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Research article

Boundedness of Gaussian Bessel potentials and fractional derivatives on variable Gaussian Besov−Lipschitz spaces

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Abstract: In this paper, following [6], we study the regularity properties of Bessel potentials and Bessel fractional derivatives in the context of variable Gaussian Besov−Lipschitz spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, which were defined and studied in [9], under certain conditions on $p(\cdot)$ and $q(\cdot)$.

Keywords: Bessel potential; fractional derivative; variable exponent; Besov−Lipschitz; Gaussian measure
Mathematics Subject Classification: Primary 42B25, 42B35; Secondary 46E30, 47G10

1. Introduction

Before [9] the study of Besov−Lipschitz and Triebel−Lizorkin Gaussian spaces had not been extended to the context of variable exponents since there was no condition on the exponent functions $p(\cdot)$ and $q(\cdot)$ that allowed proving the independence with respect to the integer $k > \alpha$ in the definition of these spaces.

By considering the condition on $p(\cdot)$ introduced in [3] and the condition on $q(\cdot)$ presented in [4] to obtain the variable Hardy's inequalities, it was possible to prove in [9] that the definition of the spaces is independent of k and in this way extend the results in [8] to the variable context.

This condition on $p(\cdot)$ is closely related to the Gaussian measure. Examples of a large family of exponent (non-constant) functions that satisfy this condition are shown in [3].

Once we have the Gaussian Besov−Lipschitz spaces with these conditions, we were able to obtain the boundedness of Gaussian Bessel potentials and fractional derivatives and thus extend the results in [6] to the variable context.

By changing the conditions on the exponents $p(\cdot)$ and $q(\cdot)$, we would first have to prove that the spaces corresponding to these new conditions are well defined, obtaining properties analogous to those of [9], and then study the boundedness of the operators.

When trying to extend these results to other function spaces, for example, for Laguerre or Jacobi measures, we would need conditions on the exponent $p(\cdot)$ that are associated with those measures, then study the properties of the new spaces and finally obtain the boundedness of the operators.

In a future work we will study the boundedness of these operators on Gaussian Triebel−Lizorkin spaces with variable exponents and thus extend the respective results obtained in [6] for Triebel−Lizorkin.

Our work provides new function spaces that can be the habitat for solutions of partial differential equations or differential integral equations. We still need to study the boundedness of the Gaussian Riesz potentials and their fractional derivative in these spaces as well as the Littlewood−Paley Gaussian function g.

For a deeper dive into these and other related topics, see [1, 5, 12].

In classical harmonic analysis we study the notions of semigroups, covering lemmas, maximal functions, Littlewood−Paley functions, spectral multipliers, fractional integrals and derivatives, singular integrals, etc., in the Lebesgue measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$ and consider the Laplacian operator, $\Delta_x = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$.

The Bessel potential of order $\beta > 0$, $\mathcal{J}_\beta$, is defined as

$$\mathcal{J}_\beta = (I - \Delta_x)^{-\beta/2}, \text{ with } \mathcal{J}_0 = I. \tag{1.1}$$

For $f \in L^p(\mathbb{R}^d)$, $\mathcal{J}_\beta(f) = G_\beta * f$, where the kernel G_β is given by,

$$G_\beta(x) = \frac{1}{(4\pi)^{\beta/2}} \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-\pi\|x\|^2/t} e^{-t/4\pi} t^{(-d+\beta)/2} \frac{dt}{t}, x \in \mathbb{R}^d$$

where for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$ is the Euclidean norm on $\mathbb{R}^d$. *The Potential spaces* $\mathcal{L}^p_\beta(\mathbb{R}^d)$ are defined by $\mathcal{L}^p_\beta(\mathbb{R}^d) = \mathcal{J}_\beta(L^p(\mathbb{R}^d))$, for
$\beta \geq 0$ and $1 \leq p \leq \infty$.
It is easy to see that $G_\beta \in L^1(\mathbb{R}^d)$, which implies that $\|\mathcal{J}_\beta(f)\|_p \leq \|f\|_p$, $1 \leq p \leq \infty$.

Therefore, $\mathcal{L}^p_\beta(\mathbb{R}^d)$ is continuously embedded in $L^p(\mathbb{R}^d)$.

These spaces generalize the Sobolev space $L^p_k(\mathbb{R}^d)$, in the sense that, for $1 < p < \infty$ and $k \in \mathbb{N}$, the potential space $\mathcal{L}^p_k(\mathbb{R}^d)$ is equivalent to the Sobolev space $L^p_k(\mathbb{R}^d)$.

In Gaussian harmonic analysis, we consider *the Ornstein−Uhlenbeck* second-order differential operator,

$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle$, where $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_d})$.

In the Gaussian context, the Hermite polynomials are orthogonal with respect to the Gaussian measure and also are eigenfunctions of the Ornstein-Uhlenbeck operator L.

The Gaussian Bessel potential (or fractional integral) of order $\beta > 0$, which we also denote $\mathcal{J}_\beta$, is defined formally as

$$\mathcal{J}_\beta = \left(I + \sqrt{-L}\right)^{-\beta}, \tag{1.2}$$

and *Gaussian Bessel fractional derivative* $\mathcal{D}^\beta$ of order $\beta > 0$, is defined by

$$\mathcal{D}^\beta = \left(I + \sqrt{-L}\right)^\beta,$$

which means that for the Hermite polynomials, we have

$$\mathcal{J}_\beta h_\nu(x) = \frac{1}{\left(1 + \sqrt{|\nu|}\right)^\beta} h_\nu(x) \tag{1.3}$$

and

$$\mathcal{D}^\beta h_\nu(x) = \left(1 + \sqrt{|\nu|}\right)^\beta h_\nu(x). \tag{1.4}$$

Meyer's theorem allows us to extend Gaussian Bessel potentials to a bounded operator on $L^p(\gamma_d)$, $1 < p < \infty$.
Also, from (1.4), we conclude that, $\mathcal{D}^\beta$ (as a good derivative), is not a bounded operator on $L^p(\gamma_d)$, $1 \leq p \leq \infty$.

Gaussian Bessel potentials have the integral representation

$$\mathcal{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^\beta e^{-t} P_t f(x) \frac{dt}{t}, \tag{1.5}$$

where $\{P_t\}_{t\geq 0}$ is the Poisson−Hermite semigroup.

On the other hand, let k be the smallest integer greater than β; then the fractional derivative $\mathcal{D}^\beta$ has the integral representation

$$\mathcal{D}^\beta f(x) = \frac{1}{c_\beta^k} \int_0^\infty t^{-\beta-1} (e^{-t} P_t - I)^k f(x)\, dt, \tag{1.6}$$

with $c_\beta^k = \displaystyle\int_0^\infty u^{-\beta-1}(e^{-u} - 1)^k du$.

There are significant differences between classical and Gaussian harmonic analysis, namely: Lebesgue measure is a doubling, translation-invariant measure. Semigroups associated with Lebesgue measure are convolution semigroups. Gaussian measure does not satisfy any of these properties. For details, see [11].

In [9], replacing p and q with measurable functions $p(\cdot), q(\cdot)$ taking values in $[1, \infty]$ and satisfying suitable regularity conditions, we define and study the structure of Besov−Lipschitz spaces $B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$ with variable exponents with respect to the Gaussian measure, following [8, 11].

In this paper, we generalize some results in [6] for $\mathcal{J}_\beta$ and $\mathcal{D}^\beta$ on $B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$. To do this, we present three sections:

- In Section 2, we give the preliminaries in the Gaussian setting and some background on variable spaces with respect to a Borel measure μ.
- In Section 3, we obtain the boundedness of $\mathcal{J}_\beta$ and $\mathcal{D}^\beta$ on $B^\alpha_{p(\cdot),q(\cdot)}(\gamma_d)$.
- In Section 4, we give some conclusions.

2. Preliminaries

The Gaussian measure on $\mathbb{R}^d$ is given by

$$\gamma_d(x) = \frac{e^{-\|x\|^2}}{\pi^{d/2}}dx,\ x \in \mathbb{R}^d. \tag{2.1}$$

For $\nu = (\nu_1, ..., \nu_d) \in \mathbb{Z}^d$ such that $\nu_i \geq 0, i = 1, \cdots, d$, we consider $\nu! = \prod_{i=1}^{d} \nu_i!$, $|\nu| = \sum_{i=1}^{d} \nu_i$, $\partial_i = \frac{\partial}{\partial x_i}$, with $1 \leq i \leq d$ and $\partial^\nu = \partial_1^{\nu_1} ... \partial_d^{\nu_d}$. Then:

- The *normalized Hermite polynomials of order ν in d variables* are defined by,

$$h_\nu(x) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \prod_{i=1}^{d} (-1)^{\nu_i} e^{x_i^2} \frac{\partial^{\nu_i}}{\partial_i^{\nu_i}}(e^{-x_i^2}). \tag{2.2}$$

- The *Ornstein−Uhlenbeck semigroup* $\{T_t\}_{t\geq 0}$ is defined by

$$\begin{aligned} T_t f(x) &= \frac{1}{(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy) \\ &= \frac{1}{\pi^{d/2}(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} f(y)dy. \end{aligned} \tag{2.3}$$

- *Poisson−Hermite semigroup* $\{P_t\}_{t\geq 0}$ by

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds), \tag{2.4}$$

  where for all $t > 0$,

$$\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds \tag{2.5}$$

  is the *one-sided stable measure on* $(0, \infty)$ of order $1/2$.

As usual, we use the notation

$$u(x,t) = P_t f(x) \quad \text{and} \quad u^{(k)}(x,t) = \frac{\partial^k}{\partial t^k} P_t f(x).$$

Now, let us obtain some background on variable Lebesgue spaces with respect to a Borel measure μ.

Let $\Omega \subset \mathbb{R}^d$, a μ-measurable function $p(\cdot) : \Omega \to [1, \infty]$ is an *exponent function*. The set of exponent functions is denoted by $\mathcal{P}(\Omega, \mu)$. For $E \subset \Omega$ we set

$$p_-(E) = \operatorname{ess\,inf}_{x \in E} p(x) \ , \ p_+(E) = \operatorname{ess\,sup}_{x \in E} p(x).$$

Also $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, $p_+ = p_+(\Omega)$, and $p_- = p_-(\Omega)$.

Definition 2.1. *Let $E \subset \mathbb{R}^d$ and $\alpha(\cdot) : E \to \mathbb{R}$. We say that:*

i) *$\alpha(\cdot)$ is locally log-Hölder continuous if there exists a constant $C_1 > 0$ such that*

$$|\alpha(x) - \alpha(y)| \quad \leq \quad \frac{C_1}{log(e + \frac{1}{|x-y|})}$$

for all $x, y \in E$. Denoted by $\alpha(\cdot) \in LH_0(E)$.

ii) *$\alpha(\cdot)$ is log-Hölder continuous at infinity with base point at $x_0 \in \mathbb{R}^d$, if there exist constants $\alpha_\infty \in \mathbb{R}$ and $C_2 > 0$ such that*

$$|\alpha(x) - \alpha_\infty| \quad \leq \quad \frac{C_2}{log(e + |x - x_0|)}$$

for all $x \in E$. Denoted by $\alpha(\cdot) \in LH_\infty(E)$.

iii) *$\alpha(\cdot)$ is log-Hölder continuous if both conditions are satisfied. In this case, we say $\alpha(\cdot) \in LH(E)$.*

Definition 2.2. *For a μ-measurable function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ (an extended real-valued function), we define the modular*

$$\rho_{p(\cdot),\mu}(f) = \int_{\mathbb{R}^d \setminus \Omega_\infty} |f(x)|^{p(x)} \mu(dx) + \|f\|_{L^\infty(\Omega_\infty,\mu)}. \tag{2.6}$$

The variable exponent Lebesgue space on $\mathbb{R}^d$, $L^{p(\cdot)}(\mu)$ is the set of μ-measurable functions f such that there exists $\lambda > 0$ with $\rho_{p(\cdot),\mu}(f/\lambda) < \infty$, with norm given by

$$\|f\|_{p(\cdot),\mu} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot),\mu}(f/\lambda) \leq 1 \right\}. \tag{2.7}$$

Theorem 2.1. *(Minkowski's integral inequality for variable Lebesgue spaces) Given μ and ν complete σ-finite measures on X and Y respectively, $p \in \mathcal{P}(X, \mu)$. Let $f : X \times Y \to \overline{\mathbb{R}}$ be measurable with respect to the product measure on $X \times Y$, such that for almost every $y \in Y$, $f(\cdot, y) \in L^{p(\cdot)}(X, \mu)$. Then*

$$\left\| \int_Y f(\cdot, y) d\nu(y) \right\|_{p(\cdot),\mu} \leq C \int_Y \|f(\cdot, y)\|_{p(\cdot),\mu} d\nu(y). \tag{2.8}$$

Proof. See [9]. □

In the rest of the paper, μ represents the Haar measure $\mu(dt) = \dfrac{dt}{t}$ on $\mathbb{R}^+$.

Now, $\mathcal{M}_{0,\infty}$ denotes the set of measurable functions $p(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

i) $0 \le p_- \le p_+ < \infty$.

ii$_0$) There exists $p(0) = \lim_{x\to 0} p(x)$ and $|p(x) - p(0)| \le \frac{A}{\ln(1/x)}, 0 < x \le 1/2$.

ii$_\infty$) There exists $p(\infty) = \lim_{x\to\infty} p(x)$ and $|p(x) - p(\infty)| \le \frac{A}{\ln(x)}, x > 2$.

$\mathcal{P}_{0,\infty}$ is the set of functions $p(\cdot) \in \mathcal{M}_{0,\infty}$ such that $p_- \ge 1$.

Next, we present the Hardy inequalities associated with the exponent $q(\cdot) \in \mathcal{P}_{0,\infty}$ and the measure μ.

Theorem 2.2. *Let $q(\cdot) \in \mathcal{P}_{0,\infty}$ and $r > 0$, then*

$$\left\| t^{-r} \int_0^t g(y)dy \right\|_{q(\cdot),\mu} \le C_{r,q(\cdot)} \left\| y^{-r+1} g \right\|_{q(\cdot),\mu} \tag{2.9}$$

and

$$\left\| t^{r} \int_t^\infty g(y)dy \right\|_{q(\cdot),\mu} \le C_{r,q(\cdot)} \left\| y^{r+1} g \right\|_{q(\cdot),\mu}. \tag{2.10}$$

Proof. See [9]. □

Also, we need the classical Hardy's inequalities; see [10]

$$\int_0^{+\infty} \left(\int_0^x g(y)dy \right)^p x^{-r-1} dx \le \frac{p}{r} \int_0^{+\infty} (yg(y))^p y^{-r-1} dy, \tag{2.11}$$

and

$$\int_0^{+\infty} \left(\int_x^\infty g(y)dy \right)^p x^{r-1} dx \le \frac{p}{r} \int_0^{+\infty} (yg(y))^p y^{r-1} dy, \tag{2.12}$$

with $g \ge 0, p \ge 1$ and $r > 0$.

In what follows, we only consider variable exponent Lebesgue spaces, $L^{p(\cdot)}(\gamma_d)$ with respect to the Gaussian measure γ_d.

Remark 2.1. *The families $\{T_t\}_{t\ge 0}$, $\{P_t\}_{t\ge 0}$ and $\{\mathcal{J}_\beta\}_{\beta\ge 0}$ are bounded on $L^{p(\cdot)}(\gamma_d)$, for $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \le p_+ < \infty$. For the proof see [7].*

The next condition was introduced by E. Dalmasso and R. Scotto in [3].

Definition 2.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d)$, we say that $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d)$ if there exist constants $C_{\gamma_d} > 0$ and $p_\infty \ge 1$ such that*

$$|p(x) - p_\infty| \le \frac{C_{\gamma_d}}{\|x\|^2}, \tag{2.13}$$

for $x \in \mathbb{R}^d, x \ne \mathbf{0}$.

Example 2.1. *There exist non-constant functions in $\mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d)$, by considering*
$p(x) = p_\infty + \dfrac{A}{(e + \|x\|)^q}$, $x \in \mathbb{R}^d$, *for any $p_\infty \ge 1, A \ge 0$ and $q \ge 2$.*

Remark 2.2. *It is easy to see that $\mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \subset LH_\infty(\mathbb{R}^d)$.*

Finally, we need some technical results

Lemma 2.1. *Given $k \in \mathbb{N}$ and $t > 0$ then $\mu_t^{(1/2)}$ satisfies*

$$\int_0^{+\infty} \left| \frac{\partial^k \mu_t^{(1/2)}}{\partial t^k} \right| (ds) \le \frac{C_k}{t^k}. \tag{2.14}$$

For the proof see [11].

Lemma 2.2. *Given an integer $k \geq 0$, let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $f \in L^{p(\cdot)}(\gamma_d)$, then*

$$\left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot),\gamma_d} \le C_{p(\cdot)} \left\| \frac{\partial^k}{\partial s^k} P_s f \right\|_{p(\cdot),\gamma_d},$$

for $0 < s < t < +\infty$. Moreover,

$$\left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p(\cdot),\gamma_d} \le \frac{C_{k,p(\cdot)}}{t^k} \|f\|_{p(\cdot),\gamma_d}, \quad t > 0. \tag{2.15}$$

For the proof see [9].

The variable Gaussian Besov−Lipschitz space were defined in [9], following [6, 10].

Definition 2.4. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Let $\alpha \geq 0$ and k be the smallest integer greater than α. The variable Gaussian Besov−Lipschitz space $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, is the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ such that*

$$\left\| t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} < \infty, \tag{2.16}$$

with norm

$$\|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}} = \|f\|_{p(\cdot),\gamma_d} + \left\| t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu}. \tag{2.17}$$

The variable Gaussian Besov−Lipschitz space $B^{\alpha}_{p(\cdot),\infty}(\gamma_d)$, is the set of functions $f \in L^{p(\cdot)}(\gamma_d)$ for which there exists a constant A such that

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \le A t^{-k+\alpha}, \forall t > 0,$$

and the norm

$$\|f\|_{B^{\alpha}_{p(\cdot),\infty}} = \|f\|_{p(\cdot),\gamma_d} + A_k(f), \tag{2.18}$$

where $A_k(f)$ is the smallest constant A in the above inequality.

One of the main results in [9] was that the definition of $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ is independent on the integer $k > \alpha$.
For more details about the definition of variable Gaussian Besov−Lipschitz spaces, we refer to [9].

Additionally, in [9] we obtained some inclusion relations between variable Gaussian Besov−Lipschitz spaces. These results are analogous to Proposition 10, page 153 in [10].

Proposition 2.1. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{P}_{0,\infty}$. The inclusion $B^{\alpha_1}_{p(\cdot),q_1(\cdot)}(\gamma_d) \subset B^{\alpha_2}_{p(\cdot),q_2(\cdot)}(\gamma_d)$ holds, i.e., $\|f\|_{B^{\alpha_2}_{p(\cdot),q_2(\cdot)}} \leq C\|f\|_{B^{\alpha_1}_{p(\cdot),q_1(\cdot)}}$ if:*

i) $\alpha_1 > \alpha_2 > 0$ ($q_1(\cdot)$ and $q_2(\cdot)$ do not need to be related), or
ii) If $\alpha_1 = \alpha_2$ and $q_1(t) \leq q_2(t)$ a.e.

As usual in this theory, C represents a constant that is not necessarily the same in each occurrence.

3. Results

The main results of the paper are the regularity properties of the Gaussian Bessel potentials and the Gaussian Bessel fractional derivatives on variable Gaussian Besov−Lipschitz spaces.

Let us start considering the regularity properties of the Gaussian Bessel potentials. In the following theorem we consider their action on $B^{\alpha}_{p(\cdot),\infty}(\gamma_d)$ spaces, which is analogous to Theorem 4 in [6].

Theorem 3.1. *Let $\alpha \geq 0, \beta > 0$ then for $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$. Then, the Gaussian Bessel potential $\mathcal{J}_\beta$ is bounded from $B^{\alpha}_{p(\cdot),\infty}(\gamma_d)$ into $B^{\alpha+\beta}_{p(\cdot),\infty}(\gamma_d)$.*

Proof. Let $k > \alpha + \beta$ a fixed integer and $f \in B^{\alpha}_{p(\cdot),\infty}(\gamma_d)$, then $\mathcal{J}_\beta f \in L^{p(\cdot)}(\gamma_d)$ (see [7]). By using the representation (1.5), the dominated convergence theorem, and the chain's rule, we obtain

$$\frac{\partial^k}{\partial t^k} P_t(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta e^{-s} u^{(k)}(x, t+s) \frac{ds}{s}.$$

Then, using Minkowski's integral inequality (2.8), and splitting the integral into two

$$\begin{aligned} \left\| \frac{\partial^k}{\partial t^k} P_t(\mathcal{J}_\beta f) \right\|_{p(\cdot),\gamma_d} &\leq \frac{C}{\Gamma(\beta)} \int_0^t s^\beta e^{-s} \left\| u^{(k)}(\cdot, t+s) \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \\ &\quad + \frac{C}{\Gamma(\beta)} \int_t^\infty s^\beta e^{-s} \left\| u^{(k)}(\cdot, t+s) \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \\ &= (I) + (II). \end{aligned}$$

Now, proceeding as in [6] by Lemma 2.2 since $t + s > t$

$$\begin{aligned} (I) &= \frac{C}{\Gamma(\beta)} \int_0^t s^\beta e^{-s} \left\| u^{(k)}(\cdot, t+s) \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \\ &\leq \frac{C}{\Gamma(\beta)} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p(\cdot),\gamma_d} \int_0^t e^{-s} s^{\beta-1} ds \\ &\leq \frac{C}{\Gamma(\beta)} A_k(f) t^{-k+\alpha} \int_0^t s^{\beta-1} ds, \text{ since } f \in B^{\alpha}_{p(\cdot),\infty}(\gamma_d) \\ &= \frac{C}{\Gamma(\beta)\beta} t^\beta A_k(f) t^{-k+\alpha} \\ &= C_\beta A_k(f) t^{-k+\alpha+\beta}. \end{aligned}$$

On the other hand, as $k > \alpha + \beta$ and again by Lemma 2.2, since $t + s > s$

$$(II) = \frac{C}{\Gamma(\beta)} \int_t^\infty s^\beta e^{-s} \left\| u^{(k)}(\cdot, t+s) \right\|_{p(\cdot),\gamma_d} \frac{ds}{s}$$

$$
\begin{aligned}
&\leq \frac{C}{\Gamma(\beta)} \int_t^\infty s^\beta e^{-s} \left\| \frac{\partial^k P_s f}{\partial s^k} \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \\
&\leq C\frac{A_k(f)}{\Gamma(\beta)} \int_t^\infty s^{-k+\alpha+\beta-1} ds, \text{ since } f \in B^{\alpha}_{p(\cdot),\infty}(\gamma_d) \\
&= C\frac{A_k(f)}{\Gamma(\beta)(k-(\alpha+\beta))} t^{-k+\alpha+\beta} \\
&= C_{k,\alpha,\beta} A_k(f) t^{-k+\alpha+\beta}.
\end{aligned}
$$

Thus,

$$
\left\| \frac{\partial^k}{\partial t^k} P_t(\mathcal{J}_\beta f) \right\|_{p(\cdot),\gamma_d} \leq CA_k(f) t^{-k+\alpha+\beta}, \text{for all } t > 0,
$$

which implies that $\mathcal{J}_\beta f \in B^{\alpha+\beta}_{p(\cdot),\infty}(\gamma_d)$ and $A_k(\mathcal{J}_\beta f) \leq CA_k(f)$.
Moreover, by Remark 2.1,

$$
\begin{aligned}
\left\|\mathcal{J}_\beta f\right\|_{B^{\alpha+\beta}_{p(\cdot),\infty}} &= \left\|\mathcal{J}_\beta f\right\|_{p(\cdot),\gamma_d} + A_k(\mathcal{J}_\beta f) \\
&\leq C\|f\|_{p(\cdot),\gamma_d} + CA_k(f) \leq C\|f\|_{B^{\alpha}_{p(\cdot),\infty}}.
\end{aligned}
$$

□

In the next theorem we consider the action of Gaussian Bessel potentials on $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ spaces. It is analogous to Theorem 2.4 (i) of [8].

Theorem 3.2. *Let $\alpha \geq 0$, $\beta > 0$, $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Then, the Gaussian Bessel potential $\mathcal{J}_\beta$ is bounded from $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ into $B^{\alpha+\beta}_{p(\cdot),q(\cdot)}(\gamma_d)$.*

Proof. Let $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ then $\mathcal{J}_\beta f \in L^{p(\cdot)}(\gamma_d)$ since $\mathcal{J}_\beta$ is bounded on $L^{p(\cdot)}(\gamma_d)$.
Let denote $U(x,t) = P_t \mathcal{J}_\beta f(x)$. Using the representation (2.4) of P_t and the semigroup property

$$
U(x, t_1 + t_2) = P_{t_1}(P_{t_2}(\mathcal{J}_\beta f))(x) = \int_0^{+\infty} T_s(P_{t_2}(\mathcal{J}_\beta f))(x) \mu^{\frac{1}{2}}_{t_1}(ds).
$$

Now, fix k and l as integer greater than α and β, respectively. By using the dominated convergence theorem, differentiating k times with respect to t_2 and l times with respect to t_1, we obtain

$$
\frac{\partial^{k+l} U(x, t_1 + t_2)}{\partial (t_1 + t_2)^{k+l}} = \int_0^{+\infty} T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f))(x) \frac{\partial^l}{\partial t_1^l} \mu^{\frac{1}{2}}_{t_1}(ds).
$$

So, taking $t = t_1 + t_2$, we obtain

$$
\frac{\partial^{k+l} U(x,t)}{\partial t^{k+l}} = \int_0^{+\infty} T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f))(x) \frac{\partial^l}{\partial t_1^l} \mu^{\frac{1}{2}}_{t_1}(ds),
$$

therefore, by using inequality (2.8), the boundedness of T_s on $L^{p(\cdot)}(\gamma_d)$ and Lemma 2.1

$$
\left\| \frac{\partial^{k+l} U(\cdot,t)}{\partial t^{k+l}} \right\|_{p(\cdot),\gamma_d} \leq C \int_0^{+\infty} \left\| T_s(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f)) \right\|_{p(\cdot),\gamma_d} \left| \frac{\partial^l}{\partial t_1^l} \mu^{\frac{1}{2}}_{t_1}(ds) \right|
$$

$$\leq C\left\|\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathcal{J}_\beta f)\right\|_{p(\cdot),\gamma_d}\int_0^{+\infty}\left|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{\frac{1}{2}}(ds)\right|$$
$$\leq C(t_1)^{-l}\left\|\frac{\partial^k}{\partial t_2^k}P_{t_2}\mathcal{J}_\beta f\right\|_{p(\cdot),\gamma_d}. \tag{3.1}$$

On the other hand, by the chain's rule

$$\frac{\partial^k P_t}{\partial t^k}(\mathcal{J}_\beta f)(x) = \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\frac{\partial^k P_{t+s}f(x)}{\partial t^k}\frac{ds}{s} = \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}\frac{ds}{s},$$

and again by inequality (2.8)

$$\left\|\frac{\partial^k P_t}{\partial t^k}(\mathcal{J}_\beta f)\right\|_{p(\cdot),\gamma_d} \leq \frac{C}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\left\|\frac{\partial^k P_{t+s}f}{\partial (t+s)^k}\right\|_{p(\cdot),\gamma_d}\frac{ds}{s}. \tag{3.2}$$

Now, since the definition of $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ is independent of the integer $k > \alpha$, take $k > \alpha + \beta$ and $l > \beta$; then $k + l > \alpha + 2\beta > \alpha + \beta$.
We will show that $\mathcal{J}_\beta f \in B^{\alpha+\beta}_{p(\cdot),q(\cdot)}(\gamma_d)$.

In fact, taking $t_1 = t_2 = t/2$ in (3.1) and by (3.2), we obtain

$$\begin{aligned}
&\left\|t^{k+l-(\alpha+\beta)}\left\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}\\
&\leq C\left\|t^{k+l-(\alpha+\beta)}\left\|\frac{\partial^k P_{\frac{t}{2}}}{\partial (\frac{t}{2})^k}(\mathcal{J}_\beta f)\right\|_{p(\cdot),\gamma_d}(\frac{t}{2})^{-l}\right\|_{q(\cdot),\mu}\\
&\leq \frac{C}{\Gamma(\beta)}\left\|t^{k-(\alpha+\beta)}\left(\int_0^{+\infty} s^\beta e^{-s}\left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial (s+\frac{t}{2})^k}\right\|_{p(\cdot),\gamma_d}\frac{ds}{s}\right)\right\|_{q(\cdot),\mu}\\
&\leq \frac{C}{\Gamma(\beta)}\left\|t^{k-(\alpha+\beta)}\left(\int_0^{t} s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial (s+\frac{t}{2})^k}\right\|_{p(\cdot),\gamma_d}\frac{ds}{s}\right)\right\|_{q(\cdot),\mu}\\
&\qquad+\frac{C}{\Gamma(\beta)}\left\|t^{k-(\alpha+\beta)}\left(\int_t^{+\infty} s^\beta \left\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial (s+\frac{t}{2})^k}\right\|_{p(\cdot),\gamma_d}\frac{ds}{s}\right)\right\|_{q(\cdot),\mu}\\
&= I + II.
\end{aligned}$$

Thus, by Lemma 2.2 (since $s + \frac{t}{2} > \frac{t}{2}$) and the change of variables $u = t/2$, we have

$$\begin{aligned}
I &\leq \frac{C}{\Gamma(\beta)}\left\|t^{k-(\alpha+\beta)}\left(\int_0^{t} s^\beta \left\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial (\frac{t}{2})^k}\right\|_{p(\cdot),\gamma_d}\frac{ds}{s}\right)\right\|_{q(\cdot),\mu}\\
&= C_{k,\alpha,\beta}\left\|u^{k-\alpha}\left\|\frac{\partial^k P_u f}{\partial u^k}\right\|_{p(\cdot),\gamma_d}\right\|_{q(\cdot),\mu}.
\end{aligned}$$

Finally, by Lemma 2.2 (since $s + \frac{t}{2} > s$) and Hardy's inequality (2.10), with $r = k - (\alpha + \beta)$ and $g(s) = s^{\beta-1} \left\| \frac{\partial^k P_s f}{\partial s^k} \right\|_{p(\cdot),\gamma_d}$, we have

$$\begin{aligned} II &\leq \frac{C}{\Gamma(\beta)} \left\| t^{k-(\alpha+\beta)} \left(\int_t^{+\infty} s^{\beta} \left\| \frac{\partial^k P_s f}{\partial s^k} \right\|_{p(\cdot),\gamma_d} \frac{ds}{s} \right) \right\|_{q(\cdot),\mu} \\ &= \frac{C}{\Gamma(\beta)} \left\| t^{k-(\alpha+\beta)} \left(\int_t^{+\infty} s^{\beta-1} \left\| \frac{\partial^k P_s f}{\partial s^k} \right\|_{p(\cdot),\gamma_d} ds \right) \right\|_{q(\cdot),\mu} \\ &\leq \frac{C}{\Gamma(\beta)} \left\| s^{k-(\alpha+\beta)+1} \cdot s^{\beta-1} \left\| \frac{\partial^k P_s f}{\partial s^k} \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} \\ &= C_{k,\alpha,\beta} \left\| s^{k-\alpha} \left\| \frac{\partial^k}{\partial s^k} P_s f \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu}. \end{aligned}$$

Therefore, $\mathcal{J}_\beta f \in B^{\alpha+\beta}_{p(\cdot),q(\cdot)}(\gamma_d)$.

Moreover, $\left\| \mathcal{J}_\beta f \right\|_{B^{\alpha+\beta}_{p(\cdot),q(\cdot)}} \leq C \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}}$. □

Now, we will study the action of the Bessel fractional derivative $\mathcal{D}^\beta$ on variable Gaussian Besov−Lipschitz spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$. We will use the representation (1.6) of the Bessel fractional derivative and Hardy's inequalities.

First, we need to consider the forward differences. Remember, for a given function f, the k-th order forward difference of f starting at t with increment s is defined as

$$\Delta_s^k(f,t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t + (k-j)s).$$

The forward differences have the following properties (see Appendix 10.9 in [11]): We will need the following technical result.

Lemma 3.1. *For any positive integer k*

i) $\Delta_s^k(f,t) = \Delta_s^{k-1}(\Delta_s(f,\cdot),t) = \Delta_s(\Delta_s^{k-1}(f,\cdot),t)$

ii) $\Delta_s^k(f,t) = \int_t^{t+s} \int_{v_1}^{v_1+s} ... \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k) dv_k dv_{k-1} ... dv_2 dv_1.$

For any positive integer k,

$$\frac{\partial}{\partial s}(\Delta_s^k(f,t)) = k\, \Delta_s^{k-1}(f', t+s), \tag{3.3}$$

and for any integer $j > 0$*,*

$$\frac{\partial^j}{\partial t^j}(\Delta_s^k(f,t)) = \Delta_s^k(f^{(j)},t). \tag{3.4}$$

Additionally, we obtain the next result.

Lemma 3.2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d)$, $f \in L^{p(\cdot)}(\gamma_d)$ and $k, n \in \mathbb{N}$, then*

$$\left\|\Delta_s^k(u^{(n)}, t)\right\|_{p(\cdot),\gamma_d} \leq C_{k,p(\cdot)} s^k \left\|u^{(k+n)}(\cdot, t)\right\|_{p(\cdot),\gamma_d}.$$

Proof. By Lemma 3.1 ii), we have

$$\Delta_s^k(u^{(n)}(x,\cdot), t) = \int_t^{t+s} \int_{v_1}^{v_1+s} ... \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} u^{(k+n)}(x, v_k) dv_k dv_{k-1} ... dv_2 dv_1,$$

thus, by using inequality (2.8) and Lemma 2.2 k-times, respectively, we obtain

$$\begin{aligned}\left\|\Delta_s^k(u^{(n)}, t)\right\|_{p(\cdot),\gamma_d} &\leq C^k \int_t^{t+s} \int_{v_1}^{v_1+s} ... \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} \left\|u^{(k+n)}(\cdot, v_k)\right\|_{p(\cdot),\gamma_d} dv_k dv_{k-1} ... dv_2 dv_1 \\ &\leq C^k (C_{p(\cdot)})^k s^k \left\|u^{(k+n)}(\cdot, t)\right\|_{p(\cdot),\gamma_d} = C_{k,p(\cdot)} s^k \left\|\frac{\partial^{k+n}}{\partial t^{k+n}} u(\cdot, t)\right\|_{p(\cdot),\gamma_d}.\end{aligned}$$

□

We are ready to consider the action of Gaussian Bessel fractional derivatives on $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ spaces. This is the analogous result to Theorem 8 in [6].

Remark 3.1. *By semigroup property of $\{P_t\}$, we have*

$$\begin{aligned}(e^{-t}P_t - I)^k f(x) &= \sum_{j=0}^{k} \binom{k}{j} (e^{-t}P_t)^{k-j}(-I)^j f(x) \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^j v(x, (k-j)t) = \Delta_t^k(v(x,\cdot), 0),\end{aligned}$$

where $v(x,t) = e^{-t}u(x,t)$.

Theorem 3.3. *Let $0 < \beta < \alpha$, $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$. Then, the Gaussian Bessel fractional derivative $\mathcal{D}_\beta$ is bounded from $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ into $B^{\alpha-\beta}_{p(\cdot),q(\cdot)}(\gamma_d)$.*

Proof. Let $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$ and $k \in \mathbb{N}$ such that $k-1 \leq \beta < k$. Then by Remark 3.1, inequality (2.11), the fundamental theorem of calculus, and Lemma 3.1,

$$\begin{aligned}|\mathcal{D}_\beta f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta_s^k(v(x,\cdot), 0)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \left|\frac{\partial}{\partial r} \Delta_r^k(v(x,\cdot), 0)\right| dr\, ds \\ &\leq \frac{k}{\beta c_\beta} \int_0^{+\infty} r^{-\beta} |\Delta_r^{k-1}(v'(x,\cdot), r)| dr,\end{aligned}$$

and by inequality (2.8), we obtain

$$\left\|\mathcal{D}_\beta f\right\|_{p(\cdot),\gamma_d} \leq \frac{k}{\beta c_\beta} C \int_0^{+\infty} r^{-\beta} \left\|\Delta_r^{k-1}(v', r)\right\|_{p(\cdot),\gamma_d} dr.$$

Again, by Lemma 3.1 and inequality (2.8),

$$\left\|\Delta_r^{k-1}(v',r)\right\|_{p(\cdot),\gamma_d} \leq C^k \int_r^{2r}\int_{v_1}^{v_1+r}\cdots\int_{v_{k-2}}^{v_{k-2}+r}\left\|v^{(k)}(\cdot,v_{k-1})\right\|_{p(\cdot),\gamma_d} dv_{k-1}dv_{k-2}...dv_2dv_1,$$

and also by Leibnitz's differentiation rule,

$$\begin{aligned}\left\|v^{(k)}(\cdot,v_{k-1})\right\|_{p(\cdot),\gamma_d} &= \left\|\sum_{j=0}^{k}\binom{k}{j}(e^{-v_{k-1}})^{(j)}u^{(k-j)}(\cdot,v_{k-1})\right\|_{p(\cdot),\gamma_d}\\ &\leq \sum_{j=0}^{k}\binom{k}{j}e^{-v_{k-1}}\left\|u^{(k-j)}(\cdot,v_{k-1})\right\|_{p(\cdot),\gamma_d}.\end{aligned}$$

Thus, by Lemma 2.2, since $p(\cdot)\in\mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)\cap LH_0(\mathbb{R}^d)$,

$$\begin{aligned}&\left\|\Delta_r^{k-1}(v',r)\right\|_{p(\cdot),\gamma_d}\\ &\quad\leq C_{k,p(\cdot)}\sum_{j=0}^{k}\binom{k}{j}r^{k-1}e^{-r}\left\|u^{(k-j)}(\cdot,r)\right\|_{p(\cdot),\gamma_d}.\end{aligned}$$

Therefore, by using the boundedness of P_t on $L^{p(\cdot)}(\gamma_d)$ for $p(\cdot)\in\mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)\cap LH_0(\mathbb{R}^d)$, see [7],

$$\begin{aligned}\left\|\mathcal{D}_\beta f\right\|_{p(\cdot),\gamma_d} &\leq \frac{k}{\beta c_\beta}C_{k,p(\cdot)}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty}r^{k-\beta-1}e^{-r}\left\|u^{(k-j)}(\cdot,r)\right\|_{p(\cdot),\gamma_d}dr\\ &= C_{k,p(\cdot)}\frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty}r^{(k-j)-(\beta-j)-1}e^{-r}\left\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_rf\right\|_{p(\cdot),\gamma_d}dr\\ &\quad + C_{k,p(\cdot)}\frac{k}{\beta c_\beta}\int_0^{+\infty}r^{k-\beta-1}e^{-r}\|P_rf\|_{p(\cdot),\gamma_d}dr\\ &\leq C_{k,p(\cdot)}\frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty}r^{(k-j)-(\beta-j)-1}\left\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_rf\right\|_{p(\cdot),\gamma_d}dr\\ &\quad + C_{k,p(\cdot)}\frac{k}{\beta c_\beta}\int_0^{+\infty}r^{k-\beta-1}e^{-r}\|f\|_{p(\cdot),\gamma_d}dr.\end{aligned}$$

Thus,

$$\begin{aligned}\left\|\mathcal{D}_\beta f\right\|_{p(\cdot),\gamma_d} &\leq C_{k,p(\cdot)}\frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty}r^{k-j-(\beta-j)}\left\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_rf\right\|_{p(\cdot),\gamma_d}\frac{dr}{r}\\ &\quad +C_{k,p(\cdot)}\frac{k\Gamma(k-\beta)}{\beta c_\beta}\|f\|_{p(\cdot),\gamma_d}\leq C_1\|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}},\end{aligned}$$

since $f\in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)\subset B^{\beta-j}_{p(\cdot),1}(\gamma_d)$ as $\alpha>\beta>\beta-j\geq 0$, for $j\in\{0,...,k-1\}$, by Proposition 2.1, since $p(\cdot)\in\mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)\cap LH_0(\mathbb{R}^d)$ and $q(\cdot)\in\mathcal{P}_{0,\infty}$.
Hence, $\mathcal{D}_\beta f\in L^{p(\cdot)}(\gamma_d)$.

On the other hand, it is clear that

$$P_t(e^{-s}P_s - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-s(k-j)} u(x, t + (k-j)s).$$

Let $n \in \mathbb{N}$ such that $n - 1 \leq \alpha < n$, then

$$\begin{aligned} \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f)(x) &= \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{-s(k-j)} u^{(n)}(x, t + (k-j)s) ds \\ &= \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \Delta_s^k (w(x, \cdot), t) ds, \end{aligned}$$

where $w(x, t) = e^{-t} u^{(n)}(x, t)$. Thus, by using the fundamental theorem of calculus,

$$\frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f)(x) = \frac{e^t}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \frac{\partial}{\partial r} \Delta_r^k (w(x, \cdot), t) dr\, ds.$$

Then, by inequality (2.11), and Lemma 3.1,

$$\left| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f)(x) \right| \leq \frac{k e^t}{c_\beta \beta} \int_0^{+\infty} r^{-\beta} |\Delta_r^{k-1} (w'(x, \cdot), t + r)| dr.$$

Now, proceeding as above, using again Lemma 3.1 and Leibnitz's differentiation rule

$$\left\| \Delta_r^{k-1} (w', t + r) \right\|_{p(\cdot), \gamma_d} \leq C_{k, p(\cdot)} \sum_{j=0}^{k} \binom{k}{j} r^{k-1} e^{-(t+r)} \left\| u^{(k+n-j)}(\cdot, t + r) \right\|_{p(\cdot), \gamma_d},$$

and by inequality (2.8) we obtain that

$$\left\| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \right\|_{p(\cdot), \gamma_d} \leq C_{k, p(\cdot)} \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \left\| u^{(k+n-j)}(\cdot, t + r) \right\|_{p(\cdot), \gamma_d} dr.$$

Therefore,

$$\begin{aligned} &\left\| t^{n-(\alpha-\beta)} \left\| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \right\|_{p(\cdot), \gamma_d} \right\|_{q(\cdot), \mu} \\ &\leq C_{k, p(\cdot)} \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} \left\| t^{n-(\alpha-\beta)} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \left\| u^{(k+n-j)}(\cdot, t + r) \right\|_{p(\cdot), \gamma_d} dr \right\|_{q(\cdot), \mu}. \end{aligned}$$

Now, for each $1 \leq j \leq k$, $0 < \alpha - \beta + k - j \leq \alpha$. Then by Lemma 2.2 and Proposition 2.1, since $p(\cdot) \in \mathcal{P}_{\gamma_d}^{\infty}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$,

$$\begin{aligned} &\left\| t^{n-(\alpha-\beta)} \int_0^{\infty} r^{k-\beta-1} e^{-r} \left\| u^{(k+n-j)}(\cdot, t + r) \right\|_{p(\cdot), \gamma_d} dr \right\|_{q(\cdot), \mu} \\ &\leq C_{p(\cdot)} \Gamma(k - \beta) \left\| t^{n+(k-j)-(\alpha-\beta+k-j)} \left\| u^{(n+k-j)}(\cdot, t) \right\|_{p(\cdot), \gamma_d} \right\|_{q(\cdot), \mu} \leq C_{p(\cdot), \beta, \alpha, k} \|f\|_{B^{\alpha}_{p(\cdot), q(\cdot)}}, \end{aligned}$$

since $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d) \subset B^{\alpha-\beta+(k-j)}_{p(\cdot),q(\cdot)}(\gamma_d)$, for $j \in \{1, \cdots, k\}$.

Finally, we study the case $j = 0$,

$$\begin{aligned}
&\left\| t^{n-(\alpha-\beta)} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \left\| u^{(n+k)}(\cdot, t+r) \right\|_{p(\cdot),\gamma_d} dr \right\|_{q(\cdot),\mu} \\
&\quad \leq \left\| t^{n-(\alpha-\beta)} \int_0^{t} r^{k-\beta-1} e^{-r} \left\| u^{(n+k)}(\cdot, t+r) \right\|_{p(\cdot),\gamma_d} dr \right\|_{q(\cdot),\mu} \\
&\qquad + \left\| t^{n-(\alpha-\beta)} \int_t^{+\infty} r^{k-\beta-1} e^{-r} \left\| u^{(n+k)}(\cdot, t+r) \right\|_{p(\cdot),\gamma_d} dr \right\|_{q(\cdot),\mu} \\
&\quad = (I) + (II).
\end{aligned}$$

By Lemma 2.2,

$$\begin{aligned}
(I) &\leq C_{p(\cdot)} \left\| t^{n-(\alpha-\beta)} \int_0^{t} r^{k-\beta-1} \left\| u^{(n+k)}(\cdot, t) \right\|_{p(\cdot),\gamma_d} dr \right\|_{q(\cdot),\mu} \\
&= \frac{C_{p(\cdot)}}{k-\beta} \left\| t^{n+k-\alpha} \left\| u^{(n+k)}(\cdot, t) \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu},
\end{aligned}$$

and by using again Lemma 2.2 and inequality (2.10) since $q(\cdot) \in \mathcal{P}_{0,\infty}$,

$$\begin{aligned}
(II) &\leq C_{p(\cdot)} \left\| t^{n-(\alpha-\beta)} \int_t^{+\infty} r^{k-\beta-1} \left\| u^{(n+k)}(\cdot, r) \right\|_{p(\cdot),\gamma_d} dr \right\|_{q(\cdot),\mu} \\
&\leq C_{p(\cdot)} C_{q(\cdot)} \left\| r^{n+k-\alpha} \left\| u^{(n+k)}(\cdot, r) \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu}.
\end{aligned}$$

Therefore,

$$\left\| t^{n-(\alpha-\beta)} \left\| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} < +\infty,$$

since $f \in B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$. Thus, $\mathcal{D}_\beta f \in B^{\alpha-\beta}_{p(\cdot),q(\cdot)}(\gamma_d)$.

Moreover, from all the above inequalities, we conclude that

$$\begin{aligned}
\left\| \mathcal{D}_\beta f \right\|_{B^{\alpha-\beta}_{p(\cdot),q(\cdot)}} &= \left\| \mathcal{D}_\beta f \right\|_{p(\cdot),\gamma_d} + \left\| t^{n-(\alpha-\beta)} \left\| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}_\beta f) \right\|_{p(\cdot),\gamma_d} \right\|_{q(\cdot),\mu} \\
&\leq C_1 \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}} + C_{k,p(\cdot)} \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} C_2 \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}} \\
&= C_1 \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}} + C_{k,p(\cdot)} \frac{k}{c_\beta \beta} 2^k C_2 \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}} \\
&= C \|f\|_{B^{\alpha}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

□

Remark 3.2. *The boundedness of $\mathcal{D}_\beta$, for $0 < \beta < \alpha$, only uses the fact that $\alpha - \beta > 0$ and $\beta > 0$, It does not matter how close β is to the ends of the interval $(0, \alpha)$ or the dimension d.*

The boundedness of Gaussian Riesz potentials and Riesz fractional derivatives on variable Gaussian Besov−Lipschitz spaces and the regularity of all these operators on variable Gaussian Triebel−Lizorkin spaces, which were also defined in [9], will be considered in a forthcoming paper.

4. Conclusions

(i) Theorems 3.1–3.3, extend some results obtained in [6] when we go from constant exponent to variable exponent settings if the exponent functions $p(\cdot)$, $q(\cdot)$ satisfy the regularity conditions $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ and $q(\cdot) \in \mathcal{P}_{0,\infty}$.
(ii) The key to the proof of theorems is the generalization of Minkowski's integral inequality and Hardy's inequality to the context of variable exponents as well as Lemmas 2.1 and 2.2.
(iii) The boundedness of $\{T_t\}_{t\geq 0}$, $\{P_t\}_{t\geq 0}$, and $\{\mathcal{J}_\beta\}_{\beta\geq 0}$ on $L^{p(\cdot)}(\gamma_d)$ was also necessary to obtain the results.
(iv) From the properties obtained in [9] for the Besov−Lipschitz spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, it was shown that these form a decreasing family of spaces (in the sense of inclusion) that are continuously immersed in $L^{p(\cdot)}(\gamma_d)$.
(v) Boundedness of the Gaussian Bessel potentials $\mathcal{J}_\beta$ and fractional derivative $\mathcal{D}_\beta$ together with inclusion properties between the Besov−Lipschitz spaces $B^{\alpha}_{p(\cdot),q(\cdot)}(\gamma_d)$, indicate that when applying the Bessel potential to a function, it gains regularity while when applying the fractional derivative, it loses regularity (as a good derivative).
(vi) Currently, we do not have practical applications for the Gaussian measure, apart from having generalized the study done for a constant exponent, but we consider that there may be applications in the future in the field of partial differential equations, as occurs for the Lebesgue measure.

Author contributions

All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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