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Research article

Optimal classes of memory-type estimators of population mean for temporal surveys

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Abstract: In this article, we explore how to efficiently estimate the population mean utilizing past and current sample information through exponentially weighted moving average (EWMA) statistics in temporal surveys. We propose some optimal classes of memory-type estimators of population mean for temporal surveys within the framework of simple random sampling (SRS). We derive the expressions for the bias and mean square error (MSE) of the suggested estimators up to first-order approximation. We compare the traditional and newly introduced memory-type estimators and establish the efficiency conditions. Moreover, we conduct a thorough simulation study using real and artificial populations to refine our theoretical outcomes. The simulation results show that studying past and current sample data increase the efficiency of the proposed estimators.

Keywords: mean square error; exponentially weighted moving average; simple random sampling; mean estimation; efficiency
Mathematics Subject Classification: 62D05

1. Introduction

During a sampling survey, it is likely common to have auxiliary information available about a variable that is either correlated positively or correlated negatively with the main variable. This auxiliary information can help to optimize the variance or mean square error of the developed estimators. The ratio and product estimators can provide precise results, depending on whether the auxiliary variable x is correlated positively or negatively with the main variable y. In simple random sampling (SRS), if the auxiliary information is correlated positively with the main variable, the conventional ratio estimator is commonly utilized to estimate the population mean. The ratio estimator was defined by [1] as:

$$\beth_r = \bar{y}\frac{\bar{X}}{\bar{x}}, \tag{1.1}$$

where $\bar{y}$ is the sample mean of the main variable y; $\bar{X}$ and $\bar{x}$ denote the population mean and sample mean of the auxiliary variable x, respectively. The approximate mean square error (MSE) expression for the ratio estimator $\beth_r$ is given below as:

$$MSE(\beth_r) \approx \Pi\bar{Y}^2(C_x^2 + C_y^2 - 2\rho_{xy}C_xC_y), \tag{1.2}$$

where

$$\Pi = 1/n.$$

Here, n denotes the size of the sample; $\bar{Y}$ denotes the population mean of the main variable;

$$C_x = S_x/\bar{X}$$

and

$$C_y = S_y/\bar{Y}$$

are the population coefficients of variation for the auxiliary and main variables, respectively; S_x and S_y are the population standard deviations of the auxiliary and main variables; and ρ_{xy} is the population correlation coefficient between the variables x and y.

According to [2], if the main and auxiliary variables are positively correlated but the regression line passes away from the origin, then the regression estimator provides efficient results compared with the ratio estimator. The regression estimator is defined as

$$\beth_{reg} = \bar{y} + \beta(\bar{X} - \bar{x}), \tag{1.3}$$

where β is an appropriately selected scalar. The approximate minimum MSE at

$$\beta_{(opt)} = \rho_{xy}(C_y\bar{Y}/C_x\bar{X})$$

for the regression estimator $\beth_{reg}$ is given as:

$$min.MSE(\beth_{reg}) \approx \Pi\bar{Y}^2C_y^2(1 - \rho_{xy}^2). \tag{1.4}$$

The power ratio estimator introduced by [3] was an improved form of the ratio estimator to improve the accuracy and is defined as

$$\beth_s = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)^{\theta}, \tag{1.5}$$

where θ is an appropriately selected scalar. The approximate minimum MSE at

$$\theta_{(opt)} = \rho_{xy}(C_y/C_x)$$

of the estimator $\beth_s$ is given below as:

$$min.MSE(\beth_s) \approx \Pi\bar{Y}^2C_y^2(1 - \rho_{xy}^2). \tag{1.6}$$

To increase the efficiency of the estimator, Searls [4] suggested pre-multiplying a constant in the estimator. Motivated by his novel approach, the Searls type regression estimator (STRE) and Searls type power ratio estimators (STPRE) are given as

$$\daleth_1 = \alpha_1 \bar{y} + \beta_1(\bar{X} - \bar{x}), \tag{1.7}$$

$$\text{and } \daleth_2 = \alpha_2 \bar{y} \left(\frac{\bar{X}}{\bar{x}} \right)^{\beta_2}, \tag{1.8}$$

where α_j, $j = 1, 2$ and β_j are appropriately selected scalars. The approximate minimum MSE at

$$\alpha_{j(opt)} = B_j / A_j, \quad \beta_{1(opt)} = \alpha_{1(opt)} \rho_{xy} (C_y \bar{Y} / C_x \bar{X})$$

and

$$\beta_{2(opt)} = \rho_{xy} (C_y / C_x)$$

of the estimator $\daleth_j$ is given below as:

$$min.MSE(\daleth_j) \approx \bar{Y}^2 \left(1 - \frac{B_j^2}{A_j} \right), \tag{1.9}$$

where

$$A_1 = 1 + \Pi C_y^2 (1 - \rho_{xy}^2), \quad B_1 = 1, \quad A_2 = 1 + \Pi C_y^2 + \beta_2 (2\beta_2 + 1) \Pi C_x^2 - 4\beta_2 \Pi \rho_{xy} C_x C_y$$

and

$$B_2 = 1 + \frac{\beta_2(\beta_2 + 1)}{2} \Pi C_x^2 - \beta_2 \Pi \rho_{xy} C_x C_y.$$

The reader interested in the estimation of population means may refer to some latest studies. Audu et al. [5] introduced modified robust regression estimators using multi-auxiliary information. Kumar et al. [6] measured the impact of correlated measuremet errors using some efficient classes of estimators. Bhushan and Kumar [7] proposed some efficient logarithmic estimators under stratified sampling. Yadav and Prasad [8] proposed imprecise mean estimation procedure in survey sampling.

It is widely recognized that leveraging auxiliary information effectively may decrease the variance or MSE of the designed estimators. Typically, the main and auxiliary variables are derived only from the present sample y_t at time t. However, there is potential to enhance the performance of the estimator by incorporating data from the current sample and more recent and earlier samples. For example, taking $\bar{y}_{t-1}$ from time $t - 1$, $\bar{y}_{t-2}$ from time $t - 2$ and so on can enhance the accuracy of the designed estimators. This becomes especially critical when the surveys are either performed quarterly or monthly, or annually. The memory-type ratio and product estimators for the population mean based on the exponentially weighted moving average (EWMA) were first introduced under SRS by [9], under stratified sampling by [10], and under ranked-based sampling by [11]. Bhushan et al. [12] examined EWMA-based memory-type logarithmic estimators under SRS. Kumar et al. [13] investigated the optimal class of memory-type imputation methods for time-based surveys using EWMA statistics. Yahya et al. [14] introduced an EWMA-based regression imputation estimating the population mean in surveys with nonresponses. The objectives of this study are described below.

(i) To suggest some optimal classes of memory-type estimators of population means for time-scaled surveys utilizing EWMA statistics;
(ii) To obtain the efficiency conditions by comparing the proposed memory-type estimators with the traditional and existing memory-type estimators;
(iii) To present a comprehensive empirical study utilizing artificial and real datasets;
(iv) To conclude the performance of the suggested estimators in relation to the conventional ones.

Classical statistics is extended by neutrosophic statistics, which is used when the data originates from uncertain circumstances or complicated processes. As a generalization of the intuitionistic fuzzy set, Smarandache [15] examined the neutrosophic set. Neutrosophic measures, neutrosophic integrals, and neutrosophic probability were all covered by [16]. Yazdi et al. [17] used uncertainty modeling to evaluate the risk of digitalized process systems. Zarei et al. [18] advocated an expert judgment and uncertainty analysis in sociotechnical systems. From the studies above, it is evident that uncertainty is an important issue to be considered in the estimation of parameters, especially in time-scaled surveys. Therefore, in future studies, we intend to extend the current study under uncertainties.

The structure of this paper is as follows. The existing memory-type estimators with their MSE expressions are discussed in Section 2. The optimal classes of memory-type estimators with their properties are proposed in Section 3. In Section 4, the algebraic comparison of the proposed optimal classes of memory-type estimators with their conventional counterparts is presented. Section 5 is devoted to the empirical study consisting of a simulation study and application to real data. The conclusion is given in Section 6.

2. Existing memory-type estimators

The EWMA is a statistical technique that analyzes and interprets time series data. It was initially developed by [19] in the paper titled "control chart tests based on geometric moving averages". For time $t > 1$, the EWMA statistics for the main and auxiliary variables are given as:

$$Z_t = \omega\bar{y}_t + (1-\omega)Z_{t-1} \quad \text{and} \quad Q_t = \omega\bar{x}_t + (1-\omega)Q_{t-1}, \tag{2.1}$$

where $\bar{y}_t$ and $\bar{x}_t$ are the means of the sample data at time t, and ω is the weight parameter or smoothing constant given to the observations, which range from 0 to 1. A higher value of ω indicates that more weight is given to the current observation compared with older observations. Conversely, a lower value of ω assigns greater weight to older observations, reducing the weight given to the present one. Whenever ω equals 1, the current data are fully weighted, resulting in the EWMA statistic being identical to the sample mean. The notation Z_{t-1} indicates the past observations of the statistic. The starting value (Z_0) of Z_{t-1} is the expected mean or the average of the prior sample.

Here, (Z_t, Q_t) are unbiased estimators for the population means ($\bar{Y}$, $\bar{X}$), respectively. The mean and variance of Z_t are, respectively, given as:

$$E(Z_t) = \bar{Y} \quad \text{and} \quad V(Z_t) = \frac{\sigma_y^2}{n}\left[\frac{\omega}{2-\omega}\left\{1-(1-\omega)^{2t}\right\}\right]. \tag{2.2}$$

Similarly, the mean and variance of Q_t can be stated.

To obtain the properties of the proposed memory-type estimators, we consider some error terms along with their expectations as follows:

$$\left.\begin{aligned} \epsilon_0 &= \frac{Z_t - \bar{Y}}{\bar{Y}}, \\ \epsilon_1 &= \frac{Q_t - \bar{X}}{\bar{X}}, \\ \text{such that } E(\epsilon_0) &= E(\epsilon_1) = 0, \\ E(\epsilon_0^2) &= \xi \varPi C_y^2, \\ E(\epsilon_1^2) &= \xi \varPi C_x^2, \\ \text{and } E(\epsilon_0 \epsilon_1) &= \xi \varPi \rho_{xy} C_x C_y, \end{aligned}\right\} \tag{2.3}$$

where

$$\xi = \omega/(2 - \omega).$$

By employing the past and present observations, the EWMA statistic enhances the efficiency of the memory-type estimators. Considering this fact, Noor-ul-Amin [9] gave the idea of estimating the population mean using the EWMA statistic and developed the memory-type ratio estimator within the framework of SRS as

$$\daleth_r^m = Z_t \frac{\bar{X}}{Q_t}. \tag{2.4}$$

By substituting from (2.3) in (2.4), using Taylor series, expanding and retaining error terms up to the power two, we have the approximate MSE expression

$$MSE(\daleth_r^m) \approx \bar{Y}^2 \xi \varPi (C_x^2 + C_y^2 - 2\rho_{xy} C_x C_y). \tag{2.5}$$

Following [2], we adapt the memory-type regression estimator as follows

$$\daleth_{reg}^m = Z_t + \beta(\bar{X} - Q_t). \tag{2.6}$$

The approximate expression for the minimum MSE at $\beta_{(opt)}$ for the estimator $\daleth_{reg}^m$ is given as

$$min.MSE(\daleth_{reg}^m) \approx \bar{Y}^2 \xi \varPi C_y^2 (1 - \rho_{xy}^2). \tag{2.7}$$

Following [3], we adapt the memory-type power ratio estimator within the framework of SRS as follows

$$\daleth_s^m = Z_t \left(\frac{\bar{X}}{Q_t}\right)^{\theta}, \tag{2.8}$$

where θ is an appropriately selected scalar. The approximate minimum MSE at

$$\theta_{opt} = \rho_{xy}(C_y/C_x)$$

is given below as

$$min.MSE(\daleth_s^m) \approx \bar{Y}^2 \xi \varPi C_y^2 (1 - \rho_{xy}^2). \tag{2.9}$$

The work [12] suggested the following memory-type logarithmic estimator under SRS

$$\beth_l^m = Z_t \left\{ 1 + \log\left(\frac{Q_t}{\bar{X}}\right) \right\}^{\kappa}, \tag{2.10}$$

where κ is an appropriately selected scalar. The approximate minimum MSE of [12] estimator $\beth_l^m$ at

$$\kappa_{(opt)} = -\rho_{xy}(C_y/C_x)$$

is given by:

$$min.MSE(\beth_l^m) \approx \bar{Y}^2 \xi \Pi C_y^2 (1 - \rho_{xy}^2). \tag{2.11}$$

It is worth mentioning that the adapted memory-type power ratio estimator $\beth_s^m$ and the memory-type logarithmic estimator $\beth_l^m$ suggested by [12] achieved the minimum MSE of the adapted memory-type estimator $\beth_{reg}^m$, which is the best linear unbiased estimator in the category of memory-type estimators. The memory-type estimators based on EWMA statistics are particularly suitable for surveys or datasets where detecting trends or tracking gradual changes over time is critical. Here are a few examples:

(i) Tracking the progression of diseases or infections, such as weekly influenza cases or daily COVID-19 positivity rates, to identify emerging trends;
(ii) Monitoring air quality indices, water pollution levels, or temperature changes where gradual shifts or seasonality are important to capture;
(iii) Analyzing shifts in consumers' preferences or satisfaction over time in response to changing market conditions or campaigns;
(iv) Observing trends in unemployment rates, inflation rates, or stock market indices to identify underlying patterns despite short-term fluctuations;
(v) Tracking students' performance or behavioral metrics in longitudinal studies to understand patterns of improvement or decline;
(vi) Monitoring defect rates or production metrics where early detection of deviations can prevent significant issues.

These settings benefit from EWMA because it effectively smooths out short-term noise while emphasizing recent observations, making it ideal for trend analysis and early detection of shifts.

3. Proposed memory-type estimators

To estimate the population mean in time-scaled surveys, the existing memory-type estimators merely achieve the MSE of the memory-type regression estimator, which is the best linear unbiased estimator itself in the category of memory-type estimators. This motivated us to provide efficient estimators of the population mean for time-scaled surveys under SRS. Along the lines of [9, 12], we extend the work of [20] and propose some optimal classes of memory-type estimators under SRS as follows:

$$\beth_1^m = \phi_1 Z_t + \psi_1(\bar{X} - Q_t) \tag{3.1}$$

and

$$\beth_2^m = \phi_2 Z_t \left(\frac{\bar{X}}{Q_t}\right)^{\psi_2}, \tag{3.2}$$

where ϕ_i and ψ_i; $i = 1, 2$ are appropriately selected scalars.

Remark 3.1. *The proposed optimal classes of memory-type estimators $\beth_1^m$ and $\beth_2^m$ will reduce into the memory-type regression estimator $\beth_{reg}^m$ and the memory-type power ratio estimator $\beth_s^m$ for $\phi_1 = 1$ and $\phi_2 = 1$, respectively.*

Remark 3.2. *The memory-type estimators, while effective in leveraging past information to enhance the estimation efficiency, are limited by their robustness on accurate prior data. Any bias or inaccuracies in the past data can adversely affect the performance, and the method may not adapt well to rapidly changing or non-stationary datasets.*

Theorem 3.1. *The approximate minimum MSE of the optimal class of memory-type estimator $\beth_1^m$ at*

$$\phi_{1(opt)} = 1/\{1 + \xi \Pi C_y^2(1 - \rho_{xy}^2)\}$$

and

$$\psi_{1(opt)} = (\bar{Y}/\bar{X})\rho_{xy}(C_y/C_x)\phi_{1(opt)}$$

is given by:

$$min.MSE(\beth_1^m) \approx \bar{Y}^2(1 - \phi_{1(opt)}) \approx \bar{Y}^2\left(1 - \frac{Q_1^2}{P_1}\right), \tag{3.3}$$

where

$$P_1 = 1 + \Pi\xi C_y^2(1 - \rho_{xy}^2) \quad and \quad Q_1 = 1.$$

Proof. The proposed optimal class of the memory-type estimator $\beth_1^m$ is given as

$$\beth_1^m = \phi_1 Z_t + \psi_1(\bar{X} - Q_t). \tag{3.4}$$

Utilizing the notations discussed in (2.3), we can write the estimator $\beth_1^m$ as

$$\beth_1^m = \phi_1\bar{Y}(1 + \epsilon_0) + \psi_1\{\bar{X} - \bar{X}(1 + \epsilon_1)\}. \tag{3.5}$$

Subtracting $\bar{Y}$ from both sides of the expression above, we have

$$\beth_1^m - \bar{Y} = (\phi_1 - 1)\bar{Y} + \phi_1\bar{Y}\epsilon_0 - \psi_1\bar{X}\epsilon_1. \tag{3.6}$$

By taking the expectation on both sides of (3.6), we get

$$E(\beth_1^m - \bar{Y}) = E\{(\phi_1 - 1)\bar{Y} + \phi_1\bar{Y}\epsilon_0 - \psi_1\bar{X}\epsilon_1\}, \tag{3.7}$$

$$E(\beth_1^m - \bar{Y}) = (\phi_1 - 1)\bar{Y} + \phi_1\bar{Y}E(\epsilon_0) - \psi_1\bar{X}E(\epsilon_1). \tag{3.8}$$

Using the notation discussed in (2.3), we get the approximate bias of estimator $\beth_1^m$ as follows:

$$Bias(\beth_1^m) \approx (\phi_1 - 1)\bar{Y}. \tag{3.9}$$

Taking the square and expectation on both sides of (3.6), we have

$$E(\beth_1^m - \bar{Y})^2 = E((\phi_1 - 1)\bar{Y} + \phi_1 \bar{Y}\epsilon_0 - \psi_1 \bar{X}\epsilon_1)^2, \tag{3.10}$$

$$MSE(\beth_1^m) = E\left\{ \begin{array}{l} (\phi_1 - 1)^2\bar{Y}^2 + \phi_1^2\bar{Y}^2\epsilon_0^2 + \psi_1^2\bar{X}^2\epsilon_1^2 + 2\phi_1(\phi_1 - 1)\bar{Y}^2\epsilon_0 \\ -2\phi_1\psi_1\bar{X}\bar{Y}\epsilon_0\epsilon_1 - 2\psi_1(\phi_1 - 1)\bar{X}\bar{Y}\epsilon_1 \end{array} \right\}, \tag{3.11}$$

$$MSE(\beth_1^m) = \left\{ \begin{array}{l} (\phi_1 - 1)^2\bar{Y}^2 + \phi_1^2\bar{Y}^2E(\epsilon_0^2) + \psi_1^2\bar{X}^2E(\epsilon_1^2) + 2\phi_1(\phi_1 - 1)\bar{Y}^2E(\epsilon_0) \\ -2\phi_1\psi_1\bar{X}\bar{Y}E(\epsilon_0\epsilon_1) - 2\psi_1(\phi_1 - 1)\bar{X}\bar{Y}E(\epsilon_1) \end{array} \right\}. \tag{3.12}$$

Using the notations discussed in (2.3), we get the approximate MSE of the estimator $\beth_1^m$ as follows:

$$MSE(\beth_1^m) \approx (\phi_1 - 1)^2\bar{Y}^2 + \phi_1^2\bar{Y}^2\xi\Pi C_y^2 + \psi_1^2\bar{X}^2\xi\Pi C_x^2 - 2\phi_1\psi_1\bar{X}\bar{Y}\xi\Pi\rho_{xy}C_xC_y. \tag{3.13}$$

The optimum value of ϕ_1 and ψ_1 can be obtained by minimizing (3.13) with respect to ϕ_1 and ψ_1 as

$$\frac{\partial MSE(\beth_1^m)}{\partial\phi_1} = 0 \implies \phi_{1(opt)} = \frac{1}{1 + \Pi\xi C_y^2(1 - \rho_{xy}^2)} = \frac{Q_1}{P_1}(\text{say}), \tag{3.14}$$

$$\frac{\partial MSE(\beth_1^m)}{\partial\psi_1} = 0 \implies \psi_{1(opt)} = \rho_{xy}\frac{\bar{Y}C_y}{\bar{X}C_x}\phi_{1(opt)}, \tag{3.15}$$

where

$$P_1 = 1 + \Pi\xi C_y^2(1 - \rho_{xy}^2) \quad \text{and} \quad Q_1 = 1.$$

Putting the optimum values $\phi_{1(opt)}$ and $\psi_{1(opt)}$ in (3.13) provides the approximate minimum MSE as follows:

$$min.MSE(\beth_1^m) \approx \bar{Y}^2(1 - \phi_{1(opt)}) \approx \bar{Y}^2\left(1 - \frac{Q_1^2}{P_1}\right). \tag{3.16}$$

This completes the proof. □

Theorem 3.2. *The approximate minimum MSE of the optimal class of memory-type estimator $\beth_2^m$ is given by:*

$$min.MSE(\beth_2^m) \approx \bar{Y}^2\left(1 - \frac{Q_2^2}{P_2}\right), \tag{3.17}$$

where

$$P_2 = 1 + \xi\Pi C_y^2 + \psi_2(2\psi_2 + 1)\xi\Pi C_x^2 - 4\psi_2\xi\Pi\rho_{xy}C_xC_y$$

and

$$Q_2 = 1 + \frac{\psi_2(\psi_2 + 1)}{2}\xi\Pi C_x^2 - \psi_2\xi\Pi\rho_{xy}C_xC_y.$$

Proof. The proposed optimal class of the memory-type estimator $\beth_2^m$ is given as

$$\beth_2^m = \phi_2 Z_t\left(\frac{\bar{X}}{Q_t}\right)^{\psi_2}. \tag{3.18}$$

Utilizing the notation discussed in (2.3), we can write the estimator $\beth_2^m$ as

$$\beth_2^m = \phi_2\bar{Y}(1 + \epsilon_0)\left\{\frac{\bar{X}}{\bar{X}(1 + \epsilon_1)}\right\}^{\psi_2}, \tag{3.19}$$

$$\beth_2^m = \phi_2 \bar{Y}(1+\epsilon_0)(1+\epsilon_1)^{-\psi_2}. \tag{3.20}$$

Using Taylor series, expanding and retaining the error terms up to a power of two, we get the approximate expression as follows:

$$\beth_2^m \approx \phi_2 \bar{Y}\left\{1+\epsilon_0-\psi_2\epsilon_1-\psi_2\epsilon_0\epsilon_1+\frac{\psi_2(\psi_2+1)}{2}\epsilon_1^2\right\}. \tag{3.21}$$

Subtracting $\bar{Y}$ from both sides of (3.21), we get

$$\beth_2^m - \bar{Y} \approx \bar{Y}\left[\phi_2\left\{1+\epsilon_0-\psi_2\epsilon_1-\psi_2\epsilon_0\epsilon_1+\frac{\psi_2(\psi_2+1)}{2}\epsilon_1^2\right\}-1\right]. \tag{3.22}$$

Taking the expectation on both sides of (3.22) and by utilizing the notations discussed in (2.3), we get the approximate bias of the estimator $\beth_2^m$ as follows:

$$Bias(\beth_2^m) \approx \bar{Y}\left[\phi_2\left\{1+\frac{\psi_2(\psi_2+1)}{2}\xi\Pi C_x^2-\psi_2\xi\Pi\rho_{xy}C_xC_y\right\}-1\right]. \tag{3.23}$$

Taking the square and expectation on both sides of (3.22) and using the notations discussed in (2.3), we get the approximate MSE of the estimator $\beth_2^m$ as follows:

$$E(\beth_2^m - \bar{Y})^2 \approx \bar{Y}^2 E\left[\phi_2\left\{1+\epsilon_0-\psi_2\epsilon_1-\psi_2\epsilon_0\epsilon_1+\frac{\psi_2(\psi_2+1)}{2}\epsilon_1^2\right\}-1\right]^2, \tag{3.24}$$

$$MSE(\beth_2^m) \approx \bar{Y}^2\left[\begin{array}{c} 1+\phi_2^2\left\{1+\xi\Pi C_y^2+\psi_2(2\psi_2+1)\xi\Pi C_x^2-4\psi_2\xi\Pi\rho_{xy}C_xC_y\right\} \\ -2\phi_2\left\{1+\frac{\psi_2(\psi_2+1)}{2}\xi\Pi C_x^2-\psi_2\xi\Pi\rho_{xy}C_xC_y\right\} \end{array}\right], \tag{3.25}$$

$$MSE(\beth_2^m) \approx \bar{Y}^2(1+\phi_2^2P_2-2\phi_2Q_2), \tag{3.26}$$

where

$$P_2 = 1+\xi\Pi C_y^2+\psi_2(2\psi_2+1)\xi\Pi C_x^2-4\psi_2\xi\Pi\rho_{xy}C_xC_y$$

and

$$Q_2 = 1+\frac{\psi_2(\psi_2+1)}{2}\xi\Pi C_x^2-\psi_2\xi\Pi\rho_{xy}C_xC_y.$$

The optimum value of ϕ_2 can be obtained by minimizing (3.26) with respect to ϕ_2 as

$$\frac{\partial MSE(\beth_2^m)}{\partial\phi_2}=0 \implies \phi_{2(opt)}=\frac{Q_2}{P_2}. \tag{3.27}$$

Putting the value of $\phi_{2(opt)}$ in (3.26), we get the approximate minimum MSE of the estimator $\beth_2^m$ as follows:

$$min.MSE(\beth_2^m) \approx \bar{Y}^2\left(1-\frac{Q_2^2}{P_2}\right). \tag{3.28}$$

Remark 3.3. *The simultaneous minimization of ϕ_2 and ψ_2 is tedious. Therefore, the optimum value of ψ_2 can be obtained by putting*

$$\phi_2 = 1$$

in the estimator $\beth_2^m$ and then minimizing its MSE, which is readily given as

$$\psi_{2(opt)} = \rho_{xy}C_y/C_x.$$

□

4. Algebraic comparison

To establish the efficiency conditions, we conducted an algebraic comparison of the MSE of the proposed optimal classes of memory-type estimators $\beth_i^m; i = 1, 2$ with the MSE of the available traditional and memory-type estimators.

(1) The proposed optimal classes of memory-type estimators dominate the conventional mean estimator if

$$\begin{aligned} min.MSE(\beth_i^m) &< V(\bar{y}), \\ \bar{Y}^2\left(1 - \frac{Q_i^2}{P_i}\right) &< \Pi\bar{Y}^2C_y^2, \\ 1 - \frac{Q_i^2}{P_i} &< \Pi C_y^2, \\ \frac{Q_i^2}{P_i} &> 1 - \Pi C_y^2. \end{aligned} \tag{4.1}$$

(2) The proposed optimal classes of memory-type estimators dominate the conventional ratio estimator if

$$\begin{aligned} min.MSE(\beth_i^m) &< MSE(\beth_r), \\ \frac{Q_i^2}{P_i} &> 1 - \Pi(C_x^2 + C_y^2 - 2\rho_{xy}C_xC_y). \end{aligned} \tag{4.2}$$

(3) The proposed optimal classes of memory-type estimators dominate the conventional regression estimator or power ratio estimator if

$$\begin{aligned} min.MSE(\beth_i^m) &< min.MSE(\beth_*), \quad \text{where } \beth_* = \beth_{reg}, \beth_s, \\ \frac{Q_i^2}{P_i} &> 1 - \Pi C_y^2(1 - \rho_{xy}^2). \end{aligned} \tag{4.3}$$

(4) The proposed optimal classes of memory-type estimators dominate the STREs and STPREs if

$$\begin{aligned} min.MSE(\beth_i^m) &< min.MSE(\beth_i), \quad i = 1, 2, \\ \frac{Q_i^2}{P_i} &> \frac{B_i^2}{A_i}. \end{aligned} \tag{4.4}$$

(5) The proposed optimal classes of memory-type estimators dominate the memory-type ratio estimator if

$$\begin{aligned} minMSE(\beth_i^m) &< MSE(\beth_r^m), \\ \frac{Q_i^2}{P_i} &> 1 - \xi\Pi(C_x^2 + C_y^2 - 2\rho_{xy}C_xC_y). \end{aligned} \tag{4.5}$$

(6) The proposed optimal classes of memory-type estimators dominate the memory-type regression estimator or memory-type power ratio estimator or memory-type logarithmic estimator if

$$min.MSE(\beth_i^m) < min.MSE(\beth_*^m), \quad \text{where } \beth_*^m = \beth_{reg}^m, \beth_s^m, \beth_l^m,$$

$$\frac{Q_i^2}{P_i} > 1 - \xi \Pi C_y^2 (1 - \rho_{xy}^2). \tag{4.6}$$

The proposed optimal classes of memory-type estimators will outperform the conventional and memory-type counterparts.

5. Empirical study

To justify the theoretical findings, we performed an empirical study in two parts. In the first part, we performed a simulation study utilizing two hypothetical populations, while in the second part, we performed a real data application utilizing some real datasets.

5.1. Simulation study

To access the performance of the proposed memory-type estimators, a simulation study was conducted on hypothetically generated populations. To generate hypothetical populations, we have followed [21] and considered the following models

$$y = 9.6 + \sqrt{(1 - \rho_{xy}^2)}\, y^* + \rho_{xy} \left(S_y / S_x\right) x^*$$

and

$$x = 5.2 + x^*,$$

where the independent variates x^* and y^* follow different distributions. The steps of the simulation are elaborated below:

(i) Draw a normal population with $N = 1000$ units utilizing $x^* \sim N(9, 25)$ and $y^* \sim N(11, 29)$, and a Weibull population with $N = 1000$ units utilizing $x^* \sim Weibull\ (0.40, 4.05)$ and $y^* \sim Weibull\ (0.38, 3.07)$ with acceptable chosen values of correlation coefficients such as 0.15, 0.35, 0.55, 0.75, and 0.95.

(ii) Draw a sample of sizes n= 25, 50, 100, 200, and 400 using SRS without replacement from these populations and compute the necessary statistics.

(iii) Employing 30,000 iterations, calculate the MSE and relative efficiency (RE) of the estimators for the selected samples with the help of the formulae given in (5.1) and (5.2), respectively, as follows:

$$MSE = \frac{1}{30,000} \sum_{i=1}^{30,000} (\beth_i^* - \bar{Y})^2, \tag{5.1}$$

$$RE = \frac{MSE(\bar{y})}{MSE(\beth^*)}, \tag{5.2}$$

where $\beth^*=\bar{y}$, $\beth_r$, $\beth_{reg}$, $\beth_s$, $\beth_1$, $\beth_2$, $\beth_r^m$, $\beth_{reg}^m$, $\beth_s^m$, $\beth_l^m$, $\beth_1^m$, and $\beth_2^m$.

The MSE and RE findings for the normal and Weibull populations are exhibited in Tables 1–4. After carefully observing the findings in Tables 1–4, we can make the following interpretations:

(i) From Tables 1–4, it is noticed that the MSE and RE of the suggested memory-type estimators are the minimum and maximum concerning the traditional mean estimator $\bar{y}$, the traditional ratio estimator $\beth_r$, the regression estimator $\beth_{reg}$, the power ratio estimator $\beth_s$, the STRE $\beth_1$, the STPRE $\beth_2$, the memory-type ratio estimator $\beth_r^m$, the memory-type regression estimator $\beth_{reg}^m$, the memory-type power ratio estimator $\beth_s^m$, and the memory-type logarithmic estimator $\beth_l^m$ for both the normal and Weibull populations. As per the findings given in Tables 1–4, the utilization of auxiliary information increases an estimator's efficiency.

(ii) From Tables 1–4, for a fixed value of ω, it is observed that there is an inverse relationship between the MSE of an estimator and the correlation coefficient ρ_{xy} but a proportional relationship between the RE of an estimator and the correlation coefficient ρ_{xy}, i.e., as the value of ρ_{xy} increases, the MSE and RE of an estimator decreases and increases, respectively.

Table 1. MSE of traditional and memory-type estimators employing a normal population.

							$\omega = 0.1$				$\omega = 0.5$				$\omega = 0.9$			
ρ_{xy}	n	$\bar{y}$	$\beth_r$	$\beth_{reg}/\beth_s$	$\beth_1$	$\beth_2$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$
0.15	25	24.64	34.44	22.32	19.33	19.16	1.81	1.18	1.16	1.16	11.48	7.44	7.03	7.01	28.17	18.27	16.17	16.05
	50	12.32	15.66	11.42	10.72	10.66	0.82	0.60	0.59	0.59	5.22	3.80	3.72	3.71	12.81	9.34	8.87	8.82
	100	6.15	7.61	5.76	5.59	5.57	0.40	0.30	0.30	0.30	2.54	1.92	1.90	1.90	6.23	4.71	4.60	4.59
	200	3.07	3.76	2.89	2.85	2.85	0.19	0.16	0.15	0.15	1.25	0.97	0.96	0.96	3.07	2.37	2.34	2.33
	400	1.53	1.86	1.46	1.44	1.44	0.09	0.08	0.07	0.07	0.62	0.49	0.48	0.48	1.53	1.19	1.18	1.18
0.35	25	24.64	24.51	19.44	17.10	16.95	1.29	1.02	1.01	1.01	8.17	6.48	6.16	6.14	20.05	15.90	14.27	14.17
	50	12.32	11.59	9.94	9.41	9.35	0.61	0.53	0.52	0.52	3.86	3.31	3.25	3.24	9.48	8.14	7.77	7.73
	100	6.15	5.68	5.02	4.89	4.87	0.29	0.26	0.26	0.26	1.89	1.67	1.65	1.65	4.65	4.10	4.02	4.01
	200	3.07	2.81	2.52	2.49	2.48	0.14	0.14	0.13	0.13	0.93	0.84	0.83	0.83	2.30	2.06	2.04	2.04
	400	1.53	1.40	1.26	1.25	1.25	0.07	0.06	0.06	0.06	0.46	0.42	0.42	0.42	1.14	1.05	1.03	1.02
0.55	25	24.64	17.08	15.14	13.64	13.65	0.89	0.79	0.79	0.79	5.69	5.04	4.84	4.85	13.97	12.39	11.34	11.36
	50	12.32	8.22	7.74	7.41	7.40	0.43	0.40	0.40	0.40	2.74	2.58	2.54	2.54	6.73	6.33	6.11	6.10
	100	6.15	4.04	3.90	3.83	3.82	0.21	0.21	0.20	0.20	1.34	1.30	1.29	1.29	3.31	3.19	3.14	3.14
	200	3.07	2.00	1.96	1.94	1.94	0.11	0.11	0.10	0.10	0.67	0.66	0.65	0.65	1.64	1.60	1.59	1.59
	400	1.53	1.00	0.98	0.98	0.97	0.05	0.05	0.05	0.05	0.33	0.32	0.32	0.32	0.81	0.80	0.80	0.80
0.75	25	24.64	10.72	9.46	8.82	8.98	0.56	0.50	0.49	0.49	3.57	3.15	3.07	3.09	8.77	7.74	7.29	7.41
	50	12.32	5.14	4.83	4.69	4.72	0.27	0.25	0.25	0.25	1.71	1.61	1.59	1.59	4.21	3.95	3.86	3.88
	100	6.15	2.53	2.43	2.40	2.41	0.13	0.13	0.12	0.12	0.84	0.81	0.80	0.81	2.07	1.99	1.97	1.97
	200	3.07	1.25	1.22	1.21	1.21	0.06	0.06	0.06	0.06	0.41	0.40	0.40	0.40	1.02	1.00	0.99	0.99
	400	1.53	0.62	0.61	0.61	0.61	0.03	0.03	0.03	0.03	0.20	0.20	0.20	0.20	0.51	0.51	0.50	0.50
0.95	25	24.64	7.98	2.18	2.13	1.42	0.42	0.12	0.11	0.11	2.66	0.73	0.72	0.71	6.53	1.78	1.75	1.75
	50	12.32	1.79	1.11	1.10	1.10	0.09	0.06	0.05	0.05	0.59	0.37	0.37	0.37	1.46	0.91	0.90	0.90
	100	6.15	0.87	0.57	0.56	0.56	0.04	0.03	0.03	0.03	0.29	0.19	0.18	0.18	0.71	0.46	0.45	0.45
	200	3.07	0.42	0.28	0.28	0.28	0.02	0.01	0.01	0.01	0.14	0.09	0.09	0.09	0.35	0.24	0.23	0.23
	400	1.53	0.21	0.14	0.14	0.14	0.01	0.01	0.00	0.00	0.07	0.04	0.04	0.04	0.17	0.11	0.11	0.11

(iii) Tables 2–4 show that using ω to assign a weight to the current and past observations can improve the efficiency of the proposed memory-type estimators. Usually, a higher ω value provides less weight to the past information, and thus the EWMA-based memory-type estimators rely on the current information to estimate the population mean. Tables 2–4 show that for ω=0.9, the memory-type estimators obtain the least gain over the corresponding conventional counterparts. However, Tables 2–4 show that decreasing the value of ω, which assigns greater weight to the past information, progressively increases the efficiency of the suggested memory-type estimators.

Table 2. RE of traditional and memory-type estimators employing a normal population.

							$\omega = 0.1$				$\omega = 0.5$				$\omega = 0.9$			
ρ_{xy}	n	$\bar{y}$	$\beth_r$	$\beth_{reg}/\beth_s$	$\beth_1$	$\beth_2$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$
0.15	25	1.00	0.72	1.10	1.27	1.29	13.59	20.97	21.21	21.22	2.15	3.31	3.50	3.52	0.87	1.35	1.52	1.54
	50	1.00	0.79	1.08	1.15	1.16	14.95	20.50	20.57	20.58	2.36	3.24	3.31	3.31	0.96	1.32	1.39	1.40
	100	1.00	0.81	1.07	1.10	1.10	15.34	20.28	20.31	20.31	2.42	3.20	3.23	3.24	0.99	1.30	1.34	1.34
	200	1.00	0.82	1.06	1.08	1.08	15.53	20.17	20.18	20.19	2.45	3.18	3.20	3.20	1.00	1.28	1.31	1.31
	400	1.00	0.82	1.06	1.07	1.07	15.63	20.12	20.13	20.13	2.47	3.18	3.18	3.18	1.01	1.29	1.30	1.30
0.35	25	1.00	1.01	1.27	1.44	1.45	19.10	24.08	24.32	24.33	3.02	3.80	4.00	4.01	1.23	1.55	1.73	1.74
	50	1.00	1.06	1.24	1.31	1.32	20.19	23.53	23.61	23.62	3.19	3.72	3.79	3.80	1.30	1.51	1.58	1.59
	100	1.00	1.08	1.22	1.26	1.26	20.56	23.27	23.31	23.31	3.25	3.67	3.71	3.71	1.32	1.50	1.53	1.53
	200	1.00	1.09	1.22	1.23	1.24	20.75	23.15	23.16	23.16	3.28	3.65	3.67	3.67	1.33	1.49	1.50	1.51
	400	1.00	1.10	1.22	1.22	1.22	20.84	23.09	23.10	23.10	3.29	3.65	3.65	3.65	1.34	1.49	1.50	1.50
0.55	25	1.00	1.44	1.63	1.81	1.80	27.41	30.92	31.17	31.14	4.33	4.88	5.08	5.08	1.76	1.99	2.17	2.17
	50	1.00	1.50	1.59	1.66	1.66	28.46	30.24	30.31	30.31	4.49	4.77	4.85	4.85	1.83	1.95	2.02	2.02
	100	1.00	1.52	1.57	1.61	1.61	28.87	29.90	29.94	29.94	4.56	4.72	4.75	4.76	1.86	1.92	1.96	1.96
	200	1.00	1.53	1.57	1.58	1.58	29.07	29.74	29.76	29.76	4.59	4.70	4.71	4.71	1.87	1.91	1.93	1.93
	400	1.00	1.54	1.56	1.57	1.57	29.18	29.68	29.68	29.68	4.61	4.69	4.69	4.69	1.88	1.91	1.92	1.92
0.75	25	1.00	2.30	2.60	2.79	2.74	43.68	49.48	49.75	49.64	6.90	7.81	8.03	7.95	2.81	3.18	3.38	3.32
	50	1.00	2.39	2.55	2.62	2.61	45.48	48.46	48.54	48.52	7.18	7.65	7.72	7.71	2.93	3.12	3.19	3.17
	100	1.00	2.43	2.52	2.55	2.55	46.11	47.93	47.97	47.96	7.28	7.57	7.60	7.59	2.97	3.08	3.12	3.11
	200	1.00	2.44	2.51	2.53	2.52	46.43	47.69	47.71	47.70	7.33	7.53	7.55	7.54	2.99	3.07	3.08	3.08
	400	1.00	2.45	2.50	2.51	2.51	46.60	47.59	47.60	47.59	7.36	7.51	7.52	7.52	3.00	3.06	3.07	3.07
0.95	25	1.00	3.09	11.30	11.53	17.31	58.63	214.79	215.17	213.89	9.26	33.91	34.18	34.59	3.77	13.82	14.05	11.79
	50	1.00	6.87	11.08	11.16	11.14	130.60	210.61	210.68	210.59	20.62	33.25	33.33	33.30	8.40	13.55	13.62	13.56
	100	1.00	7.07	10.96	10.99	10.97	134.25	208.24	208.27	208.24	21.20	32.88	32.91	32.89	8.64	13.40	13.43	13.40
	200	1.00	7.16	10.91	10.92	10.91	136.01	207.23	207.24	207.23	21.48	32.72	32.73	32.72	8.75	13.33	13.35	13.33
	400	1.00	7.21	10.88	10.89	10.88	136.92	206.76	206.76	206.76	21.62	32.65	32.65	32.65	8.80	13.30	13.31	13.30

Table 3. MSE of traditional and memory-type estimators employing a Weibull population.

							$\omega = 0.1$				$\omega = 0.5$				$\omega = 0.9$			
ρ_{xy}	n	$\bar{y}$	$\beth_r$	$\beth_{reg}/\beth_s$	$\beth_1$	$\beth_2$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$
0.15	25	76.16	78.78	58.9	49.29	49.63	4.15	3.10	3.07	3.07	26.26	19.63	18.40	18.44	64.45	48.19	41.51	41.76
	50	38.18	43.41	33.97	30.39	30.38	2.28	1.79	1.78	1.78	14.47	11.32	10.89	10.89	35.52	27.79	25.34	25.33
	100	19.27	23.06	18.27	17.21	17.18	1.21	0.96	0.96	0.96	7.69	6.09	5.97	5.96	18.86	14.95	14.23	14.21
	200	9.54	11.76	9.28	9.01	9.00	0.62	0.49	0.49	0.49	3.92	3.09	3.06	3.06	9.62	7.59	7.41	7.40
	400	4.78	4.78	4.70	4.63	4.63	0.31	0.25	0.25	0.25	1.99	1.57	1.56	1.56	4.89	3.84	3.80	3.80
0.35	25	76.16	54.65	38.80	34.16	34.45	2.88	2.04	2.03	2.03	18.22	12.93	12.36	12.40	44.71	31.75	28.55	28.76
	50	38.18	31.66	24.84	22.89	23.02	1.67	1.31	1.30	1.30	10.55	8.28	8.05	8.06	25.90	20.32	19.00	19.08
	100	19.27	17.31	14.70	14.03	14.03	0.91	0.77	0.77	0.77	5.77	4.90	4.82	4.82	14.16	12.03	11.57	11.58
	200	9.54	8.96	7.97	7.78	7.77	0.47	0.42	0.42	0.42	2.99	2.66	2.63	2.63	7.33	6.52	6.39	6.39
	400	4.78	4.59	4.18	4.13	4.13	0.24	0.22	0.22	0.22	1.53	1.39	1.39	1.39	3.76	3.42	3.39	3.38
0.55	25	76.16	37.94	25.71	23.41	23.49	2.00	1.35	1.35	1.35	12.65	8.57	8.29	8.31	31.04	21.03	19.46	19.52
	50	38.18	22.62	17.07	16.11	16.22	1.19	0.90	0.90	0.90	7.54	5.69	5.58	5.59	18.51	13.97	13.31	13.39
	100	19.27	12.60	10.59	10.23	10.27	0.66	0.56	0.56	0.56	4.20	3.53	3.49	3.49	10.31	8.66	8.42	8.45
	200	9.54	6.60	6.02	5.91	5.92	0.35	0.32	0.32	0.32	2.20	2.01	1.99	2.00	5.4	4.93	4.85	4.86
	400	4.78	3.40	3.27	3.24	3.24	0.18	0.17	0.17	0.17	1.13	1.09	1.09	1.09	2.78	2.68	2.65	2.66
0.75	25	76.16	24.29	15.43	14.46	14.39	1.28	0.81	0.81	0.81	8.10	5.14	5.03	5.02	19.88	12.62	11.97	11.92
	50	38.18	14.73	10.29	9.90	9.95	0.78	0.54	0.54	0.54	4.91	3.43	3.38	3.39	12.05	8.42	8.16	8.19
	100	19.27	8.32	6.42	6.29	6.32	0.44	0.34	0.34	0.34	2.77	2.14	2.13	2.13	6.80	5.26	5.16	5.169
	200	9.54	4.39	3.71	3.66	3.68	0.23	0.20	0.19	0.19	1.46	1.24	1.23	1.23	3.59	3.03	3.00	3.01
	400	4.78	2.27	2.04	2.03	2.03	0.12	0.11	0.11	0.11	0.76	0.68	0.68	0.68	1.86	1.67	1.66	1.66
0.95	25	76.16	10.01	4.57	4.44	4.23	0.53	0.24	0.24	0.24	3.34	1.52	1.51	1.49	8.19	3.74	3.65	3.51
	50	38.18	6.17	2.85	2.81	2.75	0.32	0.15	0.15	0.15	2.06	0.95	0.94	0.94	5.05	2.33	2.30	2.26
	100	19.27	3.54	1.65	1.63	1.62	0.19	0.09	0.09	0.09	1.18	0.55	0.55	0.55	2.90	1.35	1.34	1.33
	200	9.54	1.89	0.89	0.89	0.89	0.10	0.05	0.05	0.05	0.63	0.30	0.30	0.30	1.55	0.73	0.73	0.73
	400	4.78	0.98	0.47	0.46	0.46	0.05	0.02	0.02	0.02	0.33	0.16	0.16	0.16	0.80	0.38	0.38	0.38

Table 4. RE of traditional and memory-type estimators employing a Weibull population.

							$\omega = 0.1$				$\omega = 0.5$				$\omega = 0.9$			
ρ_{xy}	n	$\bar{y}$	$\beth_r$	$\beth_{reg}/\beth_s$	$\beth_1$	$\beth_2$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$	$\beth_r^m$	$\beth_{reg}^m/\beth_s^m/\beth_l^m$	$\beth_1^m$	$\beth_2^m$
0.15	25	1.00	0.97	1.29	1.55	1.53	18.37	24.57	24.83	24.82	2.90	3.88	4.14	4.13	1.18	1.58	1.83	1.82
	50	1.00	0.88	1.12	1.26	1.26	16.71	21.36	21.49	21.49	2.64	3.37	3.51	3.51	1.07	1.37	1.51	1.51
	100	1.00	0.84	1.05	1.12	1.12	15.88	20.04	20.10	20.10	2.51	3.16	3.23	3.23	1.02	1.29	1.35	1.36
	200	1.00	0.81	1.03	1.06	1.06	15.42	19.54	19.57	19.57	2.43	3.09	3.12	3.12	0.99	1.26	1.29	1.29
	400	1.00	0.8	1.02	1.03	1.03	15.21	19.35	19.36	19.36	2.40	3.05	3.07	3.07	0.98	1.24	1.26	1.26
0.35	25	1.00	1.39	1.96	2.23	2.21	26.48	37.30	37.57	37.56	4.18	5.89	6.16	6.14	1.70	2.40	2.67	2.65
	50	1.00	1.21	1.54	1.67	1.66	22.91	29.21	29.34	29.33	3.62	4.61	4.74	4.74	1.47	1.88	2.01	2.00
	100	1.00	1.11	1.31	1.37	1.37	21.15	24.91	24.97	24.97	3.34	3.93	4.00	4.00	1.36	1.60	1.67	1.66
	200	1.00	1.06	1.20	1.23	1.23	20.22	22.75	22.78	22.78	3.19	3.59	3.62	3.62	1.30	1.46	1.49	1.49
	400	1.00	1.04	1.14	1.16	1.16	19.78	21.73	21.74	21.75	3.12	3.43	3.45	3.45	1.27	1.40	1.41	1.41
0.55	25	1.00	2.01	2.96	3.25	3.24	38.14	56.29	56.59	56.57	6.02	8.89	9.19	9.17	2.45	3.62	3.91	3.90
	50	1.00	1.69	2.24	2.37	2.35	32.07	42.49	42.62	42.61	5.06	6.71	6.84	6.83	2.06	2.73	2.87	2.85
	100	1.00	1.53	1.82	1.88	1.88	29.05	34.59	34.65	34.64	4.59	5.46	5.52	5.52	1.87	2.22	2.29	2.28
	200	1.00	1.45	1.58	1.61	1.61	27.47	30.11	30.14	30.14	4.34	4.75	4.78	4.78	1.77	1.94	1.97	1.96
	400	1.00	1.41	1.46	1.48	1.48	27.72	27.78	27.79	27.79	4.22	4.39	4.40	4.40	1.72	1.79	1.80	1.80
0.75	25	1.00	3.14	4.94	5.27	5.29	59.57	93.79	94.13	94.13	9.41	14.81	15.14	15.16	3.83	6.03	6.37	6.39
	50	1.00	2.59	3.71	3.86	3.84	49.26	70.52	70.67	70.65	7.78	11.14	11.28	11.26	3.17	4.54	4.68	4.66
	100	1.00	2.32	3.00	3.06	3.05	44.02	56.99	57.05	57.04	6.95	9.00	9.06	9.05	2.83	3.67	3.73	3.72
	200	1.00	2.17	2.57	2.60	2.60	41.29	48.92	48.95	48.94	6.52	7.72	7.75	7.75	2.66	3.15	3.18	3.17
	400	1.00	2.11	2.35	2.36	2.36	40.00	44.59	44.61	44.60	6.32	7.04	7.05	7.05	2.57	2.87	2.88	2.88
0.95	25	1.00	7.61	16.67	17.15	18.02	144.55	316.78	317.26	317.96	22.82	50.02	50.50	51.25	9.30	20.38	20.86	21.69
	50	1.00	6.19	13.42	13.60	13.91	117.52	254.92	255.10	255.37	18.56	40.25	40.44	40.72	7.56	16.40	16.58	16.88
	100	1.00	5.44	11.71	11.78	11.86	103.30	222.45	222.52	222.60	16.31	35.12	35.20	35.27	6.65	14.31	14.39	14.46
	200	1.00	5.05	10.72	10.75	10.76	95.89	203.59	203.62	203.63	15.14	32.15	32.18	32.19	6.17	13.10	13.13	13.14
	400	1.00	4.86	10.27	10.29	10.28	92.34	195.18	195.19	195.19	14.58	30.82	30.83	30.83	5.94	12.56	12.57	12.57

(iv) For a fixed value of ρ_{xy} and ω, it is observed that the MSE of an estimator reduces as the sample size increases. As an example, from the findings of the normal population of Table 1, for a fixed $\omega = 0.5$ and $\rho_{xy} = 0.35$, the MSE of the estimator $\beth_1^m$ is 1.65 when $n = 100$ and the MSE of the estimator $\beth_1^m$ is 0.83 when $n = 200$.

(v) From Tables 1–3, it is observed that the MSE of a memory-type estimator increases when the value of the smoothing constant ω increases. Additionally, as the value of ω varies, less weight is assigned to the past information compared with the information that is currently being used, increasing the MSE of the memory-type estimator.

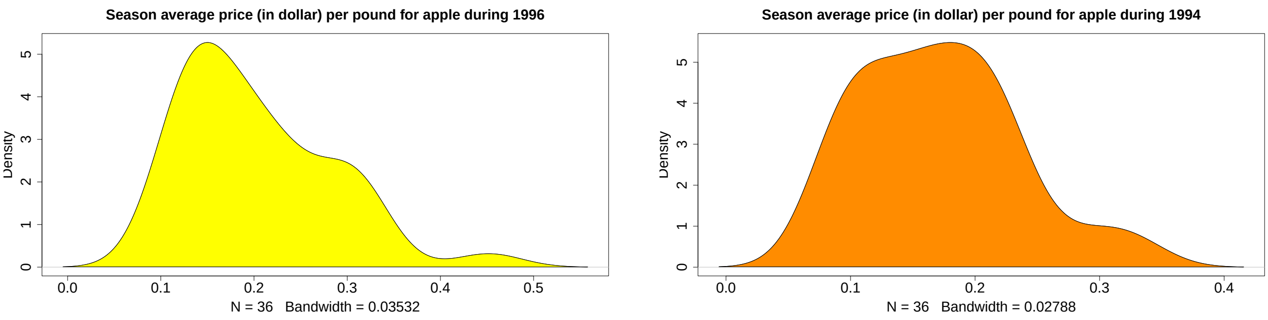
5.2. *Application to real data*

We tested the performance of the proposed estimators utilizing a real dataset taken from [22]. The data are based on the seasonal average price (in dollars) per pound for apple crops in different states of the USA during 1994–1996. The main variable y is taken as the seasonal average price in 1996, while the auxiliary variable x is taken as the seasonal average price in 1994. The density plots of the main and auxiliary variables are provided in Figure 1. The important population characteristics required to compute the MSE of different estimators are given as

$$N = 36, \quad n = 20, \quad \bar{Y} = 0.2032, \quad \bar{X} = 0.1708, \quad S_x^2 = 0.0040, \quad S_y^2 = 0.0064$$

and

$$\rho_{xy} = 0.8577.$$



(**a**) (**b**)

Figure 1. The density plot of the seasonal average price of apples per pound for (a) study variables; (b) auxiliary variables.

From this populations, we selected 20 samples, each of size 5, utilizing SRS without replacement, as reported in Table 5. By taking the smoothing constant $\omega = 0.2$, the memory-type ratio estimator $\sqsupset_r^m$, the memory-type regression estimator $\sqsupset_{reg}^m$, the memory-type power ratio estimator $\sqsupset_s^m$, the memory-type logarithmic estimator $\sqsupset_l^m$, and the optimal classes of estimators $\sqsupset_i^m,\ i = 1, 2$ are computed for each sample and are listed in Table 5. Furthermore, with the help of the respective MSE, the MSEs of different estimators were calculated and are given as follows:

$$MSE(\sqsupset_r^m) = 0.00003887, \quad MSE(\sqsupset_{reg}^m) = 0.00003794, \quad MSE(\sqsupset_s^m) = 0.00003794,$$
$$MSE(\sqsupset_l^m) = 0.00003794, \quad MSE(\sqsupset_1^m) = 0.00003790, \quad MSE(\sqsupset_2^m) = 0.00003788.$$

The results of the real data illustration revealed that using information from the current and previous sample in the form of EWMA statistics increases the efficiency of the proposed optimal classes of memory-type estimators.

Table 5. Evaluation of the memory-type estimators using real data for $\omega = 0.2$.

	Sample of y					$\bar{y}_t$	Sample of x					$\bar{x}_t$	Z_t	Q_t	$\sqsupset_r^m$	$\sqsupset_{reg}^m$	$\sqsupset_s^m$	$\sqsupset_l^m$	$\sqsupset_1^m$	$\sqsupset_2^m$
1	0.185	0.221	0.126	0.314	0.117	0.1926	0.121	0.095	0.130	0.181	0.226	0.1506	0.2011	0.1668	0.2060	0.2055	0.2056	0.2056	0.2053	0.2054
2	0.316	0.312	0.194	0.117	0.180	0.2238	0.181	0.310	0.206	0.095	0.157	0.1898	0.2056	0.1714	0.2050	0.2051	0.2051	0.2051	0.2049	0.2049
3	0.306	0.312	0.235	0.246	0.125	0.2448	0.101	0.174	0.332	0.217	0.157	0.1962	0.2134	0.1764	0.2068	0.2075	0.2074	0.2075	0.2073	0.2072
4	0.194	0.235	0.122	0.279	0.198	0.2056	0.157	0.216	0.244	0.139	0.206	0.1924	0.2118	0.1796	0.2016	0.2024	0.2024	0.2026	0.2022	0.2022
5	0.157	0.279	0.160	0.324	0.185	0.2210	0.088	0.101	0.217	0.244	0.157	0.1614	0.2136	0.1760	0.2075	0.2081	0.2080	0.2081	0.2079	0.2078
6	0.316	0.198	0.221	0.204	0.235	0.2348	0.209	0.200	0.139	0.216	0.164	0.1856	0.2178	0.1779	0.2092	0.2102	0.2100	0.2101	0.2100	0.2097
7	0.194	0.204	0.133	0.264	0.223	0.2036	0.332	0.078	0.230	0.130	0.090	0.1720	0.2150	0.1767	0.2079	0.2086	0.2085	0.2086	0.2084	0.2082
8	0.312	0.324	0.292	0.158	0.228	0.2628	0.283	0.216	0.173	0.138	0.104	0.1828	0.2246	0.1779	0.2157	0.2169	0.2164	0.2166	0.2167	0.2162
9	0.125	0.133	0.160	0.228	0.316	0.1924	0.090	0.107	0.283	0.174	0.244	0.1796	0.2182	0.1782	0.2092	0.2102	0.2099	0.2101	0.2100	0.2097
10	0.142	0.160	0.246	0.133	0.198	0.1758	0.133	0.121	0.101	0.219	0.310	0.1768	0.2097	0.1779	0.2014	0.2021	0.2021	0.2023	0.2019	0.2019
11	0.204	0.111	0.306	0.130	0.142	0.1786	0.164	0.078	0.219	0.200	0.168	0.1658	0.2035	0.1755	0.1982	0.1985	0.1986	0.1987	0.1983	0.1984
12	0.126	0.175	0.158	0.130	0.314	0.1806	0.216	0.088	0.118	0.173	0.200	0.1590	0.1989	0.1722	0.1974	0.1975	0.1975	0.1975	0.1973	0.1973
13	0.279	0.126	0.117	0.292	0.312	0.2252	0.332	0.195	0.086	0.157	0.101	0.1742	0.2042	0.1726	0.2021	0.2023	0.2023	0.2023	0.2021	0.2021
14	0.223	0.312	0.160	0.246	0.158	0.2198	0.138	0.133	0.095	0.121	0.173	0.1320	0.2073	0.1645	0.2154	0.2143	0.2147	0.2148	0.2141	0.2145
15	0.175	0.126	0.223	0.314	0.246	0.2168	0.209	0.206	0.121	0.103	0.088	0.1454	0.2092	0.1607	0.2225	0.2203	0.2213	0.2217	0.2201	0.2211
16	0.223	0.194	0.452	0.122	0.314	0.2610	0.209	0.165	0.173	0.164	0.217	0.1856	0.2196	0.1657	0.2265	0.2252	0.2259	0.2260	0.2250	0.2256
17	0.324	0.306	0.125	0.160	0.452	0.2734	0.121	0.283	0.107	0.118	0.104	0.1466	0.2304	0.1619	0.2432	0.2402	0.2421	0.2424	0.2399	0.2418
18	0.292	0.246	0.175	0.142	0.264	0.2238	0.165	0.217	0.310	0.226	0.121	0.2078	0.2291	0.1711	0.2288	0.2289	0.2288	0.2288	0.2287	0.2286
19	0.246	0.198	0.279	0.223	0.160	0.2212	0.164	0.198	0.101	0.332	0.283	0.2156	0.2275	0.1800	0.2160	0.2176	0.2170	0.2172	0.2174	0.2167
20	0.173	0.130	0.223	0.235	0.452	0.2426	0.164	0.157	0.157	0.217	0.101	0.1592	0.2305	0.1758	0.2240	0.2251	0.2246	0.2247	0.2249	0.2243

6. Conclusions

In this article, we suggest some optimal classes of memory-type estimators to estimate the population mean in SRS framework by utilizing information from the previous and the current sample in the form of EWMA statistics. We obtained the approximate bias and MSE expressions for the suggested optimal classes of memory-type estimators. An algebraic comparison of the proposed optimal classes of memory-type estimators in relation to the existing estimators was carried out to develop the efficiency conditions. A comprehensive simulation study was conducted on two artificially generated populations with normal and Weibull distributions. The results of the simulation study are shown in Tables 1–4, which show the superiority of the proposed optimal classes of memory-type estimators over the conventional ones. Furthermore, the superiority of the proposed estimators was also investigated through an application to real data. The results reported in Table 5 were found to be in favor of the proposed estimators. Thus, the proposed optimal classes of memory-type estimators can be recommended to survey professionals dealing with the problem of population mean estimation in the time-scaled surveys under SRS.

Author contributions

Anoop Kumar: methodology, simulation study, review and editing, and supervision; Renu Kumari: writing original manuscript and software; Abdullah Mohammed Alomair: project administration and financial support. All authors have read and agreed to the published version of the manuscript.

Use of generative AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

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