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*Research article*

# Bayesian estimation and prediction for linear exponential models using ordered moving extremes ranked set sampling in medical data

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**Abstract:** Our study aimed to compare ordered ranked set sampling with moving extremes ranked set sampling in the context of type II censoring. We focused on deriving Bayesian estimations and predictions using the linear exponential model. This analysis included various loss functions, such as squared error, Al-Bayyati, and general entropy. To evaluate the efficiency of the estimators we produced, we assessed their mean squared error and relative absolute bias. Additionally, we provide Bayesian point and interval predictions for the ordered future lifetime, considering both squared error and general entropy loss functions. To ensure the accuracy and effectiveness of these estimation and prediction methods, we conducted numerical tests using Monte Carlo simulations. Finally, we illustrated these theoretical concepts with a practical example that utilized real-world medical data.

**Keywords:** ordered moving extremes ranked set sampling; Bayes estimation; squared error loss function; Al-Bayyati loss function; general entropy loss function; Bayesian prediction
**Mathematics Subject Classification:** 62E15, 62F25, 62G30, 62G32, 62M20

## 1. Introduction

Cost-effective sampling is a vital consideration in certain experiments, particularly when measuring a characteristic of interest that is expensive or time-consuming. Ranked set sampling (RSS) offers an efficient strategy to achieve observational economy by enhancing precision per sampling unit. Initially introduced by McIntyre [30], RSS was designed to improve the accuracy of the sample mean as an estimator of the population mean. Chen et al. [17] explored the theory and applications of RSS, while other studies by Chen et al. [18, 19] identified various parametric estimations using RSS. Ali et al. [6] applied different inference methods for linear exponential distribution based on extreme RSS. Mohie El-Din et al. [22–25] conducted Bayesian estimation and prediction based on ordered RSS under type

II censored samples, demonstrating that the estimations and predictions derived from ordered RSS (ORSS) outperform those based on simple random samples (SRS). Recently, Bhushan et al. [11, 12] investigated modified classes of estimators using RSS, providing valuable insights into the development of new estimation techniques. Additionally, Newer et al. [31] utilized various applications of RSS to derive inferences for the parameters of the Nadarajah-Haghighi distribution. However, the effectiveness of an estimator relies heavily on the accuracy of the ranking process. To address this issue, Al-Odat and Al-Saleh [3] introduced moving extremes ranked set sampling (MERSS), which aims to minimize ranking errors while preserving the optimality of the original ranked set method. The MERSS scheme can be outlined as follows:

1 Select $m_1$ SRSs of sizes 1, 2, ..., $m_1$.
2 Rank the elements of each sample through visual inspection or another cost-effective method, without directly measuring the characteristic of interest.
3 Accurately measure the maximum ordered observation from each set: the first set, the second set, the third set, and so forth.
4 Repeat steps 1–3 for an additional $m_2$ sets of sizes 1, 2, ..., $m_2$, this time measuring the minimum ordered observations instead of the maximum ones.

The total sample size for MERSS is $n = m_1 + m_2$. The measured MERSS units are denoted as $\omega_{MERSS} = (\omega_{1:1}, \omega_{2:2}, \ldots, \omega_{m_1:m_1}, \omega_{1:1}, \omega_{1:2}, \ldots, \omega_{1:m_2})$. Balakrishnan and Li [9] introduced the concept of ORSS by utilizing order statistics derived from independent and non-identically distributed (IND) random variables. They established optimal linear inference based on ORSS, where the values $\omega_{i:i}$ are arranged in increasing order. For convenience, we denote the first $r$ observations from type II censored ordered MERSS (OMERSS) as $\mathbf{z}_{OMERSS} = (z_1 \leq z_2 \leq \ldots \leq z_r)$. The literature is abundant with studies focused on parameter estimation for various distributions using MERSS. For instance, Abu-Dayyeh and Al Sawi [1] examined the modified maximum likelihood estimator (MLE) of the mean for the exponential distribution under MERSS. Similarly, Al-Omari and Al-Hadhrami [2] investigated the properties of Bayesian estimators for the population mean of the normal distribution using MERSS. Furthermore, the authors in [4, 5] analyzed the MLE of the mean for both exponential and normal distributions under MERSS, demonstrating that the MLEs derived from MERSS consistently outperformed those obtained through SRS in quantitative terms. For a more thorough understanding of MERSS, refer to the works of [17, 18].

In applied statistics and reliability analysis, linear exponential distribution (LED) finds numerous applications. Broadbent [13] and Carbone et al. [15] utilized LED to examine survival patterns of patients with plasmacytic myeloma. Also known as the linear failure rate distribution, LED is well-regarded for modeling lifetime data in both reliability and medical studies, particularly effective in representing processes characterized by an increasing linear failure rate. In survival analysis, models featuring bathtub-shaped failure rates are highly sought after. However, LED falls short in providing an appropriate parametric fit for phenomena exhibiting decreasing, nonlinearly increasing, or non-monotonic failure rates—characteristics commonly observed in firmware reliability modeling and biological studies. For further insights into these topics, refer to the works of Lai and Xie [27], Lai et al. [28] and Zhang et al. [32]. The two-scale parameter LED, denoted as $(\alpha, \lambda)$, is defined by the following probability density function (pdf) and cumulative distribution function (cdf):

$$f(y) = (\alpha + \lambda y) \exp\left(-\alpha y - \frac{\lambda}{2} y^2\right), \quad y \geq 0, \ (\alpha, \lambda \geq 0), \tag{1.1}$$

$$F(y) = 1 - \exp\left(-\alpha y - \frac{\lambda}{2}y^2\right), \quad y \geq 0, \ (\alpha, \lambda \geq 0). \tag{1.2}$$

Notably, LED represents the distribution of the minimum of two independent random variables, $v$ and $w$, where $v$ follows an exponential distribution with parameter $\alpha$, and $w$ follows a Rayleigh distribution with parameter $\lambda$. Consequently, LED encompasses both exponential and Rayleigh distributions as special cases:
- When $\lambda = 0$, LED simplifies to an exponential distribution.
- When $\alpha = 0$, it reduces to a Rayleigh distribution.

Testing how products fail under normal operating conditions is essential for reliability analysis. However, thorough testing can be extremely costly, especially for products known for their high reliability. When testing becomes too expensive and the product shows a strong level of reliability, it may not be practical to use complete samples for statistical analysis. To address these challenges of time and cost, statisticians have introduced various censoring methods. In life or quality tests, a random sample $y_1, y_2, \ldots, y_n$ is taken from a distribution with cdf $F(y)$. Instead of waiting for all $n$ samples to fail, the experiment is stopped after the $r$th failure, where $1 < r < n$. The order statistics of the data can be arranged as $y_{1:n} \leq y_{2:n} \leq \cdots \leq y_{r:n}$. This approach is known as type II censored data, where only the smallest observed values are recorded. Type II censoring is particularly beneficial when waiting for all $n$ individuals to fail would take a long time, helping to save both time and resources. The number of censored samples is decided before the experiment starts. For more in-depth discussions, you can refer to sources [16, 29, 30] and other cited works within those references.

In many real-world scenarios, the ability to predict future observations is crucial. This is especially important in industrial applications and survival studies, where predicting ordered random variables helps assess the likelihood of producing defective items. In lifetime testing, both interval and point predictions are vital for identifying the most effective censoring methods. By analyzing the number of failed items during a life-test experiment, we can estimate the failure times of the remaining objects. Additionally, these predictions can guide decisions on whether to accelerate the life test and when the failure of a future item might signal the end of the test. The statistical literature shows that prediction problems have garnered significant interest from researchers in both theoretical and practical areas, as highlighted in the works of Mohie El-Din et al. [22, 24, 25].

Our study aims to compare ORSS with OMERSS under a type II censoring scheme to derive Bayesian estimations for the parameters of the linear exponential model. We will also develop Bayesian point and interval predictions for the ordered future lifetime, taking into account both symmetric and asymmetric loss functions. The estimation and prediction results obtained from ORSS will be compared with those derived from OMERSS. To illustrate these theoretical concepts, we will use a specific example from the medical field that incorporates real-life data. The structure of this paper is organized as follows: Section 2 presents essential preliminary results relevant to our study. Section 3 details the derivation of Bayesian estimators for the parameters under various loss functions. Section 4 focuses on establishing Bayesian prediction bounds for LED using OMERSS. Section 5 includes a Monte Carlo simulation and a discussion to evaluate the accuracy of the estimation and prediction methods, along with an application to a real dataset. Section 6 concludes the study.

## 2. Preliminary results

Here, let $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_r)$, where $z_1 \leq z_2 \leq \ldots \leq z_r$, represent the informative type II censored sample consisting of the first $r$ observations (out of $n \geq r$) from OMERSS. Utilizing results for order statistics from IND random variables (refer to Balakrishnan [8]), the joint density function (JDF) of $\mathbf{z}$ can be expressed as:

$$\zeta(\vartheta|\mathbf{z}) = \frac{1}{\iota!}\sum_{\mathbf{p}}\left(\prod_{\kappa=1}^{r} g_{i_\kappa:j_\kappa}(z_\kappa)\prod_{\kappa=r+1}^{n}\left[1 - G_{i_\kappa:j_\kappa}(z_r)\right]\right), \tag{2.1}$$

where $\iota = n - r$ and $\sum_{\mathbf{p}}$ signifies the summation over all $n! = (m_1 + m_2)!$ permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$. It is clear that the JDF can be expressed as

$$\zeta(\vartheta|\mathbf{z}) = \frac{1}{\iota!}\ \mathrm{Per}(\nu_r), \tag{2.2}$$

where $\mathrm{Per}(\nu) = \sum_{\mathbf{p}}\prod_{j=1}^{n} c_{j,i_j}$ represents the permanent of a real matrix $\nu = (c_{i,j})$ of size $n \times n$,

$$\nu_r = \begin{pmatrix} g_{1:1}(z_1) & \ldots & g_{m_1:m_1}(z_1) & g_{1:1}(z_1) & \ldots & g_{1:m_2}(z_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{1:1}(z_r) & \ldots & g_{m_1:m_1}(z_r) & g_{1:1}(z_r) & \ldots & g_{1:m_2}(z_r) \\ 1 - G_{1:1}(z_r) & \ldots & 1 - G_{m_1:m_1}(z_r) & 1 - G_{1:1}(z_r) & \ldots & 1 - G_{1:m_2}(z_r) \end{pmatrix} \}(n-r) \text{ rows.} \tag{2.3}$$

Assuming that $Z_i$ follows the same distribution as the $i$th order statistic $\omega_{i:j}$, where $i, j = 1, 2, \ldots, m_\varsigma$ and $\varsigma = 1, 2$ with $i \leq j$ for for an SRS of size $m_\varsigma$, the pdf and cdf of $Z_j$ are given by (see [7, 20]):

$$\begin{aligned} g_{i:j}(z) &= \sum_{k_1=0}^{i-1} a_{k_1,i}(j)(1 - F(z))^{j+k_1-i} f(z) \\ &= (\alpha + \lambda z)\sum_{k_1=0}^{i-1} a_{k_1,i}(j)\exp\left(-\alpha(j + k_1 - i + 1)z - \lambda(j + k_1 - i + 1)\frac{z^2}{2}\right), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} G_{i:j}(z) &= 1 - \sum_{k_2=1}^{i}\widetilde{a}_{k_2,i}(j)\,(1 - F(z))^{j+k_2-i} \\ &= 1 - \sum_{k_2=1}^{i}\widetilde{a}_{k_2,i}(j)\exp\left(-\alpha(j + k_2 - i)z - \lambda(j + k_2 - i)\frac{z^2}{2}\right), \end{aligned} \tag{2.5}$$

respectively, where

$$a_{k_1,i}(j) = (-1)^{k_1}\binom{i-1}{k_1}a_{i,j}, \quad a_{i,j} = i\binom{j}{i}, \quad \text{and} \quad \widetilde{a}_{k_2,i}(j) = \frac{a_{k_2-1,i}(j)}{(j + k_2 - i)}. \tag{2.6}$$

By substituting the expressions from (2.4) and (2.5) into (2.1), it follows that the likelihood function (LF) for type II censored OMERSS can be represented as

$$\zeta(\vartheta|\mathbf{z}) \propto \sum_{\mathbf{p}}\left\{\prod_{\kappa=1}^{r}(\alpha + \lambda z_\kappa)\sum_{k_1=0}^{i_\kappa-1} a_{k_1,i_\kappa}(j_\kappa)\exp\left[-\alpha(j_\kappa + k_1 - i_\kappa + 1)z_\kappa - \lambda(j_\kappa + k_1 - i_\kappa + 1)\frac{z_\kappa^2}{2}\right]\right.$$

$$\times \prod_{\kappa=r+1}^{n} \sum_{k_2=1}^{i_\kappa} \widetilde{a}_{k_2,i_\kappa}(j_\kappa) \exp\left[-\alpha(j_\kappa + k_2 - i_\kappa)z_\kappa - \lambda(j_\kappa + k_2 - i_\kappa)\frac{z_\kappa^2}{2}\right]\Bigg\}. \tag{2.7}$$

By utilizing the following relations:

$$\prod_{\kappa=1}^{r} \sum_{k_1=0}^{i_\kappa-1} \triangle_{k_1}(i_\kappa) = \sum_{t_1=0}^{i_1-1}\sum_{t_2=0}^{i_2-1}\cdots\sum_{t_r=0}^{i_r-1}\prod_{\kappa=1}^{r} \triangle_{t_\kappa}(i_\kappa), \tag{2.8}$$

$$\prod_{\kappa=r+1}^{n} \sum_{k_2=1}^{i_\kappa} \nabla_{k_2}(i_\kappa) = \sum_{\nu_{r+1}=1}^{i_{r+1}}\sum_{\nu_{r+2}=1}^{i_{r+2}}\cdots\sum_{\nu_n=1}^{i_n}\prod_{\kappa=r+1}^{n} \nabla_{\nu_\kappa}(i_\kappa), \tag{2.9}$$

where $i_\kappa$ $(\kappa = 1, 2, \ldots, n)$ are positive integers satisfying $i_1 < i_2 < \ldots < i_n$, we can directly express the LF as

$$\zeta(\alpha, \lambda|\mathbf{z}) \propto \sum_{\mathbf{p}} \sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n} \Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})\alpha^{r-\varepsilon}\beta^{\varepsilon} \exp\left[-\alpha\Psi_{\underline{t},\underline{\nu}}(\mathbf{z}) - \lambda\Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right], \tag{2.10}$$

where $\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n} = \sum_{t_1=0}^{i_1-1}\sum_{t_2=0}^{i_2-1}\cdots\sum_{t_r=0}^{i_r-1}\cdot\sum_{\nu_{r+1}=1}^{i_{r+1}}\sum_{\nu_{r+2}=1}^{i_{r+2}}\cdots\sum_{\nu_n=1}^{i_n}\sum_{\varepsilon=0}^{r}$, $\underline{t} = (t_1, ..., t_r)$, $\underline{\nu} = (\nu_{r+1}, ..., \nu_n)$,

$$\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i}) = \left(\prod_{\kappa=1}^{r} a_{t_\kappa,i_\kappa}(j_\kappa)\right)\left(\prod_{\kappa=r+1}^{n} \widetilde{a}_{\nu_\kappa,i_\kappa}(j_\kappa)\right)\left(\sum_{J_1=1}^{r-\varepsilon+1} z_{J_1} \sum_{J_2=J_1+1}^{r-\varepsilon+2} z_{J_2} \cdots \sum_{J_\varepsilon=J_{\varepsilon-1}+1}^{r} z_{J_\varepsilon}\right),$$

$$\Psi_{\underline{t},\underline{\nu}}(\mathbf{z}) = \sum_{\kappa=1}^{r}(j_\kappa + t_\kappa - i_\kappa + 1)z_\kappa + \sum_{\kappa=r+1}^{n}(j_\kappa + \nu_\kappa - i_\kappa)z_r. \tag{2.11}$$

## 3. Bayes estimation

In this section, we illustrate how to derive the Bayesian estimators for the unknown parameters $\alpha$ and $\lambda$. We will explore Bayesian estimation methods under the assumption that the random variable $(\mathfrak{I}, \aleph)$ follows an exponential prior distribution. Using the realizations $(\alpha, \lambda)$, we will estimate the posterior density of $(\alpha, \lambda)$ based on the observed sample $\mathbf{z}$. The prior distribution can be determined as follows:

$$\hbar(\alpha, \lambda) = \eta\rho \exp(-\alpha\eta - \lambda\rho), \quad \alpha, \lambda > 0, \ (\eta, \rho > 0). \tag{3.1}$$

The hyperparameters $(\eta, \rho)$ were selected to reflect our prior knowledge of $(\alpha, \lambda)$. It is important to highlight that we chose an exponential family prior due to its flexibility and simplicity, which allows it to capture a wide range of the experimenter's prior beliefs. For more details, see [10]. The posterior density of $(\alpha, \lambda)$ can be expressed as follows, utilizing the likelihood function from Eq (2.10) and the prior density from Eq (3.1):

$$\mathfrak{R}(\alpha, \lambda|\mathbf{z}) = \frac{1}{\Bbbk(\eta, \rho)}\sum_{\mathbf{p}} \sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n} \Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})\alpha^{r-\varepsilon}\beta^{\varepsilon} \exp\left[-\alpha\left(\eta + \Psi_{\underline{t},\underline{\nu}}(\mathbf{z})\right) - \lambda\left(\rho + \Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)\right], \tag{3.2}$$

where $\Bbbk(\eta, \rho)$ is the normalization constant,

$$\Bbbk(\eta, \rho) = \sum_{\mathbf{p}} \sum_{\underline{t}, \underline{\nu}, \varepsilon}^{r,n} \frac{\Omega_{\underline{t}, \underline{\nu}, \varepsilon}(\underline{i}) \Gamma(r - \varepsilon + 1) \Gamma(\varepsilon + 1)}{\left(\eta + \Psi_{\underline{t}, \underline{\nu}}(\mathbf{z})\right)^{r-\varepsilon+1} \left(\rho + \Psi_{\underline{t}, \underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+1}}.$$

Choosing an appropriate loss function is crucial in Bayesian analysis. To comprehensively evaluate Bayesian estimates, we consider several types of loss functions: squared error (SR), linear exponential (LINEX), Al-Bayyati (AB), and general entropy (GE). The definitions of the SR, AB, and GE loss functions for the model parameter $\Theta$ are as follows:

$$\ell_{SR}(\Theta, \tilde{\Theta}) = \left(\Theta - \tilde{\Theta}\right)^2,$$

$$\ell_{LINEX}(\Theta, \tilde{\Theta}) = e^{c(\tilde{\Theta} - \Theta)} - c(\tilde{\Theta} - \Theta) - 1, \quad c \neq 0,$$

$$\ell_{AB}(\Theta, \tilde{\Theta}) = \Theta^a \left(\Theta - \tilde{\Theta}\right)^2, \quad a \in \mathbb{R},$$

$$\ell_{GE}(\Theta, \tilde{\Theta}) = \left(\frac{\tilde{\Theta}}{\Theta}\right)^a - a \ln\left(\frac{\tilde{\Theta}}{\Theta}\right) - 1,$$

where $\tilde{\Theta}$ is the estimate of $\Theta$, and $a$ in the AB and GE loss functions indicates the degree of asymmetry. In Bayesian analysis, defining an appropriate loss function is essential for determining the optimal estimate of an unknown parameter. Numerous loss functions have been developed in the literature to accommodate various loss structures. For further information, see Calabria and Pulcini [14] and EL-Sagheer et al. [21]. When $a > 0$, a positive error (overestimation) ($\tilde{\Theta} > \Theta$) results in more significant consequences than a negative error (underestimation), leading to a highly skewed asymmetric loss function. Conversely, if $a < 0$, the opposite is true. When $a$ is close to zero, the estimates derived from the asymmetric loss function and the squared error (SR) loss function tend to be quite similar. As a result, asymmetric loss functions are particularly suitable for lifetime modeling. For instance, overestimating the survival function and failure rate function is often more detrimental than underestimating them. The Bayesian estimate of the model parameter $T = T(\Theta)$, based on the SR, AB, LINEX, and GE loss functions, can generally be expressed as follows:

$$\tilde{T}_{SR} = E(T(\Theta)|\mathbf{z}) = \int_{\Theta} T(\Theta) \mathfrak{R}(\Theta|\mathbf{z}) d\Theta,$$

where $\mathfrak{R}(\Theta|\mathbf{z})$ is the posterior distribution of $\Theta$ given the observed data $\mathbf{z}$,

$$\tilde{T}_{AB} = \frac{E((T(\Theta))^{a+1}|\mathbf{z})}{E((T(\Theta))^a|\mathbf{z})} = \frac{\int_{\Theta} (T(\Theta))^{a+1} \mathfrak{R}(\Theta|\mathbf{z}) d\Theta}{\int_{\Theta} (T(\Theta))^a \mathfrak{R}(\Theta|\mathbf{z}) d\Theta},$$

where $a$ is the asymmetry parameter for the AB loss function,

$$\tilde{T}_{LINEX} = \frac{-1}{c} \log\left\{E\left[\exp(-cT(\Theta))|\mathbf{z}\right]\right\} = \frac{-1}{c} \log\left\{\int_{\Theta} \exp[-cT(\Theta)] \mathfrak{R}(\Theta|\mathbf{z}) d\Theta\right\},$$

where $c$ is the asymmetry parameter for the LINEX loss function,

$$\tilde{T}_{GE} = \left(E\left((T(\Theta))^{-a}|\mathbf{z}\right)\right)^{-1/a} = \left(\int_{\Theta}(T(\Theta))^{-a}\mathfrak{R}(\Theta|\mathbf{z})d\Theta\right)^{-1/a},$$

where $a$ is the asymmetry parameter for the GE loss function. These expressions represent the Bayesian estimates under various loss functions, taking into account the posterior distribution $\mathfrak{R}(\Theta|\mathbf{z})$ of the unknown parameter $\Theta$ based on the observed data.

Under the SR loss function, the Bayes estimators of the unknown parameters $(\alpha, \lambda)$ are given by their respective posterior means. Specifically:

$$\tilde{\alpha}_{SR} = \frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\frac{\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})\Gamma(r-\varepsilon+2)\Gamma(\varepsilon+1)}{\left(\eta+\Psi_{\underline{t},\underline{\nu}}(\mathbf{z})\right)^{r-\varepsilon+2}\left(\rho+\Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+1}}, \tag{3.3}$$

$$\tilde{\lambda}_{SR} = \frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\frac{\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})\Gamma(r-\varepsilon+1)\Gamma(\varepsilon+2)}{\left(\eta+\Psi_{\underline{t},\underline{\nu}}(\mathbf{z})\right)^{r-\varepsilon+1}\left(\rho+\Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+2}}. \tag{3.4}$$

Under the LINEX, AB, and GE loss functions, the Bayes estimators of the parameters $(\alpha, \lambda)$ are as follows:

$$\tilde{\alpha}_{LINEX} = \frac{-1}{c}\log\left[\frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\frac{\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})\Gamma(r-\varepsilon+1)\Gamma(\varepsilon+1)}{\left(c+\eta+\Psi_{\underline{t},\underline{\nu}}(\mathbf{z})\right)^{r-\varepsilon+1}\left(\rho+\Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+1}}\right], \tag{3.5}$$

$$\tilde{\lambda}_{LINEX} = \frac{-1}{c}\log\left[\frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\frac{\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})\Gamma(r-\varepsilon+1)\Gamma(\varepsilon+1)}{\left(\eta+\Psi_{\underline{t},\underline{\nu}}(\mathbf{z})\right)^{r-\varepsilon+1}\left(c+\rho+\Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+1}}\right], \tag{3.6}$$

$$\tilde{\alpha}_{AB} = \frac{\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})Q_{\underline{t},\underline{\nu},\varepsilon}(a+1)}{\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})Q_{\underline{t},\underline{\nu},\varepsilon}(a)}, \tag{3.7}$$

$$\tilde{\lambda}_{AB} = \frac{\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})H_{\underline{t},\underline{\nu},\varepsilon}(a+1)}{\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon}^{r,n}\Omega_{\underline{t},\underline{\nu},\varepsilon}(\underline{i})H_{\underline{t},\underline{\nu},\varepsilon}(a)}, \tag{3.8}$$

where

$$Q_{\underline{t},\underline{\nu},\varepsilon}(a) = \frac{\Gamma(r-\varepsilon+a+1)\Gamma(\varepsilon+1)}{\left(\eta+\Psi_{\underline{t},\underline{\nu}}(\mathbf{z})\right)^{r-\varepsilon+a+1}\left(\rho+\Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+1}},$$

$$H_{\underline{t},\underline{v},\varepsilon}(a) = \frac{\Gamma(r-\varepsilon+1)\,\Gamma(\varepsilon+a+1)}{\Big(\eta+\Psi_{\underline{t},\underline{v}}(\mathbf{z})\Big)^{r-\varepsilon+1}\left(\rho+\Psi_{\underline{t},\underline{v}}\left(\frac{\mathbf{z}^2}{2}\right)\right)^{\varepsilon+a+1}},$$

$$\tilde{\alpha}_{GE} = \left(\frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{v},\varepsilon}^{r,n}\Omega_{\underline{t},\underline{v},\varepsilon}(\underline{i})Q_{\underline{t},\underline{v},\varepsilon}(-a)\right)^{-1/a}, \tag{3.9}$$

$$\tilde{\lambda}_{GE} = \left(\frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{v},\varepsilon}^{r,n}\Omega_{\underline{t},\underline{v},\varepsilon}(\underline{i})H_{\underline{t},\underline{v},\varepsilon}(-a)\right)^{-1/a}. \tag{3.10}$$

## 4. Bayes prediction

Based on the observed OMERSS in the two sample cases, we present the Bayesian prediction distribution for future order statistics.

### 4.1. Bayesian prediction bounds

Let $y_s$, for $s = 1, \ldots, r$, represent a future independent type II censored sample from the same population. Assume that $y_1 \le y_2 \le \ldots \le y_r$ constitutes a type II censored sample of size $r$ obtained from a life test involving $n$ items. Our objective is to develop a method for providing a Bayesian prediction for the $s$th ordered lifetime $y_s$ of a future sample of size $r$, based on the observed sample $\mathbf{z}$ from the same population. The pdf of $y_s$ is expressed as follows:

$$\phi_{s:n}(y_s|\alpha,\lambda) = \sum_{\iota=0}^{s-1}\sum_{\varsigma=0}^{1} a_{\iota,s}(n)\alpha^{1-\varsigma}\lambda^{\varsigma}y_s^{\varsigma}\exp\left(-\alpha(n+\iota-s+1)y_s-\lambda(n+\iota-s+1)\frac{y_s^2}{2}\right). \tag{4.1}$$

According to the definition of the Bayes predictive pdf of $y_s$,

$$\phi^*_{s:n}(y_s|\mathbf{z}) = \int_{\Theta}\phi_{s:n}(y_s|\Theta)\Re(\Theta|\mathbf{z})d\Theta. \tag{4.2}$$

In light of this, along with (4.1) and (3.2), we have

$$\phi^*_{s:n}(y_s|\mathbf{z}) = \frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{v},\varepsilon,\iota,\varsigma}^{r,n}\frac{\Omega^*_{\underline{t},\underline{v},\varepsilon,\iota,\varsigma}(\underline{i},n)y_s^{\varsigma}}{\Big(\eta+\Psi_{\underline{t},\underline{v}}(\mathbf{z})+\varrho y_s\Big)^{r-\varepsilon-\varsigma+2}\left(\rho+\Psi_{\underline{t},\underline{v}}\left(\frac{\mathbf{z}^2}{2}\right)+\varrho\frac{y_s^2}{2}\right)^{\varepsilon+\varsigma+1}}, \tag{4.3}$$

where $\sum_{\underline{t},\underline{v},\varepsilon,\iota,\varsigma}^{r,n} = \sum_{\underline{t},\underline{v},\varepsilon}^{r,n}\sum_{\iota=0}^{s-1}\sum_{\varsigma=0}^{1}$, $\Omega^*_{\underline{t},\underline{v},\varepsilon,\iota,\varsigma}(\underline{i},n) = \Omega_{\underline{t},\underline{v},\varepsilon}(\underline{i})a_{\iota,s}(n)\Gamma(r-\varepsilon-\varsigma+2)\,\Gamma(\varepsilon+\varsigma+1)$, and $\varrho = (n+\iota-s+1)$.

To establish the prediction boundaries for $y_s$, it is essential to determine the predictive survival function $p[y_s > \upsilon|\mathbf{z}]$. Therefore, based on Eq (4.3), we have:

$$\begin{aligned} p[y_s > \upsilon|\mathbf{z}] &= \int_{\upsilon}^{\infty} \phi^*_{s:n}(y_s|\mathbf{z})dy_s \\ &= \frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}^{r,n} \Omega^*_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(\underline{i},n)\Phi^{(0,0)}_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(\upsilon), \quad \upsilon > 0, \end{aligned} \tag{4.4}$$

where $\Phi^{(o,c)}_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(\upsilon) = \int_{\upsilon}^{\infty} e^{(\varsigma+o)\log y_s - cy_s}\Big(\eta + \Psi_{\underline{t},\underline{\nu}}(\mathbf{z}) + \varrho y_s\Big)^{-(r-\varepsilon-\varsigma+2)}\left(\rho + \Psi_{\underline{t},\underline{\nu}}\left(\frac{\mathbf{z}^2}{2}\right) + \varrho\frac{y_s^2}{2}\right)^{-(\varepsilon+\varsigma+1)} dy_s.$

A two-sided $100\pi\%$ predictive interval for $y_s$ $(1 \le s \le r)$ is expressed as $p[K(\mathbf{z}) < y_s < H(\mathbf{z})] = \pi$, where $K(\mathbf{z})$ and $H(\mathbf{z})$ represent the lower and upper confidence limits, respectively. These limits satisfy the conditions: $p[K(\mathbf{z}) < y_s] = \frac{1+\pi}{2}$ and $p[y_s > H(\mathbf{z})] = \frac{1-\pi}{2}$. In this context, obtaining analytical solutions for these limits is also challenging, which necessitates the use of an appropriate simulation method.

### 4.2. Bayesian predictor of $y_s$

With $\phi^*_{s:n}(y_s|\mathbf{z})$ provided by (4.3), the two-sample Bayesian predictive pdf of $y_s$ under the SR function can be expressed as

$$\begin{aligned} \tilde{y}_s^{SR} &= \int_0^{\infty} y_s\phi^*_{s:n}(y_s|\mathbf{z})dy_s \\ &= \frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}^{r,n} \Omega^*_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(\underline{i},n)\Phi^{(1,0)}_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(0). \end{aligned} \tag{4.5}$$

Additionally, the following expressions represent the Bayes point predictors of $y_s$ under the asymmetric LINEX and GE loss functions, respectively:

$$\begin{aligned} \tilde{y}_s^{LINEX} &= \frac{-1}{c}\log\left[\int_0^{\infty} \exp(-cy_s)\phi^*_{s:n}(y_s|\mathbf{z})dy_s\right] \\ &= \frac{-1}{c}\log\left[\frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}^{r,n} \Omega^*_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(\underline{i},n)\Phi^{(0,c)}_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(0)\right], \end{aligned} \tag{4.6}$$

$$\begin{aligned} \tilde{y}_s^{GE} &= \left(\int_0^{\infty} y_s^{-a}\phi^*_{s:n}(y_s|\mathbf{z})dy_s\right)^{-1/a} \\ &= \left(\frac{1}{\Bbbk(\eta,\rho)}\sum_{\mathbf{p}}\sum_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}^{r,n} \Omega^*_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(\underline{i},n)\Phi^{(-a,0)}_{\underline{t},\underline{\nu},\varepsilon,\iota,\varsigma}(0)\right)^{-1/a}. \end{aligned} \tag{4.7}$$

## 5. Illustrative example

### 5.1. Simulation study

To conduct Bayesian estimation and derive two-sample Bayesian prediction intervals based on type II censoring for the ORSS and OMERSS procedures, a simulation study is carried out following these steps:

1) Set the hyperparameter values $(\eta, \rho) = (0.5, 0.3)$ to generate $\alpha = 0.4999$ and $\lambda = 0.2998$ from $\hbar(\alpha, \lambda)$.
2) Generate $n(= 3, 4, 5)$ SRSs from the LED using the transformation: $Z_{i:j} = \sqrt{\left(\frac{\alpha}{\lambda}\right)^2 - \frac{2}{\lambda}\ln(1 - U_{i:j})} - \frac{\alpha}{\lambda}$, for $i, j = 1, \ldots, n$, where $U_{i:j}$ follows a uniform distribution on the interval $(0, 1)$. Next, apply the procedures for RSS as described in [31] and for MERSS with $m_1(= 1, 2, 3)$ and $m_2(= 2)$ as outlined in Section 1. This process will yield the RSS and MERSS, which can then be ordered to obtain the ORSS and OMERSS, respectively.
3) Calculate the Bayes estimates of $\alpha$ and $\lambda$ using the SR, AB, LINEX, and GE loss functions as outlined in Section 3.
4) Repeat Steps 2 and 3 $m = 1000$ times to calculate the average estimates $(\mathrm{AV}(\widehat{\vartheta}) = \frac{1}{m}\sum_{i=1}^{m}\widehat{\vartheta}_i)$, the mean squared error $(\mathrm{MSE}(\widehat{\vartheta}) = \frac{1}{m}\sum_{i=1}^{m}(\vartheta - \widehat{\vartheta}_i)^2)$, and the relative absolute bias $(\mathrm{RAB}(\widehat{\vartheta}) = \frac{1}{m}\sum_{i=1}^{m}\frac{|\widehat{\vartheta}_i - \vartheta|}{\vartheta})$, where $\widehat{\vartheta}$ represents an estimate of $\vartheta = (\alpha, \lambda)$.
5) Display the computational results of the Bayes estimates derived based on the ORSS and OMERSS methods in Table 1.
6) For the two-sample Bayesian prediction of $y_\imath$, where $\imath = 1, \cdots, s$, set the hyperparameter values $(\eta, \rho) = (1.5, 1.3)$ to generate $\alpha = 1.5005$ and $\lambda = 1.3052$ from $\hbar(\alpha, \lambda)$. Subsequently, obtain ORSS and OMERSS samples of size $n = 3, 4, 5$ from LED.
7) Use the results from Section 3 to construct (95%) two-sample Bayesian prediction intervals and the Bayes predictive estimate for $y_\imath$, where $\imath = 1, \cdots, s$, under the SR, GE, and LINEX functions. The results are summarized in Table 2 for ORSS and OMERSS, respectively.

### 5.2. *Discussion*

The numerical results from the simulation studies, as shown in Tables 1–3, reveal the following insights:

1) The AVs, APRs, MSEs, and RABs for ORSS and OMERSS decrease as $r$ and $n$ increase.
2) For fixed $r$ and $n$, the AVs, MSEs, and RABs of the GE and LINEX loss functions increase with increasing $a$ and $c$.
3) The Bayes estimates of $\lambda$ and $\alpha$ from OMERSS outperform those from ORSS in terms of AVs, APRs, MSEs, and RABs.
4) The Bayes estimates derived from the LINEX, AB, and GE (with $c = -0.1,\ a = -0.1, -0.6$) loss functions for $\lambda$ and $\alpha$ are superior to those obtained using the LINEX, AB, and GE (with $c = 0.1,\ a = 0.3, -1.6$) loss functions, based on AVs, MSEs, and RABs.
5) The Bayes estimates for $\lambda$ from both ORSS and OMERSS are more accurate than those for $\alpha$ in terms of AVs, APRs, MSEs, and RABs.
6) According to Table 2, the prediction estimators $\tilde{y}_s$ and their prediction intervals increase as $s$ rises for fixed $n$ and $r$ but decrease as $n$ and $r$ increase for fixed $s$, applicable to both ORSS and OMERSS.
7) The predictive intervals from OMERSS demonstrate better results compared to those from ORSS in all scenarios analyzed.
8) The Bayes predictive estimates for $\tilde{y}_s,\ s = 1, \ldots, r$, based on the LINEX and GE (with $c =$

$-0.1,\ a = -0.6$) loss functions outperform those from LINEX and GE (with $c = 0.1,\ a = -1.6$) and from the SR loss function.

9) The Bayes predictive estimates obtained from OMERSS are more effective than those derived from ORSS.
10) The choice of small sample sizes $n = 3, 4, 5$ was deliberate and based on several considerations:
   - Practical relevance: In many real-world applications, especially in medical and reliability studies, obtaining large sample sizes can be challenging due to cost, time, and ethical constraints. Our choice of smaller sample sizes reflects the practical limitations often encountered in such fields.
   - Demonstration of methodology: The primary goal of our simulation study is to demonstrate the effectiveness and applicability of our Bayesian estimation and prediction methods under type II censoring. Using smaller sample sizes allows us to clearly illustrate the methodology and its performance in a controlled setting.
   - Computational efficiency: Smaller sample sizes enable us to conduct a more extensive simulation study with a larger number of replications (1000 replications in our case). This ensures that our results are statistically robust and reliable, even with smaller sample sizes.
   - Comparative analysis: By using smaller sample sizes, we can effectively compare the performance of our methods (ORSS and OMERSS) under different conditions. This comparative analysis is crucial for understanding the strengths and limitations of our approaches.
11) The series involved in the derivation of Bayesian estimators under various loss functions (e.g., squared error, linear exponential, Al-Bayyati, and general entropy) have been verified for convergence. This ensures that the estimators are well-defined and reliable.
12) The series used in the calculation of Bayesian prediction bounds and predictive distributions have also been checked for convergence. This ensures that the prediction intervals and estimates are accurate and meaningful.
13) In our simulation study, we have ensured that the series used in the calculations are convergent for the sample sizes and parameters considered. This guarantees the robustness and reliability of our simulation results.

Based on the observations mentioned, we suggest utilizing the Bayes GE (with $a = -0.6$) loss function for estimating the parameters, as well as the Bayes GE (with $a = -0.6$) loss function for predicting future ordered observations in the context of OMERSS under type II censoring.

The traditional sampling methods such as SRS may not be as efficient as RSS and MERSS in terms of reducing the mean squared error and relative absolute bias (Al-Saleh and Al-Hadrami [4]; Chen et al. [17]). Maximum likelihood estimation, while widely used, can be less robust in the presence of censored data compared to our Bayesian approaches (Lai and Xie [27]). Other Bayesian methods may not incorporate the same loss functions or prior distributions, which can affect the accuracy and reliability of the estimates (Berger [10]). Our methods, particularly under type II censoring, demonstrate superior performance in these aspects.

**Table 1.** Bayesian estimates of $\alpha$ and $\lambda$, along with their associated AVs, MSEs, and RABs, using both ORSS (Case I) and OMERSS (Case II).

| | | | | LINEX | | AB | | GE | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $c = 0.1$ | $c = -0.1$ | $a = 0.3$ | $a = -0.1$ | $a = -1.6$ | $a = -0.6$ |
| | | | AV<br>MSE | AV<br>MSE | AV<br>MSE | AV<br>MSE | AV<br>MSE | AV<br>MSE | AV<br>MSE |
| $n$ | $r$ | $\vartheta$ | RAB | RAB | RAB | RAB | RAB | RAB | RAB |
| $3^{\mathrm{I}}$ | 2 | $\alpha$ | 1.3680 | 1.5110 | 1.2540 | 1.4820 | 1.3280 | 1.4710 | 1.2950 |
| | | | 0.8273 | 1.1490 | 0.6165 | 1.0060 | 0.7720 | 0.9924 | 0.7252 |
| | | | 1.0740 | 1.2800 | 0.9155 | 1.2280 | 1.0250 | 1.2130 | 0.9839 |
| | | $\lambda$ | 0.6945 | 0.8142 | 0.6099 | 0.8732 | 0.6328 | 0.8420 | 0.5895 |
| | | | 0.0449 | 0.1096 | 0.0171 | 0.1489 | 0.0239 | 0.1259 | 0.0139 |
| | | | 0.3895 | 0.6270 | 0.2257 | 0.7436 | 0.2706 | 0.6814 | 0.1966 |
| $4^{\mathrm{I}}$ | 2 | $\alpha$ | 2.1370 | 2.3430 | 1.9740 | 2.2390 | 2.1020 | 2.2340 | 2.0710 |
| | | | 2.7940 | 3.6090 | 2.2330 | 3.1090 | 2.6920 | 3.0960 | 2.1090 |
| | | | 2.2080 | 2.5160 | 1.9640 | 2.3610 | 2.1560 | 2.3530 | 2.1090 |
| | | $\lambda$ | 0.6806 | 0.8129 | 0.5912 | 0.8722 | 0.8722 | 0.8392 | 0.5698 |
| | | | 0.0414 | 0.1116 | 0.0143 | 0.1504 | 0.1504 | 0.1261 | 0.0119 |
| | | | 0.3616 | 0.6261 | 0.1893 | 0.7449 | 0.7449 | 0.6789 | 0.1665 |
| | 3 | $\alpha$ | 1.1740 | 1.2610 | 1.0990 | 1.2640 | 1.1420 | 1.2530 | 1.1160 |
| | | | 0.4658 | 0.5987 | 0.3672 | 0.5634 | 0.4354 | 0.5550 | 0.4103 |
| | | | 0.7961 | 0.9157 | 0.6960 | 0.9084 | 0.7600 | 0.8960 | 0.7311 |
| | | $\lambda$ | 0.7552 | 0.8797 | 0.6643 | 0.9329 | 0.6926 | 0.9010 | 0.6490 |
| | | | 0.0794 | 0.1666 | 0.0370 | 0.2075 | 0.0497 | 0.1794 | 0.0339 |
| | | | 0.5146 | 0.7618 | 0.3371 | 0.8671 | 0.3938 | 0.8033 | 0.3136 |
| $5^{\mathrm{I}}$ | 2 | $\alpha$ | 2.8320 | 3.0820 | 2.6300 | 2.9230 | 2.8010 | 2.9200 | 2.7720 |
| | | | 5.7150 | 7.1350 | 4.7230 | 6.1290 | 5.5800 | 6.1200 | 5.4520 |
| | | | 3.2510 | 3.6260 | 2.9550 | 3.3880 | 3.2050 | 3.3830 | 3.1610 |
| | | $\lambda$ | 0.6509 | 0.7807 | 0.5650 | 0.8413 | 0.5869 | 0.8091 | 0.5416 |
| | | | 0.0302 | 0.0919 | 0.0089 | 0.1276 | 0.0138 | 0.1059 | 0.0073 |
| | | | 0.3028 | 0.5625 | 0.1419 | 0.6839 | 0.1810 | 0.6195 | 0.1244 |
| | 3 | $\alpha$ | 1.7010 | 1.8210 | 1.5980 | 1.7850 | 1.6720 | 1.7780 | 1.6460 |
| | | | 1.4200 | 1.7390 | 1.1760 | 1.5940 | 1.3630 | 1.5840 | 1.3130 |
| | | | 1.5520 | 1.7310 | 1.3980 | 1.6770 | 1.5080 | 1.6660 | 1.4710 |
| | | $\lambda$ | 0.7701 | 0.9229 | 0.6653 | 0.9728 | 0.7002 | 0.9369 | 0.6512 |
| | | | 0.0857 | 0.1980 | 0.0362 | 0.2396 | 0.0515 | 0.2059 | 0.0337 |
| | | | 0.5386 | 0.8438 | 0.3302 | 0.9436 | 0.3996 | 0.8718 | 0.3045 |
| | 4 | $\alpha$ | 1.0620 | 1.1240 | 1.0060 | 1.1370 | 1.0350 | 1.1280 | 1.0130 |
| | | | 0.3076 | 0.3774 | 0.2516 | 0.3691 | 0.2885 | 0.3631 | 0.2730 |
| | | | 0.6422 | 0.7215 | 0.5725 | 0.7276 | 0.6151 | 0.7173 | 0.5941 |
| | | $\lambda$ | 0.8042 | 0.9290 | 0.7100 | 0.9770 | 0.7424 | 0.9454 | 0.6995 |
| | | | 0.1202 | 0.2262 | 0.0635 | 0.2648 | 0.0834 | 0.2333 | 0.0627 |

*Continued on the next page*

| | | | | LINEX | | AB | | GE | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | $c = 0.1$ | $c = -0.1$ | $a = 0.3$ | $a = -0.1$ | $a = -1.6$ | $a = -0.6$ |
| | | | AV | AV | AV | AV | AV | AV | AV |
| | | | MSE | MSE | MSE | MSE | MSE | MSE | MSE |
| $n$ | $r$ | $\vartheta$ | RAB | RAB | RAB | RAB | RAB | RAB | RAB |
| | | | 0.6181 | 0.8635 | 0.4367 | 0.9554 | 0.5020 | 0.8929 | 0.4243 |
| $3^{II}$ | 2 | $\alpha$ | 0.8300 | 0.9062 | 0.7681 | 0.9230 | 0.7970 | 0.9120 | 0.7709 |
| | | | 0.2489 | 0.3483 | 0.1884 | 0.3049 | 0.2342 | 0.3011 | 0.2218 |
| | | | 0.5291 | 0.6069 | 0.4745 | 0.5770 | 0.5190 | 0.5739 | 0.5106 |
| | | $\lambda$ | 0.6085 | 0.6944 | 0.5449 | 0.7557 | 0.5573 | 0.7302 | 0.5212 |
| | | | 0.0222 | 0.0549 | 0.0091 | 0.0830 | 0.0119 | 0.0691 | 0.0079 |
| | | | 0.2614 | 0.4211 | 0.1561 | 0.5296 | 0.1812 | 0.4814 | 0.1427 |
| $4^{II}$ | 2 | $\alpha$ | 1.1580 | 1.2630 | 1.0740 | 1.2470 | 1.1270 | 1.2400 | 1.1010 |
| | | | 0.6057 | 0.8304 | 0.4594 | 0.7223 | 0.5703 | 0.7166 | 0.5387 |
| | | | 0.8624 | 1.0020 | 0.7556 | 0.9543 | 0.8347 | 0.9493 | 0.8096 |
| | | $\lambda$ | 0.6199 | 0.7208 | 0.5481 | 0.7862 | 0.5631 | 0.7575 | 0.5229 |
| | | | 0.0202 | 0.0580 | 0.0064 | 0.0909 | 0.0090 | 0.0745 | 0.0051 |
| | | | 0.2489 | 0.4479 | 0.1254 | 0.5757 | 0.1505 | 0.5189 | 0.1153 |
| | 3 | $\alpha$ | 0.6660 | 0.7079 | 0.6297 | 0.7331 | 0.6421 | 0.7251 | 0.6231 |
| | | | 0.1484 | 0.1795 | 0.1280 | 0.1622 | 0.1463 | 0.1625 | 0.1440 |
| | | | 0.4514 | 0.4786 | 0.4327 | 0.4501 | 0.4562 | 0.4532 | 0.4589 |
| | | $\lambda$ | 0.5992 | 0.6784 | 0.5394 | 0.7383 | 0.5504 | 0.7140 | 0.5162 |
| | | | 0.0250 | 0.0558 | 0.0121 | 0.0812 | 0.0153 | 0.0683 | 0.0115 |
| | | | 0.2660 | 0.4052 | 0.1763 | 0.5025 | 0.2003 | 0.4582 | 0.1701 |
| $5^{II}$ | 2 | $\alpha$ | 1.7390 | 1.8940 | 1.6160 | 1.8250 | 1.7090 | 1.8200 | 1.6830 |
| | | | 1.9170 | 2.5250 | 1.5150 | 2.1390 | 1.8470 | 2.1320 | 1.7810 |
| | | | 1.6270 | 1.8540 | 1.4490 | 1.7450 | 1.5880 | 1.7400 | 1.5520 |
| | | $\lambda$ | 0.6200 | 0.7293 | 0.5443 | 0.7957 | 0.5606 | 0.7657 | 0.5186 |
| | | | 0.0205 | 0.0620 | 0.0062 | 0.0960 | 0.0090 | 0.0785 | 0.0052 |
| | | | 0.2428 | 0.4616 | 0.1176 | 0.5946 | 0.1423 | 0.5344 | 0.1136 |
| | 3 | $\alpha$ | 0.9904 | 1.0520 | 0.9368 | 1.0580 | 0.9662 | 1.0520 | 0.9462 |
| | | | 0.4222 | 0.5321 | 0.3437 | 0.4801 | 0.4050 | 0.4779 | 0.3892 |
| | | | 0.6660 | 0.7385 | 0.6075 | 0.7143 | 0.6527 | 0.7124 | 0.6406 |
| | | $\lambda$ | 0.6743 | 0.7815 | 0.5966 | 0.8419 | 0.6161 | 0.8123 | 0.5753 |
| | | | 0.0412 | 0.0960 | 0.0171 | 0.1327 | 0.0230 | 0.1121 | 0.0144 |
| | | | 0.3584 | 0.5699 | 0.2128 | 0.6880 | 0.2506 | 0.6293 | 0.1915 |
| | 4 | $\alpha$ | 0.5462 | 0.5705 | 0.5239 | 0.5991 | 0.5270 | 0.5919 | 0.5122 |
| | | | 0.1077 | 0.1139 | 0.1041 | 0.1009 | 0.1119 | 0.1028 | 0.1149 |
| | | | 0.4101 | 0.4123 | 0.4106 | 0.3817 | 0.4233 | 0.3876 | 0.4326 |
| | | $\lambda$ | 0.6186 | 0.6901 | 0.5623 | 0.7441 | 0.5736 | 0.7217 | 0.5422 |
| | | | 0.0392 | 0.0743 | 0.0216 | 0.0980 | 0.0268 | 0.0847 | 0.0209 |
| | | | 0.3290 | 0.4517 | 0.2429 | 0.5294 | 0.2701 | 0.4904 | 0.2377 |

**Table 2.** Bayesian predictive estimators and bounds for $y_s$, $s = 1, \cdots, r$, using ORSS (Case I) and OMERSS (Case II).

| | | | | $\tilde{y}_s$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | SR | LINEX | | GE | | | |
| Case | $n$ | $r$ | $s$ | | $c = 0.1$ | $c = -0.1$ | $a = -1.6$ | $a = -0.6$ | $\epsilon_L$ | $\epsilon_U$ |
| I | 3 | 2 | 1 | 0.2671 | 0.2631 | 0.2712 | 0.3400 | 0.2186 | 0.1671 | 0.3671 |
| | | | 2 | 0.7324 | 0.7165 | 0.7496 | 0.8502 | 0.6551 | 0.6324 | 0.8324 |
| | 4 | 2 | 1 | 0.2345 | 0.2316 | 0.2375 | 0.2967 | 0.1927 | 0.1345 | 0.3345 |
| | | | 2 | 0.6518 | 0.6398 | 0.6646 | 0.7524 | 0.5850 | 0.5518 | 0.7518 |
| | | 3 | 1 | 0.2978 | 0.2936 | 0.3022 | 0.3705 | 0.2475 | 0.1978 | 0.3978 |
| | | | 2 | 0.6702 | 0.6593 | 0.6816 | 0.7604 | 0.6086 | 0.5702 | 0.7702 |
| | | | 3 | 1.2880 | 1.2590 | 1.3200 | 1.4180 | 1.2030 | 1.1880 | 1.3880 |
| | 5 | 2 | 1 | 0.1702 | 0.1687 | 0.1718 | 0.2155 | 0.1399 | 0.0702 | 0.2702 |
| | | | 2 | 0.4896 | 0.4827 | 0.4969 | 0.5664 | 0.4386 | 0.3896 | 0.5896 |
| | | 3 | 1 | 0.1386 | 0.1376 | 0.1397 | 0.1752 | 0.1141 | 0.0386 | 0.2354 |
| | | | 2 | 0.3354 | 0.3323 | 0.3386 | 0.3856 | 0.3017 | 0.5993 | 0.2386 |
| | | | 3 | 0.6993 | 0.6893 | 0.7098 | 0.7790 | 0.6468 | 0.4354 | 0.7993 |
| | | 4 | 1 | 0.3052 | 0.3019 | 0.3086 | 0.3640 | 0.2621 | 0.2052 | 0.5011 |
| | | | 2 | 0.6011 | 0.5948 | 0.6076 | 0.6611 | 0.5586 | 0.8554 | 1.4150 |
| | | | 3 | 0.9554 | 0.9446 | 0.9667 | 1.0210 | 0.9103 | 0.4052 | 0.7011 |
| | | | 4 | 1.5150 | 1.4910 | 1.5410 | 1.6070 | 1.4560 | 1.0550 | 1.6150 |
| II | 3 | 2 | 1 | 0.5255 | 0.5129 | 0.5391 | 0.6498 | 0.4406 | 0.4255 | 0.6255 |
| | | | 2 | 1.2980 | 1.2560 | 1.3460 | 1.4790 | 1.1800 | 1.1980 | 1.3980 |
| | 4 | 2 | 1 | 0.4187 | 0.4100 | 0.4279 | 0.5243 | 0.3474 | 0.3187 | 0.5187 |
| | | | 2 | 1.0860 | 1.0540 | 1.1210 | 1.2470 | 0.9802 | 0.9855 | 1.1860 |
| | | 3 | 1 | 0.3111 | 0.3059 | 0.3166 | 0.3944 | 0.2556 | 0.2111 | 0.4111 |
| | | | 2 | 0.7109 | 0.6965 | 0.7264 | 0.8210 | 0.6385 | 0.6109 | 0.8109 |
| | | | 3 | 1.3860 | 1.3460 | 1.4330 | 1.5520 | 1.2800 | 1.2860 | 1.4860 |
| | 5 | 2 | 1 | 0.2911 | 0.2867 | 0.2958 | 0.3681 | 0.2395 | 0.1911 | 0.3911 |
| | | | 2 | 0.7981 | 0.7801 | 0.8176 | 0.9216 | 0.7166 | 0.6981 | 0.8981 |
| | | 3 | 1 | 0.3746 | 0.3687 | 0.3807 | 0.4569 | 0.3168 | 0.2746 | 0.4746 |
| | | | 2 | 0.8109 | 0.7966 | 0.8262 | 0.9098 | 0.7441 | 0.7109 | 0.9109 |
| | | | 3 | 1.5020 | 1.4640 | 1.5440 | 1.6450 | 1.4100 | 1.4020 | 1.6020 |
| | | 4 | 1 | 0.2564 | 0.2533 | 0.2596 | 0.3181 | 0.2139 | 0.1564 | 0.3564 |
| | | | 2 | 0.5463 | 0.5391 | 0.5538 | 0.6189 | 0.4970 | 0.4463 | 0.6463 |
| | | | 3 | 0.9225 | 0.9080 | 0.9380 | 1.0120 | 0.8637 | 0.8225 | 1.0230 |
| | | | 4 | 1.5570 | 1.5210 | 1.5980 | 1.6910 | 1.4720 | 1.4570 | 1.6570 |

Notice that $L_\epsilon = |\epsilon_U - \epsilon_L| = 0.2$ is the length prediction bound.

**Table 3.** The RSS and MERSS derived from actual data.

| Scheme I | | | | | | Scheme II | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\underline{1.2}$ | 1.5 | 1.8 | 1.8 | 2.2 | 3.0 | $\underline{1.2}$ | | | |
| 1.2 | $\underline{1.5}$ | 1.6 | 1.6 | 1.9 | 2.2 | 1.2 | $\underline{1.5}$ | | |
| 1.4 | 1.5 | $\underline{1.7}$ | 2.2 | 2.7 | 4.1 | 1.4 | 1.5 | $\underline{1.7}$ | |
| 1.6 | 1.6 | 1.7 | $\underline{2.2}$ | 2.3 | 3.0 | 1.6 | 1.6 | 1.7 | $\underline{2.2}$ |
| 1.4 | 1.5 | 1.6 | 1.8 | $\underline{2.7}$ | 3.0 | $\underline{1.4}$ | | | |
| 1.3 | 1.4 | 1.6 | 1.7 | 1.9 | $\underline{2.0}$ | $\underline{1.3}$ | 1.4 | | |

The SR loss function is commonly used due to its simplicity and ease of mathematical tractability. The LINEX loss function is chosen for its ability to handle asymmetric losses, which is particularly useful in scenarios where overestimation and underestimation have different costs. The AB loss function is selected for its flexibility in incorporating different degrees of asymmetry, making it suitable for a wide range of applications. The GE loss function is chosen for its robustness and ability to handle different types of data distributions. We have compared these loss functions with alternative ones, such as the absolute error loss function. The absolute error loss function is simpler but less flexible in handling asymmetric losses compared to the LINEX and AB loss functions. The SR loss function, while widely used, may not be as robust as the GE loss function in certain scenarios. Our selection of loss functions is justified by their ability to provide more accurate and reliable estimates in the context of our study. All series involved in the equations and calculations presented in this manuscript have been checked for convergence. This ensures the validity and reliability of the Bayesian estimators and prediction bounds derived in our study.

### *5.3. Application of LED distribution to real data*

In this section, we employ a real dataset from the medical domain to demonstrate the theoretical results presented in earlier sections. The dataset consists of the times reported by a group of 20 patients who received an analgesic and subsequently felt better. The data, derived from Gross and Clark [26], includes the following values: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, and 2.0. To evaluate whether LED distribution with cdf (1.2) is a suitable fit for the data, we perform a Kolmogorov-Smirnov (K-S) test and calculate the associated p-value. From the data and cdf (1.2), we estimate the scale parameters $\lambda$ and $\alpha$ by maximizing the likelihood function, resulting in $\widehat{\lambda} = 0.0240$ and $\widehat{\alpha} = 0.4900$. The K-S test statistic is computed to be 0.2340, with a corresponding p-value of 0.3307. The computed p-value of over 0.05 suggests a strong fit between the actual dataset and LED with cdf (1.2). This is further illustrated in Figure 1, which displays the histogram, empirical cdf, and pdf (1.1) along with cdf (1.2) of the dataset, indicating a good correspondence between the fitted LED and the empirical data. We have explored the potential application of our findings to other types of distributions. For instance, the Weibull distribution is widely used in reliability and survival analysis due to its flexibility in modeling different types of failure rates. The gamma distribution is commonly applied in fields such as queuing theory and hydrology, while the log-normal distribution is useful in modeling skewed data in economics and biology. By adapting our Bayesian estimation and prediction methods to LED, we can enhance the precision and reliability of statistical analyses in these areas.

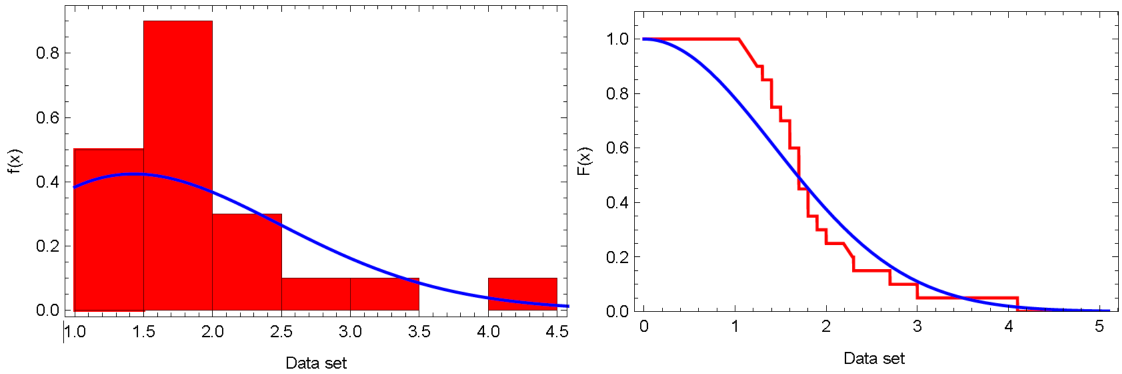


**Figure 1.** Histogram and empirical cdf (in red) compared to the pdf and cdf (in blue) of the LED for the given dataset.

In the RSS method (Scheme I), we begin by randomly selecting $n = 6$ SRSs, each of equal size. The items within each sample are ranked based on the variable of interest. The smallest observation is measured from the first set. Subsequently, a second SRS of size 6 is randomly selected without replacement from the remaining data, and the second smallest observation is recorded from this set. This process continues until the sixth SRS of size 6 is selected without replacement. As a result, we obtain the RSS values: (1.2, 1.5, 1.7, 2.2, 2.7, 2.0). For the MERSS method (Scheme II), we first randomly select $m_1 = 4$ SRSs of sizes 1, 2,..., 4. The maximum ordered observation from each of these sets is accurately measured. We then repeat this procedure for an additional $m_2 = 2$ sets of sizes 1 and 2, this time measuring the minimum ordered observations instead of the maximum ones. The total sample size for MERSS is $n = 6$, yielding the MERSS values: (1.2, 1.5, 1.7, 2.2, 1.4, 1.3). Consequently, we can derive ORSS and OMERSS by arranging the RSS and MERSS values in ascending order. Table 4 presents the Bayesian estimates of the parameters $\alpha$ and $\lambda$ for both the ORSS (Case I) and OMERSS (Case II) methods. The estimates are provided for different censoring levels $r = 2, 3, 4$ and sample size $n = 6$. The table includes estimates under various loss functions: LINEX, AB, and GE. The results indicate that the OMERSS method generally provides more accurate estimates (lower MSE and RAB) compared to the ORSS method, especially for higher censoring levels. This suggests that OMERSS is more robust and reliable for parameter estimation in the context of the linear exponential model. Table 5 presents the Bayesian predictive estimators and bounds for the real dataset using both the ORSS (Case I) and OMERSS (Case II) methods. The table includes predictive estimates and bounds for different censoring levels $r$ and sample size $n = 6$. The results show that the OMERSS method generally provides more accurate predictive estimates and tighter prediction bounds compared to the ORSS method. This is evident from the lower values of $\epsilon_L$ and $\epsilon_U$ for OMERSS, indicating better predictive performance. The predictive estimates under different loss functions (GE and LINEX) also demonstrate the robustness of the OMERSS method in predicting future observations.

**Table 4.** Bayesian estimates based on the real dataset for $n = 6$, $\eta = 0.5$, and $\rho = 0.3$, using both ORSS (Case I) and OMERSS (Case II).

| | | | | LINEX | | AB | | GE | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $r$ | $\vartheta$ | | $c = 0.1$ | $c = -0.1$ | $a = 0.3$ | $a = -0.1$ | $a = -1.6$ | $a = -0.6$ |
| $6^{\mathrm{I}}$ | 2 | $\alpha$ | 0.6121 | 0.6579 | 0.5711 | 0.7109 | 0.5749 | 0.6926 | 0.5494 |
| | | $\lambda$ | 0.7953 | 0.8707 | 0.7292 | 0.9200 | 0.7481 | 0.8968 | 0.7158 |
| | 3 | $\alpha$ | 0.3882 | 0.4103 | 0.3680 | 0.4639 | 0.3602 | 0.4491 | 0.3412 |
| | | $\lambda$ | 0.7195 | 0.7624 | 0.6793 | 0.8017 | 0.6878 | 0.7873 | 0.6657 |
| | 4 | $\alpha$ | 0.1989 | 0.2062 | 0.1921 | 0.2463 | 0.1820 | 0.2372 | 0.1704 |
| | | $\lambda$ | 0.4782 | 0.4908 | 0.4660 | 0.5144 | 0.4644 | 0.5092 | 0.4541 |
| $6^{\mathrm{II}}$ | 2 | $\alpha$ | 1.3988 | 1.5016 | 1.3132 | 1.4793 | 1.3717 | 1.4757 | 1.3465 |
| | | $\lambda$ | 0.5981 | 0.7015 | 0.5261 | 0.7711 | 0.5398 | 0.7417 | 0.4985 |
| | 3 | $\alpha$ | 0.9885 | 1.0441 | 0.9383 | 1.0570 | 0.9640 | 1.0507 | 0.9434 |
| | | $\lambda$ | 0.7294 | 0.8739 | 0.6307 | 0.9311 | 0.6605 | 0.8956 | 0.6120 |
| | 4 | $\alpha$ | 0.2642 | 0.2744 | 0.2546 | 0.3156 | 0.2453 | 0.3059 | 0.2324 |
| | | $\lambda$ | 0.6583 | 0.7021 | 0.6185 | 0.7464 | 0.6250 | 0.7314 | 0.6015 |

**Table 5.** Bayesian predictive estimators and bounds for $n = 6$, $\eta = 0.5$, and $\rho = 0.3$ based on the real dataset, using ORSS (Case I) and OMERSS (Case II).

| | | | | $\tilde{y}_s$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | SR | LINEX | | GE | | | |
| Case | $n$ | $r$ | $s$ | | $c = 0.1$ | $c = -0.1$ | $a = -1.6$ | $a = -0.6$ | $\epsilon_L$ | $\epsilon_U$ |
| I | 6 | 2 | 1 | 0.6849 | 0.6709 | 0.6995 | 0.7988 | 0.6006 | 0.5849 | 0.7849 |
| | | | 2 | 1.5260 | 1.4900 | 1.5650 | 1.6630 | 1.4320 | 1.4260 | 1.6260 |
| | | 3 | 1 | 0.5227 | 0.5148 | 0.5309 | 0.6074 | 0.4596 | 0.4227 | 0.6227 |
| | | | 2 | 1.0270 | 1.0120 | 1.0430 | 1.1120 | 0.9668 | 0.9270 | 1.1270 |
| | | | 3 | 1.7460 | 1.7130 | 1.7810 | 1.8550 | 1.6730 | 1.6460 | 1.8460 |
| | | 4 | 1 | 0.5546 | 0.5465 | 0.5628 | 0.6361 | 0.4932 | 0.4546 | 0.6546 |
| | | | 2 | 1.0150 | 1.0020 | 1.0290 | 1.0900 | 0.9620 | 0.9154 | 1.1150 |
| | | | 3 | 1.5250 | 1.5050 | 1.5460 | 1.6030 | 1.4720 | 1.4250 | 1.6250 |
| | | | 4 | 2.2810 | 2.2410 | 2.3250 | 2.3850 | 2.2130 | 2.1810 | 2.3810 |
| II | 6 | 2 | 1 | 1.0030 | 0.9751 | 1.0340 | 1.1600 | 0.8900 | 0.9032 | 1.1030 |
| | | | 2 | 2.1620 | 2.0890 | 2.2490 | 2.3620 | 2.0320 | 2.0620 | 2.2620 |
| | | 3 | 1 | 0.2395 | 0.2364 | 0.2428 | 0.3043 | 0.1964 | 0.1395 | 0.3395 |
| | | | 2 | 0.5593 | 0.5502 | 0.5689 | 0.6467 | 0.5017 | 0.4593 | 0.6593 |
| | | | 3 | 1.1160 | 1.0900 | 1.1460 | 1.2510 | 1.0300 | 1.0160 | 1.2160 |
| | | 4 | 1 | 0.2956 | 0.2917 | 0.2995 | 0.3628 | 0.2489 | 0.1956 | 0.3956 |
| | | | 2 | 0.6117 | 0.6031 | 0.6208 | 0.6896 | 0.5595 | 0.5117 | 0.7117 |
| | | | 3 | 1.0090 | 0.9917 | 1.0270 | 1.1050 | 0.9468 | 0.9089 | 1.1090 |
| | | | 4 | 1.6600 | 1.6180 | 1.7090 | 1.8060 | 1.5700 | 1.5600 | 1.7600 |

Notice that $L_\epsilon = |\epsilon_U - \epsilon_L| = 0.2$ is the length prediction bound.

## 6. Conclusions

The performance of Bayes estimations and predictions based on ORSS and OMERSS for the linear exponential distribution has been examined. OMERSS offers several practical advantages, including improved estimation accuracy, reduced mean squared error, and enhanced robustness in the presence of censored data. OMERSS can be more efficient than traditional sampling methods such as SRS and RSS, particularly in scenarios where the data is skewed or follows a complex distribution. The MERSS technique allows for more flexible and adaptive sampling, which can lead to better performance in various applications. Additionally, OMERSS can provide more reliable estimates in situations where the data is subject to type II censoring, making it a valuable tool for statistical inference in a wide range of fields. Under type II censoring, the Bayes estimates of the unknown parameters $\alpha$ and $\lambda$—using SR, AB, LINEX, and GE loss functions—and the predictive estimates for future observations have been established. A real dataset from the medical field has been utilized to illustrate the theoretical concepts discussed in this study. Additionally, a Monte Carlo simulation was performed to assess the effectiveness and accuracy of the estimation and prediction methods. Based on the numerical findings, we recommend using the Bayes method for estimation and prediction based on OMERSS in the context of the specified censoring scenarios.

## Author contributions

Haidy A. Newer: Writing–original draft, Writing–review and editing, Validation, Investigation, Formal analysis, Methodology, Software, Visualization, Supervision; Bader S Alanazi: Funding acquisition. All authors have approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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