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Research article

Integral expressions of solutions to higher order λ-weighted Dirac equations valued in the parameter dependent Clifford algebra

Xiaojing Du, Xiaotong Liang and Yonghong Xie*

School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China

* **Correspondence:** Email: xyh1973@126.com.

Abstract: First, some important properties of functions valued in the parameter dependent Clifford algebra $\mathscr{B}_n(2, \alpha_j, \gamma_{ij})$ were studied. Second, we obtained Cauchy-Pompeiu formulae for functions valued in $\mathscr{B}_n(2, \alpha_j, \gamma_{ij})$ and the integral representation of solutions to the higher order λ-weighted Dirac equation, respectively. Finally, the integral representation of solutions to bilateral higher order λ-weighted Dirac equations was derived.

Keywords: integral expression of solutions; higher order λ-weighted Dirac equations; parameter dependent Clifford algebras
Mathematics Subject Classification: 30E20, 35G35, 45E05

1. Introduction

In 1878, Clifford algebra was defined in [1]. In 1982, Brackx et al. [2] generalized some results of the complex analysis to Clifford analysis. Malonek and Ren [3] studied the Almansi-type decomposition theorems for the k-order monogenic functions and k-order λ-weighted monogenic functions in 2002. In the unweighted case, the star-like condition of the domain is needed. This fact accounts for the greater generality of the decomposition in the weighted case, which indeed holds in any domain. When $k = 1$, the origin of the notion of λ-weighted monogenic functions is given. In 2017, García et al. [4] studied an integral representation for the solution to the sandwich Dirac equation in Clifford analysis. Yang et al. [5] obtained the Cauchy theorem for the solution to the k-order Dirac equation with α-weight in 2018, where $k > 0$ is an integer and α is a nonzero real number. In 2020, Blaya et al. [6] gave the integral representation for the solution of the bilateral higher-order Dirac equation and proved some properties for Cauchy and Teodorescu transforms. In 2022, Peláez et al. [7] took the sum of the left Dirac operator multiplied by α and the right Dirac operator multiplied by β as a new operator, and studied the integral representation of solutions to higher-order new operators, where α, β are real numbers. In 2023, Dinh [8] introduced (α, β)-monogenic functions and isotonic functions, where α, β are real numbers and $\alpha \neq \beta$; they gave the integral representation formulae of these functions respectively by using the new proof method and proved the series representation of polynomial Dirac equations. In 2024, Gao et al. [9] got an integral representation for the solution of the bilateral higher order Dirac equation with α-weight, where α is a nonzero real number. Liu et al. [10] investigated some Riemann-Hilbert boundary value problems for perturbed Dirac operators in the Clifford algebra $Cl(V_{3,3})$. D. A. Santiesteban et al. [11] examined well-posed boundary value problems for second-order elliptic systems of partial differential equations in bounded regular domains of Euclidean spaces.

In 2008, Clifford algebras depending on parameters emerged as an extension of the classical Clifford algebra. Its applications in partial differential equations were introduced by Tutschke and Vanegas [12]. In 2012, Di Teodoro et al. [13] studied solutions for the first order homogeneous meta-Dirac equation and then gave a solution of the inhomogeneous equation by using Fubini′s theorem. In 2013, the integral representations for the meta-Dirac operator of n-order and its conjugate operators of n-order are derived by Balderrama et al. [14]. In 2014, some achievements of hypercomplex analysis were expounded and some of its development trends were presented in reference [15]. Ariza et al. [16] gave the integral formulae to solutions for second order elliptic Dirac equation in 2015. In 2017, Ariza García et al. [17] obtained the correlation between first-order differential operators and q-Dirac operators, with the aim of studying initial value problems, where q is a n-dimensional vector. In 2021, Cuong et al. [18] studied the integral expression of monogenic functions in the Clifford algebra depending on three parameters and solved two boundary value problems related to this function.

Based on the above work, we have conducted certain work with the aim of extending the results from the classical Clifford algebra to the framework of parameter dependent Clifford algebra. In Section 2, we investigate some important properties of functions valued in this Clifford algebra. In Section 3, integral representations for $\mathfrak{p}$-order λ-weighted monogenic functions and right $\mathfrak{q}$-order λ-weighted monogenic functions are derived. Furthermore, in Section 4, we present an integral representation for $(\mathfrak{p} + \mathfrak{q})$-order λ-weighted monogenic functions. Finally, Section 5 contains the conclusion and discussion of this paper. This paper mainly generalizes some results of references [5, 9].

2. Preliminaries

In this section, we present some basic results on the parameter dependent Clifford analysis, meanwhile, we prove some important properties of some functions valued in the parameter dependent Clifford algebra.

2.1. Some basic results on the parameter dependent Clifford analysis

Suppose that α_j, $\gamma_{ij} = \gamma_{ji}$ are nonnegative real numbers for $i, j = 1, 2, \ldots, n, i \neq j$, the set of base element is $\{e_0 = 1, e_1, \ldots, e_n\}$, and the base element satisfies the following multiplication rule

$$\begin{cases} e_j^2 = -\alpha_j, \\ e_i e_j + e_j e_i = 2\gamma_{ij}. \end{cases} \tag{2.1}$$

From this, we obtain a parameter dependent Clifford algebra $\mathcal{B}_n(2, \alpha_j, \gamma_{ij})$ which is generated by the structural relationship (2.1). Every element of the algebra is of the form $c = \sum_{A_1} c_{A_1} e_{A_1}$, $c_{A_1} \in \mathbf{R}$, where $A_1 := \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$, $j_1 < j_2 < \cdots < j_k$, $e_{A_1} = e_{j_1} \cdots e_{j_k}$, and $e_0 = e_\varnothing = 1$. As indices we use the elements A_1 of the set containing the ordered subsets of $\{1, 2, \ldots, n\}$, with the empty subset corresponding to the index 0. The set A_1 runs over all the possible ordered sets $A_1 = \{1 \le j_1 < \ldots < j_k \le n\}$, or $A_1 = \varnothing$. The dimension of this algebra is 2^n.

Let $\mathbf{N}^*$ be the set of positive integers. If $1 \le j \le n$ and $j \in \mathbf{N}^*$, the base element satisfies the involution $\overline{e_j} = -e_j$. If $e_A = e_{h_1 \cdots h_r} = e_{h_1} \cdots e_{h_r}$, then $\overline{e_A} = \overline{e_{h_r}} \cdots \overline{e_{h_1}} = (-1)^r e_{h_r} \cdots e_{h_1}$. For any $\xi = \sum_{A_1} \xi_{A_1} e_{A_1} \in \mathcal{B}_n(2, \alpha_j, \gamma_{ij})$, we define $\overline{\xi} = \sum_{A_1} \xi_{A_1} \overline{e_{A_1}}$, $|\xi|^2 = \sum_{A_1} \xi_{A_1}^2$, where $\xi_{A_1} \in \mathbf{R}$.

The Euclidean Clifford algebra $\mathcal{B}_n(2, 1, 0)$ is one of the special cases of $\mathcal{B}_n(2, \alpha_j, \gamma_{ij})$.

The function $f : \Omega \to \mathcal{B}_n(2, \alpha_j, \gamma_{ij})$ is denoted by $f(x) = \sum_A f_{A_1}(x) e_{A_1}$, where $f_{A_1}(x)$ is a real-valued function and Ω is an open connected bounded domain in $\mathbf{R}^n$. f is a r-times continuously differentiable function, which means f_{A_1} is a r-times continuously differentiable function, where $r \in \mathbf{N}^*$. The set consisting of the r-times continuously differentiable function is denoted by $\mathbf{F}^r(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$.

When $f \in \mathbf{F}^1(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, Dirac operators and its conjugate operators acting on function f are defined respectively as follows:

$$D_x f = \sum_{k=1}^n e_k \frac{\partial f}{\partial x_k}, \ f D_x = \sum_{k=1}^n \frac{\partial f}{\partial x_k} e_k, \ \overline{D_x} f = \sum_{k=1}^n \overline{e_k} \frac{\partial f}{\partial x_k}, \ f \overline{D_x} = \sum_{k=1}^n \frac{\partial f}{\partial x_k} \overline{e_k}.$$

After a direct calculation, we have

$$D_x \overline{D_x} = \overline{D_x} D_x = \sum_{j=1}^n \alpha_j \partial_j^2 - 2 \sum_{1 \le i < j \le n} \gamma_{ij} \partial_i \partial_j,$$

the corresponding quadratic form is

$$\sum_{j=1}^n \alpha_j \xi_j^2 - 2 \sum_{1 \le i < j \le n} \gamma_{ij} \xi_i \xi_j, \tag{2.2}$$

which has a coefficient matrix

$$B = \begin{pmatrix} \alpha_1 & -\gamma_{12} & \cdots & -\gamma_{1n} \\ -\gamma_{12} & \alpha_2 & \cdots & -\gamma_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ -\gamma_{1n} & -\gamma_{2n} & \cdots & \alpha_n \end{pmatrix}. \tag{2.3}$$

Denote

$$B_1 = \alpha_1, B_2 = \begin{pmatrix} \alpha_1 & -\gamma_{12} \\ -\gamma_{12} & \alpha_2 \end{pmatrix}, B_3 = \begin{pmatrix} \alpha_1 & -\gamma_{12} & -\gamma_{13} \\ -\gamma_{12} & \alpha_2 & -\gamma_{12} \\ -\gamma_{13} & -\gamma_{12} & \alpha_3 \end{pmatrix}, \cdots, B_n = B.$$

See references [19, 20]. By using the Sylvester′s criterion, (2.2) is a positive definite quadratic form if and only if the determinant of each B_j is a positive number for all $j = 1, 2, \ldots, n$, i.e.,

$$det(B_j) > 0. \tag{2.4}$$

In this situation, $D_x \overline{D_x} = \overline{D_x} D_x$ becomes an elliptic Dirac operator, so we denote $D_x \overline{D_x} = \overline{D_x} D_x$ by $\widetilde{\Delta}_n$.

Suppose that (2.4) holds in this paper, then the inverse matrix of matrix B exists and can be represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}, \tag{2.5}$$

where $a_{ij} = a_{ji}$, $i, j = 1, 2, \ldots, n$.

See reference [12]. For two points $x = (x_1, \cdots, x_n)$ and $\zeta = (\zeta_1, \cdots, \zeta_n)$ in $\mathbf{R}^n$, $x \neq \zeta$, the representation of the non-Euclidean distance ρ as follows:

$$\rho^2 := \rho^2(x, \zeta) = \sum_{i,j=1}^{n} a_{ij}(x_i - \zeta_i)(x_j - \zeta_j), \tag{2.6}$$

the representation of the Euclidean distance is $\iota = |x - \zeta|$.

See reference [14]. Suppose that for some $Y \in \mathbf{R}^n$ and Y satisfying $|Y| = 1$, we denote $x - \xi = \iota Y$, then the infimum of $\rho(Y, 0)$ for all Y is positive, i.e., $\rho^2(Y, 0) \geq c_0 > 0$, where c_0 is a constant, so $\rho^2(x, \xi) \geq c_0 \iota^2$.

For $f \in \mathbf{F}^1(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, the first-order λ-weighted Dirac operators acting on the function f are defined as follows:

$$D_x^\lambda f = \rho_x^{-\lambda} H(x)(D_x f), \ f D_x^\lambda = (f D_x) \rho_x^{-\lambda} H(x),$$

where $\rho_x = \left(\sum_{i,j=1}^n a_{ij} x_i x_j \right)^{\frac{1}{2}}$, $H(x) = \sum_{i,j=1}^n \overline{e_i} a_{ij} x_j$, and λ is a fixed nonzero real number.

Definition 2.1. *[12] Suppose* $f \in \mathbf{F}^1(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, *then a solution* f *of the Dirac equation* $D_x f(x) = 0$ *(*$f(x) D_x = 0$*) is called a left (right) monogenic function.*

See reference [12]. We know that $\rho_x^{-n} H(x)$ is not only a left monogenic function but also a right monogenic function.

Definition 2.2. *Suppose* $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $\mathfrak{p}, \mathfrak{q}$ *are positive integers.*

(i) A solution f *of the* $\mathfrak{p}$*-order* λ*-weighted Dirac equation* $(D_x^\lambda)^{\mathfrak{p}} f(x) = 0$ *is called a left* $\mathfrak{p}$*-order* λ*-weighted monogenic function, where* $(D_x^\lambda)^{\mathfrak{p}} = D_x^\lambda \circ \cdots \circ D_x^\lambda$ *(*$\mathfrak{p}$*-times),* $\circ$ *is a composite operation of operators.*

(ii) A solution f *of the right* $\mathfrak{q}$*-order* λ*-weighted equation* $f(x)(D_x^\lambda)^{\mathfrak{q}} = 0$ *is called a right* $\mathfrak{q}$*-order* λ*-weighted monogenic function, where* $(D_x^\lambda)^{\mathfrak{q}} = D_x^\lambda \circ \cdots \circ D_x^\lambda$ *(*$\mathfrak{q}$*-times).*

Remark 2.1. *(i) When* $\mathfrak{p} = 1 (\mathfrak{q} = 1)$ *in Definition 2.2, a solution* f *of the* λ*-weighted Dirac equation* $D_x^\lambda f(x) = 0$ $(f(x) D_x^\lambda = 0)$ *is called a left (right)* λ*-weighted monogenic function.*

(ii) A left $\mathfrak{p}$*-order* λ*-weighted monogenic function can be called a* $\mathfrak{p}$*-order* λ*-weighted monogenic function for short. A left* λ*-weighted monogenic function can be called a* λ*-weighted monogenic function for short.*

Definition 2.3. *Suppose* $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $\mathfrak{p}$ *and* $\mathfrak{q}$ *are positive integers, then the solution* f *of equation* $\left((D_x^\lambda)^{\mathfrak{p}} f(x)\right)(D_x^\lambda)^{\mathfrak{q}} = 0$ *is called a* $(\mathfrak{p} + \mathfrak{q})$*-order* λ*-weighted monogenic function.*

If f is a left $\mathfrak{p}$-order λ-weighted monogenic function, then f is a $(\mathfrak{p}+\mathfrak{q})$-order monogenic λ-weighted function.

Remark 2.2. *$\rho_x^{-n}H(x)$, $\rho_x^{-n+(\mathfrak{p}-1)\lambda}H(x)$, and $\rho_x^{-n+(\mathfrak{p}+\mathfrak{q}-1)\lambda}H(x)$ are $(\mathfrak{p}+\mathfrak{q})$-order λ-weighted monogenic functions, where λ is a fixed nonzero real number.*

For $x \in \partial\Omega$, its outer unit normal vector is $N(x) = (N(x_1), \ldots, N(x_n)) = (N_1, \ldots, N_n)$, $d\sigma_x = \sum_{i=1}^{n} N_i e_i d\mu$ is the Clifford-algebra-valued measure element of $\partial\Omega$, $d\mu$ represents the scalar measure element of $\partial\Omega$, and $\partial\Omega$ is a sufficiently smooth boundary.

Lemma 2.1. *[12] Suppose $f, g \in \mathbf{F}^1(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, then*

$$\int_{\partial\Omega} f(x) d\sigma_x g(x) = \int_{\Omega} \Big[(f(x)D_x)g(x) + f(x)(D_x g(x))\Big] dx,$$

where $dx = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

Similar to the proof of the theorem in reference [21], we can prove that Lemma 2.2 holds.

Lemma 2.2. *Suppose f, $g \in \mathbf{F}^1(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, then*

$$D_x(f(x)g(x)) = (D_x f(x))g(x) + \sum_{k=1}^{n} e_k f(x) \frac{\partial g(x)}{\partial x_k},$$

$$(f(x)g(x))D_x = \sum_{k=1}^{n} \frac{\partial f(x)}{\partial x_k} g(x) e_k + f(x)(g(x)D_x).$$

Proof. We suppose that $f(x) = \sum_{A_1} f_{A_1}(x) e_{A_1}$, $g(x) = \sum_{A_2} g_{A_2}(x) e_{A_2}$, where $f_{A_1}(x)$ and $g_{A_2}(x)$ are real-valued functions, then

$$\begin{aligned}
D_x(f(x)g(x)) &= \sum_{k=1}^{n} e_k \frac{\partial\big(f(x)g(x)\big)}{\partial x_k} = \sum_{k=1}^{n} e_k \frac{\partial\Big[\Big(\sum_{A_1} f_{A_1}(x)e_{A_1}\Big)\Big(\sum_{A_2} g_{A_2}(x)e_{A_2}\Big)\Big]}{\partial x_k} \\
&= \sum_{k=1}^{n} e_k \sum_{A_1}\sum_{A_2} \frac{\partial\big(f_{A_1}(x)g_{A_2}(x)\big)}{\partial x_k} e_{A_1}e_{A_2} = \sum_{k=1}^{n} e_k \sum_{A_1}\sum_{A_2} \Big[\frac{\partial f_{A_1}(x)}{\partial x_k} g_{A_2}(x) + f_{A_1}(x)\frac{\partial g_{A_2}(x)}{\partial x_k}\Big] e_{A_1}e_{A_2} \\
&= \sum_{k=1}^{n} e_k \frac{\partial \sum_{A_1} f_{A_1}(x)e_{A_1}}{\partial x_k} \sum_{A_2} g_{A_2}(x)e_{A_2} + \sum_{k=1}^{n} e_k \sum_{A_1} f_{A_1}(x)e_{A_1} \frac{\partial \sum_{A_2} g_{A_2}(x)e_{A_2}}{\partial x_k} \\
&= (D_x f(x))g(x) + \sum_{k=1}^{n} e_k f(x) \frac{\partial g(x)}{\partial x_k}.
\end{aligned}$$

Similarly, we can prove the other equality. □

2.2. Some important properties of some functions valued in $\mathcal{B}_n(2, \alpha_j, \gamma_{ij})$

Suppose that $M_s^{\lambda}(x) = E_s(x)\rho_x^{-\lambda}H(x)$, where $E_s(x) = \frac{C_s}{\rho_x^{n-s\lambda}}$, $C_s = \frac{1}{\omega_n \lambda^{s-1}(s-1)!}$, $s \in \mathbf{N}^*$, ω_n represents the Euclidean surface measure of the unit sphere.

Proposition 2.1. *When $s > 1$, we have*

$$D_x M_s^{\lambda}(x) = M_s^{\lambda}(x) D_x = E_{s-1}(x).$$

Proof. Since $AB = E$ and E is the identity matrix, we obtain that for $m, k = 1, 2, \ldots, n$,

$$\alpha_k a_{km} - \sum_{i=1, i\neq k}^{n} \gamma_{ik} a_{im} = \begin{cases} 0, & m \neq k, \\ 1, & m = k, \end{cases}$$

then

$$\sum_{i,k=1}^{n} e_k \overline{e_i} a_{mk} a_{im} = \sum_{k=1}^{n} e_k \overline{e_k} a_{mk} a_{km} + \sum_{i,k=1,k<i}^{n} e_k \overline{e_i} a_{mk} a_{im} + \sum_{i,k=1,k>i}^{n} e_k \overline{e_i} a_{mk} a_{im}$$

$$= \sum_{k=1}^{n} e_k \overline{e_k} a_{mk} a_{km} + \sum_{i,k=1,k<i}^{n} e_k \overline{e_i} a_{mk} a_{im} + \sum_{j,l=1,j<l}^{n} e_l \overline{e_j} a_{ml} a_{jm} = \sum_{k=1}^{n} e_k \overline{e_k} a_{mk} a_{km} + \sum_{i,k=1,k<i}^{n} (e_k \overline{e_i} + e_i \overline{e_k}) a_{mk} a_{im}$$

$$= \sum_{k=1}^{n} \alpha_k a_{mk} a_{km} - 2 \sum_{i,k=1,k<i}^{n} \gamma_{ik} a_{mk} a_{im} = \sum_{k=1}^{n} \alpha_k a_{mk} a_{km} - \sum_{i,k=1,k\neq i}^{n} \gamma_{ik} a_{mk} a_{im}$$

$$= \sum_{k=1}^{n} \Big(\alpha_k a_{mk} - \sum_{i=1,k\neq i}^{n} \gamma_{ik} a_{im} \Big) a_{km} = \sum_{k=1}^{n} \delta_{km} a_{km} = a_{mm},$$

therefore,

$$\sum_{i,k,m=1}^{n} e_k \overline{e_i} a_{mk} a_{im} x_m^2 = \sum_{m=1}^{n} a_{mm} x_m^2.$$

Also,

$$\sum_{m,i,j,k=1,j\neq m}^{n} e_k \overline{e_i} a_{mk} a_{ij} x_m x_j = \sum_{m,i,j,k=1,j<m}^{n} e_k \overline{e_i} a_{mk} a_{ij} x_m x_j + \sum_{m,i,j,k=1,j<m}^{n} e_k \overline{e_i} a_{jk} a_{im} x_j x_m$$

$$= \sum_{m,j,k=1,j<m}^{n} e_k \overline{e_k} (a_{mk} a_{kj} + a_{jk} a_{km}) x_m x_j + \sum_{m,i,j,k=1,i<k,j<m}^{n} e_k \overline{e_i} (a_{mk} a_{ij} + a_{jk} a_{im}) x_m x_j$$

$$+ \sum_{m,i,j,k=1,i<k,j<m}^{n} e_i \overline{e_k} (a_{mk} a_{ij} + a_{jk} a_{im}) x_m x_j$$

$$= \sum_{m,j,k=1,j<m}^{n} \alpha_k (a_{mk} a_{kj} + a_{jk} a_{km}) x_m x_j - 2 \sum_{m,i,j,k=1,i<k,j<m}^{n} \gamma_{ik} (a_{mk} a_{ij} + a_{jk} a_{im}) x_m x_j$$

$$= \sum_{m,j,k=1,j<m}^{n} \alpha_k (a_{mk} a_{kj} + a_{jk} a_{km}) x_m x_j - \sum_{m,i,j,k=1,i\neq k,j<m}^{n} \gamma_{ik} (a_{mk} a_{ij} + a_{jk} a_{im}) x_m x_j$$

$$= \sum_{m,k,j=1,j<m}^{n} \Big(\alpha_k a_{mk} - \sum_{i=1,i\neq k}^{n} \gamma_{ik} a_{im} \Big) a_{kj} x_m x_j + \sum_{m,k,j=1,j<m}^{n} \Big(\alpha_k a_{jk} - \sum_{i=1,i\neq k}^{n} \gamma_{ik} a_{ij} \Big) a_{km} x_m x_j$$

$$= \sum_{m,k,j=1,j<m}^{n} \delta_{km} a_{kj} x_m x_j + \sum_{m,k,j=1,j<m}^{n} \delta_{kj} a_{km} x_m x_j = \sum_{m,j=1,j\neq m}^{n} a_{mj} x_m x_j.$$

Consequently,

$$\overline{H(x)}H(x) = \Big(\sum_{i,j=1}^{n} e_i a_{ij} x_j\Big)\Big(\sum_{k,m=1}^{n} \overline{e_k} a_{km} x_m\Big) = \Big(\sum_{i,k,m=1}^{n} e_k\overline{e_i} a_{mk} a_{im} x_m^2\Big) + \Big(\sum_{m,i,j,k=1,m\neq j}^{n} e_k\overline{e_i} a_{mk} a_{ij} x_m x_j\Big)$$

$$= \sum_{m=1}^{n} a_{mm} x_m^2 + \sum_{j,m=1,m\neq j}^{n} a_{mj} x_m x_j = \rho_x^2.$$

By $\overline{H(x)} = -H(x)$, we can conclude that $H(x)\overline{H(x)} = -H^2(x) = \rho_x^2$.

By $AB = E$, we have

$$D_x H(x) = \sum_{i,j=1}^{n} e_i a_{ji} \overline{e_j} = \sum_{i=1}^{n} a_{ii} e_i \overline{e_i} + \sum_{i,j=1,i<j}^{n} a_{ij}\big(e_i\overline{e_j} + e_j\overline{e_i}\big)$$

$$= \sum_{j=1}^{n} a_{jj}\alpha_j - \sum_{i,j=1,i\neq j}^{n} a_{ij}\gamma_{ij} = \sum_{j=1}^{n}\Big(a_{jj}\alpha_j - \sum_{i=1,i\neq j}^{n} a_{ij}\gamma_{ij}\Big) = n.$$

Similarly, we can prove that $H(x)D = n$.

By equalities $\overline{H(x)}H(x) = \rho_x^2$ and $D_x H(x) = n$, we can conclude that

$$D_x M_s^{\lambda}(x) = C_s\Big[\Big(D_x \rho_x^{-n+(s-1)\lambda}\Big)H(x) + \rho_x^{-n+(s-1)\lambda}\Big(D_x H(x)\Big)\Big]$$

$$= C_s\Big[\sum_{k=1}^{n} e_k \frac{\partial\Big(\sum_{i,j=1}^{n} a_{ij}x_i x_j\Big)^{\frac{-n+(s-1)\lambda}{2}}}{\partial x_k} H(x) + n\rho_x^{-n+(s-1)\lambda}\Big]$$

$$= C_s\Big[\sum_{k=1}^{n} e_k \frac{-n+(s-1)\lambda}{2}\Big(\sum_{i,j=1}^{n} a_{ij}x_i x_j\Big)^{\frac{-n+(s-1)\lambda}{2}-1}\Big(\sum_{i=1}^{n} a_{ik}x_i + \sum_{j=1}^{n} a_{kj}x_j\Big)H(x) + n\rho_x^{-n+(s-1)\lambda}\Big]$$

$$= C_s\Big[\sum_{k=1}^{n} e_k\Big(-n+(s-1)\lambda\Big)\rho_x^{-n+(s-1)\lambda-2}\sum_{i=1}^{n} a_{ik}x_i H(x) + n\rho_x^{-n+(s-1)\lambda}\Big]$$

$$= C_s\Big[\Big(-n+(s-1)\lambda\Big)\rho_x^{-n+(s-1)\lambda-2}\sum_{i,k=1}^{n} a_{ik}x_i e_k H(x) + n\rho_x^{-n+(s-1)\lambda}\Big]$$

$$= C_s\Big[\Big(-n+(s-1)\lambda\Big)\rho_x^{-n+(s-1)\lambda-2}\overline{H(x)}H(x) + n\rho_x^{-n+(s-1)\lambda}\Big]$$

$$= C_s(s-1)\lambda\rho_x^{-n+(s-1)\lambda} = E_{s-1}(x).$$

Similarly, we have $M_s^{\lambda}(x)D_x = E_{s-1}(x)$. □

Proposition 2.2. *Let $f \in \mathbf{F}^k(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $k \in \mathbf{N}^*$, $s = 1, 2, \ldots, k$.*

(1) *Suppose that f is a solution of the Dirac equation $D_x f = 0$, then*

$$(D_x^{\lambda})^s\Big(\rho_x^{k\lambda} f(x)\Big) = \frac{k!}{(k-s)!}\lambda^s \rho_x^{(k-s)\lambda} f(x).$$

(2) *Suppose that f is a solution of the Dirac equation $f D_x = 0$, then*

$$\Big(\rho_x^{k\lambda} f(x)\Big)(D_x^{\lambda})^s = \frac{k!}{(k-s)!}\lambda^s \rho_x^{(k-s)\lambda} f(x).$$

Proof. (1) When $s = 1$, by using the equality $H(x)\overline{H(x)} = \rho_x^2$, it is easy to deduce that

$$D_x\Big(\rho_x^{k\lambda} f(x)\Big) = (D\rho_x^{k\lambda})f(x) + \sum_{m=1}^{n} e_m \rho_x^{k\lambda} \frac{\partial f(x)}{\partial x_m} = \sum_{m=1}^{n} e_m \frac{\partial\Big(\sum\limits_{i,j=1}^{n} a_{ij}x_ix_j\Big)^{\frac{k\lambda}{2}}}{\partial x_m} f(x) + \rho_x^{k\lambda}\Big(Df(x)\Big)$$
$$= \sum_{m=1}^{n} e_m \frac{k\lambda}{2}\rho_x^{k\lambda-2}\Big(\sum_{i=1}^{n} a_{im}x_i + \sum_{j=1}^{n} a_{mj}x_j\Big)f(x) = k\lambda\rho_x^{k\lambda-2}\Big(\sum_{m,i=1}^{n} a_{im}x_ie_m\Big)f(x) = k\lambda\rho_x^{k\lambda-2}\overline{H(x)}f(x),$$

and

$$D_x^\lambda\Big(\rho_x^{k\lambda} f(x)\Big) = \rho_x^{-\lambda}H(x)\Big[D_x\Big(\rho_x^{k\lambda}f(x)\Big)\Big] = k\lambda\rho_x^{(k-1)\lambda}f(x).$$

We suppose that $(D_x^\lambda)^{s-1}\Big(\rho_x^{k\lambda}f(x)\Big) = \frac{k!}{(k-s+1)!}\lambda^{s-1}\rho_x^{(k-s+1)\lambda}f(x)$ holds, then

$$(D_x^\lambda)^s\Big(\rho_x^{k\lambda}f(x)\Big) = D_x^\lambda\Big(\frac{k!}{(k-s+1)!}\lambda^{s-1}\rho_x^{(k-s+1)\lambda}f(x)\Big) = \frac{k!}{(k-s+1)!}\lambda^{s-1}\rho_x^{-\lambda}H(x)\Big[D_x\Big(\rho_x^{(k-s+1)\lambda}f(x)\Big)\Big]$$
$$=\frac{k!}{(k-s+1)!}\lambda^{s-1}\rho_x^{-\lambda}H(x)\Big[(k-s+1)\lambda\rho_x^{(k-s+1)\lambda-2}\overline{H(x)}f(x)\Big] = \frac{k!}{(k-s)!}\lambda^s\rho_x^{(k-s)\lambda}f(x).$$

According to the mathematical induction, we get the conclusion.

Similarly, we can prove that (2) holds □

Proposition 2.3. *Suppose* $1 \le s \le \mathfrak{p}$, $1 \le t \le \mathfrak{q}$, *and* $1 \le k \le \mathfrak{p} + \mathfrak{q}$, *where* $s, t, k, \mathfrak{p}, \mathfrak{q} \in \mathbf{N}^*$.

(1) *Let* $f \in \mathbf{F}^{\mathfrak{p}}(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$ *be a solution of the equation* $D_xf = 0$. *Then,* $\rho_x^{(\mathfrak{p}-s)\lambda}f(x)$ *is a solution of the* $\mathfrak{p}$*-order* λ*-weighted Dirac equation* $(D_x^\lambda)^{\mathfrak{p}}f(x) = 0$.

(2) *Let* $f \in \mathbf{F}^{\mathfrak{q}}(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$ *be a solution of the right equation* $fD_x = 0$. *Then,* $\rho_x^{(\mathfrak{q}-t)\lambda}f(x)$ *is a solution of the right* $\mathfrak{q}$*-order* λ*-weighted Dirac equation* $f(x)(D_x^\lambda)^{\mathfrak{q}} = 0$.

(3) *Let* $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\Omega, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$ *be a solution of the equation system* $D_xf = 0$ *and* $fD_x = 0$. *Then,* $\rho_x^{(\mathfrak{p}+\mathfrak{q}-k)\lambda}f(x)$ *is a solution of the* $(\mathfrak{p} + \mathfrak{q})$*-order* λ*-weighted Dirac equation* $\Big((D_x^\lambda)^{\mathfrak{p}}f(x)\Big)(D_x^\lambda)^{\mathfrak{q}} = 0$.

Proof. By Proposition 2.2, (1) and (2) hold.

(3) (i) When $\mathfrak{q} \le k \le \mathfrak{p} + \mathfrak{q}$, i.e., $1 \le k - \mathfrak{q} \le \mathfrak{p}$, by (1) in Proposition 2.3, we conclude that $\rho_x^{(\mathfrak{p}-(k-\mathfrak{q}))\lambda}f(x)$ satisfies equation $(D_x^\lambda)^{\mathfrak{p}}\Big(\rho_x^{(\mathfrak{p}-(k-\mathfrak{q}))\lambda}f(x)\Big) = 0$. Therefore, (3) is clearly valid.

(ii) When $1 < k \le \mathfrak{q}$, as f satisfies condition $D_xf = 0$ and based on (1) in Proposition 2.2, we can deduce that

$$(D_x^\lambda)^{\mathfrak{p}}\Big(\rho_x^{(\mathfrak{p}+\mathfrak{q}-k)\lambda}f(x)\Big) = \frac{(\mathfrak{p}+\mathfrak{q}-k)!}{(\mathfrak{q}-k)!}\lambda^{\mathfrak{p}}\rho_x^{(\mathfrak{q}-k)\lambda}f(x).$$

As f satisfies condition $fD_x = 0$ and based on (2) in Proposition 2.3, we obtain

$$\Big[(D_x^\lambda)^{\mathfrak{p}}\Big(\rho_x^{(\mathfrak{p}+\mathfrak{q}-k)\lambda}f(x)\Big)\Big](D_x^\lambda)^{\mathfrak{q}} = \frac{(\mathfrak{p}+\mathfrak{q}-k)!}{(\mathfrak{q}-k)!}\lambda^{\mathfrak{p}}\Big[\Big(\rho_x^{(\mathfrak{q}-k)\lambda}f(x)\Big)(D_x^\lambda)^{\mathfrak{q}}\Big] = 0,$$

therefore, (3) is established. □

Theorem 2.1. *Let* $1 \le s \le \mathfrak{p}$, $1 \le t \le \mathfrak{q}$, $1 \le k \le \mathfrak{p} + \mathfrak{q}$, *where* $s, t, k, \mathfrak{p}, \mathfrak{q} \in \mathbf{N}^*$.

(1) $E_{\mathfrak{p}}(x)\rho_x^{-s\lambda}H(x)$ *is a solution of the* $\mathfrak{p}$*-order* λ*-weighted Dirac equation* $(D_x^\lambda)^{\mathfrak{p}}f(x) = 0$.

(2) $E_{\mathfrak{q}}(x)\rho_x^{-t\lambda}H(x)$ *is a solution of the right* $\mathfrak{q}$*-order* λ*-weighted Dirac equation* $f(x)(D_x^\lambda)^{\mathfrak{q}} = 0$.

(3) $E_{\mathfrak{p}+\mathfrak{q}}(x)\rho_x^{-k\lambda}H(x)$ *is a solution of the* $(\mathfrak{p}+\mathfrak{q})$*-order* λ*-weighted Dirac equation* $\Big((D_x^\lambda)^{\mathfrak{p}}f(x)\Big)(D_x^\lambda)^{\mathfrak{q}}=0$.

Proof. By Proposition 2.3, (1) and (2) hold.

(3) It is obvious that

$$E_{\mathfrak{p}+\mathfrak{q}}(x)\rho_x^{-k\lambda}H(x)=\frac{C_{\mathfrak{p}+\mathfrak{q}}}{\rho_x^{n-(\mathfrak{p}+\mathfrak{q})\lambda}}\rho_x^{-k\lambda}H(x)=\rho_x^{(\mathfrak{p}+\mathfrak{q}-k)\lambda}C_{\mathfrak{p}+\mathfrak{q}}\frac{H(x)}{\rho_x^n}.$$

By Proposition 2.3 and the equality $D_x(\rho_x^{-n}H(x))=(\rho_x^{-n}H(x))D_x=0$, we conclude that (3) in Theorem 2.1 is established. □

3. Integral representations for the solutions of the $\mathfrak{p}$-order λ-weighted Dirac equation and right $\mathfrak{q}$-order λ-weighted Dirac equation

In this section, we prove two Cauchy-Pompeiu integral formulae for functions valued in $\mathcal{B}_n(2,\alpha_j,\gamma_{ij})$, and obtain the Cauchy integral formulae for the null solution to higher order λ-weighted Dirac operators as their corollary, respectively.

In this paper, we denote $\{x|y_0=x+x_0\in\Omega\}$ as $\Omega_{x_0}^*$, for any $x_0\in\Omega$.

Theorem 3.1. *Let* $\mathfrak{p},\mathfrak{q}\in\mathbf{N}^*$, $s=0,1,\ldots,\mathfrak{p}$; $r=0,1,\ldots,\mathfrak{q}$.

(1) *If* $f\in\mathbf{F}^{\mathfrak{p}}(\overline{\Omega},\mathcal{B}_n(2,\alpha_j,\gamma_{ij}))$, *then for any* $x_0\in\overline{\Omega}$, *when* $0<\lambda<\frac{1}{\mathfrak{p}}$, $(D_x^\lambda)^s f(y_0)$ *is a bounded function in* $\overline{\Omega_{x_0}^*}$.

(2) *If* $f\in\mathbf{F}^{\mathfrak{q}}(\overline{\Omega},\mathcal{B}_n(2,\alpha_j,\gamma_{ij}))$, *then for any* $x_0\in\overline{\Omega}$, *when* $0<\lambda<\frac{1}{\mathfrak{q}}$, $f(y_0)(D_x^\lambda)^r$ *is a bounded function in* $\overline{\Omega_{x_0}^*}$.

Proof. (1) When $s=0$, as $f\in\mathbf{F}^{\mathfrak{p}}(\overline{\Omega},\mathcal{B}_n(2,\alpha_j,\gamma_{ij}))$, $(D_x^\lambda)^0 f(y_0)$ is a bounded function in $\overline{\Omega_{x_0}^*}$.

When $s=1,2,\ldots,\mathfrak{p}$, we denote $H(x)f_s(x)$ by $g_s(x)$, and let

$$\begin{aligned}
f_1(x)&=D_xf(y_0),\\
f_2(x)&=-\lambda f_1(x)+D_xg_1(x),\\
f_3(x)&=-2\lambda f_2(x)+D_xg_2(x),\\
&\cdots\\
f_{\mathfrak{p}-1}(x)&=-(\mathfrak{p}-2)\lambda f_{p-2}(x)+D_xg_{\mathfrak{p}-2}(x),\\
f_{\mathfrak{p}}(x)&=-(\mathfrak{p}-1)\lambda f_{\mathfrak{p}-1}(x)+D_xg_{\mathfrak{p}-1}(x).
\end{aligned}$$

As $f\in\mathbf{F}^{\mathfrak{p}}(\overline{\Omega},\mathcal{B}_n(2,\alpha_j,\gamma_{ij}))$, $f_1,f_2,...,f_{\mathfrak{p}}$ are bounded functions in $\overline{\Omega_{x_0}^*}$.

When $s=1$, we have $D_x^\lambda f(y_0)=\rho_x^{-\lambda}H(x)\Big(D_xf(y_0)\Big)=\rho_x^{-\lambda}g_1(x)$.

We suppose that $t<\mathfrak{p}$, $t\in\mathbf{N}^*$, and $(D_x^\lambda)^t f(y_0)=\rho_x^{-t\lambda}g_t(x)$, then

$$\begin{aligned}
(D_x^\lambda)^{t+1}f(y_0)&=\rho_x^{-\lambda}H(x)\Big[D_x\Big(\rho_x^{-t\lambda}g_t(x)\Big)\Big]\\
=&\rho_x^{-\lambda}H(x)\Big[\Big((-t\lambda)\rho_x^{-t\lambda-2}\overline{H(x)}\Big)g_t(x)+\rho_x^{-t\lambda}\Big(D_xg_t(x)\Big)\Big]\\
=&\rho_x^{-(t+1)\lambda}H(x)\Big[-t\lambda f_t(x)+D_x\Big(g_t(x)\Big)\Big]=\rho_x^{-(t+1)\lambda}g_{t+1}(x).
\end{aligned}$$

According to the mathematical induction, we get

$$(D_x^\lambda)^s f(y_0)=\rho_x^{-s\lambda}g_s(x).$$

So for any $x_0 \in \overline{\Omega}$, if $0 < \lambda < \frac{1}{s}$, then we conclude that $(D_x^\lambda)^s f(y_0)$ is bounded in $\overline{\Omega_{x_0}^*}$.

Hence, for any $x_0 \in \overline{\Omega}$, when $\lambda \in \bigcap_{s=1}^{\mathfrak{p}} \left(0, \frac{1}{s}\right) = \left(0, \frac{1}{\mathfrak{p}}\right)$, we conclude that $(D_x^\lambda)^s f(y_0)$ is bounded in $\overline{\Omega_{x_0}^*}$, where $s = 1, 2, \ldots, \mathfrak{p}$.

Similarly, we can prove that (2) holds. □

Theorem 3.2. *Suppose that $f \in \mathbf{F}^{\mathfrak{p}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $\mathfrak{p} \le n$, $0 < \lambda < \frac{1}{\mathfrak{p}}$, $\mathfrak{p} \in \mathbf{N}^*$, for arbitrary $x_0 \in \Omega$, then we have*

$$c_1(\alpha_j, \gamma_{ij}) f(x_0) = \sum_{s=1}^{\mathfrak{p}} (-1)^s \int_{\partial\Omega_{x_0}^*} M_s^\lambda(x) d\sigma_x \Big((D_x^\lambda)^{s-1} f(y_0)\Big) - (-1)^{\mathfrak{p}} \int_{\Omega_{x_0}^*} E_{\mathfrak{p}}(x) \Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big) dx, \tag{3.1}$$

where $c_1(\alpha_j, \gamma_{ij})$ is a Clifford constant. If $c_1(\alpha_j, \gamma_{ij})$ has a single inverse element, this formula is called the Cauchy-Pompeiu integral formula of the function $f \in \mathbf{F}^{\mathfrak{p}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$.

Proof. For any $x_0 \in \Omega$, we have $0 + x_0 \in \Omega$, so $0 \in \Omega_{x_0}^*$.

We can choose an arbitrarily small positive number δ, and make a small ball $B_\delta = \{x : |x| < \delta\}$ such that $\overline{B_\delta}$ is a subset of $\Omega_{x_0}^*$.

For any $s = 2, 3, \ldots, \mathfrak{p}$, by Lemma 2.1, Proposition 2.1, and Theorem 3.1, we have

$$\begin{aligned}
&\int_{\partial\Omega_{x_0}^*} M_s^\lambda(x) d\sigma_x \Big((D_x^\lambda)^{s-1} f(y_0)\Big) - \lim_{\delta\to 0} \int_{\partial B_\delta} M_s^\lambda(x) d\sigma_x \Big((D_x^\lambda)^{s-1} f(y_0)\Big) \\
=&\lim_{\delta\to 0} \int_{\Omega_{x_0}^* \setminus B_\delta} \Big(M_s^\lambda(x) D_x\Big)\Big((D_x^\lambda)^{s-1} f(y_0)\Big) dx + \lim_{\delta\to 0} \int_{\Omega_{x_0}^* \setminus B_\delta} M_s^\lambda(x) \Big[D_x\Big((D_x^\lambda)^{s-1} f(y_0)\Big)\Big] dx \\
=&\lim_{\delta\to 0} \int_{\Omega_{x_0}^* \setminus B_\delta} E_{s-1}(x) \Big((D_x^\lambda)^{s-1} f(y_0)\Big) dx + \lim_{\delta\to 0} \int_{\Omega_{x_0}^* \setminus B_\delta} E_s(x) \Big((D_x^\lambda)^{s} f(y_0)\Big) dx \\
=&\int_{\Omega_{x_0}^*} E_{s-1}(x) \Big((D_x^\lambda)^{s-1} f(y_0)\Big) dx + \int_{\Omega_{x_0}^*} E_s(x) \Big((D_x^\lambda)^{s} f(y_0)\Big) dx.
\end{aligned}$$

From Theorem 3.1, it can be derived that $(D_x^\lambda)^{s-1} f(y_0)$ is a bounded function in $\overline{\Omega_{x_0}^*}$, then $|(D_x^\lambda)^{s-1} f(y_0)| \le M_1$.

For $x \in \partial B_\delta$, suppose that $x = \delta X$, where $X \in \partial B_1 = \{X : |X| = 1\}$, $d\mu = \delta^{n-1} d\mu_1$, $d\mu_1$ is the surface element of the unit sphere ∂B_1, and since $\rho_x^2 \ge c_0 \delta^2$, we can obtain

$$\begin{aligned}
&\left| \int_{\partial B_\delta} M_s^\lambda(x) d\sigma_x \Big((D_x^\lambda)^{s-1} f(y_0)\Big) \right| \\
\le& M_1 \int_{\partial B_\delta} |E_s(x)| \rho_x^{1-\lambda} |d\sigma_x| = M_1 \int_{\partial B_\delta} \frac{|C_s|}{\omega_n \rho_x^{n-s\lambda}} \rho_x^{1-\lambda} d\mu = M_1 \int_{\partial B_\delta} \frac{|C_s|}{\omega_n \rho_x^{n-1-(s-1)\lambda}} \delta^{n-1} d\mu_1 \\
\le& M_2 \int_{\partial B_\delta} \frac{1}{\delta^{n-1-(s-1)\lambda}} \delta^{n-1} d\mu_1 = M_2 \int_{\partial B_\delta} \frac{1}{\delta^{-(s-1)\lambda}} d\mu_1 \le M_3 \delta^{(s-1)\lambda+1} \le M_3 \delta^2,
\end{aligned}$$

where $M_i > 0$ are constants, $i = 1, 2, 3$, then we can conclude that

$$\lim_{\delta\to 0} \int_{\partial B_\delta} M_s^\lambda(x) d\sigma_x \Big((D_x^\lambda)^{s-1} f(y_0)\Big) = 0.$$

Hence,

$$\int_{\partial\Omega^*_{x_0}} M^\lambda_s(x)d\sigma_x\Big((D^\lambda_x)^{s-1}f(y_0)\Big) = \int_{\Omega^*_{x_0}} E_{s-1}(x)\Big(f(y_0)(D^\lambda_x)^{s-1}\Big)dx + \int_{\Omega^*_{x_0}} E_s(x)\Big((D^\lambda_x)^s f(y_0)\Big)dx, \tag{3.2}$$

where $s = 2, 3, \ldots, \mathfrak{p}$.

For $s = 1$, by Lemma 2.1 and the equality $\Big(\rho_x^{-n}H(x)\Big)D = 0$, we have

$$\begin{aligned}
&\int_{\partial\Omega^*_{x_0}} M^\lambda_1(x)d\sigma_x f(y_0) - \lim_{\delta\to 0}\int_{\partial B_\delta} M^\lambda_1(x)d\sigma_x f(y_0)\\
=&\lim_{\delta\to 0}\int_{\Omega^*_{x_0}\backslash B_\delta}\Big(M^\lambda_1(x)D_x\Big)f(y_0)dx + \lim_{\delta\to 0}\int_{\Omega^*_{x_0}\backslash B_\delta} M^\lambda_1(x)\Big(D_x f(y_0)\Big)dx\\
=&\lim_{\delta\to 0}\int_{\Omega^*_{x_0}\backslash B_\delta}\Big[\Big(\frac{H(x)}{\omega_n\rho_x^n}\Big)D_x\Big]f(y_0)dx + \lim_{\delta\to 0}\int_{\Omega^*_{x_0}\backslash B_\delta} E_1(x)\Big(D^\lambda_x f(y_0)\Big)dx = \int_{\Omega^*_{x_0}} E_1(x)\Big(D^\lambda_x f(y_0)\Big)dx.
\end{aligned}$$

We can calculate that

$$\lim_{\delta\to 0}\int_{\partial B_\delta}\frac{1}{\omega_n\rho_x^{n-1}}d\mu = \lim_{\delta\to 0}\int_{\partial B_1}\frac{1}{\omega_n\delta^{n-1}\Big(\sum\limits_{i,j=1}^n a_{ij}X_iX_j\Big)^{\frac{n-1}{2}}}\delta^{n-1}d\mu_1 = \int_{\partial B_1}\frac{1}{\omega_n\Big(\sum\limits_{i,j=1}^n a_{ij}X_iX_j\Big)^{\frac{n-1}{2}}}d\mu_1 = c_1(\alpha_j,\gamma_{ij}), \tag{3.3}$$

we can conclude that $c_1(\alpha_j, \gamma_{ij})$ is a Clifford constant, and $c_1(\alpha_j, \gamma_{ij})$ does not depend on δ but only on the values of the parameters α_j and γ_{ij}; see Remark 2.6 in reference [14].

Hence,

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\partial B_\delta} M^\lambda_1(x)d\sigma_x f(y_0) = \lim_{\delta\to 0}\int_{\partial B_\delta}\frac{H(x)}{\omega_n\rho_x^n}\frac{H(x)}{\rho_x}f(y_0)d\mu\\
=&\lim_{\delta\to 0}\int_{\partial B_\delta}\frac{-1}{\omega_n\rho_x^{n-1}}\Big(f(y_0) - f(x_0)\Big)d\mu + \lim_{\delta\to 0}\int_{\partial B_\delta}\frac{-1}{\omega_n\rho_x^{n-1}}f(x_0)d\mu\\
=&-\Big[\lim_{\delta\to 0}\int_{\partial B_\delta}\frac{1}{\omega_n\rho_x^{n-1}}d\mu_1\Big]f(x_0) = -c_1(\alpha_j,\gamma_{ij})f(x_0),
\end{aligned}$$

therefore,

$$\int_{\partial\Omega^*_{x_0}} M^\lambda_1(x)d\sigma_x f(y_0) + c_1(\alpha_j,\gamma_{ij})f(x_0) = \int_{\Omega^*_{x_0}} E_1(x)\Big(D^\lambda_x f(y_0)\Big)dx. \tag{3.4}$$

By Equalities (3.2) and (3.4), we have

$$\begin{aligned}
&\sum_{s=2}^{\mathfrak{p}}(-1)^s\int_{\partial\Omega^*_{x_0}} M^\lambda_s(x)d\sigma_x\Big((D^\lambda_x)^{s-1}f(y_0)\Big) - \int_{\partial\Omega^*_{x_0}} M^\lambda_1(x)d\sigma_x f(y_0) - c_1(\alpha_j,\gamma_{ij})f(x_0)\\
=&(-1)^{\mathfrak{p}}\int_{\Omega^*_{x_0}} E_{\mathfrak{p}}(x)\Big((D^\lambda_x)^{\mathfrak{p}}f(y_0)\Big)dx + (-1)^{\mathfrak{p}}\int_{\Omega^*_{x_0}} E_{\mathfrak{p}-1}(x)\Big((D^\lambda_x)^{\mathfrak{p}-1}f(y_0)\Big)dx\\
&+ (-1)^{\mathfrak{p}-1}\int_{\Omega^*_{x_0}} E_{\mathfrak{p}-1}(x)\Big((D^\lambda_x)^{\mathfrak{p}-1}f(y_0)\Big)dx + (-1)^{\mathfrak{p}-1}\int_{\Omega^*_{x_0}} E_{\mathfrak{p}-2}(x)\Big((D^\lambda_x)^{\mathfrak{p}-2}f(y_0)\Big)dx\\
&+\cdots+\int_{\Omega^*_{x_0}} E_2(x)\Big((D^\lambda_x)^2 f(y_0)\Big)dx + \int_{\Omega^*_{x_0}} E_1(x)\Big(D^\lambda_x f(y_0)\Big)dx - \int_{\Omega^*_{x_0}} E_1(x)\Big(D^\lambda_x f(y_0)\Big)dx\\
=&(-1)^{\mathfrak{p}}\int_{\Omega^*_{x_0}} E_{\mathfrak{p}}(x)\Big((D^\lambda_x)^{\mathfrak{p}}f(y_0)\Big)dx.
\end{aligned}$$

Consequently, we prove that the conclusion holds. □

Remark 3.1. *When $c_1(\alpha_j, \gamma_{ij})$ is not required to be invertible, the value of $f(x_0)$ is not uniquely determined by the integral transform.*

Corollary 3.1. *Suppose that $f \in \mathbf{F}^{\mathfrak{p}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$ is a solution of the equation $(D_x^\lambda)^{\mathfrak{p}} f(y_0) = 0$ in $\overline{\Omega^*_{x_0}}$, $\mathfrak{p} \le n$, $0 < \lambda < \frac{1}{\mathfrak{p}}$, $\mathfrak{p} \in \mathbf{N}^*$, for arbitrary $x_0 \in \Omega$, then we have*

$$c_1(\alpha_j, \gamma_{ij}) f(x_0) = \sum_{s=1}^{\mathfrak{p}} (-1)^s \int_{\partial\Omega^*_{x_0}} M_s^\lambda(x) d\sigma_x \Big((D_x^\lambda)^{s-1} f(y_0) \Big), \tag{3.5}$$

where $c_1(\alpha_j, \gamma_{ij})$ is a Clifford constant. If $c_1(\alpha_j, \gamma_{ij})$ has a single inverse element, this formula is called the Cauchy integral formula of the $\mathfrak{p}$-order λ-weighted monogenic function.

Theorem 3.3. *Suppose that $f \in \mathbf{F}^{\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $\mathfrak{q} \le n$, $0 < \lambda < \frac{1}{\mathfrak{q}}$, $\mathfrak{q} \in \mathbf{N}^*$, for arbitrary $x_0 \in \Omega$, then we have*

$$f(x_0) c_1(\alpha_j, \gamma_{ij}) = \sum_{r=1}^{\mathfrak{q}} (-1)^r \int_{\partial\Omega^*_{x_0}} \Big(f(y_0) (D_x^\lambda)^{r-1} \Big) d\sigma_x M_r^\lambda(x) - (-1)^{\mathfrak{q}} \int_{\Omega^*_{x_0}} E_{\mathfrak{q}}(x) \Big(f(y_0) (D_x^\lambda)^{\mathfrak{q}} \Big) dx, \tag{3.6}$$

where $c_1(\alpha_j, \gamma_{ij})$ is a Clifford constant. If $c_1(\alpha_j, \gamma_{ij})$ has a single inverse element, this formula is called the Cauchy-Pompeiu integral formula of the function $f \in \mathbf{F}^{\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$.

Proof. Similar to the proof of Theorem 3.2, we can prove Theorem 3.3. □

Corollary 3.2. *Suppose that $f \in \mathbf{F}^{\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$ is a solution of right $\mathfrak{q}$-order λ-weighted Dirac equation $f(y_0)(D_x^\lambda)^{\mathfrak{q}} = 0$ in $\overline{\Omega^*_{x_0}}$, $\mathfrak{q} \le n$, $0 < \lambda < \frac{1}{\mathfrak{q}}$, $\mathfrak{q} \in \mathbf{N}^*$, for arbitrary $x_0 \in \Omega$, then we have*

$$f(x_0) c_1(\alpha_j, \gamma_{ij}) = \sum_{r=1}^{\mathfrak{q}} (-1)^r \int_{\partial\Omega^*_{x_0}} \Big(f(y_0) (D_x^\lambda)^{r-1} \Big) d\sigma_x M_r^\lambda(x), \tag{3.7}$$

where $c_1(\alpha_j, \gamma_{ij})$ is a Clifford constant. If $c_1(\alpha_j, \gamma_{ij})$ has a single inverse element, this formula is called the Cauchy integral formula of the right $\mathfrak{q}$-order λ-weighted monogenic function.

4. The integral representation for the $(\mathfrak{p} + \mathfrak{q})$-order λ-weighted monogenic function

In this section, we obtain the integral representation for the $(\mathfrak{p} + \mathfrak{q})$-order λ-weighted monogenic function.

Theorem 4.1. *Let $\mathfrak{p}, \mathfrak{q} \in \mathbf{N}^*$, $s = 0, 1, \ldots, \mathfrak{p}$; $r = 0, 1, \ldots, \mathfrak{q}$.*

Suppose that $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, when $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}}$, then for arbitrary $x_0 \in \overline{\Omega}$, $\Big((D_x^\lambda)^s f(y_0) \Big) (D_x^\lambda)^r$ is a bounded function in $\overline{\Omega^*_{x_0}}$.

Proof. (i) For arbitrary $x_0 \in \overline{\Omega}$, when $s = 0$ and $r = 0, 1, \ldots, \mathfrak{q}$, from Theorem 3.1 and the inequality $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}} < \frac{1}{\mathfrak{q}}$, it follows that $f(y_0)(D_x^\lambda)^r$ is a bounded function in $\overline{\Omega^*_{x_0}}$.

(ii) When $s = 1, 2, \ldots, \mathfrak{p}$, and $r = 1, \ldots, \mathfrak{q}$, we denote $H(x)f_{s,0}(x)$ by $g_{s,0}(x)$, we denote $f_{s,r}(x)H(x)$ by $g'_{s,r}(x)$, and let

$$\begin{aligned}
&f_{1,0}(x) = D_x f(y_0),\\
&f_{2,0}(x) = -\lambda f_{1,0}(x) + D_x g_{1,0}(x),\\
&f_{3,0}(x) = -2\lambda f_{2,0}(x) + D_x g_{2,0}(x),\\
&\quad \cdots\\
&f_{s,0}(x) = -(s-1)\lambda f_{s-1,0}(x) + D_x g_{s-1,0}(x),\\
&f_{s,1}(x) = -s\lambda \rho_x^{-2} g_{s,0}(x)\overline{H(x)} + g_{s,0}(x)D_x,\\
&f_{s,2}(x) = -(s+1)\lambda f_{s,1}(x) + g'_{s,1}(x)D_x,\\
&f_{s,3}(x) = -(s+2)\lambda f_{s,2}(x) + g'_{s,2}(x)D_x,\\
&\quad \cdots\\
&f_{s,r}(x) = -(s+r-1)\lambda f_{s,r-1}(x) + g'_{s,r-1}(x)D_x.
\end{aligned}$$

As $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $f_{1,0}, \ldots, f_{s,0}, f_{s,1}, \ldots, f_{s,r}$ are bounded functions in $\overline{\Omega^*_{x_0}}$.

(a) When $s = 1$ and $r = 0$, by directly calculating, we can obtain

$$D^\lambda_x f(y_0) = \rho_x^{-\lambda} H(x)\Big(D_x f(y_0)\Big) = \rho_x^{-\lambda} g_{1,0}(x).$$

When $s = 2, \ldots, \mathfrak{p}$ and $r = 0$, we suppose that $s = t$, where $t < \mathfrak{p}$, $t \in \mathbf{N}^*$, and

$$(D^\lambda_x)^t f(y_0) = \rho_x^{-t\lambda} g_{t,0}(x),$$

then

$$\begin{aligned}
(D^\lambda_x)^{t+1} f(y_0) &= D^\lambda_x\Big(\rho_x^{-t\lambda} g_{t,0}(x)\Big) = \rho_x^{-\lambda} H(x)\Big[D_x\Big(\rho_x^{-t\lambda} g_{t,0}(x)\Big)\Big]\\
&=\rho_x^{-\lambda} H(x)\Big[\Big(-t\lambda\rho_x^{-t\lambda-2}\overline{H(x)}\Big)H(x)f_{t,0}(x) + \rho_x^{-t\lambda}\Big(D_x g_{t,0}(x)\Big)\Big]\\
&=\rho_x^{-(t+1)\lambda} H(x)\Big(-t\lambda f_{t,0}(x) + D_x g_{t,0}(x)\Big) = \rho_x^{-(t+1)\lambda} g_{t+1,0}(x).
\end{aligned}$$

According to the mathematical induction, we have

$$(D^\lambda_x)^s f(y_0) = \rho_x^{-s\lambda} g_{s,0}(x). \tag{4.1}$$

For any $x_0 \in \overline{\Omega}$, when $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}}$, we conclude that $0 < \lambda \le \frac{1}{s}$, so $(D^\lambda)^s f(y_0)$ is a bounded function in $\overline{\Omega^*_{x_0}}$.

(b) When $s = 1, \ldots, \mathfrak{p}$ and $r = 1$, by Equality (4.1), we have

$$\begin{aligned}
&\Big((D^\lambda_x)^s f(y_0)\Big)D^\lambda_x = \Big[\Big(\rho_x^{-s\lambda} g_{s,0}(x)\Big)D_x\Big]\rho_x^{-\lambda}H(x) = \Big[-s\lambda\rho_x^{-s\lambda-2} g_{s,0}(x)\overline{H(x)} + \rho_x^{-s\lambda}\Big(g_{s,0}(x)D_x\Big)\Big]\rho_x^{-\lambda}H(x)\\
&=\Big(-s\lambda\rho_x^{-2} g_{s,0}(x)\overline{H(x)} + g_{s,0}(x)D_x\Big)\rho_x^{-(s+1)\lambda}H(x) = f_{s,1}(x)\rho_x^{-(s+1)\lambda}H(x) = g'_{s,1}(x)\rho_x^{-(s+1)\lambda}.
\end{aligned}$$

When $s = 1, \ldots, \mathfrak{p}$ and $r = 2, \ldots, \mathfrak{q}$, we suppose that $r = l$, where $l < \mathfrak{q}$, $l \in \mathbf{N}^*$, and

$$\Big((D^\lambda_x)^s f(y_0)\Big)(D^\lambda_x)^l = g'_{s,l}(x)\rho_x^{-(s+l)\lambda},$$

then

$$\begin{aligned}&\Big((D_x^\lambda)^s f(y_0)\Big)(D_x^\lambda)^{l+1} = \Big[\Big(g'_{s,l}(x)\rho_x^{-(s+l)\lambda}\Big)D_x\Big]\rho_x^{-\lambda}H(x)\\ =&\Big[-(s+l)\lambda g'_{s,l}(x)\rho_x^{-(s+l)\lambda-2}\overline{H(x)} + \rho_x^{-(s+l)\lambda}\Big(g'_{s,l}(x)D_x\Big)\Big]\rho_x^{-\lambda}H(x)\\ =&\Big(-(s+l)\lambda f_{s,l}(x) + g'_{s,l}(x)D_x\Big)\rho_x^{-(s+l+1)\lambda}H(x) = f_{s,l+1}(x)\rho_x^{-(s+l+1)\lambda}H(x) = g'_{s,l+1}(x)\rho_x^{-(s+l+1)\lambda}.\end{aligned}$$

According to the mathematical induction, we have

$$\Big((D_x^\lambda)^s f(y_0)\Big)(D_x^\lambda)^r = g'_{s,r}(x)\rho_x^{-(s+r)\lambda}.$$

For arbitrary $x_0 \in \overline{\Omega}$, when $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}}$, we conclude that $0 < \lambda < \frac{1}{s+r}$, so $\Big((D_x^\lambda)^s f(y_0)\Big)(D_x^\lambda)^r$ is a bounded function in $\overline{\Omega_{x_0}^*}$, where $s = 1, 2, \ldots, \mathfrak{p}$; $r = 1, \ldots, \mathfrak{q}$.

From the above, for any $x_0 \in \overline{\Omega}$, when $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}}$, we conclude that $0 < \lambda < \frac{1}{s+r}$, it can be concluded that $\Big((D_x^\lambda)^s f(y_0)\Big)(D_x^\lambda)^r$ is a bounded function in $\overline{\Omega_{x_0}^*}$, where $s = 0, 1, \ldots, \mathfrak{p}$; $r = 0, 1, \ldots, \mathfrak{q}$. □

Theorem 4.2. *Suppose that* $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$, $\mathfrak{p}+\mathfrak{q} \le n$, $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}}$, $\mathfrak{p}, \mathfrak{q} \in \mathbf{N}^*$, *for arbitrary* $x_0 \in \Omega$, *then we have*

$$\begin{aligned}&c_1(\alpha_j,\gamma_{ij})f(x_0) = \sum_{r=1}^{\mathfrak{q}}(-1)^{\mathfrak{p}+r}\int_{\partial\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x)\\ &+\sum_{s=1}^{\mathfrak{p}}(-1)^s\int_{\partial\Omega_{x_0}^*}M_s^\lambda(x)d\sigma_x\Big((D_x^\lambda)^{s-1}f(y_0)\Big) - (-1)^{\mathfrak{p}+\mathfrak{q}}\int_{\Omega_{x_0}^*}E_{\mathfrak{p}+\mathfrak{q}}(x)\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}}\Big]dx,\end{aligned} \tag{4.2}$$

where $c_1(\alpha_j, \gamma_{ij})$ *is a Clifford constant. If* $c_1(\alpha_j, \gamma_{ij})$ *has a single inverse element, this formula is called the Cauchy-Pompeiu integral formula of the function* $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$.

Proof. We can conclude that Theorem 4.2 holds by applying Theorem 3.2, once we prove that the following equality holds, that is,

$$\begin{aligned}&\sum_{r=1}^{\mathfrak{q}}(-1)^{\mathfrak{p}+r}\int_{\partial\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x) - (-1)^{\mathfrak{p}+\mathfrak{q}}\int_{\Omega_{x_0}^*}E_{\mathfrak{p}+\mathfrak{q}}(x)\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}}\Big]dx\\ =&-(-1)^{\mathfrak{p}}\int_{\Omega_{x_0}^*}E_{\mathfrak{p}}(x)\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)dx.\end{aligned}$$

For any $x_0 \in \Omega$, we know that $0 + x_0 \in \Omega$, so $0 \in \Omega_{x_0}^*$. We can choose an arbitrarily small positive number δ and make a small ball $B_\delta = \{x : |x| < \delta\}$ such that $\overline{B_\delta}$ is a subset of $\Omega_{x_0}^*$.

When $r = 1, 2, 3, \ldots, \mathfrak{q}$, by Lemma 2.1 and Proposition 2.1, we have

$$\begin{aligned}&\int_{\partial\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x) - \lim_{\delta\to 0}\int_{\partial B_\delta}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x)\\ =&\lim_{\delta\to 0}\int_{\Omega_{x_0}^*\setminus B_\delta}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r}\Big]E_{\mathfrak{p}+r}(x)dx + \lim_{\delta\to 0}\int_{\Omega_{x_0}^*\setminus B_\delta}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]E_{\mathfrak{p}+r-1}(x)dx\\ =&\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r}\Big]E_{\mathfrak{p}+r}(x)dx + \int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]E_{\mathfrak{p}+r-1}(x)dx.\end{aligned}$$

By Theorem 4.1, it follows that $\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}$ is a bounded function in $\overline{\Omega_{x_0}^*}$, then $\Big|\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big| \leq M_4$, hence,

$$\begin{aligned}
&\Big|\int_{\partial B_\delta}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x)\Big|\\
\leq& M_4\int_{\partial B_\delta}|E_{\mathfrak{p}+r}(x)|\rho_x^{1-\lambda}|d\sigma_x| = M_4\int_{\partial B_\delta}\frac{|C_{\mathfrak{p}+r}|}{\rho_x^{n-(\mathfrak{p}+r)\lambda}}\rho_x^{1-\lambda}d\mu\\
\leq& M_5\int_{\partial B_\delta}\frac{1}{\delta^{-(\mathfrak{p}+r-1)\lambda}}d\mu_1 \leq M_6\delta^{(\mathfrak{p}+r-1)\lambda+1} \leq M_6\delta^2,
\end{aligned}$$

where $M_i > 0$ are positive constants, $i = 4, 5, 6$, and we can conclude that

$$\lim_{\delta\to 0}\int_{\partial B_\delta}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x) = 0.$$

Hence,

$$\begin{aligned}
&\int_{\partial\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x)\\
=&\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r}\Big]E_{\mathfrak{p}+r}(x)dx + \int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]E_{\mathfrak{p}+r-1}(x)dx.
\end{aligned}\tag{4.3}$$

By Equality (4.3), we can deduce that

$$\begin{aligned}
&\sum_{r=1}^{\mathfrak{q}}(-1)^{\mathfrak{p}+r}\int_{\partial\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x)\\
=&(-1)^{\mathfrak{p}+\mathfrak{q}}\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}}\Big]E_{\mathfrak{p}+\mathfrak{q}}(x)dx + (-1)^{\mathfrak{p}+\mathfrak{q}}\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}-1}\Big]E_{\mathfrak{p}+\mathfrak{q}-1}(x)dx\\
&+(-1)^{\mathfrak{p}+\mathfrak{q}-1}\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}-1}\Big]E_{\mathfrak{p}+\mathfrak{q}-1}(x)dx + (-1)^{\mathfrak{p}+\mathfrak{q}-1}\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}-2}\Big]E_{\mathfrak{p}+\mathfrak{q}-2}(x)dx\\
&+\cdots+(-1)^{\mathfrak{p}+1}\int_{\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)D_x^\lambda\Big]E_{\mathfrak{p}+1}(x)dx + (-1)^{\mathfrak{p}+1}\int_{\Omega_{x_0}^*}\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)E_{\mathfrak{p}}(x)dx\\
=&(-1)^{\mathfrak{p}+\mathfrak{q}}\int_{\Omega_{x_0}^*}E_{\mathfrak{p}+\mathfrak{q}}(x)\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{\mathfrak{q}}\Big]dx - (-1)^{\mathfrak{p}}\int_{\Omega_{x_0}^*}E_{\mathfrak{p}}(x)\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)dx.
\end{aligned}$$

We complete the proof. □

Corollary 4.1. *Suppose that* $f \in \mathbf{F}^{\mathfrak{p}+\mathfrak{q}}(\overline{\Omega}, \mathcal{B}_n(2, \alpha_j, \gamma_{ij}))$ *is a solution of the equation* $\Big((D^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D^\lambda)^{\mathfrak{q}}{=}0$ *in* $\overline{\Omega_{x_0}^*}$, $\mathfrak{p}+\mathfrak{q} \leq n$, $0 < \lambda < \frac{1}{\mathfrak{p}+\mathfrak{q}}$, $\mathfrak{p}, \mathfrak{q} \in \mathbf{N}^*$, *for arbitrary* $x_0 \in \Omega$, *then we have*

$$\begin{aligned}
&c_1(\alpha_j, \gamma_{ij})f(x_0)\\
=&\sum_{r=1}^{\mathfrak{q}}(-1)^{\mathfrak{p}+r}\int_{\partial\Omega_{x_0}^*}\Big[\Big((D_x^\lambda)^{\mathfrak{p}} f(y_0)\Big)(D_x^\lambda)^{r-1}\Big]d\sigma_x M_{\mathfrak{p}+r}^\lambda(x) + \sum_{s=1}^{\mathfrak{p}}(-1)^s\int_{\partial\Omega_{x_0}^*}M_s^\lambda(x)d\sigma_x\Big((D_x^\lambda)^{s-1}f(y_0)\Big),
\end{aligned}\tag{4.4}$$

where $c_1(\alpha_j, \gamma_{ij})$ *is a Clifford constant. If* $c_1(\alpha_j, \gamma_{ij})$ *has a single inverse element, this formula is called the Cauchy integral formula of the* $(\mathfrak{p}+\mathfrak{q})$*-order* λ*-weighted monogenic function.*

Remark 4.1. *Theorem 4.1 is used to prove Theorem 4.2. As* $\mathfrak{p} \in \mathbf{N}^*$*, where* $\mathbf{N}^*$ *is a set of positive integers, there is no direct relationship between Theorems 3.3 and 4.2. However, when* $\mathfrak{p} = 0$ *in Theorem 4.2, if* $\sum_{s=1}^{\mathfrak{p}}(-1)^s \int_{\partial\Omega^*_{x_0}} M_s^{\lambda}(x)d\sigma_x\Big((D_x^{\lambda})^{s-1}f(y_0)\Big) = 0$ *in Equality* (4.2)*, then the right end of the equality in Theorem 4.2 is reduced to the right end of the equality in Theorem 3.3. When* $\mathfrak{q} = 0$ *in Theorem 4.2, if* $\sum_{r=1}^{\mathfrak{q}}(-1)^{\mathfrak{p}+r} \int_{\partial\Omega^*_{x_0}} \Big[\Big((D_x^{\lambda})^{\mathfrak{p}}f(y_0)\Big)(D_x^{\lambda})^{r-1}\Big]d\sigma_x M^{\lambda}_{\mathfrak{p}+r}(x) = 0$ *in Equality* (4.2)*, then the equality in Theorem 4.2 is reduced to the equality in Theorem 3.2.*

5. Conclusions

In recent years, the integral representations for the solution to the higher order Dirac equation in $\mathcal{B}_n(2, \alpha_j, \gamma_{ij})$ have been studied, which generalize the integral representation in the classical Clifford algebra. In this paper, we not only prove three Cauchy-Pompeiu integral formulae for functions valued in the dependent parameter Clifford algebra, but also obtain integral representations for three different higher order λ-weighted monogenic functions.

If $\mathcal{B}_n(2, \alpha_j, \gamma_{ij}) = \mathcal{B}_n(2, 1, 0)$, then Corollary 3.1 in this paper is reduced to one result of Theorem 3.7 in reference [5], that is,

Theorem 5.1. *[5] Suppose that* $\Omega \subseteq \mathbf{R}^n$ *is a domain,* $\Omega^* := \{x|y_0 = x + x_0 \in \Omega\}$*,* $H_j(x) = \frac{A_j}{|x|^{n-j\alpha}}$*,* $A_j = \frac{(-1)^{j-1}}{\omega_n\alpha^{j-1}(j-1)!}$*,* $0 < \alpha < \frac{1}{k}$*. If* $f(x + x_0)$ *is a k-monogenic function with α-weight in* Ω^**, for arbitrary* $x_0 \in \Omega$*, then we have*

$$f(x_0) = \sum_{j=1}^{k}(-1)^{j-1}\int_{\partial\Omega^*} H_j(x)|x|^{-\alpha}xd\sigma_x\Big((D_x^{\alpha})^{j-1}f(x + x_0)\Big). \tag{5.1}$$

If $\mathcal{B}_n(2, \alpha_j, \gamma_{ij}) = \mathcal{B}_n(2, 1, 0)$, Corollary 4.1 in this paper is reduced to Corollary 3.5 in [9], that is,

Theorem 5.2. *[9] Suppose* $f \in C^r(\Omega, Cl_{0,n}(\mathbf{R}))$*, where* $r \geq p + q$*,* $n \geq p + q$*,* $\Omega \subseteq \mathbf{R}^n$ *is a domain,* $\Omega^* := \{x|y_0 = x + x_0 \in \Omega\}$*,* $H_{p+j}(x) = \frac{A_{p+j}}{|x|^{n-(p+j)\alpha}}$*,* $A_{p+j} = \frac{(-1)^{p+j-1}}{\omega_n\alpha^{p+j-1}(p+j-1)!}$*,* $0 < \alpha < \frac{1}{p+q}$*. If* $f(x + x_0)$ *is a* (p, q)*-monogenic function with α-weight in* Ω^**, then for any* $x_0 \in \Omega$*, we have*

$$\begin{aligned} f(x_0) = &\sum_{j=1}^{q}(-1)^{p+j}\int_{\partial\Omega^*}\Big((D_x^{\alpha})^p f(x + x_0)\Big)(D_x^{\alpha})^{j-1}d\sigma_x\Big(x|x|^{-\alpha}H_{p+j}(x)\Big) \\ &+ \sum_{j=1}^{p}(-1)^j\int_{\partial\Omega^*} H_j(x)|x|^{-\alpha}xd\sigma_x\Big((D_x^{\alpha})^{j-1}f(x + x_0)\Big). \end{aligned} \tag{5.2}$$

With the method of the Clifford analytic approach and Newton embedding method, reference [10] proved the existence and uniqueness of solutions of the nonlinear Riemann-Hilbert problems. For a k-vector field F_k, reference [11] obtained the solution of boundary value problems for the associated with the equations $(D_x)^{2s-1}(F_k)D_x = f_k$, where $f_k \in \mathbf{F}(\Omega, \mathcal{B}_m^{(k)}(2, 1, 0))$, $\mathcal{B}_m^{(k)}(2, 1, 0)$ is the space of pseudo-scalars in the classical Clifford algebra $\mathcal{B}_m(2, 1, 0)$. We hope to solve the boundary value problem related to the equation $(D_x)^{2s-1}(F_k)D_x = f_k$ in the dependent parameter Clifford algebra in our future work.

Author contributions

Xiaojing Du: Conceptualization, Writing-original draft, Writing-review and editing; Xiaotong Liang: Validation and Writing-review; Yonghong Xie: Supervision, Validation and Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state that there is no conflicts of interest in this paper.

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