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*Research article*

# Properties of fractional generalized entropy in ordered variables and symmetry testing

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**Abstract:** Uncertainty measures are widely used in various statistical applications, including hypothesis testing and characterizations. Numerous generalizations of information measures with different extensions have been developed. Inspired by this, our study introduced the principle of the fractional generalized entropy measure and investigated its properties through stochastic comparisons and characterizations using order statistics and upper random variables. We explored the monotonicity and symmetry properties of the fractional generalized entropy, emphasizing conditions under which it uniquely identified the parent distribution. In the case of distributions that were completely continuous, The symmetrical nature of order statistics suggested that symmetry of the underpinning distribution. Based on the fractional generalized entropy measure in non-parametric estimate of order statistics, a new test for the symmetry hypothesis was put forward. This test offered the supremacy of not requiring the symmetry center to be specified. Additionally, an example of real-world data was shown to illustrate how the suggested technique might be applied.

**Keywords:** entropy; hazard rate function; stochastic order; order statistics; upper record values; symmetry testing
**Mathematics Subject Classification:** 60E15, 62G10, 94A17

## 1. Introduction

The measurement of a probability distribution's uncertainty has attracted a lot of attention in recent decades. Shannon [43] established the idea of entropy, which is a key measure of relevance in information theory. In many scientific fields, the entropy function measure is a helpful instrument. In the continuous domain, uncertainty measures have undergone a number of modifications. Differential entropy is the name given to the continuous case of entropy. Throughout the entire work, $Y$ stands for a random variable (r.v.) follows an entirely continuously cumulative distribution function (CDF) $F$ and a matching probability density function (PDF) $f$. The entropy model of the Shannon differential measure is represented as follows

$$\Psi(Y) = -\int_{-\infty}^{\infty} f(y) \ln f(y) dy. \tag{1.1}$$

The literature has presented potential substitute measurements of information. On fact, a measure of uncertainty akin to $\Psi(Y)$ was presented by Rao et al. [40]; located to the side of right-hand of (1.1), the survival function in the form $\overline{F}(y) = 1 - F(y) = P(Y > y)$ takes the location of the PDF $f$. This is generally recognized as the cumulative residual entropy measure (CRE), and its definition for an r.v. that is not negative is

$$R\Psi(Y) = -\int_{-\infty}^{\infty} \overline{F}(y) \ln \overline{F}(y) dy = \int_{-\infty}^{\infty} \overline{F}(y)\Phi(y) dy, \tag{1.2}$$

seeing that the function of the cumulative hazard rate is expressed as

$$\Phi(y) = \left[-\ln \overline{F}(y)\right] = \int_{0}^{y} \eta(u) du, y \geq 0, \tag{1.3}$$

and the hazard rate function form is $\eta(v) = \frac{f(v)}{\overline{F}(v)}$, $v \geq 0$. In keeping with this, Di Crescenzo and Longobardi [16] established and examined cumulative entropy as a comparable measure. The CDF $F(y) = P(Y \leq y)$ is used to define this, i.e., (also see Navarro et al. [34]) by

$$C\Psi(Y) = -\int_{0}^{\kappa} F(y) \ln F(y) dy, \tag{1.4}$$

where $Y$'s support is denoted by $(0, \kappa)$. It is usually quite interesting to see how Shannon entropy is generated for different fields. Under the discrete case, the author created a novel entropy in [46] using fractional calculus, which is the fractional entropy function given by

$$F\Psi(p_1, p_2, ..., p_n) = \sum_{i=1}^{n} p_i(-\ln p_i)^{\delta}, 0 \leq \delta \leq 1. \tag{1.5}$$

It is non-additive, concave, and positive fractional entropy. It also fulfills Lesche and the thermodynamic case of stability in a physical sense. Furthermore, compared to a single entropy value, the research described in [27] shows that the measure of the fractional entropy model has a better vulnerability to the signal development, enabling the revelation of additional features and information about the underlying system.

Recently, under the continuous case, fractional versions of a number of different information measures have been put out. Xiong et al. [48] have examined and discussed a number of aspects of the cumulative fractional entropy model, including its limitations, how it is related to stochastic ordering and empirical estimation, its change and adaptation under linear transformations, and its numerous connections to other functions, which is given by

$$CFGR\Psi(Y)_{\delta} = \int_{-\infty}^{\infty} \overline{F}(y)[\Phi(y)]^{\delta} dy, 0 \leq \delta \leq 1, \tag{1.6}$$

where $\Phi(y)$ is defined in (1.3). The CRE measure was expanded to the generalization measure of the fractional cumulative residual entropy model as a notable study by Di Crescenzo et al. [17] as follows:

$$FGR\Psi(Y)_\theta = Q(\theta) \int_{-\infty}^{\infty} \overline{F}(y)[\Phi(y)]^\theta dy, \tag{1.7}$$

where $\Phi(y)$ is defined in (1.3), and $Q(\theta) = \frac{1}{\Gamma(\theta+1)}$, $\theta \geq 0$. The cumulative residual generalized entropy, developed by Psarrakos and Navarro [38], is identified with $FGR\Psi(Y)_n$ if $\theta$ is a positive integer, such as $\theta = n \in \mathbb{N}$, see also Psarrakos and Toomaj [39] for more features. The dispersion measure that has a strong association with the upper record values of a set of not dependent, identically distributed r.v.'s is $FGR\Psi(Y)_n$, it should be noted. Furthermore, it is associated with the relevation transform and the interepoch intervals of a non-homogeneous Poisson process (for some new findings on this measure, see, for example, Toomaj and Di Crescenzo [45] and sources therewith).

Additional findings for the fractional generalized model of the cumulative residual entropy version were examined by Alomani and Kayid [6]. Using this model, they carried out several stochastic comparisons and found correlations between it and other reliability measures and a few well-known stochastic orders. Furthermore, they demonstrated that, given an appropriate prior distribution function, the measure equals the Bayesian risk of a mean residual lifespan.

A basic premise of statistical analysis is that the population being studied has a symmetric underlying distribution. Regression models, for example, require a symmetric distribution for errors. As a result, we need to rigorously verify the symmetry assumption. Presume that $S_Y$ is $F$'s support. Additionally, suppose that there is a mean $\mu$ such that, for every $y \in S_Y$, $F(\mu - z) + F(\mu + z) = 1$. In this case, $Y$ is said to have a symmetrical distribution surrounding $\mu$. In probability and statistics, symmetry is a basic structural assumption that may be applied to a wide range of issues. In the literature, several elements of symmetry in probability distributions have been studied in detail. Numerous writers have characterized symmetric distributions using order statistics and other ordered data sets (like sequential order statistics and record values). By way of illustration, Balakrishnan and Selvitella [10] demonstrated that, for a given sample of size $n$ and some of fixed $r = 1, \ldots, n$, $Y_{r:n} \overset{D}{=} Y_{n-r+1:n}$, provided and only if $F$ is considered to be symmetric about 0, with noting that $\overset{D}{=}$ indicates that the distribution of the two r.v.s is identical. Furthermore, Ahmadi [2] provided several novel descriptions of symmetric distributions to be continuous using $k$-records. Mahdizadeh and Zamanzade used ranked set samplings to estimate the symmetrical distribution function non-parametrically [28]. Generally speaking, based on a distribution's unique features, criteria may be created to assess whether or not it is symmetric. Goodness-of-fit testing is therefore used to test for the symmetry; for instance, see Dai et al. [15] and Bozin et al. [12].

Several symmetry tests have been proposed in the literature based on uncertainty measurements of ordered variables. Xiong et al. [49] utilized the measure of extropy model of the upper and lower $k$-records to test for symmetry. Jose and Sathar [25] examined the measure of extropy of the upper and lower $k$-records in the context of $nth$ symmetry. Gupta and Chaudhary [23] recently characterized continual symmetric distributions using the extropy measure of the record values. For additional discussions, see Park [37], Noughabi [35], Noughabi and Jarrahiferiz [36], and Husseiny [24].

Throughout this article, we will address the stochastic orders that are remembered later with the goal of offering appropriate comparisons. Let's presuming that $Y_1$ and $Y_2$ are two r.v.'s with corresponding PDFs $f_1$ and $f_2$, and CDFs of $F_1$ and $F_2$ with the continuous left inverses $F_1^{-1}(y) = \inf\{v : F_1(v) \geq y\}$

and $F_2^{-1}(y) = \inf\{v : F_2(v) \geq y\}$ , $0 < y < 1$, respectively. Consequently, for all $y \geq 0$, less than $Y_2$ is $Y_1$:

1) in the order of the likelihood of ratio, indicated by $Y_1 \leq_{Lr} Y_2$, if $\frac{f_1(y)}{f_2(y)}$ is decreasing in $y$.
2) in the order of the hazard of rate, indicated by $Y_1 \leq_{hrt} Y_2$, if $\eta_{Y_1}(y) \geq \eta_{Y_1}(y)$, for all $y$.
3) in the order of the usual of stochastic, indicated by $Y_1 \leq_{St} Y_2$, if $\overline{F}_1(y) \leq \overline{F}_2(y)$.
4) in the order of the super-additive, indicated by $Y_1 \leq_{Su-A} Y_2$, if $F_2^{-1}F_1(y)$ is super-additive.
5) in the order of the dispersive, indicated by $Y_1 \leq_{Disp} Y_2$, if $F_2^{-1}F_1(y) - y$ is increasing in $y \geq 0$.

For information on their primary characteristics, we direct the reader to Shaked and Shanthikumar's book [42].

*Work motivation*

Many physical and financial r.v.'s, such as stock returns, chromatographic separation, asset pricing, and nuclear resonance spectroscopy, are predicated on the symmetrical distribution as a fundamental premise. Because symmetrical distribution provides a consistent substructure for role-modeling, measuring and evaluating data that is both physical and financial, it makes statistical analysis easier. We should mention that testing for symmetry has a rich history and is among the earliest classical non-parametric topics. A number of writers have looked at it based on characterization results. A solution to the symmetry question is often crucial for many topics in the social sciences, computer science, engineering, and econometrics, as noted by Jozefczyk [26]. Thus, the information gathered in this article might help address your query by keeping an eye on some. Thus, by keeping an eye on a few basic symmetry characteristics of the measures of uncertainty of the specified distribution, the findings in this study could help address that query. As mentioned in different references, the extropy measure is used to test symmetry. Therefore, we chose another measure of information to be implemented in the symmetry test and compared the behaviors of these measures.

This article aims to present the concept of the fractional generalized entropy model and study its features in the context of the r.v.'s, with a discussion on symmetric continuous distributions. The remaining content of the paper is organized as follows: Section 2 discloses the continuous case of the fractional generalized entropy model. Using order statistics and upper records, we provide stochastic comparisons, characterizations, and monotonic properties. In Section 3, we discuss some characteristics of symmetry and its testing based on non-parametric estimation of order statistics.

## 2. Fractional generalized entropy measure

In this section, we disclose the continuous case of the fractional generalized entropy. It is noted in the literature of information theory that entropy functions appear and their properties are studied, then the cumulative entropy or cumulative residual appears and is studied. Here, we take the concept of the measure of the fractional generalized model of the cumulative residual entropy and define the fractional generalized model of the entropy as follows, drawing inspiration from the characteristics of CRE in Eq (1.2), fractional generalized version of the residual cumulative entropy measure in Eq (1.7), and the fractional entropy in Eq (1.5), and we obtain:

$$FG\Psi(Y)_\theta = Q(\theta) \int_{-\infty}^{\infty} f(y)[-\ln f(y)]^\theta dy = Q(\theta) \int_{-\infty}^{\infty} f(y)[\phi(y)]^\theta dy, \tag{2.1}$$

with noting that $\phi(y) = [-\ln f(y)]$, and $Q(\theta) = \frac{1}{\Gamma(\theta+1)}$, $\theta \geq 0$.

We shall talk about some stochastic order of the fractional generalized model of the entropy measure in the subsequent case. We may examine the following results as Shaked and Shanthikumar's Theorem 4.B.2 [42] indicating that if $Y_1 \leq_{St} Y_2$, then $Y_1 \leq_{Su-A} Y_2$ implies $Y_1 \leq_{Disp} Y_2$.

**Lemma 2.1.** *Provided that $Y_1 \leq_{Disp} Y_2$, then $FG\Psi(Y_1)_\theta \leq FG\Psi(Y_2)_\theta$.*

*Proof.* It is clear that, from (2.1), we obtain:

$$FG\Psi(Y)_\theta = Q(\theta)\int_{-\infty}^{\infty} f(y)[\phi(y)]^\theta dy = Q(\theta)\int_0^1 [\phi(F^{-1}(u))]^\theta du.$$

If $Y_1 \leq_{Disp} Y_2$, then, we obtain $f_1(F_1^{-1}(u)) \geq f_2(F_2^{-1}(u))$ for all $u \in (0,1)$. Therefore,

$$FG\Psi(Y_1)_\theta = Q(\theta)\int_0^1 [\phi(F_1^{-1}(u))]^\theta du \leq Q(\theta)\int_0^1 [\phi(F_2^{-1}(u))]^\theta = FG\Psi(Y_2)_\theta.$$

□

Note that, Alomani and Kayid [6] showed that if $Y_1 \leq_{Disp} Y_2$, then $FGR\Psi(Y_1)_\theta \leq FGR\Psi(Y_2)_\theta$.

Assume that the observations $Y_1, ..., Y_n$ have identical distributions and are independent, with CDF $F$ and PDF $f$. $Y_{1:n} \leq Y_{2:n} \leq \ \leq Y_{n:n}$ represents the sample's order statistics. Shaked and Shanthikumar's Theorem 3.B.26 [42] asserts that $Y_{1,i:n} \leq_{Disp} Y_{2,i:n}$, $i = 1, 2, ..., n$, if $Y_1 \leq_{Disp} Y_2$. Thus, we can readily arrive at the following conclusion based on Lemma 2.1.

**Proposition 2.1.** *Provided that $Y_1 \leq_{Disp} Y_2$, then $FG\Psi(Y_{1;i:n})_\theta \leq FG\Psi(Y_{2;i:n})_\theta$.*

When an observation $Y_j$ has a value that is considered to be bigger than that of all earlier observations, it is referred to as an upper r.v.'s record; thus, $Y_j$ is considered to be an upper record r.v., if $Y_j > Y_i$ for all $i < j$. An analogous definition may be given for records from lower r.v. Belzunce et al. [11] demonstrated that if $Y_1 \leq_{Disp} Y_2$, then $U_n^{Y_1} \leq_{Disp} U_n^{Y_2}$, where $U_n^{Y_1}$ and $U_n^{Y_2}$ are the *nth* upper records r.v.'s of $Y_1$ and $Y_2$, correspondingly. Based on Lemma 2.1, we immediately arrive at the following outcome.

**Proposition 2.2.** *Provided that $Y_1 \leq_{Disp} Y_2$, then $FG\Psi(U_n^{Y_1})_\theta \leq FG\Psi(U_n^{Y_2})_\theta$.*

### *2.1. Redesigned characterizations using ordered variables*

The PDF of a sample of size $n$ with an underlying distribution $Y$ that contains the *rth* order statistic $Y_{r:n}$, $1 \leq r \leq n$, is obtained by

$$f_{r:n}(y) = \frac{1}{\beta_g(r, n-r+1)} F^{r-1}(y)\overline{F}^{n-r}(y) f(y), \tag{2.2}$$

with nothing that $\beta_g(r, n-r+1) = \frac{\Gamma(r)\Gamma(n-r+1)}{\Gamma(n+1)}$.

The primary findings in this part will be demonstrated by the Stone–Weierstrass Theorem's corollary (Aliprantis and Burkinshaw, [3]), which is the following result.

**Lemma 2.2.** *If $\zeta$ is a continual function on the interval $[0, 1]$ with the condition that $\int_0^1 x^n\zeta(x)dx = 0$ for $n \geq 0$, then $\zeta(x) = 0$ for every $x \in [0, 1]$.*

Now, we demonstrate in the upcoming theorem that the features of the fractional generalized entropy information of $Y_{r:n}$ may be used to characterize the parent distribution.

**Theorem 2.1.** *Given two PDFs $f_1$, $f_2$ with corresponding CDFs $F_1$, $F_2$ of the r.v., $Y_1$ and $Y_2$, respectively. For a specified value of r, where $1 \le r \le n$, and $\theta \ge 0$, we observe the following:*

$$Y_1 \overset{D}{=} Y_2 \Longleftrightarrow FG\Psi(Y_{1;r:n})_\theta = FG\Psi(Y_{2;r:n})_\theta, \forall n \ge r,$$

where

$$FG\Psi(Y_{i;r:n})_\theta = Q(\theta)\int_0^1 (1-u)^{r-1}u^m\left[A(u) - \ln f_i(F_i^{-1}(1-u))\right]^\theta du,$$

$A(u) = \left[-\ln \frac{1}{\beta_g(r,n-r+1)}(1-u)^{r-1}u^m\right]$, and under the condition $\left|\frac{A(u)}{-\ln f_i(F_i^{-1}(1-u))}\right| < 1$, $m = n-r$, $i = 1, 2$.

*Proof.* The necessity is inconsequential; therefore, it is essential to demonstrate the sufficiency aspect. From (2.1) and (2.2), with $m = n-r$, suppose that $FG\Psi(Y_{1;r:n})_\theta = FG\Psi(Y_{2;r:n})_\theta$, and this situation can be stated as

$$\int_{-\infty}^{\infty} F_1^{r-1}(y)\overline{F}_1^m(y)f_1(y)\left[-\ln\frac{1}{\beta_g(r,n-r+1)}F_1^{r-1}(y)\overline{F}_1^m(y)f_1(y)\right]^\theta dy$$
$$= \int_{-\infty}^{\infty} F_2^{r-1}(y)\overline{F}_2^m(y)f_2(y)\left[-\ln\frac{1}{\beta_g(r,n-r+1)}F_2^{r-1}(y)\overline{F}_2^m(y)f_2(y)\right]^\theta dy.$$

Use $u = \overline{F}_1(y)$ and $u = \overline{F}_2(y)$ on the previous equation, respectively. As a result, it can be concluded that

$$\begin{aligned}&\int_0^1 (1-u)^{r-1}\left[-\ln\frac{1}{\beta_g(r,n-r+1)}(1-u)^{r-1}u^m f_1(F_1^{-1}(1-u))\right]^\theta u^m du\\ &= \int_0^1 (1-u)^{r-1}\left[-\ln\frac{1}{\beta_g(r,n-r+1)}(1-u)^{r-1}u^m f_2(F_2^{-1}(1-u))\right]^\theta u^m du.\end{aligned} \tag{2.3}$$

Since $\theta \ge 0$, we can use the generalized binomial theorem for non-negative real exponents of the following expression:

$$\begin{aligned}\left[-\ln\frac{1}{\beta_g(r,n-r+1)}(1-u)^{r-1}u^m f(F^{-1}(1-u))\right]^\theta &= \sum_{k=0}^{\infty}\binom{\theta}{k}\left[-\ln\frac{1}{\beta_g(r,n-r+1)}(1-u)^{r-1}u^m\right]^k\\ &\quad\times\left[-\ln f(F^{-1}(1-u))\right]^{\theta-k}\\ &= \sum_{k=0}^{\infty}\binom{\theta}{k}A(u)^k\left[-\ln f(F^{-1}(1-u))\right]^{\theta-k},\end{aligned}$$

where $A(u) = \left[-\ln\frac{1}{\beta_g(r,n-r+1)}(1-u)^{r-1}u^m\right]$. The series converges if: $\left|\frac{A(u)}{-\ln f(F^{-1}(1-u))}\right| < 1$, which ensures that the ratio of $A(u)$ to $-\ln f(F^{-1}(1-u))$ lies within the radius of convergence of the binomial series expansion. This condition ensures that the infinite series representation is valid for non-negative real $\theta$. Then, we can write (2.3) as

$$\sum_{k=0}^{\infty}\binom{\theta}{k}\int_0^1 A(u)^k(1-u)^{r-1}\left[(-\log f_1(F_1^{-1}(1-u)))^{\theta-k}-(-\log f_2(F_2^{-1}(1-u)))^{\theta-k}\right]u^m\,du=0.$$

Thus, the given condition becomes:

$$\sum_{k=0}^{\infty}\binom{\theta}{k}\int_0^1 B(u;k)u^m\,du=0,$$

where $B(u;k) = A(u)^k(1-u)^{r-1}\left[(-\log f_1(F_1^{-1}(1-u)))^{\theta-k}-(-\log f_2(F_2^{-1}(1-u)))^{\theta-k}\right]$. Assuming that interchanging the sum and integral is justified (which is valid here since the sum is finite), we have:

$$\int_0^1\left(\sum_{k=0}^{\infty}\binom{\theta}{k}B(u;k)\right)u^m\,du=0.$$

Define: $\omega(u)=\sum_{k=0}^{\infty}\binom{\theta}{k}B(u;k)$. Then, the condition simplifies to:

$$\int_0^1\omega(u)u^m\,du=0.$$

From Lemma 2.2, we conclude: $\omega(u)=0$, for every $u\in[0,1]$. Recall that:

$$\omega(u)=\sum_{k=0}^{\infty}\binom{\theta}{k}B(u;k)=\sum_{k=0}^{\infty}\binom{\theta}{k}A(u)^k(1-u)^{r-1}[(-\log f_1(F_1^{-1}(1-u)))^{\theta-k} - (-\log f_2(F_2^{-1}(1-u)))^{\theta-k}].$$

Factor out $(1-u)^{r-1}$:

$$\omega(u)=(1-u)^{r-1}\sum_{k=0}^{\infty}\binom{\theta}{k}A(u)^k\left[(-\log f_1(F_1^{-1}(1-u)))^{\theta-k}-(-\log f_2(F_2^{-1}(1-u)))^{\theta-k}\right].$$

Set

$$x_1(u)=-\log f_1(F_1^{-1}(1-u)),\quad x_2(u)=-\log f_2(F_2^{-1}(1-u)).$$

Then,

$$\omega(u)=(1-u)^{r-1}\left[\sum_{k=0}^{\infty}\binom{\theta}{k}A(u)^k x_1(u)^{\theta-k}-\sum_{k=0}^{\infty}\binom{\theta}{k}A(u)^k x_2(u)^{\theta-k}\right].$$

Observe that

$$\sum_{k=0}^{\infty}\binom{\theta}{k}A(u)^k x^{\theta-k}=(A(u)+x)^{\theta}.$$

Applying this to our context:

$$\omega(u)=(1-u)^{r-1}\left[(A(u)+x_1(u))^{\theta}-(A(u)+x_2(u))^{\theta}\right]=0\quad\text{for all } u\in[0,1].$$

Given that $(1-u)^{r-1} \neq 0$ for $u \in [0,1)$ when $r \geq 1$, we can divide both sides by $(1-u)^{r-1}$, yielding:

$$(A(u) + x_1(u))^{\theta} - (A(u) + x_2(u))^{\theta} = 0 \quad \Rightarrow \quad (A(u) + x_1(u))^{\theta} = (A(u) + x_2(u))^{\theta}.$$

Therefore, we can see that,

$$A(u) + x_1(u) = A(u) + x_2(u) \quad \Rightarrow \quad x_1(u) = x_2(u).$$

Substitute back

$$-\log f_1(F_1^{-1}(1-u)) = -\log f_2(F_2^{-1}(1-u)) \quad \Rightarrow \quad f_1(F_1^{-1}(1-u)) = f_2(F_2^{-1}(1-u)).$$

By taking $1-u=p$, we have $f_1(F_1^{-1}(p)) = f_2(F_2^{-1}(p))$ for all $p \in [0,1]$. Thus, $(F_1^{-1})'(p) = (F_2^{-1})'(p)$ for all $p \in [0,1]$. Hence, $F_1^{-1}(p) = F_2^{-1}(p) + c_n$ for all $p \in [0,1]$, where $c_n$ is a constant. By noting that $\lim_{p\to 0} F_1^{-1}(p) = \lim_{p\to 0} F_2^{-1}(p) = q$, we have $F_1^{-1}(p) = F_2^{-1}(p)$ for all $p \in [0,1]$. Hence, the outcome ensues. □

Remember that if $\widetilde{\eta}_Y(y) = \frac{f(y)}{F(y)}$ is decreasing in $y$, then $Y$ is said to have a decreased hazard rate reversed (DHRV). The following theory addresses this issue based on the *rth* order statistics.

**Theorem 2.2.** *If $Y$ is DHRV, and $\theta$ takes an odd value, then $FG\Psi(Y_{r:n})_\theta$ is decreasing in $n$ for fixed $r$, $1 \leq r \leq n$.*

*Proof.* According to (2.1) and (2.2), it follows that

$$\begin{aligned}
\frac{FG\Psi(Y_{r:n})_\theta}{FG\Psi(Y_{r:n+1})_\theta} &= \frac{n-r+1}{n+1} \frac{\int_{-\infty}^{\infty} F^{r-1}(y)\overline{F}^{n-r}(y)f(y)\left[-\ln F^{r-1}(y)\overline{F}^{n-r}(y)f(y)\right]^{\theta} dy}{\int_{-\infty}^{\infty} F^{r-1}(y)\overline{F}^{n-r+1}(y)f(y)\left[-\ln F^{r-1}(y)\overline{F}^{n-r+1}(y)f(y)\right]^{\theta} dy} \\
&= C^* \frac{\int_0^1 \frac{1}{\beta_g(r,n-r+1)} u^{r-1}(1-u)^{n-r}\left[-\ln u^r(1-u)^{n-r}\widetilde{\eta}_Y(F^{-1}(u))\right]^{\theta} du}{\int_0^1 \frac{1}{\beta_g(r,n-r+2)} u^{r-1}(1-u)^{n-r+1}\left[-\ln u^r(1-u)^{n-r+1}\widetilde{\eta}_Y(F^{-1}(u))\right]^{\theta} du} \\
&= \frac{\mathbb{E}\left[\left(-\ln U^r(1-U)^{n-r}\widetilde{\eta}_Y(F^{-1}(U))\right)^{\theta}\right]}{\mathbb{E}\left[\left(-\ln V^r(1-V)^{n-r+1}\widetilde{\eta}_Y(F^{-1}(V))\right)^{\theta}\right]},
\end{aligned}$$

where $C^* = \frac{(n-r+1)(n+1)}{(n+1)(n-r+1)}$, $U$ and $V$ are the *rth* order statistics of uniform samples with sizes $n$ and $n+1$ respectively, with PDF's $\int_0^1 \frac{1}{\beta_g(r,n-r+1)} u^{r-1}(1-u)^{n-r} du$ and $\int_0^1 \frac{1}{\beta_g(r,n-r+2)} u^{r-1}(1-u)^{n-r+1} du$, respectively, and $0 \leq u \leq 1$. As per Shaked and Shanthikumar [42], Theorem 1.B.28, $U \geq_{hrt} V$, and hence, $U \geq_{St} V$. For $\theta$ takes an odd value, then from the premise we have that

$$\mathbb{E}\left[\left(\ln U^r(1-U)^{n-r}\widetilde{\eta}_Y(F^{-1}(U))\right)^{\theta}\right] \geq \mathbb{E}\left[\left(\ln V^r(1-V)^{n-r+1}\widetilde{\eta}_Y(F^{-1}(V))\right)^{\theta}\right],$$

which indicates that $\frac{FG\Psi(Y_{r:n})_\theta}{FG\Psi(Y_{r:n+1})_\theta} \geq 1$. This brings the proof to a close. □

Recall the DHRV property of Pareto distribution with CDF $1 - y^{-d}$, $y \geq 1$, $d > 0$, and Weibull distribution with CDF $1 - e^{-(\frac{y}{d_2})^{d_1}}$, $y \geq 0$, $d_1 > 1$, $d_2 > 0$ (i.e., when $d_1 > 1$, the Weibull distribution has an increasing hazard rate and a DHRV). Figure 1 shows the factional generalized model of the entropy measure of the $rth$ order statistics ($r = 3$) with increasing $n$ for Pareto and Weibull distribution and $\theta = 1, 3, 5, 7$, which ensures the decreasing property of Theorem 2.2 when $\theta$ is odd.

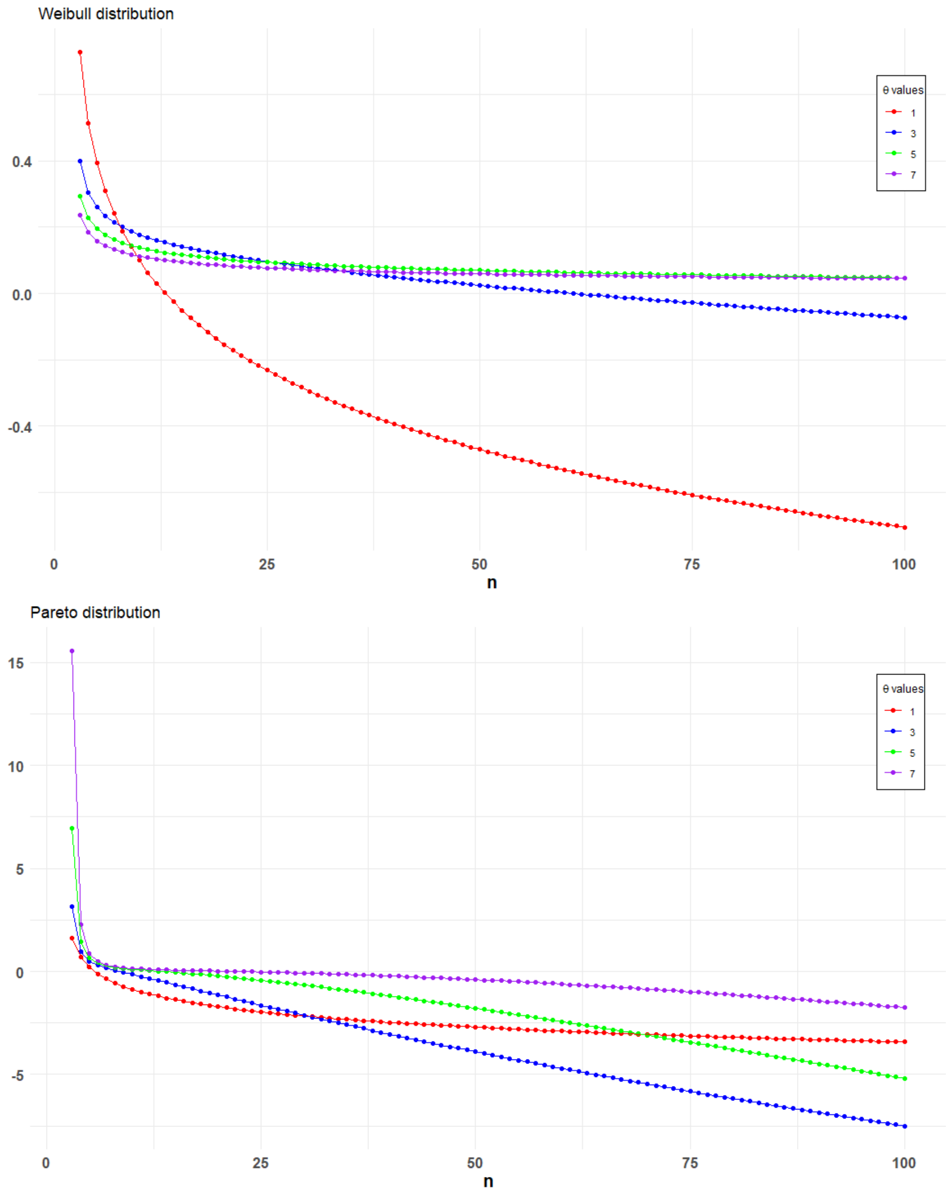


**Figure 1.** Fractional generalized entropy of $3rd$ order statistics $Y_{3:n}$ of Pareto (with parameter $d = 2$), and Weibull (with parameters $d_1 = 3$, $d_1 = 2$) distributions, with increasing $n$ and $\theta = 1, 3, 5, 7$.

*2.2. Redesigned characterizations using nth upper random variables*

If $Y(n)$ represents the component's lifespan and n minimum repairs are permitted, then the $Y(n)$ survival function is equal to the $(n+1)th$ higher record value r.v. (refer to Shaked and Shanthikumar [42]). As a result, researching the idea of record values is equivalent to researching lifetimes with few repairs. The *nth* upper r.v. $U_n^Y$ of $Y$ has the following pdf:

$$f_{U_n^Y}(y) = \frac{f(y)[-\ln \overline{F}(y)]^{n-1}}{(n-1)!}. \tag{2.4}$$

Goffman and Pedrick [21] provide the following lemma to support the impending characterization findings for fractional generalized entropy of r.v.'s.

**Lemma 2.3.** *Provide the Laguerre polynomial*

$$L_m(y) = e^y \frac{d^m}{dy^m} y^m e^{-y} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} m(m-1)...(j+1)y^j.$$

In the space $L_2(0,\infty)$, the set of Laguerre functions $\frac{1}{m!} e^{\frac{-y}{2}} L_m(y)$ for $m \geq 0$ forms a complete orthonormal system. If $g \in L_2(0,\infty)$ and $\int_0^\infty g(y) e^{\frac{-y}{2}} L_m(y)\, dy = 0$ for all $m \geq 0$, then $g$ is almost everywhere equal to zero.

**Theorem 2.3.** *Given that $U_n^{Y_1}$ and $U_n^{Y_2}$ are the nth upper records r.v.'s of $Y_1$ and $Y_2$, respectively. Assuming the conditions of Theorem 2.1, we have*

$$Y_1 \stackrel{D}{=} Y_2 \Longleftrightarrow FG\Psi(U_n^{Y_1})_\theta = FG\Psi(U_n^{Y_2})_\theta, \forall n \geq 1,$$

where $E(\ln^2 f_1(Y)) < +\infty$, $E(\ln^2 f_2(Y)) < +\infty$,

$$FG\Psi(U_n^{Y_i})_\theta = Q(\theta) \int_0^\infty e^{-u} u^{n-1} \left[ A^*(u) - \ln f_i(F_i^{-1}(1-e^{-u})) \right]^\theta du,$$

$A^*(u) = \left[ -\ln \frac{u^{n-1}}{(n-1)!} \right]$, and under the condition $\left| \frac{A^*(u)}{-\ln f_i(F_i^{-1}(1-e^{-u}))} \right| < 1$, $i = 1\ 2$.

*Proof.* The necessity is inconsequential; therefore, it is essential to demonstrate the sufficiency aspect. From (2.1) and (2.4), suppose that $FG\Psi(U_n^{Y_1})_\theta = FG\Psi(U_n^{Y_2})_\theta$, and this situation can be stated as

$$\int_{-\infty}^{\infty} [-\ln \overline{F}_1(y)]^{n-1} f_1(y) \left[ -\ln \frac{[-\ln \overline{F}_1(y)]^{n-1}}{(n-1)!} f_1(y) \right]^\theta dy$$
$$= \int_{-\infty}^{\infty} [-\ln \overline{F}_2(y)]^{n-1} f_2(y) \left[ -\ln \frac{[-\ln \overline{F}_2(y)]^{n-1}}{(n-1)!} f_2(y) \right]^\theta dy.$$

Use $u = -\ln \overline{F}_1(y)$ and $u = -\ln \overline{F}_2(y)$ on the previous equation, respectively. As a result, it can be concluded that

$$\begin{aligned} &\int_0^\infty \left[ -\ln \frac{u^{n-1}}{(n-1)!} f_1(F_1^{-1}(1-e^{-u})) \right]^\theta e^{-u} u^{n-1} du \\ &= \int_0^\infty \left[ -\ln \frac{u^{n-1}}{(n-1)!} f_2(F_2^{-1}(1-e^{-u})) \right]^\theta e^{-u} u^{n-1} du. \end{aligned} \tag{2.5}$$

Since $\theta \geq 0$, we can use the generalized binomial theorem for non-negative real exponents of the following expression:

$$\left[-\ln \frac{u^{n-1}}{(n-1)!} f(F^{-1}(1-e^{-u}))\right]^{\theta} = \sum_{k=0}^{\infty} \binom{\theta}{k} \left[-\ln \frac{u^{n-1}}{(n-1)!}\right]^{k} \times \left[-\ln f(F^{-1}(1-e^{-u}))\right]^{\theta-k}$$
$$= \sum_{k=0}^{\infty} \binom{\theta}{k} A^*(u)^k \left[-\ln f(F^{-1}(1-e^{-u}))\right]^{\theta-k},$$

where $A^*(u) = \left[-\ln \frac{u^{n-1}}{(n-1)!}\right]$. The series converges if: $\left|\frac{A^*(u)}{-\ln f_i(F_i^{-1}(1-e^{-u}))}\right| < 1$, which ensures that the ratio of $A^*(u)$ to $-\ln f_i(F_i^{-1}(1-e^{-u}))$ lies within the radius of convergence of the binomial series expansion. This condition ensures that the infinite series representation is valid for non-negative real $\theta$. Then, we can write (2.5) as

$$\sum_{k=0}^{\infty} \binom{\theta}{k} \int_0^1 A^*(u)^k \left[(-\log f_1(F_1^{-1}(1-e^{-u})))^{\theta-k} - (-\log f_2(F_2^{-1}(1-e^{-u})))^{\theta-k}\right] e^{-u} u^{n-1}\, du = 0.$$

Thus, the given condition becomes:

$$\sum_{k=0}^{\infty} \binom{\theta}{k} \int_0^1 B^*(u;k) e^{\frac{-u}{2}} L_n(u)\, du = 0,$$

where $B^*(u;k) = e^{\frac{-u}{2}} A^*(u)^k \left[(-\log f_1(F_1^{-1}(1-e^{-u})))^{\theta-k} - (-\log f_2(F_2^{-1}(1-e^{-u})))^{\theta-k}\right]$, and Lemma 2.3 gives the Laguerre polynomial, which is $L_n(u)$. Assuming that interchanging the sum and integral is justified (which is valid here since the sum is finite), we have:

$$\int_0^1 \left(\sum_{k=0}^{\infty} \binom{\theta}{k} B^*(u;k)\right) e^{\frac{-u}{2}} L_n(u)\, du = 0.$$

Define: $\omega^*(u) = \sum_{k=0}^{\infty} \binom{\theta}{k} B^*(u;k)$. Then, the condition simplifies to:

$$\int_0^{\infty} \omega^*(u) e^{\frac{-u}{2}} L_n(u)\, du = 0.$$

From Lemma 2.3, we conclude: $\omega^*(u) = 0$, for all $u \in [0,1]$. Therefore, applying the similar substitutions in Theorem 2.1, we acquire

$$-\log f_1(F_1^{-1}(1-e^{-u})) = -\log f_2(F_2^{-1}(1-e^{-u})) \quad \Rightarrow \quad f_1(F_1^{-1}(1-e^{-u})) = f_2(F_2^{-1}(1-e^{-u})),$$

for all $u \in [0,\infty]$. Or, in other word, $f_1(F_1^{-1}(p^*)) = f_2(F_2^{-1}(p^*))$, for all $1-e^{-u} = p^* \in [0,1]$. The remainder resembles that found in Theorem 2.1. The intended outcome so follows. □

The monotonic characteristics of the fractional generalized entropy of r.v.'s will be covered in the descriptions that follow.

**Theorem 2.4.** *Let $Y$ be an r.v. with CDF $F$ and PDF $f$. If $f(F^{-1}(y))$ is increasing in $y$, then $FG\Psi(U_n^Y)_\theta$ is increasing in $n$.*

*Proof.* According to (2.1) and (2.4), it follows that

$$
\begin{aligned}
\frac{FG\Psi(U_n^Y)_\theta}{FG\Psi(U_{n+1}^Y)_\theta} &= \frac{\int_{-\infty}^{\infty} \frac{f(y)[-\ln \overline{F}(y)]^{n-1}}{(n-1)!}\left[-\ln \frac{f(y)[-\ln \overline{F}(y)]^{n-1}}{(n-1)!}\right]^\theta dy}{\int_{-\infty}^{\infty} \frac{f(y)[-\ln \overline{F}(y)]^{n}}{(n)!}\left[-\ln \frac{f(y)[-\ln \overline{F}(y)]^{n}}{(n)!}\right]^\theta dy} \\
&= \frac{n!}{(n-1)!}\cdot\frac{\int_0^\infty u^{n-1}e^{-u}\left[-\ln f(F^{-1}(1-e^{-u})) - (n-1)\ln u + \ln(n-1)!\right]^\theta du}{\int_0^\infty u^{n}e^{-u}\left[-\ln f(F^{-1}(1-e^{-u})) - n\ln u + \ln n!\right]^\theta du} \\
&= n\cdot\frac{\int_0^\infty u^{n-1}e^{-u}\left[-\ln f(F^{-1}(1-e^{-u})) - (n-1)\ln u + \ln(n-1)!\right]^\theta du}{\int_0^\infty u^{n}e^{-u}\left[-\ln f(F^{-1}(1-e^{-u})) - n\ln u + \ln n!\right]^\theta du} \\
&= \frac{\int_0^\infty \frac{1}{\Gamma(n)}u^{n-1}e^{-u}\left[-\ln f(F^{-1}(1-e^{-u})) - (n-1)\ln u + \ln(n-1)!\right]^\theta du}{\int_0^\infty \frac{1}{\Gamma(n+1)}u^{n}e^{-u}\left[-\ln f(F^{-1}(1-e^{-u})) - n\ln u + \ln n!\right]^\theta du} \\
&= \frac{\mathbb{E}\left[\left(-\ln \frac{f(F^{-1}(1-e^{-U}))U^{n-1}}{(n-1)!}\right)^\theta\right]}{\mathbb{E}\left[\left(-\ln \frac{f(F^{-1}(1-e^{-V}))V^{n}}{(n)!}\right)^\theta\right]},
\end{aligned}
$$

where the r.v.'s $U$ and $V$ follows Gamma$(n,1)$ and Gamma$(n+1,1)$ distributions with PDF's $f_U(y) = \frac{y^{n-1}e^{-y}}{(n-1)!}$ and $f_V(y) = \frac{y^n e^{-y}}{n!}$, respectively. Due to the decreasing nature of $\frac{f_U(y)}{f_V(y)} = \frac{n}{y}$, we can infer that $U \leq_{Lr} V$, which implies $U \leq_{St} V$. Notice that for $x \geq 0$, $f(F^{-1}(1-e^{-x}))$ is increasing. Additionally, we have

$$
\mathbb{E}\left(-\ln \frac{f(F^{-1}(1-e^{-U}))U^{n-1}}{(n-1)!}\right)^\theta \leq \mathbb{E}\left(-\ln \frac{f(F^{-1}(1-e^{-V}))V^{n}}{(n)!}\right)^\theta,
$$

which indicates that $\frac{FG\Psi(U_n^Y)_\theta}{FG\Psi(U_{n+1}^Y)_\theta} \leq 1$. This brings the proof to a close. □

Under the *nth* record $U_n^Y$ of exponential distribution with CDF $F(y)1 - e^y$, $y \geq 0$. Figure 2 shows the factional generalized entropy with increasing $n$ and $\theta = 2, 3$, which ensure the increasing property of Theorem 2.4.

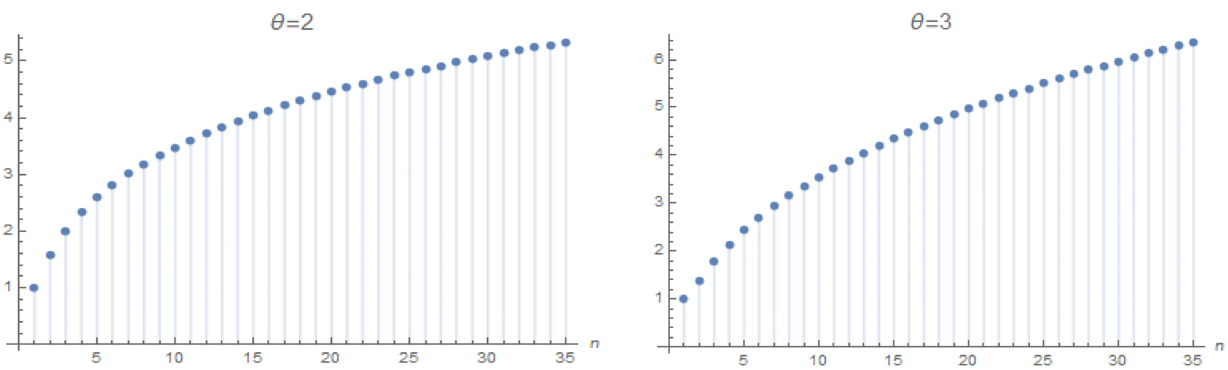


**Figure 2.** Fractional generalized entropy of *nth* record $U_n^Y$ of exponential distribution with increasing $n$ and $\theta = 2, 3$.

## 3. Characteristics symmetric in the fractional generalized entropy

When the PDF of the underlying identical besides the independent distributed r.v.'s is symmetric, several intriguing characteristics of the fractional generalized entropy of order statistics emerge. We start with two lemmas, the proof of which flows directly from the definition of $f_{r:n}$ in (2.2) and the symmetry assumption.

**Lemma 3.1.** *(Fashandi and Ahmadi [19]) With support $S_Y$, PDF $f$, and CDF $F$, and $Y$ as a continual r.v., the relationship*

$$f\left(F^{-1}(u)\right) = f\left(F^{-1}(1-u)\right) \quad \text{for all } u \in (0,1),$$

implies that $F(y)$ is symmetric about a constant $c \in S_Y$.

**Lemma 3.2.** *(Balakrishnan and Selvitella [10]) Let us assume that the order statistic $Y_{j:n}$, $j = 1, ..., n$, has a parent distribution with a PDF $f$ such that $f(\mu + y) = f(\mu - y)$, $y \geq 0$, where $\mu$ represents the mean of $Y$. Next, we have*

$$F(\mu + y) = \overline{F}(\mu - y), \qquad f_{j:n}(\mu + y) = f_{n-j+1}(\mu - y).$$

**Theorem 3.1.** *With the exception of independent distributed samples from $Y$ whose PDF is considered to be symmetric about its mean $\mu$, let $Y_1, ..., Yn$ be identical. Therefore, we have*

1) If n is considered to be odd, then, $FG\Psi(Y_{j:n})_\theta = FG\Psi(Y_{n-j+1:n})_\theta$, $j = 1, ..., n$.
2) $Y$ has a symmetric PDF if, and only if, $FG\Psi(Y_{1:n})_\theta = FG\Psi(Y_{n:n})_\theta$, $\forall n \geq 1$.

*Proof.* 1) From Lemma 3.2 and Eq (2.1), we get

$$\begin{aligned} FG\Psi(Y_{j:n})_\theta &= Q(\theta)\int_{-\infty}^{\infty} f_{j:n}(y)[-\ln f_{j:n}(y)]^\theta dy = Q(\theta)\int_{-\infty}^{\infty} f_{j:n}(\mu + y)[-\ln f_{j:n}(\mu + y)]^\theta dy \\ &= Q(\theta)\int_{-\infty}^{\infty} f_{n-j+1:n}(\mu - y)[-\ln f_{n+j-1:n}(\mu - y)]^\theta dy \\ &= Q(\theta)\int_{-\infty}^{\infty} f_{n-j+1:n}(y)[-\ln f_{n-j+1:n}(y)]^\theta dy = FG\Psi(Y_{n-j+1:n})_\theta. \end{aligned}$$

2) This theorem's first component implies the necessity. Next, we present the sufficiency. Assume $FG\Psi(Y_{1:n})_\theta = FG\Psi(Y_{n:n})_\theta$, $\forall n \geq 1$. Applying a similar approach to demonstrate Theorem 2.1's sufficiency, and from Lemma 3.1, we obtain for all $u \in (0, 1)$,

$$f(F^{-1}(1-u)) = f(F^{-1}(u)),$$

thus, $-\frac{d}{du}F^{-1}(1-u) = \frac{d}{du}F^{-1}(u)$. This implies $-F^{-1}(1-u) = F^{-1}(u) + C_n$, then, $f(-F^{-1}(u) - C_n) = f(F^{-1}(u))$, where $C_n$ is a constant, and for all $u \in (0, 1)$. Put $F^{-1}(u) = -\frac{C_n}{2} + y$, we get $f(-\frac{C_n}{2} - y) = f(-\frac{C_n}{2} + y)$, for all $y \in \mathbb{R}$, which proves the theorem.

□

**Corollary 3.1.** *In accordance with Theorem 3.1, given that $\Delta FG\Psi(Y_{q:n})_\theta = FG\Psi(Y_{q+1:n})_\theta - FG\Psi(Y_{q:n})_\theta$ is the operator for forward difference with respect to $q$, where $1 \leq q \leq n-1$. Then, $\Delta FG\Psi(Y_{j:n})_\theta = -\Delta FG\Psi(Y_{n-j:n})_\theta$, $j = 1, ..., n$.*

**Remark 3.1.** *Let $FG\Psi(Y_{1:n})_\theta - FG\Psi(Y_{n:n})_\theta$ be $\Omega_n$. If, and only if, $Y$ is symmetric, then $\Omega_n = 0$, $n = 1, 2, ....$ As a result, $\Omega_n$ may be used as a core idea of symmetry and as a symmetry test statistic.*

We may infer that the fractional generalized entropy $FG\Psi(Y_{j:n})_\theta$ at the median is always locally greatest or minimal based on the assumptions in Corollary 3.1. This can be demonstrated using the uniform distribution $U(-1, 1)$. For the fractional generalized entropy of the median ($j = 4$), when $n$ is set to 7, the minimum values of 0.463721 with $\theta = 2$ and 0.0843204 with $\theta = 4$, as well as the maximum value of 0.0259421 with $\theta = 3$, are obtained (see Figure 3).

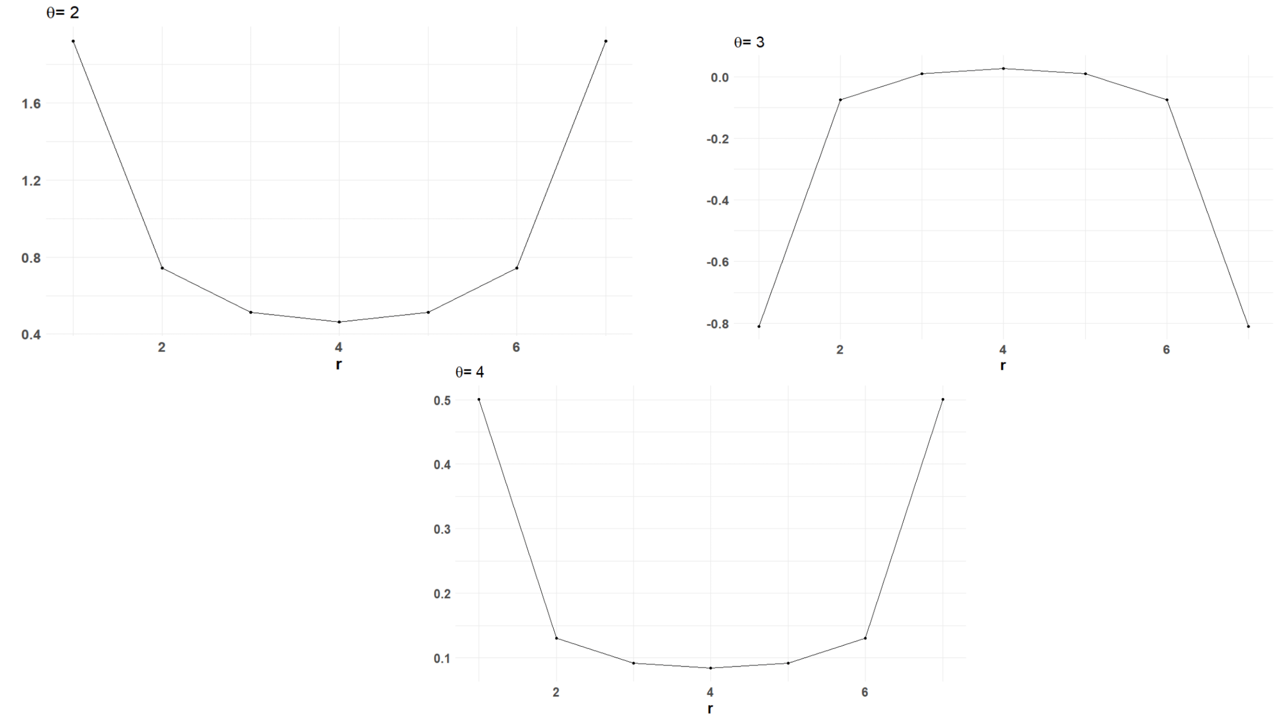


**Figure 3.** Fractional generalized entropy of the *rth* order statistics of $U(-1, 1)$ distribution.

### *3.1. Test of symmetry via non-parametric estimation*

In this part, we will discuss the non-parametric estimation form of the fractional generalized entropy which is analogy to Vasicek [47], and use it to test the symmetry. Developing statistical processes has made extensive use of the Vasicek entropy estimator of (1.1). It is provided by

$$\begin{aligned}\Psi(f_n) &= -\int_{-\infty}^{\infty} f(y)\ln f(y)dy = -\int_0^1 \ln\left[\frac{d}{d\rho}F^{-1}(\rho)\right]^{-1} d\rho \\ &= \frac{1}{n}\sum_{j=1}^{n}\ln\left[\frac{n}{2w}(Y_{(j+w)} - Y_{(j-w)})\right],\end{aligned} \tag{3.1}$$

with noting that the window positive integer size is $w < \frac{n}{2}$ and $Y_j = Y_1$ if $j < 1$ and $Y_j = Y_n$ if $j > n$.

We can rewrite $FG\Psi(Y_{1:n})_\theta$ and $FG\Psi(Y_{n:n})_\theta$, respectively, as

$$FG\Psi(Y_{1:n})_\theta = \int_0^1 n(1-u)^{n-1}\left[-\ln n(1-u)^{n-1} f(F^{-1}(y))\right]^\theta du,$$

$$FG\Psi(Y_{n:n})_\theta = \int_0^1 n(u)^{n-1}\left[-\ln n(u)^{n-1} f(F^{-1}(y))\right]^\theta du.$$

Keep in mind that Park [37] suggested a symmetry test based on the entropy of order statistics, drawing inspiration from Vasicek [47]. Therefore, we can derive the sample estimates of $FG\Psi(Y_{1:k})_\theta$ and $FG\Psi(Y_{k:k})_\theta$, based on sample of size $n$ and $k = 1, ..., \infty$, by identification to Vasicek [47], respectively, as

$$\begin{aligned}
\widehat{FG\Psi(Y_{1:k})_\theta} &= \frac{Q(\theta)}{n}\left(\sum_{j=1}^{n} k\left(1-\frac{j}{n+1}\right)^{k-1}\left[-\ln k\left(1-\frac{j}{n+1}\right)^{k-1}\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta\right) \\
&= \frac{kQ(\theta)}{n}\sum_{j=1}^{n}\left(1-\frac{j}{n+1}\right)^{k-1}\left[-\ln k\left(1-\frac{j}{n+1}\right)^{k-1}\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta,
\end{aligned}$$

$$\begin{aligned}
\widehat{FG\Psi(Y_{k:k})_\theta} &= \frac{Q(\theta)}{n}\left(\sum_{j=1}^{n} k\left(\frac{j}{n+1}\right)^{k-1}\left[-\ln k\left(\frac{j}{n+1}\right)^{k-1}\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta\right) \\
&= \frac{kQ(\theta)}{n}\sum_{j=1}^{n}\left(\frac{j}{n+1}\right)^{k-1}\left[-\ln k\left(\frac{j}{n+1}\right)^{k-1}\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta,
\end{aligned}$$

Consequently, $\widehat{\Omega_k} = \widehat{FG\Psi(Y_{1:k})_\theta} - \widehat{FG\Psi(Y_{k:k})_\theta}$, where $k = 1, 2, ..., \infty$, can be estimated using

$$\begin{aligned}
\widehat{\Omega_k} = \frac{kQ(\theta)}{n}\sum_{j=1}^{n}\Bigg\{&\left(1-\frac{j}{n+1}\right)^{k-1}\left[-\ln k\left(1-\frac{j}{n+1}\right)^{k-1}\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta \\
&-\left(\frac{j}{n+1}\right)^{k-1}\left[-\ln k\left(\frac{j}{n+1}\right)^{k-1}\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta\Bigg\}.
\end{aligned}$$

For the sake of simplicity, we simply use $k = 2$ in the following, and we suggest using

$$\begin{aligned}
\widehat{\Omega_2} = \frac{2Q(\theta)}{n}\sum_{j=1}^{n}\Bigg\{&\left(1-\frac{j}{n+1}\right)\left[-\ln 2\left(1-\frac{j}{n+1}\right)\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta \\
&-\left(\frac{j}{n+1}\right)\left[-\ln 2\left(\frac{j}{n+1}\right)\left(\frac{2w}{n(Y_{(j+w)}-Y_{(j-w)})}\right)\right]^\theta\Bigg\}.
\end{aligned}$$

This is the sample estimate of $\Omega_2 = FG\Psi(Y_{1:2})_\theta - FG\Psi(Y_{2:2})_\theta$, used to determine if the distribution of $Y$ is symmetric. Therefore, we reject the premise of symmetry because small or large values of $\Omega_2$ might be interpreted as a sign of non-symmetry.

Regretfully, the values of $\widehat{\Omega_2}$ rely on the window size $w$ in addition to the sample. Determining the precise distribution of $\widehat{\Omega_2}$ under the null hypothesis is too difficult. As a result, to ascertain its critical values, we employ Monte Carlo simulation. In accordance with earlier literature (see, for instance, McWilliams [29] and Corzo and Babativa [14]), we choose the distribution of the generalized lambda as an alternative distribution and simulate a sample of sizes $n = 20, 30, 50, 100$ from nine instances of this distribution. Thus, the modeled data is expressed as

$$y_i = \gamma_1 + \frac{u_i^{\gamma_3} - (1-u_i)^{\gamma_4}}{\gamma_2},\ 0 \le u_i \le 1,\ i = 1, 2, ..., n.$$

Table 1 lists the values of $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_4$, which were selected by McWilliams [29]. One thousand samples with sizes of 20, 30, 50, and 100 are created for each case. To select $w$, the heuristic formula used for entropy estimate, as proposed by Grzegorzewski and Wieczorkowski [22], is

$$r = [\sqrt{n} + 0.5], \tag{3.2}$$

depending on the floor value. Based on 50,000 generated samples from the standard normal distribution, the test statistic $|\widehat{\Omega_2}|$ distribution forms for $n = 25, 50, 75, 100, 150$, and $w$ selected as defined in (3.2) are displayed in Figure 4. Wolfram Mathematica (version 13) was chosen for its robust random number generation capabilities and symbolic computation, which were essential for generating the samples and calculating the test statistic. R software was used for its powerful statistical computing environment and graphics capabilities, which were utilized for the subsequent data analysis and visualization of the distributions. Therefore, we can clearly see that as the sample size increases, the distribution becomes more symmetrical.

**Table 1.** The generalized lambda distribution parameters utilized in the Monte Carlo simulation are broken down into nine categories.

| Case | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | Skewness | Kurtosis |
|---|---|---|---|---|---|---|
| 1 | 0.0000 | 0.1975 | 0.1349 | 0.1349 | 0.0000 | 3.0000 |
| 2 | -0.1167 | -0.3517 | -0.1300 | -0.1600 | 0.8000 | 11.4000 |
| 3 | 0.0000 | -1.0000 | -0.1000 | -0.1800 | 2.0000 | 21.2000 |
| 4 | 3.5865 | 0.0431 | 0.0252 | 0.0940 | 0.9000 | 4.2000 |
| 5 | 0.0000 | -1.0000 | -0.0075 | -0.0300 | 1.5000 | 7.5000 |
| 6 | 0.0000 | 1.0000 | 1.4000 | 0.2500 | 0.5000 | 2.2000 |
| 7 | 0.0000 | 1.0000 | 0.0001 | 0.1000 | 1.5000 | 5.8000 |
| 8 | 0.0000 | -1.0000 | -0.0010 | -0.1300 | 3.1600 | 23.8000 |
| 9 | 0.0000 | -1.0000 | -0.0001 | -0.1700 | 3.8800 | 40.7000 |

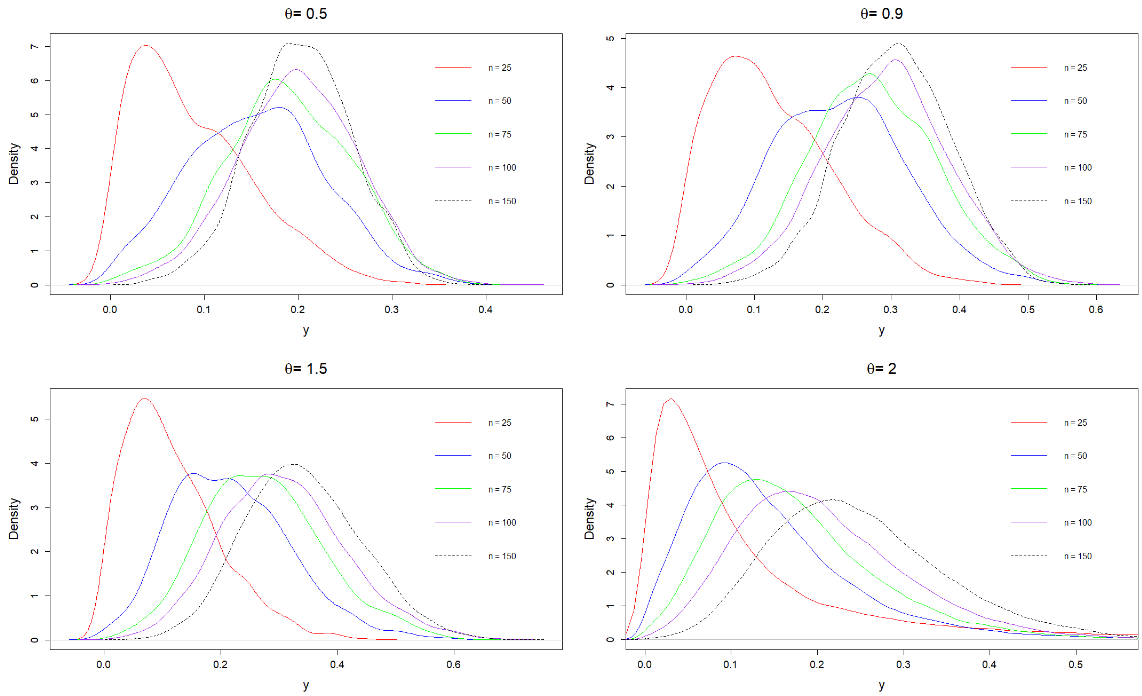


**Figure 4.** Test statistic's empirical densities were produced with 50, 000 samples from the null distribution with sizes of $n = 25, 50, 75, 100, 150$.

Tables 2 provides the precise critical values of the test statistic $|\widehat{\Omega_2}|$ for different sample sizes by a 1000-repetition Monte Carlo simulation, corresponding to the significance level $\alpha = 0.05$. Additionally, the proportion of rejections of the symmetry null hypothesis at significance level $\alpha = 0.05$ among the 1000 samples that are in the crucial range is used to determine the test's power. Table 3 displays the predicted power values for the suggested test.

**Table 2.** Critical intervals for the test statistic $|\widehat{\Omega_2}|$ at level of significance 0.05.

| $n\backslash\theta$ | 0.5 | 0.8 | 1 | 1.5 |
|---|---|---|---|---|
| 20 | (0.27845, 0.780614) | (0.568249, 0.91149) | (0.763465, 0.882621) | (0.965966, 1.16259) |
| 30 | (0.269064, 0.691) | (0.527914, 0.820641) | (0.706145,0.851285) | (0.951857, 1.10022) |
| 50 | (0.248773 , 0.539401) | (0.49003 ,0.702245 ) | (0.657996 ,0.781161 ) | (0.948808 ,1.04897 ) |
| 75 | (0.231658 , 0.451866) | (0.450263 ,0.631636 ) | (0.605813 ,0.72826 ) | (0.909779 ,0.98083 ) |
| 100 | (0.195129 , 0.384403) | (0.386637 ,0.553443 ) | (0.526217 ,0.651678 ) | (0.833306 ,0.886064 ) |
| 150 | (0.162138 , 0.317101) | (0.325805 ,0.477334 ) | (0.448865 ,0.573832 ) | (0.74274 ,0.786942 ) |

| $n\backslash\theta$ | 2 | 2.5 | 3 | 3.5 |
|---|---|---|---|---|
| 20 | (1.22615, 1.4222) | (1.42361, 1.60101) | (1.55507, 1.74352) | (1.55972, 1.82734) |
| 30 | (1.17452, 1.37511) | (1.36072, 1.56505) | (1.48666, 1.69651) | (1.52876, 1.77516) |
| 50 | (1.15663 ,1.3296 ) | (1.33021 ,1.52634 ) | (1.46184, 1.66176) | (1.52632, 1.73709) |
| 75 | (1.1149 ,1.26238 ) | (1.28865 ,1.46699 ) | (1.41209, 1.60889) | (1.48568, 1.68481) |
| 100 | (1.034 ,1.16065 ) | (1.19667 ,1.36794 ) | (1.31887, 1.51328) | (1.39654, 1.59316) |
| 150 | (0.944579 ,1.04632 ) | (1.10848 ,1.26421 ) | (1.23693, 1.41404) | (1.31222, 1.49884) |

**Table 3.** Comparison of the test's power analysis at the significance level 0.05.

| Alternative | n | $\widehat{\lvert\Omega_2\rvert}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\theta = 0.5$ | 0.8 | 1 | 1.5 | 2 | 2.5 | 3 | 4 |
| Case 1(H0) | 20 | 0.061 | 0.051 | 0.048 | 0.05 | 0.055 | 0.062 | 0.073 | 0.045 |
| | 30 | 0.061 | 0.061 | 0.052 | 0.062 | 0.058 | 0.044 | 0.055 | 0.051 |
| | 50 | 0.055 | 0.057 | 0.062 | 0.052 | 0.054 | 0.052 | 0.055 | 0.049 |
| | 100 | 0.058 | 0.06 | 0.05 | 0.052 | 0.074 | 0.056 | 0.042 | 0.063 |
| Case 2 | 20 | 0.122 | 0.107 | 0.079 | 0.176 | 0.214 | 0.198 | 0.24 | 0.223 |
| | 30 | 0.133 | 0.132 | 0.077 | 0.189 | 0.201 | 0.196 | 0.19 | 0.231 |
| | 50 | 0.166 | 0.131 | 0.092 | 0.231 | 0.222 | 0.211 | 0.201 | 0.2 |
| | 100 | 0.182 | 0.146 | 0.11 | 0.25 | 0.278 | 0.276 | 0.231 | 0.232 |
| Case 3 | 20 | 0.204 | 0.302 | 0.089 | 0.59 | 0.55 | 0.864 | 0.889 | 0.965 |
| | 30 | 0.182 | 0.295 | 0.193 | 0.814 | 0.536 | 0.977 | 0.996 | 1.000 |
| | 50 | 0.356 | 0.367 | 0.263 | 0.913 | 0.733 | 1.000 | 1.000 | 1.000 |
| | 100 | 0.274 | 0.299 | 0.539 | 0.921 | 0.987 | 1.000 | 1.000 | 0.987 |
| Case 4 | 20 | 0.361 | 0.553 | 0.112 | 0.974 | 0.982 | 0.988 | 0.991 | 0.992 |
| | 30 | 0.663 | 0.704 | 0.328 | 0.998 | 0.995 | 0.998 | 0.998 | 0.997 |
| | 50 | 0.81 | 0.83 | 0.488 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | 100 | 0.971 | 0.969 | 0.84 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| Case 5 | 20 | 1.000 | 0.999 | 0.165 | 1.000 | 1.000 | 0.921 | 0.972 | 1.000 |
| | 30 | 1.000 | 0.996 | 0.451 | 0.998 | 0.983 | 0.729 | 0.939 | 0.993 |
| | 50 | 1.000 | 1.000 | 0.651 | 0.901 | 0.898 | 0.773 | 0.869 | 0.998 |
| | 100 | 1.000 | 0.955 | 0.938 | 0.948 | 0.814 | 0.783 | 0.956 | 0.914 |
| Case 6 | 20 | 0.139 | 0.489 | 0.176 | 0.261 | 0.793 | 0.809 | 0.979 | 0.999 |
| | 30 | 0.242 | 0.621 | 0.418 | 0.432 | 0.932 | 0.982 | 0.992 | 1.000 |
| | 50 | 0.417 | 0.761 | 0.619 | 0.564 | 0.996 | 1.000 | 0.999 | 1.000 |
| | 100 | 0.54 | 0.966 | 0.832 | 0.671 | 1.000 | 1.000 | 1.000 | 1.000 |
| Case 7 | 20 | 0.76 | 0.551 | 0.165 | 0.784 | 0.734 | 0.913 | 0.865 | 0.998 |
| | 30 | 0.672 | 0.373 | 0.353 | 0.359 | 0.743 | 0.901 | 0.914 | 0.999 |
| | 50 | 0.986 | 0.377 | 0.403 | 0.538 | 0.943 | 0.908 | 0.993 | 1.000 |
| | 100 | 0.873 | 0.278 | 0.154 | 0.917 | 1.000 | 0.972 | 0.94 | 1.000 |
| Case 8 | 20 | 0.154 | 0.135 | 0.105 | 0.476 | 0.834 | 0.9 | 0.921 | 0.997 |
| | 30 | 0.076 | 0.156 | 0.196 | 0.618 | 0.94 | 0.881 | 0.952 | 1.000 |
| | 50 | 0.238 | 0.362 | 0.159 | 0.821 | 0.996 | 0.868 | 0.935 | 1.000 |
| | 100 | 0.022 | 0.983 | 0.118 | 0.998 | 1.000 | 0.859 | 0.678 | 1.000 |
| Case 9 | 20 | 0.05 | 0.18 | 0.098 | 0.547 | 0.9 | 0.822 | 0.895 | 0.999 |
| | 30 | 0.017 | 0.384 | 0.138 | 0.741 | 0.985 | 0.754 | 0.896 | 1.000 |
| | 50 | 0.058 | 0.711 | 0.109 | 0.907 | 1.000 | 0.694 | 0.852 | 1.000 |
| | 100 | 0.05 | 1.000 | 0.21 | 1.000 | 1.000 | 0.514 | 0.592 | 1.000 |

The critical values and power of our proposed test for symmetry at significance level $\alpha = 0.05$ were computed using the subsequent steps:

1) Create a sample of size n using the conventional normal distribution, and then compute the test statistics for the sample data;

2) Perform 1000 repetitions of Step 1 and establish the critical values as the 25th and 975th quantiles of the test statistics (i.e., we examined the 25th and 975th order statistics $\widehat{\Omega_2^{(25)}}$ and $\widehat{\Omega_2^{(975)}}$ and specified the critical values $\widehat{\Omega_2^{\alpha=0.05}} = \widehat{\Omega_2^{(975)}}$ and $\widehat{\Omega_2^{\alpha=0.05}} = \widehat{\Omega_2^{(975)}}$, because a $\alpha = 0.05$, $\frac{\alpha}{2} = 0.025 = \frac{25}{1000}$, $1 - \frac{\alpha}{2} = 0.975 = \frac{975}{1000}$: The null hypothesis is rejected if $\widehat{\Omega_2} < \widehat{\Omega_2^{(25)}}$ or $\widehat{\Omega_2} > \widehat{\Omega_2^{(975)}}$ and accepted if $\widehat{\Omega_2^{(25)}} < \widehat{\Omega_2} < \widehat{\Omega_2^{(975)}}$);

3) Create a sample of size n from the null distribution and determine if the test statistic's absolute value exceeds the crucial value;

4) The test's power is the rejection percentage after 1000 repetitions of Step 3.

Monte Carlo analyses are carried out to look at how well our test performs. The tests listed below are regarded as the competitors, and the power values of the suggested test are then contrasted with those of the rivals in Tables 3 and 4.

1) The McWilliams test [29] relies on $Ts^{(1)}$, the total number of runs.

2) The Baklizi test [7] relies on an adjusted runs test, as demonstrated by $Ts^{(2)}$.

3) $Ts^{(3)}$ represents the Wilcoxon Signed-Rank Test, which was proposed by Gibbons and Chakraborti [20].

4) Relies on the Wilcoxon two-sample test, represented by $Ts^{(4)}$, the Tajjudin test [44].

5) $Ts^{(5)}$ represents the Cheng and Balakrishnan test [13].

6) The Modarres test $Ts_p^{(6)}$, where $p$ is a trimming proportion, is represented as [30].

7) $Ts_{n;p}^{(7)}$ represents the Baklizi test [8], where $n$ and $p$ represent the sample size and a trimming proportion, respectively.

8) $Ts^{(8)}$, the second Baklizi test [8].

9) The Baklizi test $Ts^{(9)}$, as reported in [9].

10) $Ts^{(10)}$ represents the Corzo and Babativa [14].

11) The Noughabi and Jarrahiferiz [36] are based on $Ts^{(11)}$, which is the extropy of order statistics test.

**Table 4.** Comparison of the test's power analysis at the significance level 0.05.

| Alternative | n | $Ts^{(1)}$ | $Ts^{(2)}$ | $Ts^{(3)}$ | $Ts^{(4)}$ | $Ts^{(5)}$ | $Ts^{(6)}_{25}$ | $Ts^{(6)}_{60}$ | $Ts^{(7)}_{n;0}$ | $Ts^{(7)}_{n;0.8}$ | $Ts^{(8)}$ | $Ts^{(9)}$ | $Ts^{(10)}$ | $Ts^{(11)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Case 1(H0) | 20 | 0.046 | 0.051 | 0.043 | 0.045 | 0.048 | 0.047 | 0.046 | 0.051 | 0.055 | 0.055 | 0.046 | 0.056 | 0.049 |
| | 30 | 0.049 | 0.053 | 0.052 | 0.051 | 0.052 | 0.051 | 0.054 | 0.051 | 0.054 | 0.048 | 0.046 | 0.047 | 0.048 |
| | 50 | 0.051 | 0.054 | 0.050 | 0.052 | 0.051 | 0.049 | 0.049 | 0.049 | 0.051 | 0.058 | 0.051 | 0.046 | 0.049 |
| | 100 | 0.051 | 0.047 | 0.052 | 0.048 | 0.052 | 0.053 | 0.051 | 0.055 | 0.054 | 0.049 | 0.051 | 0.048 | 0.049 |
| Case 2 | 20 | 0.052 | 0.057 | 0.051 | 0.051 | 0.054 | 0.054 | 0.053 | 0.057 | 0.062 | 0.046 | 0.058 | 0.070 | 0.097 |
| | 30 | 0.052 | 0.051 | 0.055 | 0.056 | 0.061 | 0.053 | 0.055 | 0.051 | 0.063 | 0.058 | 0.062 | 0.061 | 0.130 |
| | 50 | 0.055 | 0.056 | 0.052 | 0.060 | 0.070 | 0.062 | 0.066 | 0.058 | 0.062 | 0.053 | 0.075 | 0.068 | 0.201 |
| | 100 | 0.054 | 0.051 | 0.055 | 0.071 | 0.091 | 0.057 | 0.062 | 0.053 | 0.066 | 0.065 | 0.106 | 0.084 | 0.324 |
| Case 3 | 20 | 0.067 | 0.075 | 0.055 | 0.079 | 0.080 | 0.079 | 0.087 | 0.057 | 0.088 | 0.070 | 0.114 | 0.112 | 0.667 |
| | 30 | 0.074 | 0.075 | 0.062 | 0.097 | 0.119 | 0.094 | 0.109 | 0.069 | 0.128 | 0.088 | 0.156 | 0.125 | 0.809 |
| | 50 | 0.089 | 0.094 | 0.064 | 0.131 | 0.204 | 0.120 | 0.153 | 0.075 | 0.145 | 0.141 | 0.253 | 0.206 | 0.920 |
| | 100 | 0.113 | 0.109 | 0.088 | 0.224 | 0.366 | 0.169 | 0.217 | 0.122 | 0.228 | 0.233 | 0.486 | 0.356 | 0.988 |
| Case 4 | 20 | 0.090 | 0.103 | 0.061 | 0.106 | 0.118 | 0.122 | 0.142 | 0.072 | 0.138 | 0.087 | 0.187 | 0.177 | 0.038 |
| | 30 | 0.114 | 0.122 | 0.070 | 0.149 | 0.219 | 0.166 | 0.199 | 0.100 | 0.229 | 0.142 | 0.287 | 0.243 | 0.071 |
| | 50 | 0.143 | 0.154 | 0.085 | 0.209 | 0.428 | 0.234 | 0.301 | 0.144 | 0.303 | 0.314 | 0.499 | 0.443 | 0.160 |
| | 100 | 0.216 | 0.209 | 0.127 | 0.385 | 0.757 | 0.406 | 0.522 | 0.333 | 0.572 | 0.595 | 0.818 | 0.750 | 0.567 |
| Case 5 | 20 | 0.115 | 0.131 | 0.067 | 0.133 | 0.155 | 0.162 | 0.190 | 0.095 | 0.165 | 0.120 | 0.254 | 0.235 | 0.992 |
| | 30 | 0.151 | 0.160 | 0.080 | 0.194 | 0.309 | 0.232 | 0.287 | 0.131 | 0.333 | 0.219 | 0.404 | 0.343 | 0.998 |
| | 50 | 0.197 | 0.213 | 0.103 | 0.287 | 0.587 | 0.342 | 0.437 | 0.230 | 0.457 | 0.455 | 0.668 | 0.602 | 1.000 |
| | 100 | 0.321 | 0.316 | 0.166 | 0.522 | 0.890 | 0.566 | 0.696 | 0.556 | 0.769 | 0.784 | 0.939 | 0.885 | 1.000 |
| Case 6 | 20 | 0.200 | 0.234 | 0.072 | 0.160 | 0.256 | 0.346 | 0.396 | 0.136 | 0.267 | 0.191 | 0.420 | 0.468 | 0.454 |
| | 30 | 0.303 | 0.330 | 0.095 | 0.231 | 0.606 | 0.558 | 0.671 | 0.256 | 0.649 | 0.469 | 0.653 | 0.715 | 0.610 |
| | 50 | 0.497 | 0.524 | 0.122 | 0.364 | 0.950 | 0.825 | 0.920 | 0.642 | 0.908 | 0.914 | 0.894 | 0.972 | 0.742 |
| | 100 | 0.782 | 0.782 | 0.198 | 0.633 | 1.000 | 0.989 | 0.998 | 0.995 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
| Case 7 | 20 | 0.311 | 0.358 | 0.096 | 0.281 | 0.421 | 0.511 | 0.578 | 0.226 | 0.330 | 0.314 | 0.593 | 0.644 | 0.997 |
| | 30 | 0.457 | 0.490 | 0.123 | 0.393 | 0.797 | 0.750 | 0.828 | 0.444 | 0.823 | 0.689 | 0.854 | 0.868 | 0.999 |
| | 50 | 0.683 | 0.707 | 0.185 | 0.600 | 0.991 | 0.941 | 0.977 | 0.860 | 0.978 | 0.980 | 0.980 | 0.994 | 1.000 |
| | 100 | 0.928 | 0.927 | 0.358 | 0.883 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| Case 8 | 20 | 0.373 | 0.426 | 0.105 | 0.330 | 0.494 | 0.594 | 0.656 | 0.295 | 0.366 | 0.389 | 0.666 | 0.715 | 0.999 |
| | 30 | 0.539 | 0.570 | 0.150 | 0.484 | 0.861 | 0.819 | 0.878 | 0.555 | 0.876 | 0.790 | 0.913 | 0.915 | 1.000 |
| | 50 | 0.761 | 0.782 | 0.233 | 0.697 | 0.996 | 0.970 | 0.989 | 0.930 | 0.991 | 0.991 | 0.993 | 0.998 | 1.000 |
| | 100 | 0.966 | 0.965 | 0.420 | 0.947 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| Case 9 | 20 | 0.399 | 0.452 | 0.112 | 0.351 | 0.530 | 0.631 | 0.696 | 0.322 | 0.359 | 0.428 | 0.692 | 0.752 | 0.998 |
| | 30 | 0.580 | 0.614 | 0.152 | 0.498 | 0.877 | 0.848 | 0.900 | 0.608 | 0.898 | 0.821 | 0.924 | 0.929 | 1.000 |
| | 50 | 0.802 | 0.821 | 0.241 | 0.725 | 0.997 | 0.979 | 0.992 | 0.953 | 0.993 | 0.995 | 0.995 | 0.999 | 1.000 |
| | 100 | 0.980 | 0.980 | 0.441 | 0.956 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

### *3.2. Real data set*

To demonstrate our process, we take into consideration the data from Cobb [18]. The dataset below includes observations of the Nile River's yearly flow at Aswan between 1871 and 1970. Plots of the data's histogram and kernel density estimation are displayed in Figure 5, and the Q-Q plot is in Figure 6.

The numbers are 1120, 1160, 963, 1210, 1160, 1160, 813, 1230, 1370, 1140, 995, 935, 1110, 994, 1020, 960, 1180, 799, 958, 1140, 1100, 1210, 1150, 1250, 1260, 1220, 1030, 1100, 774, 840, 874, 694, 940, 833, 701, 916, 692, 1020, 1050, 969, 831, 726, 456, 824, 702, 1120, 1100, 832, 764, 821, 768, 845, 864, 862, 698, 845, 744, 796, 1040, 759, 781, 865, 845, 944, 984, 897, 822, 1010, 771, 676, 649, 846, 812, 742, 801, 1040, 860, 874, 848, 890, 744, 749, 838, 1050, 918, 986, 797, 923, 975, 815, 1020, 906, 901, 1170, 912, 746, 919, 718, 714, 740.

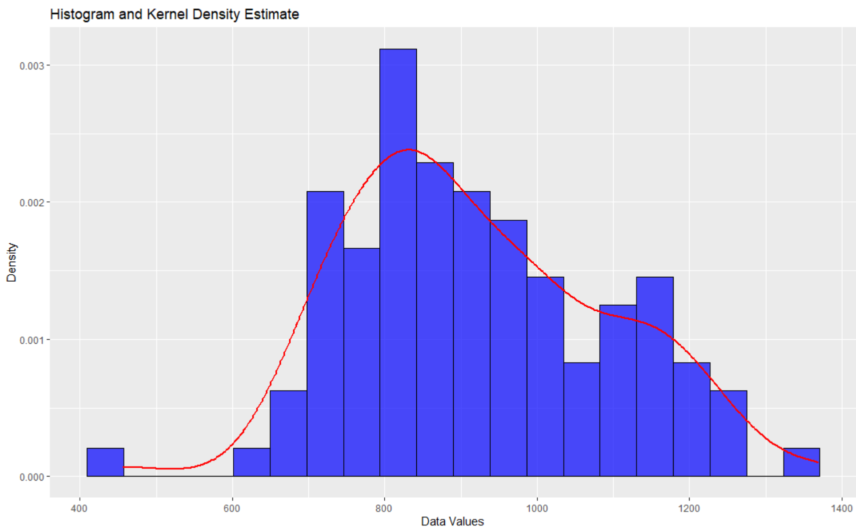


**Figure 5.** The data set's histogram and kernel density estimation.

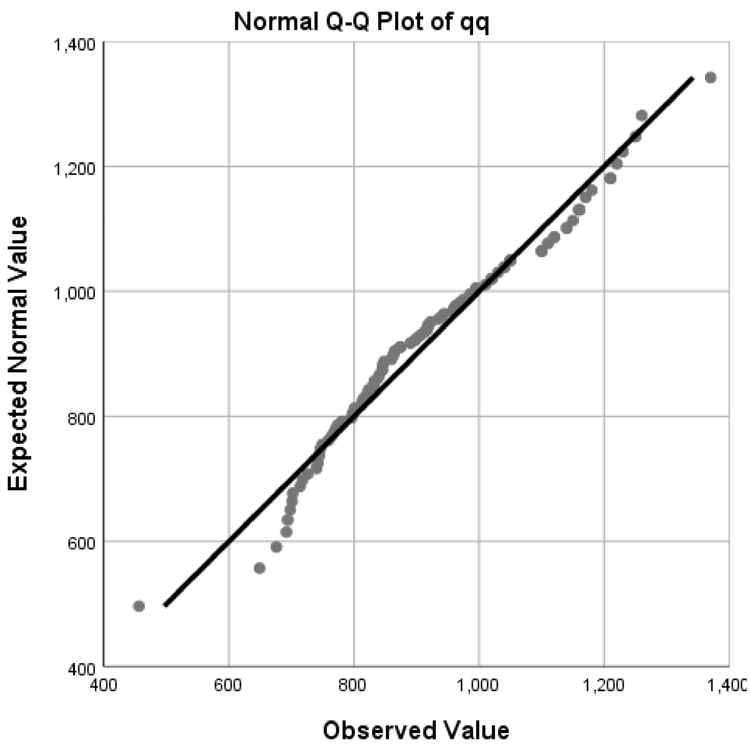


**Figure 6.** Q-Q plot for the data.

With a kurtosis of 2.695093 and a skewness of 0.3223697, the data is roughly symmetric. The symmetry hypothesis may be explored through our process. The values of the test statistic are $|\widehat{\Omega_2}|$ =

0.10484 at $\theta = 0.5$, $|\widehat{\Omega_2}| = 0.292881$ at $\theta = 0.8$, $|\widehat{\Omega_2}| = 0.509922$ at $\theta = 1$, and $|\widehat{\Omega_2}| = 1.56183$ at $\theta = 1.5$. These correspond to p-values of 0.9165028, 0.7696131, 0.6101061, and 0.118328, respectively. As a result, the symmetry hypothesis is confirmed.

## 4. Comparative analysis study

This study proposed a symmetry test statistic based on the fractional generalized entropy of order statistics, where the spacing between the first and last-order statistics of the measure is proved to be symmetric if it vanishes. After generating 50,000 samples, the plots of the PDFs of the proposed test statistic with different sample sizes show that the PDF plot becomes more symmetric as $n$ increases. Under the generalized lambda distribution with nine different cases as alternative distributions, a comparison research with eleven competitor tests was carried out, and the test statistic's performance was examined using the power values calculated by Monte Carlo simulation techniques. (noting that the $11th$ test is the extropy test statistic). As expected, the powers of all the tests in Case 1 of Tables 3 and 4 are close to 0.05, indicating a symmetric distribution. The corresponding distribution is asymmetric in the other eight examples, with the exception of cases 2 and 3, which are almost symmetric. Depending on the different values of $\theta$ in our test statistic, the power of the test varies. In general, we find that when $\theta = 2, 4$, which are even values, the test statistic gives the best results in most cases. We can interpret the optimal performance of the test statistic at $\theta = 2, 4$: the model or test performs best when $\theta$ is positive, making negative values irrelevant for the analysis. Moreover, the extropy and fractional generalized entropy tests (where $\theta$ is an even value) exhibit outstanding power, and there are notable variations in power values between the suggested tests and the rival tests. Therefore, we anticipate that the suggested test will outperform the competing tests in a wide range of practical applications. Additionally, a real-world data set has been used to assess the test procedure's ability to detect symmetric nature.

## 5. Conclusions and future work

In this consideration, we have presented the fractional generalized model of the entropy measure. Some stochastic comparisons and characterizations of the measure of order statistics and *nth* upper r.v.'s have been discussed. Furthermore, monotonic characteristics, certain symmetric qualities, and the circumstances in which the fractional generalized entropy of order statistics and r.v.'s may uniquely indicate their parent distributions have been provided. Based on the fractional generalized entropy measure of order statistics, we have examined the test of symmetry. One benefit of this test is that it eliminates the need to determine the symmetry's center. After conducting a thorough empirical investigation, we have demonstrated that the test based on fractional generalized entropy can be compared with other competing tests by changing the values of $\theta$ and that there are significant variations in the test's power values. All things considered, the simulation research indicates that our suggested test, which is based on the fractional generalized entropy of order statistics, works well, particularly when $\theta = 2, 4$, which are even values. Therefore, we anticipate that the suggested test will outperform the competing tests in a wide range of applications in real-world, which can be seen in the presented real data example. In future work, we could extend the fractional generalized entropy to other tests of hypothesis, such as the test of uniformity, as mentioned in [41]. Moreover, we could implement this model for the concomitants of ordered variables, as mentioned in [24, 31, 32]. In addition, we could connect this work with Pythagorean fuzzy information, as mentioned in [50]. See also [1, 4, 5, 33].

## Author contributions

M. S. Mohamed, M. A. Almuqrin: methodology, conceptualization, investigation, software, resources, writing-original draft, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

This essay was written without the help of artificial intelligence (AI) techniques, according to the authors.

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## Conflict of interest

The authors declare no conflict of interest.

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