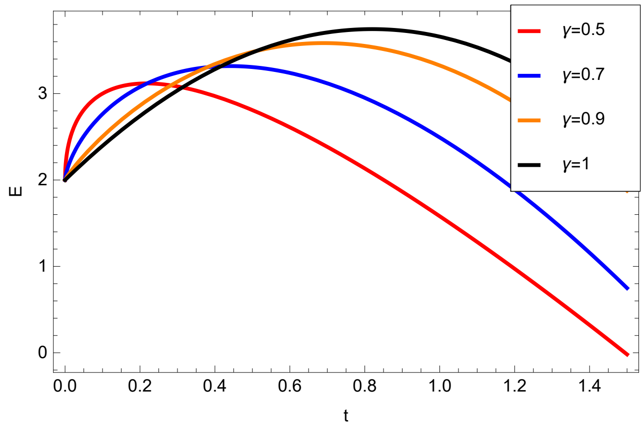

Figure 7. Exposed w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Riemann-Liouville's case.

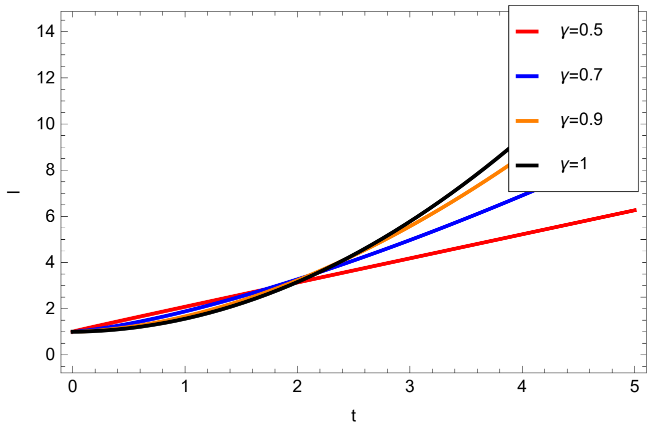

Figure 8. Infected without treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Riemann-Liouville's case.

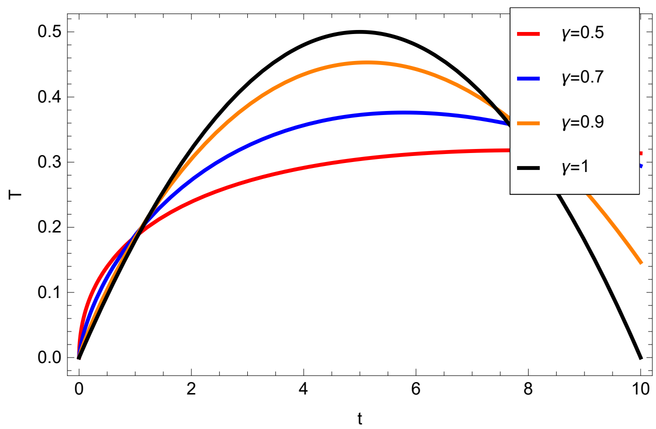

Figure 9. Infected with treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Riemann-Liouville's case.

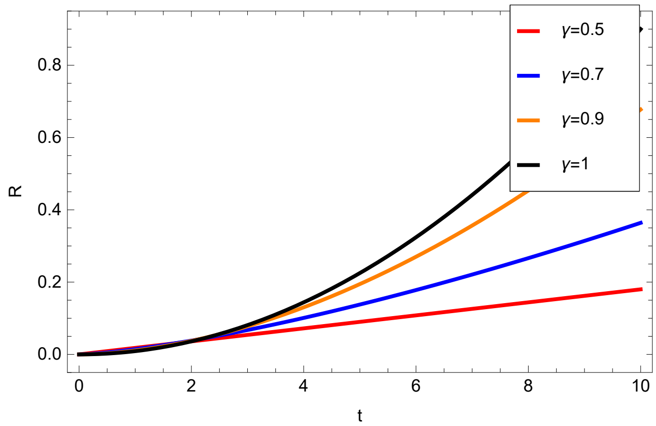

Figure 10. Recovered with treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the Riemann-Liouville's case.

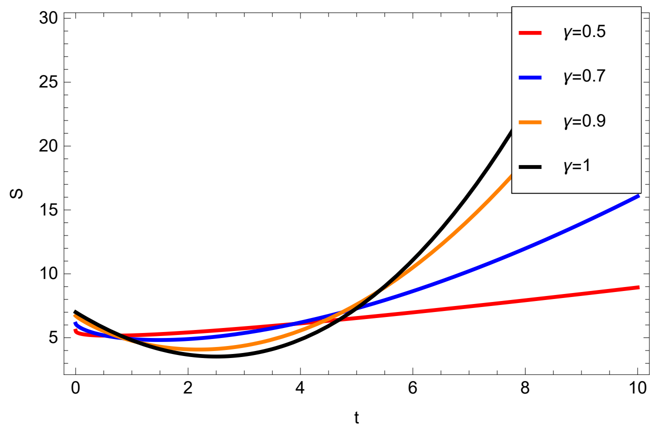

Figure 11. Susceptible w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the ABC case.

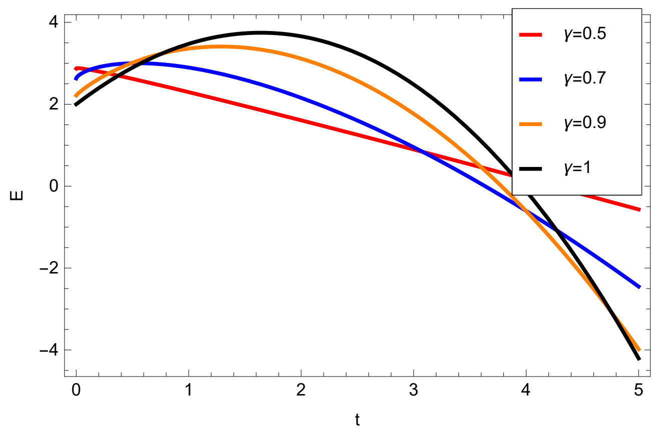

Figure 12. Exposed w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the ABC case.

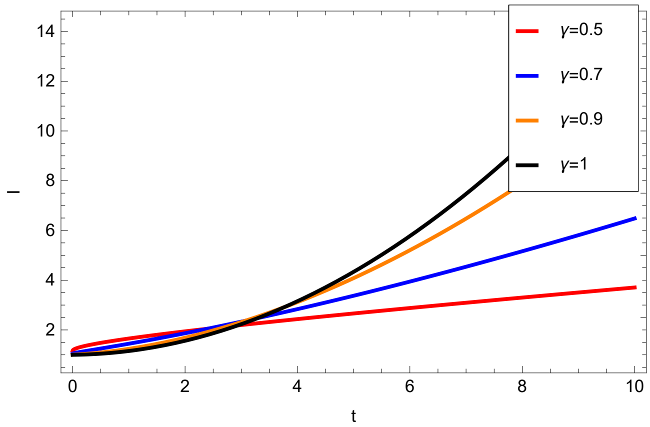

Figure 13. Infected without treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the ABC case.

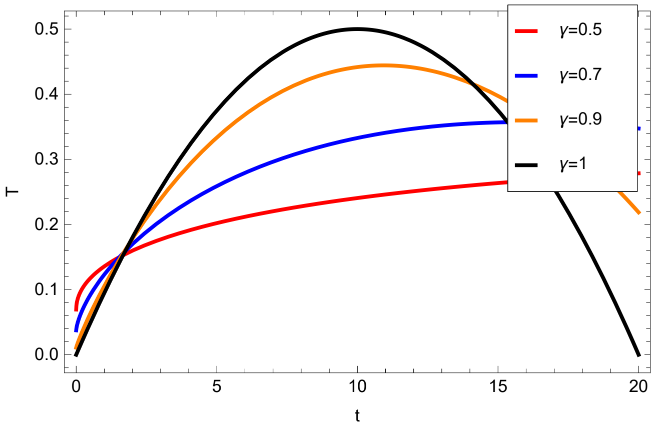

Figure 14. Infected with treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the ABC case.

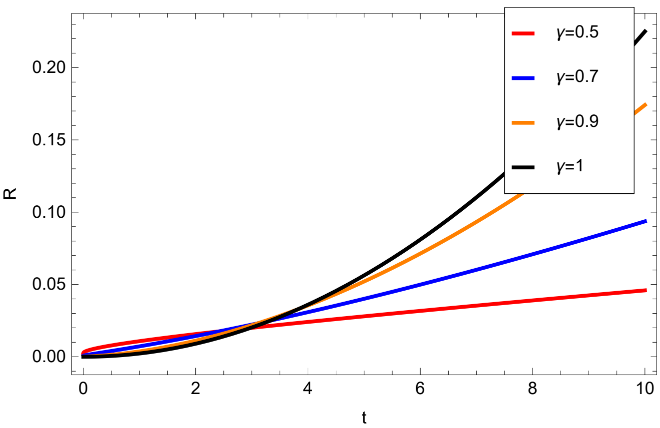

Figure 15. Recovered with treatment w.r.to t, for γ= 0.5, 0.7, 0.9, and 1 in the ABC case.

Figures 1, 6, and 11 show a rapid increase in number of susceptible people with time when the order of derivative is increases. Figures 2, 7, and 12 show a downward trend in the number of exposed people with time, particularly when approaching integer order. Figures 3, 8, and 13 show that infection will increase with time if treatment is not done properly, while Figures 4, 9, and 14 show a positive response of treatment with time, as the infection rate decreases when treatment is appropriate. From Figures 5, 10, and 15, we can see that recovery is in process; however, the recovery rate is slower than the rate of infection without treatment.

From all figures, it is clear that Caputo-Fabrizio and Atangana-Baleanu derivatives show similar tendencies with nearly equal changing rates, as opposed to Riemann-Liouville's derivatives.

From this study, we can also conclude that the fractional-order system performs better than the integer-order model in terms of understanding and predicting the phenomenon. In the fractional-order model, there was a larger number of outcomes as compared to the integer-order model, which provided only single outcome.

8. Conclusions

In this study, a fractional-order model of TB was analyzed under the general fractional derivative using Laplace transform technique with effective contact rate, treatment rate, and incomplete treatment versus efficient treatment. The existence and uniqueness of some results were investigated using the fixed-point technique. The general derivative approach was used to create some numerical findings for the suggested model. For each instance of a general derivative and for various fractional orders, the numerical findings were visually explained (see Figures 1–15). By giving the information between two numbers, the graphical depiction demonstrates that fractional-order analysis of mathematical models is substantially more accurate than integer-order analysis.

Author contributions

Manvendra Narayan Mishra: Conceptualization, supervision and writing original draft; Faten Aldosari: Data curation and software. Both the authors polished the final draft.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

No conflicts of interest to disclose about the article that is being presented.

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