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Research article

Sweeping surfaces generated by involutes of a spacelike curve with a timelike binormal in Minkowski 3-space

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Abstract: In this paper, we studied the geometry of sweeping surfaces generated by the involutes of spacelike curves with a timelike binormal in Minkowski 3-space E_1^3. First, we investigated the singularity concept, and the mean and Gaussian curvatures of these surfaces. Then, we provided the requirements for the surface to be developable (flat) and minimal. We also determined the sufficient and necessary conditions for the parameter curves of these surfaces to be geodesic and asymptotic. Moreover, we analyzed these surfaces when the parameter curves are lines of curvature on the surface. Finally, the examples of these surfaces were given and their corresponding figures were drawn.

Keywords: sweeping surface; involute curve; Minkowski space; developable and minimal surface; geodesic curve; asymptotic curve
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1. Introduction

Sweeping surfaces play an important role in both theoretical geometric research and applied fields, particularly in geometric modeling, kinematics, robotics, and computer graphics. A sweeping surface is constructed by the movement of a given planar curve, known as the profile curve (generatrix), along a predetermined path, called the spine (trajectory) curve, where the movement of the plane curve always occurs in the direction of the normal to the plane [1]. Given a profile curve $P(u)$, and a spine curve $T(v)$, the parametric representation of a sweeping surface is given as $S(u,v) = T(v) + \mathbf{M}(v) \cdot P(u)$, where $\mathbf{M}(v)$ is the so-called transformation matrix. That is, the profile curve can be rotated and scaled depending on the parameter of the trajectory curve. Due to the nature of this parametric form of sweeping surfaces, several types of other well-known surfaces, such as ruled (a profile curve of straight lines) canal (or specifically pipe), tubular, swung (surface of revolutions), and string surfaces can be formed by carefully examining the transformation matrix $\mathbf{M}(v)$ [2, 3].

However, in most of the cases, the transformation matrix is considered to be an identity matrix for simplicity, and many researchers use a local orthonormal frame moving along the trajectory curve. For example, in [4], the possibility of the construction of sweeping surfaces was examined by means of the Frenet frame and its modified version in Euclidean space. Sweeping surfaces according to the Darboux frame and rotation minimizing Darboux frame were studied in [5, 6], respectively. Further, by considering Bishop frames, the characterizations of sweeping surfaces can be found in [7], and the references therein. By utilizing the modified orthogonal frames, new sweeping surfaces were discussed in the very recent study [18].

Recent studies show that the concept of sweeping surfaces has been extended to the non-Euclidean spaces, namely Lorentz-Minkowski spaces, where the theory of curves and surfaces in these spaces were well-studied in [8, 9]. For example, timelike sweeping surfaces were studied in [10], whereas the devolopable spacelike sweeping surfaces were discussed in [11].

Additionally, researchers also benefited from the theory of associated curves to construct sweeping surfaces. The involute-evolute curve pair is one of such associated curves where the theory of these curves can be found in [12, 13] for Euclidean space, and in [14, 15] for Minkowski space. By combining the concepts of associated curve and sweeping surfaces, the involutive sweeping surfaces according to the Frenet frame in Euclidean space were defined and characterized in [16].

Apart from these, geodesic curves, asymptotic curves, and the line of curvature are important characteristic curves that lie on the surface and have been instrumental in surface analysis. The geodesic curve is defined as the shortest distance between two points on a surface. An asymptotic curve is a curve on a surface where its tangent vector at every point lies in the direction where the normal curvature is zero. Lastly, the line of curvature is a surface curve if all tangent vectors always points along with a principal direction. The studies on designing the surfaces on which a given specific curve lies as a characteristic curve can be found in [19–22] for Euclidean space and in [23–25] for Minkowski 3-space.

Motivated by the given literature, in this study, the sweeping surfaces generated by the involutes of spacelike curves with a timelike binormal and spacelike Darboux vector in Minkowski 3-space are examined. The singularity, mean curvature and Gaussian curvature of the surfaces are discussed and the necessary relations for the new sweeping surfaces to be classified as developable and minimal are provided. The necessary and sufficient conditions for the parameter curves on these surfaces to be geodesic and asymptotic are also obtained. In addition, these surfaces are studied when the parameter curves are lines of curvature. Finally, some illustrative examples of these new types of sweeping surfaces are studied.

2. Preliminaries

Let $\mathbb{R}^3$ be a 3-dimensional real vector space. For any $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$, the Lorentzian inner product of $\mathbf{p}$ and $\mathbf{q}$ is defined by

$$\langle \mathbf{p}, \mathbf{q} \rangle_L = p_1q_1 + p_2q_2 - p_3q_3. \tag{2.1}$$

The Minkowski 3-space denoted by $E_1^3 = \left(\mathbb{R}^3, \langle , \rangle_L\right)$ is defined as the real vector space $\mathbb{R}^3$ with the Lorentzian metric. The non-zero vector $\mathbf{q} \in E_1^3$ is spacelike, lightlike, or timelike if $\langle \mathbf{q}, \mathbf{q} \rangle_L > 0$, $\langle \mathbf{q}, \mathbf{q} \rangle_L = 0$, or $\langle \mathbf{q}, \mathbf{q} \rangle_L < 0$, respectively. The norm of the vector $\mathbf{q} \in E_1^3$ is defined by $\|\mathbf{q}\| = \sqrt{|\langle \mathbf{q}, \mathbf{q} \rangle_L|}$. For any two vectors $\mathbf{p}, \mathbf{q} \in E_1^3$, the vector product is defined by

$$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} -\mathbf{e_1} & -\mathbf{e_2} & \mathbf{e_3} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (p_3q_2 - p_2q_3, p_1q_3 - p_3q_1, p_1q_2 - p_2q_1),$$

where $\mathbf{e_1}, \ \mathbf{e_2}, \ \mathbf{e_3}$ is the canonical basis of E_1^3 [9, 12, 17].

Let $\varphi(s)$, $s \in I = [0, l]$, be a unit speed 3D spacelike curve with a timelike binormal in E_1^3 with curvature $\kappa(s)$ and torsion $\tau(s)$. Consider the Serret-Frenet frame $\{\mathbf{h_1}(s), \mathbf{h_2}(s), \mathbf{h_3}(s)\}$ associated with the curve $\varphi(s)$, and then the Serret-Frenet formulae is given by

$$\begin{pmatrix} \mathbf{h}'_1(s) \\ \mathbf{h}'_2(s) \\ \mathbf{h}'_3(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{h_1}(s) \\ \mathbf{h_2}(s) \\ \mathbf{h_3}(s) \end{pmatrix} = \omega(s) \times \begin{pmatrix} \mathbf{h_1}(s) \\ \mathbf{h_2}(s) \\ \mathbf{h_3}(s) \end{pmatrix},$$

where $\omega(s) = \tau(s)\mathbf{h_1}(s) - \kappa(s)\mathbf{h_3}(s)$ defines the Darboux vector of the Serret-Frenet frame.

The vectors $\mathbf{h_1}(s) = \varphi'(s)$, $\mathbf{h_2}(s) = \varphi''(s)/\|\varphi''(s)\|$, and $\mathbf{h_3}(s) = \mathbf{h_1}(s) \times \mathbf{h_2}(s)$ are called the unit tangent vector, the principal normal vector, and the binormal vector, respectively. The Serret-Frenet vector fields satisfy the following relations [17]:

$$\langle \mathbf{h_1}, \mathbf{h_1} \rangle_L = 1, \ \ \langle \mathbf{h_2}, \mathbf{h_2} \rangle_L = 1, \ \ \langle \mathbf{h_3}, \mathbf{h_3} \rangle_L = -1,$$
$$\mathbf{h_1} \times \mathbf{h_2} = \mathbf{h_3}, \ \ \mathbf{h_2} \times \mathbf{h_3} = -\mathbf{h_1}, \ \ \mathbf{h_3} \times \mathbf{h_1} = -\mathbf{h_2}.$$

Also, we have:
a) If $|\tau| > |\kappa|$, then ω is a spacelike vector and we can write

$$\begin{matrix} \kappa = \|\omega\| \sinh \phi \\ \tau = \|\omega\| \cosh \phi \end{matrix}, \quad \langle \omega, \omega \rangle_L = \|\omega\|^2 = \tau^2 - \kappa^2. \tag{2.2}$$

b) If $|\tau| < |\kappa|$, then ω is a timelike vector and we can write

$$\begin{matrix} \kappa = \|\omega\| \cosh \phi \\ \tau = \|\omega\| \sinh \phi \end{matrix}, \quad \langle \omega, \omega \rangle_L = \|\omega\|^2 = \kappa^2 - \tau^2, \tag{2.3}$$

where $\phi = \angle(\omega, \mathbf{h_3})$ [14].

Let $\varphi(s)$ and $\bar{\varphi}(s)$, $s \in I$, be two curves such that $\bar{\varphi}$ intersects the tangents of φ orthogonally. Then $\bar{\varphi}$ is called an involute of φ. An involute of a curve $\varphi(s)$ with arc length s is given by

$$\bar{\varphi}(s) = \varphi(s) + \lambda \mathbf{h_1}(s), \tag{2.4}$$

where $\lambda = c - s \neq 0$ and c is a real constant [12, 13].

If $\varphi(s)$ is a unit speed spacelike curve with a timelike binormal, then the involute curve $\bar{\varphi}(s)$ must be a spacelike curve with a spacelike or timelike binormal. On the other hand, the relations between

the Frenet frames of involute curve $\bar{\varphi}$ and evolute curve φ are given as follows [14]:
a) If $\boldsymbol{\omega}$ is a spacelike vector:

$$\begin{bmatrix} \mathbf{h}_1^* \\ \mathbf{h}_2^* \\ \mathbf{h}_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\phi & 0 & -\cosh\phi \\ -\cosh\phi & 0 & \sinh\phi \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix}. \tag{2.5}$$

b) If $\boldsymbol{\omega}$ is a timelike vector:

$$\begin{bmatrix} \mathbf{h}_1^* \\ \mathbf{h}_2^* \\ \mathbf{h}_3^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\phi & 0 & \sinh\phi \\ -\sinh\phi & 0 & \cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{bmatrix}. \tag{2.6}$$

The parametric equation of the sweeping surface along the spine curve $\varphi(s)$ can be given as follows:

$$\eta(s,t) = \varphi(s) + \mathbf{H}(s)\mathbf{x}(t), \tag{2.7}$$

where $\mathbf{x}(t) = (0, x_1(t), x_2(t))^T$ is called the planar profile (cross section); and ^{T} represents transposition, with another parameter $t \in I \subset \mathbb{R}$. The semiorthogonal matrix $\mathbf{H}(s) = \{\mathbf{h_1}(s), \mathbf{h_2}(s), \mathbf{h_3}(s)\}$ represents a moving frame along $\varphi(s)$ [4].

Let $\eta(s,t)$ be a sweeping surface in E_1^3. Then the various definitions and fundamental concepts relevant to sweeping surfaces can be presented as follows:

The normal vector is $\mathbf{U}(s,t) = \dfrac{\eta_s \times \eta_t}{\|\eta_s \times \eta_t\|}$, where $\eta_s = \dfrac{\partial \eta}{\partial s}$ and $\eta_t = \dfrac{\partial \eta}{\partial t}$.

The Gaussian and mean curvatures of $\eta(s,t)$ are defined by, respectively,

$$K = \varepsilon \frac{eg - f^2}{EG - F^2}, \quad H = \varepsilon \frac{Eg - 2Ff + Ge}{2(EG - F^2)}, \tag{2.8}$$

where

$$E = \langle \eta_s, \eta_s \rangle_L, \quad F = \langle \eta_s, \eta_t \rangle_L, \quad G = \langle \eta_t, \eta_t \rangle_L,$$
$$e = \langle \eta_{ss}, U \rangle_L, \quad f = \langle \eta_{st}, U \rangle_L, \quad g = \langle \eta_{tt}, U \rangle_L,$$

and $\langle \mathbf{U}(s,t), \mathbf{U}(s,t) \rangle_L = \varepsilon = \pm 1$ [8,9].

A surface in Minkowski 3-space E_1^3 is called a spacelike (resp. timelike) surface if the induced metric on the surface is a positive (resp. negative) definite Riemannian metric, which means the normal vector on the spacelike surface is timelike and the normal vector on the timelike surface is a spacelike vector. The surface is spacelike if $EG - F^2 > 0$ and it is timelike if $EG - F^2 < 0$ [8,9].

3. Sweeping surfaces generated by involutes of a spacelike curve with a timelike binormal

Let $\bar{\varphi}(s)$ be the involute of a regular 3D spacelike curve $\varphi(s)$ with a timelike binormal and $\left\{\mathbf{h}_1^*(s), \mathbf{h}_2^*(s), \mathbf{h}_3^*(s)\right\}$ is its Frenet frame in Minkowski 3-space E_1^3. Also, consider the spacelike Darboux vector $\boldsymbol{\omega}$ and the non-null planar profile curve. For the involute curve $\bar{\varphi}(s)$ as a spine curve and a unit speed planar profile curve $\mathbf{x}(t) = (0, x_1(t), x_2(t))^T$, the sweeping surface can be given as

$$\bar{\eta}(s,t) = \bar{\varphi}(s) + \mathbf{H}^*(s)\mathbf{x}(t) = \bar{\varphi}(s) + \mathbf{h}_2^*(s)x_1(t) + \mathbf{h}_3^* x_2(t), \tag{3.1}$$

where $\mathbf{H}^*(s) = \left\{\mathbf{h_1^*}(s), \mathbf{h_2^*}(s), \mathbf{h_3^*}(s)\right\}$ is an orthogonal matrix representing a moving frame along the involute curve $\bar{\varphi}(s)$. Then using Eqs (2.4) and (2.5), the equation of the sweeping surface $\bar{\eta}(s,t)$ can be rearranged as follows:

$$\begin{aligned}\bar{\eta}(s,t) &= \bar{\varphi}(s) + \mathbf{H}^*(s)\mathbf{x}(t) = \bar{\varphi}(s) + \mathbf{h_2^*}(s)x_1(t) + \mathbf{h_3^*}(s)x_2(t)\\ &= \varphi(s) + \lambda\mathbf{h_1}(s) + x_1(t)(\sinh\phi\mathbf{h_1}(s) - \cosh\phi\mathbf{h_3}(s))\\ &\quad + x_2(t)(-\cosh\phi\mathbf{h_1}(s) + \sinh\phi\mathbf{h_3}(s)).\end{aligned}$$

If we take the partial derivatives of $\bar{\varphi}(s)$ with respect to s and t, respectively, and using Eq (2.2), we obtain the partial differentiation with respect to s and t as follows:

$$\begin{aligned}\bar{\eta}_s &= (x_1\phi'\cosh\theta - x_2\phi'\sinh\phi)\mathbf{h_1} + \rho\mathbf{h_2} + (-x_1\phi'\sinh\phi + x_2\phi'\cosh\phi)\mathbf{h_3},\\ \bar{\eta}_t &= (x_1'\sinh\phi - x_2'\cosh\phi)\mathbf{h_1} + (-x_1'\cosh\phi + x_2'\sinh\phi)\mathbf{h_3},\end{aligned} \tag{3.2}$$

where $\rho(s,t) = \lambda\kappa - x_1\|\omega\|$.
The second-order partial derivatives of $\bar{\eta}(s,t)$ are found by

$$\begin{aligned}\bar{\eta}_{ss} &= ((x_1\phi'\cosh\phi - x_2\phi'\sinh\phi)' - \rho\kappa)\mathbf{h_1} + (x_2\phi'\|\omega\| + \rho_s)\mathbf{h_2}\\ &\quad +((-x_1\phi'\sinh\phi + x_2\phi'\cosh\phi)' + \rho\tau)\mathbf{h_3},\\ \bar{\eta}_{st} &= (x_1'\phi'\cosh\phi - x_2'\phi'\sinh\phi)\mathbf{h_1} + \rho_t\mathbf{h_2}\\ &\quad +(-x_1'\phi'\sinh\phi + x_2'\phi'\cosh\phi)\mathbf{h_3},\\ \bar{\eta}_{tt} &= (x_1''\sinh\phi - x_2''\cosh\phi)\mathbf{h_1} + (x_2''\sinh\phi - x_1''\cosh\phi)\mathbf{h_3},\end{aligned} \tag{3.3}$$

where $\rho_s = -\kappa + \lambda\kappa'$ and $\rho_t = -x_1'\|\omega\|$.
By taking the vector product of $\bar{\eta}_s$ and $\bar{\eta}_t$, we get

$$\bar{\eta}_s \times \bar{\eta}_t = \rho(x_1'\cosh\phi - x_2'\sinh\phi)\mathbf{h_1} + m\mathbf{h_2} - \rho(x_1'\sinh\phi - x_2'\cosh\phi)\mathbf{h_3}, \tag{3.4}$$

where

$$m(s,t) = \phi'(x_2x_2' - x_1x_1'). \tag{3.5}$$

Remark 3.1. *If $\phi = \operatorname{arctan}h\left(\frac{\kappa}{\tau}\right)$ is a constant angle, then the spacelike curve $\varphi(s)$ with a timelike binormal is a helix and Eq (3.5) gives $m(s,t) = 0$.*

Now, consider the cases when the planar profile curve is timelike and spacelike.
Case 1. Let the planar profile curve $\mathbf{x}(t) = (0, x_1(t), x_2(t))^T$ be a timelike curve, which means that ${x_1'}^2 - {x_2'}^2 = -1$.
Hence, Eq (3.4) gives

$$\begin{aligned}\|\bar{\eta}_s \times \bar{\eta}_t\| &= \sqrt{\left|\rho^2({x_1'}^2 - {x_2'}^2) + m^2\right|}\\ &= \sqrt{\left|m^2 - \rho^2\right|}.\end{aligned} \tag{3.6}$$

From Eqs (3.4) and (3.6), we can get the unit normal vector of $\bar{\eta}(s,t)$ as follows:

$$\mathbf{U}(s,t) = \frac{\rho(x_1'\cosh\phi - x_2'\sinh\phi)\mathbf{h_1} + m\mathbf{h_2} - \rho(x_1'\sinh\phi - x_2'\cosh\phi)\mathbf{h_3}}{\sqrt{\left|m^2 - \rho^2\right|}}. \tag{3.7}$$

By using the equations in (3.2), the components of the first fundamental form are obtained by

$$\begin{aligned} E &= (x_1^2 - x_2^2)\phi'^2 + \rho^2, \\ F &= \phi'(x_1' x_2 - x_1 x_2'), \\ G &= -x_1'^2 + x_2'^2 = 1. \end{aligned} \tag{3.8}$$

Then by the aid of Eq (3.8), we obtain $EG - F^2 = \rho^2 - m^2$. Hence, the surface is spacelike if $\rho^2 - m^2 > 0$ and the surface is timelike if $\rho^2 - m^2 < 0$.

Thus, we can give the following theorem.

Theorem 3.2. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ has singularity on the point $p = (s_0, t_0)$ if*

$$m(s_0, t_0) = \pm\rho(s_0, t_0). \tag{3.9}$$

Proof. Equation (3.6) shows that $\bar{\eta}(s,t)$ has a singularity at $p = (s_0, t_0)$ if it satisfies

$$\|\bar{\eta}_s \times \bar{\eta}_t\| = \sqrt{\left|m^2(s_0, t_0) - \rho^2(s_0, t_0)\right|}.$$

Using this equation, we get

$$m^2(s_0, t_0) - \rho^2(s_0, t_0) = 0,$$

which implies that $m(s_0, t_0) = \pm\rho(s_0, t_0)$. □

Theorem 3.3. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The Gaussian and mean curvatures of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are, respectively,*

$$\begin{aligned} K &= \frac{\varepsilon}{\left|m^2 - \rho^2\right|(\rho^2 - m^2)}\left[\rho\begin{pmatrix} \rho\,(x_1 x_1' - x_2 x_2')\,\phi'' \\ +\rho(x_1 x_2' - x_2 x_1')\phi'^2 \\ -\rho^2 x_2'\,\|\omega\| + m\,(x_2\phi'\,\|\omega\| + \rho_s) \end{pmatrix}(x_1'' x_2' - x_1' x_2'') - (\rho_t m - \rho\phi')^2\right], \\ H &= \frac{\varepsilon}{2\sqrt{\left|m^2 - \rho^2\right|}(\rho^2 - m^2)}\begin{bmatrix} \rho\left((x_1^2 - x_2^2)\phi'^2 + \rho^2\right)(x_1'' x_2' - x_1' x_2'') \\ +\rho\,(x_1 x_1' - x_2 x_2')\,\phi'' + \rho\,(x_1 x_2' - x_2 x_1')\,\phi'^2 \\ -\rho^2 x_2'\,\|\omega\| + m\,(x_2\phi'\,\|\omega\| + \rho_s) \\ -2\phi'\,(x_2 x_1' - x_1 x_2')\,(\rho_t m - \rho\phi') \end{bmatrix}. \end{aligned} \tag{3.10}$$

Proof. From Eq (3.3), we can compute the components of the second fundamental form as

$$\begin{aligned} e &= \frac{1}{\sqrt{\left|m^2 - \rho^2\right|}}\left(\rho\,(x_1 x_1' - x_2 x_2')\,\phi'' + \rho\,(x_1 x_2' - x_2 x_1')\,\phi'^2 - \rho^2 x_2'\,\|\omega\| + m\,(x_2\phi'\,\|\omega\| + \rho_s)\right), \\ f &= \frac{1}{\sqrt{\left|m^2 - \rho^2\right|}}\,(m\rho_t - \rho\phi'), \\ g &= \frac{\rho}{\sqrt{\left|m^2 - \rho^2\right|}}\,(x_1'' x_2' - x_2'' x_1'), \end{aligned} \tag{3.11}$$

where $\rho_s = \lambda\kappa' - \kappa$ and $\rho_t = -x_1' \|\omega\|$. Thus, by substituting (3.8) and (3.11) into (2.8), the Gaussian curvature and the mean curvature of the sweeping surface $\bar{\eta}(s,t)$ can be obtained as in (3.10). □

Theorem 3.4. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are lines of curvature if and only if*

$$\begin{aligned} &\phi = \textit{constant} \quad \textit{or} \quad \frac{x_1}{x_2} = \textit{constant},\\ &\phi' = \frac{m\rho_t}{\rho}. \end{aligned} \tag{3.12}$$

Proof. The parameter curves of the sweeping surface $\bar{\eta}(s,t)$ are lines of curvature if $F = f = 0$. Then, from Eqs (3.8) and (3.11), we get

$$\phi'\,(x_2x_1' - x_1x_2') = 0, \quad \text{and} \quad m\rho_t - \rho\phi' = 0. \tag{3.13}$$

Let the angle $\phi = \arctan h\left(\frac{\kappa}{\tau}\right)$ be constant. Then, the equations in (3.13) are satisfied. In this case, as $\frac{\kappa}{\tau}$ is a constant, $\varphi(s)$ is a helix. If $\frac{x_1}{x_2}$ is a constant, then the first equality in (3.13) is provided. If $\phi' = \frac{m\rho_t}{\rho}$, then the second equality in (3.13) is provided. These results show that the conditions in (3.12) must be supplied in order to satisfy the two equations in (3.13). It is obvious that the converse is true. □

Theorem 3.5. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and $\bar{\eta}(s,t)$ is a regular sweeping surface generated by the Frenet frame of $\bar{\varphi}(s)$ in E_1^3.*
i) The s parameter curve of $\bar{\eta}(s,t)$ is an asymptotic curve if

$$\rho x_1'\left(x_1\phi'' - x_2\phi'^2\right) + \rho x_2'\left(x_1\phi'^2 - x_2\phi'' - \rho\,\|\omega\|\right) + m\,(x_2\phi'\,\|\omega\| + \rho_s) = 0. \tag{3.14}$$

ii) The t parameter curve of $\bar{\eta}(s,t)$ is an asymptotic curve if

$$\rho = 0, \quad \textit{or} \quad x_2'x_1'' - x_1'x_2'' = 0. \tag{3.15}$$

Proof. i) The s parameter curve of $\bar{\eta}(s,t)$ is an asymptotic curve if $\langle \eta_{ss}, \mathbf{U}\rangle_L = 0$. Then, by using (3.3) and (3.7), we obtain (3.14).
ii) The t parameter curve of $\bar{\eta}(s,t)$ is an asymptotic curve if $\langle \eta_{tt}, \mathbf{U}\rangle_L = 0$. Then, by using (3.3) and (3.7), equation $\rho\left(x_1''x_2' - x_1'x_2''\right) = 0$ gives (3.15). □

Theorem 3.6. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and $\bar{\eta}(s,t)$ is a regular sweeping surface generated by the Frenet frame of $\bar{\varphi}(s)$ in E_1^3.*
i) The s parameter curve of $\bar{\eta}(s,t)$ is a geodesic curve if

$$\begin{aligned} &m\left((x_2\phi'\cosh\phi - x_1\phi'\sinh\phi)' + \rho\tau\right) + \rho\,(x_2\phi'\,\|\omega\| + \rho_s)\,(x_1'\sinh\phi - x_2'\cosh\phi) = 0,\\ &\rho x_1'\left(x_1\phi'^2 - x_2\phi''\right) + \rho x_2'\left(x_1\phi'' - x_2\phi'^2\right) - \rho^2 x_1'\,\|\omega\| = 0,\\ &\rho\,(x_2\phi'\,\|\omega\| + \rho_s)\,(x_1'\cosh\phi - x_2'\sinh\phi) - m\left((x_1\phi'\cosh\phi - x_2\phi'\sinh\phi)' - \rho\kappa\right) = 0. \end{aligned} \tag{3.16}$$

ii) The t parameter curve of $\bar{\eta}(s,t)$ is a geodesic curve if

$$\begin{aligned} & m=0 \quad or \quad x_1'' \cosh\phi - x_2'' \sinh\phi = 0, \\ & \rho = 0 \quad or \quad x_2'' x_2' - x_1'' x_1' = 0, \\ & x_2'' \cosh\phi - x_1'' \sinh\phi = 0. \end{aligned} \tag{3.17}$$

Proof. i) The s parameter curve of $\bar{\eta}(s,t)$ is a geodesic curve if $\mathbf{U}\times\bar{\eta}_{ss} = 0$. Then, by using (3.3) and (3.7),

$$\begin{aligned} & \left\{m\left((x_2\phi'\cosh\phi - x_1\phi'\sinh\phi)' + \rho\tau\right) + \rho\left(x_2\phi'\|\omega\| + \rho_s\right)\left(x_1'\sinh\phi - x_2'\cosh\phi\right)\right\}\mathbf{h_1} \\ & + \left\{\rho x_1'\left(x_1\phi'^2 - x_2\phi''\right) + \rho x_2'\left(x_1\phi'' - x_2\phi'^2\right) - \rho^2 x_1'\|\omega\|\right\}\mathbf{h_2} \\ & + \left\{\rho\left(x_2\phi'\|\omega\| + \rho_s\right)\left(x_1'\cosh\phi - x_2'\sinh\phi\right) - m\left((x_1\phi'\cosh\phi - x_2\phi'\sinh\phi)' - \rho\kappa\right)\right\}\mathbf{h_3} = 0, \end{aligned}$$

satisfying the equations in (3.16).
ii) The t parameter curve of $\bar{\eta}(s,t)$ is a geodesic curve if $\mathbf{U}\times\bar{\eta}_{tt} = 0$. Then, by using (3.3) and (3.7),

$$\mathbf{U}\times\bar{\eta}_{tt} = \left(m\left(x_1''\cosh\phi - x_2''\sinh\phi\right)\right)\mathbf{h_1} + \left(\rho\left(x_2''x_2' - x_1''x_1'\right)\right)\mathbf{h_2} + \left(m\left(x_2''\cosh\phi - x_1''\sinh\phi\right)\right)\mathbf{h_3} = 0,$$

satisfying the equations in (3.17). □

Now, we assume that the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ are its lines of curvature. For simplicity, we suppose that $\rho = \lambda\kappa - x_1\|\omega\| > 0$. In this case, $\|\bar{\eta}_s\times\bar{\eta}_t\| = \sqrt{\left|-\rho^2\right|} = \rho \neq 0$, which means that $\bar{\eta}(s,t)$ is a regular surface. Let us begin by examining the singular points of the sweeping surface $\bar{\eta}(s,t)$.

Theorem 3.7. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. $\bar{\eta}(s,t)$ has singularity on the point $p=(s_0,t_0)$ if*

$$x_1(s_0,t_0) = \frac{(c-s_0)\kappa(s_0)}{\sqrt{\tau^2(s_0)-\kappa^2(s_0)}}.$$

Proof. $\bar{\eta}(s,t)$ has singularity on the point $p=(s_0,t_0)$ if $\|\bar{\eta}_s\times\bar{\eta}_t\|(s_0,t_0) = 0$. Then, from Remark 3.1 and (3.9), the equation can be expressed as follows:

$$\begin{aligned} \|\bar{\eta}_s\times\bar{\eta}_t\|(s_0,t_0) &= \rho(s_0,t_0) = 0 \\ & \Rightarrow (c-s_0)\,\kappa(s_0) - x_1(t_0)\sqrt{\tau(s_0)-\kappa(s_0)} = 0 \\ & \Rightarrow x_1(t_0) = \frac{(c-s_0)\,\kappa(s_0)}{\sqrt{\tau(s_0)-\kappa(s_0)}}. \end{aligned}$$

□

Theorem 3.8. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, the Gaussian and mean curvatures of $\bar{\eta}(s,t)$ are, respectively,*

$$K = \varepsilon \frac{x_2' \left\|\omega\right\| (x_2'' x_1' - x_1'' x_2')}{\rho},$$
$$H = \varepsilon \frac{\rho(x_1'' x_2' - x_1' x_2'') - x_2' \left\|\omega\right\|}{2\rho}. \tag{3.18}$$

Proof. By utilizing Eqs (3.8), (3.11), and (3.12), we can determine the fundamental coefficients of the surface of $\bar{\eta}(s,t)$ as

$$E = \rho^2, \quad F = 0, \quad G = 1,$$
$$e = -\rho x_2' \left\|\omega\right\|, \quad f = 0, \quad g = x_1'' x_2' - x_1' x_2''. \tag{3.19}$$

By substituting these equations into Eq (2.8), the equations can be acquired. □

Theorem 3.9. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, the principal curvatures of $\bar{\eta}(s,t)$ are*

$$k_1 = -\frac{x_2' \left\|\omega\right\|}{\rho},$$
$$k_2 = x_1'' x_2' - x_2'' x_1'. \tag{3.20}$$

Proof. Let the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ be lines of curvature. Then, the principal curvatures of $\bar{\eta}(s,t)$ can be obtained as follows:

$$k_1 = \frac{e}{E} = -\frac{x_2' \left\|\omega\right\|}{\rho},$$
$$k_2 = \frac{g}{G} = x_1'' x_2' - x_2'' x_1'. \tag{3.21}$$

□

Theorem 3.10. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, the relation between the Gaussian curvature K and the mean curvature H of $\bar{\eta}(s,t)$ is given by*

$$H = \frac{1}{2}\left(\varepsilon g + \frac{K}{g}\right). \tag{3.22}$$

Proof. By utilizing Eqs (3.18) and (3.19), Eq (3.22) is obtained. □

Theorem 3.11. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, $\bar{\eta}(s,t)$ is a flat surface if*

$$x_2 = \textit{constant}, \quad \textit{or} \quad k_2 = 0. \tag{3.23}$$

Proof. The surface $\bar{\eta}(s,t)$ is a flat surface when the Gaussian curvature vanishes. Hence, Eq (3.18) gives

$$x_2' \left\|\omega\right\| (x_2'' x_1' - x_1'' x_2') = 0.$$

Since $\left\|\omega\right\| \neq 0$, by using Eq (3.20), we obtain the equations in (3.23). □

Theorem 3.12. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then $\bar{\eta}(s,t)$ is a minimal surface if*

$$x_2' = \frac{\rho g}{\|\omega\|}. \tag{3.24}$$

Proof. The surface $\bar{\eta}(s,t)$ is a flat surface when the mean curvature vanishes. Hence, Eq (3.18) gives

$$\rho(x_1'' x_2' - x_1' x_2'') - x_2' \|\omega\| = 0.$$

By using Eq (3.19), we obtain Eq (3.24). □

In the next part of this study, we examine the sweeping surfaces for a spacelike profile curve.

Case 2. Let the planar profile curve $\mathbf{x}(t) = (0, x_1(t), x_2(t))^T$ be a spacelike curve which means that $x_1'^2 - x_2'^2 = 1$.

By using Eq (3.4), we get

$$\begin{aligned} \|\bar{\eta}_s \times \bar{\eta}_t\| &= \sqrt{\left|\rho^2({x_1'}^2 - {x_2'}^2) + m^2\right|} \\ &= \sqrt{\rho^2 + m^2}. \end{aligned} \tag{3.25}$$

From Eqs (3.4) and (3.25), we can get the unit normal vector of $\bar{\eta}(s,t)$ as

$$\mathbf{U}(s,t) = \frac{\rho(x_1' \cosh\phi - x_2' \sinh\phi)\mathbf{h_1} + m\mathbf{h_2} - \rho(x_1' \sinh\phi - x_2' \cosh\phi)\mathbf{h_3}}{\sqrt{m^2 + \rho^2}}. \tag{3.26}$$

By using Eq (3.2), the components of the fundamental forms are obtained by

$$\begin{aligned} E &= (x_1^2 - x_2^2){\phi'}^2 + \rho^2, \\ F &= \phi'(x_1' x_2 - x_1 x_2'), \\ G &= -{x_1'}^2 + {x_2'}^2 = -1. \end{aligned} \tag{3.27}$$

Then by the aid of Eq (3.27), we obtain $EG - F^2 = -(\rho^2 + m^2) < 0$. Hence, the surface is timelike.

Hence, we can give the following theorem.

Theorem 3.13. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ has singularity on the point $p = (s_0, t_0)$ if*

$$m(s_0, t_0) = \rho(s_0, t_0) = 0. \tag{3.28}$$

Proof. Equation (3.25) gives that $\bar{\eta}(s,t)$ has a singularity at $p = (s_0, t_0)$ if it satisfies

$$\|\bar{\eta}_s \times \bar{\eta}_t\| = \sqrt{\rho^2(s_0,t_0) + m^2(s_0,t_0)} = 0.$$

Using this equation, we get

$$\rho^2(s_0,t_0) + m^2(s_0,t_0) = 0 \Rightarrow m(s_0,t_0) = \rho(s_0,t_0) = 0.$$

□

Theorem 3.14. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The Gaussian and mean curvatures of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are, respectively,*

$$K = \frac{1}{(m^2+\rho^2)^2}\begin{bmatrix} \rho\begin{pmatrix} -\rho(x_1x_1'-x_2x_2')\phi'' \\ -\rho(x_1x_2'-x_2x_1')\phi'^2 \\ +\rho^2 x_2'\,\|\omega\| - m(x_2\phi'\,\|\omega\|+\rho_s) \end{pmatrix}(x_1''x_2'-x_1'x_2'') \\ +(\rho_t m+\rho\phi')^2 \end{bmatrix},$$

$$H = \frac{1}{2(m^2+\rho^2)^{3/2}}\begin{bmatrix} -\rho\left((x_1^2-x_2^2)\phi'^2+\rho^2\right)(x_1''x_2'-x_1'x_2'') \\ +\rho(x_1x_1'-x_2x_2')\phi''+\rho(x_1x_2'-x_2x_1')\phi'^2 \\ -\rho^2x_2'\,\|\omega\| + m\left(x_2\phi'\,\|\omega\|+\rho_s\right) \\ +2\phi'(x_2x_1'-x_1x_2')\left(\rho_t m+\rho\phi'\right) \end{bmatrix}. \tag{3.29}$$

Proof. From Eqs (3.3) and (3.26), we can compute the components of the second fundamental form as

$$\begin{aligned} e &= \frac{1}{\sqrt{m^2+\rho^2}}\left(\rho(x_1x_1'-x_2x_2')\phi''+\rho(x_1x_2'-x_2x_1')\phi'^2-\rho^2x_2'\,\|\omega\|+m\left(x_2\phi'\,\|\omega\|+\rho_s\right)\right), \\ f &= \frac{1}{\sqrt{m^2+\rho^2}}\left(m\rho_t+\rho\phi'\right), \\ g &= \frac{\rho}{\sqrt{m^2+\rho^2}}\left(x_1''x_2'-x_2''x_1'\right). \end{aligned} \tag{3.30}$$

Thus, by substituting (3.27) and (3.30) into (2.8), the Gaussian curvature and the mean curvature of sweeping surface $\bar{\eta}(s,t)$ can be obtained as in (3.29). □

Theorem 3.15. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are lines of curvature if and only if*

$$\begin{aligned} &\phi = constant \quad or \quad \frac{x_1}{x_2} = constant, \\ &\phi' = -\frac{m\rho_t}{\rho}. \end{aligned} \tag{3.31}$$

Proof. The parameter curves of the sweeping surface $\bar{\eta}(s,t)$ are lines of curvature if $F = f = 0$. Then, using the same computations as in the proof of Theorem 3.4, from Eqs (3.27) and (3.30), we obtain the equations in (3.31). □

Theorem 3.16. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and $\bar{\eta}(s,t)$ is a regular sweeping surface generated by the Frenet frame of $\bar{\varphi}(s)$ in E_1^3.*
i) The s parameter curve of $\bar{\eta}(s,t)$ is an asymptotic curve if

$$\rho x_1'\left(x_1\phi''-x_2\phi'^2\right)+\rho x_2'\left(x_1\phi'^2-x_2\phi''-\rho\,\|\omega\|\right)+m\left(x_2\phi'\,\|\omega\|+\rho_s\right)=0. \tag{3.32}$$

ii) The t parameter curve of $\bar{\eta}(s,t)$ is an asymptotic curve if

$$\rho = 0, \quad or \quad x_1''x_2'-x_1'x_2''=0. \tag{3.33}$$

Proof. Using the same computations as in the proof of Theorem 3.5, by using (3.3) and (3.26), the equations in (3.32) and (3.33) are obtained. □

Theorem 3.17. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and $\bar{\eta}(s,t)$ is a regular sweeping surface generated by the Frenet frame of $\bar{\varphi}(s)$ in E_1^3.*
i) The s parameter curve of $\bar{\eta}(s,t)$ is a geodesic curve if

$$\begin{aligned}
&m\left((x_2\phi'\cosh\phi - x_1\phi'\sinh\phi)' + \rho\tau\right) + \rho\left(x_2\phi'\,\|\omega\| + \rho_s\right)(x_1'\sinh\phi - x_2'\cosh\phi) = 0,\\
&\rho x_1'\left(x_1\phi'^2 - x_2\phi''\right) + \rho x_2'\left(x_1\phi'' - x_2\phi'^2\right) - \rho^2 x_1'\,\|\omega\| = 0,\\
&+\rho\left(x_2\phi'\,\|\omega\| + \rho_s\right)(x_1'\cosh\phi - x_2'\sinh\phi) - m\left((x_1\phi'\cosh\phi - x_2\phi'\sinh\phi)' - \rho\kappa\right) = 0.
\end{aligned} \tag{3.34}$$

ii) The t parameter curve of $\bar{\eta}(s,t)$ is a geodesic curve if

$$\begin{aligned}
&m = 0 \quad or \quad x_1''\cosh\phi - x_2''\sinh\phi = 0,\\
&\rho = 0 \quad or \quad x_2''x_2' - x_1''x_1' = 0,\\
&x_2''\cosh\phi - x_1''\sinh\phi = 0.
\end{aligned} \tag{3.35}$$

Proof. Using the same computations as the proof of Theorem 3.6, by using (3.3) and (3.26), the equations in (3.34) and (3.35) are obtained. □

Now, assume that the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ are its lines of curvature. For simplicity, we take $\rho = \lambda\kappa - x_1\,\|\omega\| > 0$. In this case, $\|\bar{\eta}_s \times \bar{\eta}_t\| = \sqrt{\rho^2} = \rho \neq 0$, which means that $\bar{\eta}(s,t)$ is a regular surface.

Theorem 3.18. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. $\bar{\eta}(s,t)$ has singularity on the point $p = (s_0,t_0)$ if*

$$x_1(s_0,t_0) = \frac{(c - s_0)\kappa(s_0)}{\sqrt{\tau^2(s_0) - \kappa^2(s_0)}}. \tag{3.36}$$

Proof. $\bar{\eta}(s,t)$ has singularity on the point $p = (s_0,t_0)$ if $\|\bar{\eta}_s \times \bar{\eta}_t\|(s_0,t_0) = 0$. Then, from Remark 3.1 and Eq (3.28), $\rho(s_0,t_0) = 0$ satisfies (3.36). □

Theorem 3.19. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and spacelike ω Darboux vector in E_1^3. The Gaussian and mean curvatures of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are, respectively,*

$$\begin{aligned}
K &= \frac{x_2'\,\|\omega\|\,(x_1''x_2' - x_1'x_2'')}{\rho},\\
H &= \frac{-\rho(x_1''x_2' - x_1'x_2'') - x_2'\,\|\omega\|}{2\rho}.
\end{aligned} \tag{3.37}$$

Proof. By utilizing Eqs (3.27), (3.30), and (3.31), we can determine the fundamental coefficients of the surface of $\bar{\eta}(s,t)$ as

$$\begin{aligned}
&E = \rho^2, \quad F = 0, \quad G = -1,\\
&e = -\rho x_2'\,\|\omega\|, \quad f = 0, \quad g = x_1''x_2' - x_1'x_2''.
\end{aligned} \tag{3.38}$$

By substituting these equations into Eq (2.8), the equations can be acquired. □

Theorem 3.20. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, the principal curvatures of $\bar{\eta}(s,t)$ are*

$$\begin{aligned} k_1 &= -\frac{x_2' \,\|\omega\|}{\rho}, \\ k_2 &= x_1' x_2'' - x_1'' x_2'. \end{aligned} \tag{3.39}$$

Proof. Let the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ be lines of curvature. Then, the principal curvatures of $\bar{\eta}(s,t)$ are

$$\begin{aligned} k_1 &= \frac{e}{E} = -\frac{x_2' \,\|\omega\|}{\rho}, \\ k_2 &= \frac{g}{G} = x_1' x_2'' - x_1'' x_2'. \end{aligned} \tag{3.40}$$

□

Theorem 3.21. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, the relation between the Gaussian curvature K and the mean curvature H of $\bar{\eta}(s,t)$ is given by*

$$H = \frac{1}{2}\left(g - \frac{K}{g}\right). \tag{3.41}$$

Proof. By utilizing Eqs (3.37) and (3.38), Eq (3.41) is obtained. □

Theorem 3.22. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then $\bar{\eta}(s,t)$ is a flat surface if*

$$x_2 = constant, \quad or \quad k_2 = 0. \tag{3.42}$$

Proof. The surface $\bar{\eta}(s,t)$ is a flat surface when the Gaussian curvature vanishes. Hence, Eq (3.37) gives

$$x_2' \,\|\omega\|\, (x_2'' x_1' - x_1'' x_2') = 0.$$

Since $\|\omega\| \neq 0$, by using Eq (3.39), the equations in (3.42) are obtained. □

Theorem 3.23. *Let $\bar{\varphi}(s)$ be the involute of a unit speed spacelike curve $\varphi(s)$ with a timelike binormal and the parameter curves of the sweeping surface $\bar{\eta}(s,t)$ generated by the Frenet frame of $\bar{\varphi}(s)$ are its lines of curvature in E_1^3. Then, $\bar{\eta}(s,t)$ is a minimal surface if*

$$x_2' = -\frac{\rho g}{\|\omega\|}. \tag{3.43}$$

Proof. The surface $\bar{\eta}(s,t)$ is a flat surface when the mean curvature vanishes. Hence, Eq (3.37) gives

$$\rho(x_1'' x_2' - x_1' x_2'') + x_2' \,\|\omega\| = 0.$$

By using Eq (3.38), we obtain Eq (3.43). □

Example 3.24. Let us consider a spacelike curve with a timelike binormal parametrized as

$$\varphi(s) = \left(\frac{5}{3}s, \frac{4}{9}\cosh 3s, \frac{4}{9}\sinh 3s\right). \tag{3.44}$$

Then, the Frenet vectors of $\varphi(s)$ are given by

$$\begin{aligned}
&\mathbf{h_1}(s) = \left(\frac{5}{3}, \frac{4}{3}\sinh 3s, \frac{4}{3}\cosh 3s\right),\\
&\mathbf{h_2}(s) = (0, \cosh 3s, \sinh 3s),\\
&\mathbf{h_3}(s) = \left(\frac{4}{3}, \frac{5}{3}\sinh 3s, \frac{5}{3}\cosh 3s\right),\\
&\kappa(s) = 4, \quad \tau(s) = 5,\\
&\omega(s) = \tau(s)\mathbf{h_1}(s) - \kappa(s)\mathbf{h_3}(s) = (3, 0, 0).
\end{aligned} \tag{3.45}$$

Hence, using (2.2), we obtain $\sinh\phi(s) = \dfrac{4}{3}$ and $\cosh\phi(s) = \dfrac{5}{3}$. By putting $c = 0$ in (2.4), we get the involute curve of $\varphi(s)$ as

$$\bar{\varphi}(s) = \left(0, \frac{4}{9}\cosh 3s - \frac{4}{3}s\sinh 3s, \frac{4}{9}\sinh 3s - \frac{4}{3}s\cosh 3s\right), \tag{3.46}$$

with the Frenet vectors based on Eqs (2.7) and (3.1), the sweeping surfaces $\eta_1(s,t)$, and the $\bar{\eta}_1(s,t)$ generated by moving a timelike profile curve along $\varphi(s)$ and $\bar{\varphi}(s)$ expressed by the following equations:

$$\begin{aligned}
\eta_1(s,t) &= \left(\frac{5}{3}s, \frac{4}{9}\cosh 3s, \frac{4}{9}\sinh 3s\right) + (0, \sinh t, \cosh t)\begin{pmatrix} \frac{5}{3} & \frac{4}{3}\sinh 3s & \frac{4}{3}\cosh 3s \\ 0 & \cosh 3s & \sinh 3s \\ \frac{4}{3} & \frac{5}{3}\sinh 3s & \frac{5}{3}\cosh 3s \end{pmatrix}\\
&= \begin{pmatrix} \frac{5}{3}s + \frac{4}{3}\cosh t, \frac{4}{9}\cosh 3s + \sinh t\cosh 3s + \frac{5}{3}\cosh t\sinh 3s, \\ \frac{4}{9}\sinh 3s + \sinh t\sinh 3s + \frac{5}{3}\cosh t\cosh 3s \end{pmatrix},
\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}
\bar{\eta}_1(s,t) &= \left(0, \frac{4}{9}\cosh 3s - \frac{4}{3}s\sinh 3s, \frac{4}{9}\sinh 3s - \frac{4}{3}s\cosh 3s\right)\\
&\quad + (0, \sinh t, \cosh t)\begin{pmatrix} 0 & \mp\cosh 3s & \mp\sinh 3s \\ 0 & \mp\sinh 3s & \mp\cosh 3s \\ -1 & 0 & 0 \end{pmatrix}\\
&= \begin{pmatrix} -\cosh t, \frac{4}{9}\cosh 3s - \frac{4}{3}s\sinh 3s \mp \sinh t\sinh 3s, \\ \frac{4}{9}\sinh 3s - \frac{4}{3}s\cosh 3s \mp \sinh t\cosh 3s \end{pmatrix},
\end{aligned} \tag{3.48}$$

respectively. The curve $\varphi(s)$, its involute $\bar{\varphi}(s)$ as the trajectory curve, and a planar timelike profile curve are shown in Figure 1, whereas the surface $\eta_1(s,t)$ is shown in Figure 2, and the surface $\bar{\eta}_1(s,t)$ is shown in Figure 3. Finally, both surfaces $\eta_1(s,t)$ and $\bar{\eta}_1(s,t)$ are shown in Figure 4 where $-1 \leqslant s \leqslant 1$ and $-3 \leqslant t \leqslant 3$.

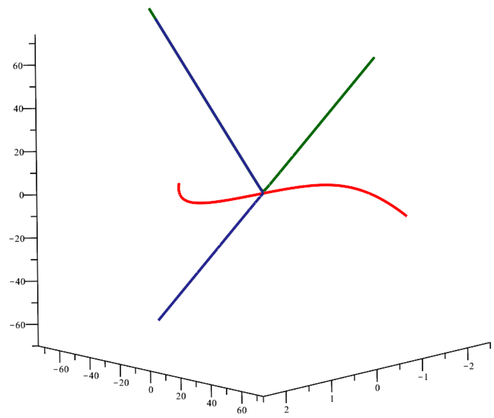

Figure 1. Timelike profile curve (dark green), the curve $\varphi(s)$ (red), and its involute curve $\bar{\varphi}(s)$ as the spine (trajectory) curve (blue).

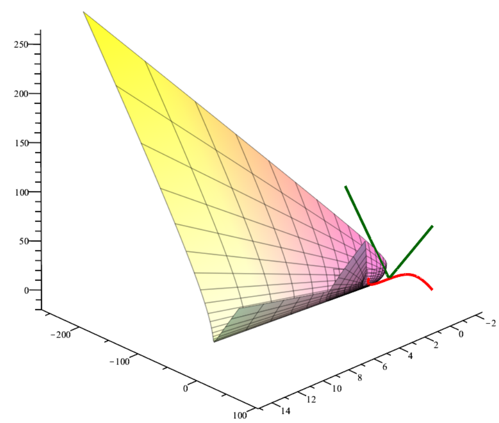

Figure 2. Sweeping surface $\eta_1(s,t)$ generated by $\varphi(s)$ (red) with the timelike profile curve (dark green).

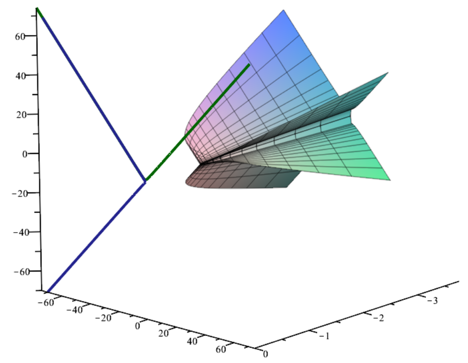

Figure 3. Sweeping surface $\bar{\eta}_1(s,t)$ generated by $\bar{\varphi}(s)$ (blue) with the timelike profile curve (dark green).

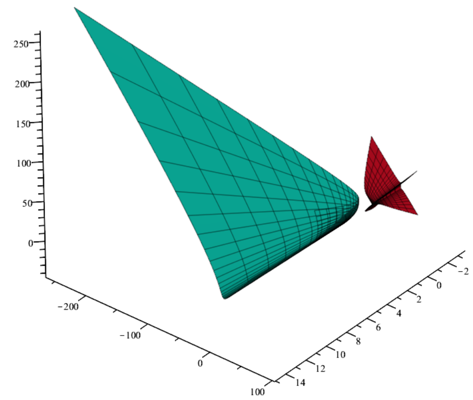

Figure 4. Sweeping surface $\eta_1(s,t)$ generated by $\varphi(s)$ (on the left) and the surface $\bar{\eta}_1(s,t)$ generated by $\bar{\varphi}(s)$ (on the right) with the timelike profile curve.

The sweeping surfaces $\eta_2(s,t)$ and $\bar{\eta}_2(s,t)$ generated by moving a spacelike profile curve along $\varphi(s)$ and $\bar{\varphi}(s)$ are expressed by the following equations:

$$\begin{aligned}\eta_2(s,t) &= \left(\frac{5}{3}s, \frac{4}{9}\cosh 3s, \frac{4}{9}\sinh 3s\right)\\ &\quad + (0,\cosh t,\sinh t)\begin{pmatrix} \frac{5}{3} & \frac{4}{3}\sinh 3s & \frac{4}{3}\cosh 3s \\ 0 & \cosh 3s & \sinh 3s \\ \frac{4}{3} & \frac{5}{3}\sinh 3s & \frac{5}{3}\cosh 3s \end{pmatrix}\\ &= \begin{pmatrix} \frac{5}{3}s + \frac{4}{3}\sinh t, \frac{4}{9}\cosh 3s + \cosh t\cosh 3s + \frac{5}{3}\sinh t\sinh 3s, \\ \frac{4}{9}\sinh 3s + \cosh t\sinh 3s + \frac{5}{3}\sinh t\cosh 3s \end{pmatrix},\end{aligned} \tag{3.49}$$

and

$$\bar{\eta}_2(s,t) = \left(0, \frac{4}{9}\cosh 3s - \frac{4}{3}s\sinh 3s, \frac{4}{9}\sinh 3s - \frac{4}{3}s\cosh 3s\right) + (0, \cosh t, \sinh t)\begin{pmatrix} 0 & \mp\cosh 3s & \mp\sinh 3s \\ 0 & \mp\sinh 3s & \mp\cosh 3s \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\sinh t, \frac{4}{9}\cosh 3s - \frac{4}{3}s\sinh 3s \mp \cosh t \sinh 3s, \\ \frac{4}{9}\sinh 3s - \frac{4}{3}s\cosh 3s \mp \cosh t\cosh 3s \end{pmatrix}. \tag{3.50}$$

The curve $\varphi(s)$, its involute $\bar{\varphi}(s)$ as the trajectory curve, and a planar spacelike profile curve are shown in Figure 5, whereas the surface $\eta_2(s,t)$ is shown in Figure 6, and the surface $\bar{\eta}_2(s,t)$ is given in Figure 7. Lastly, both of the two surfaces $\eta_2(s,t)$ and $\bar{\eta}_2(s,t)$ are shown in Figure 8 where $-1 \leqslant s \leqslant 1$ and $-3 \leqslant t \leqslant 3$.

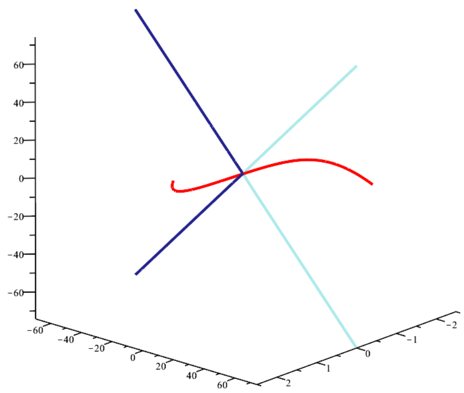

Figure 5. Spacelike profile curve (turquoise), the curve $\varphi(s)$ (red), and its involute $\bar{\varphi}(s)$ as the spine (trajectory) curve (blue).

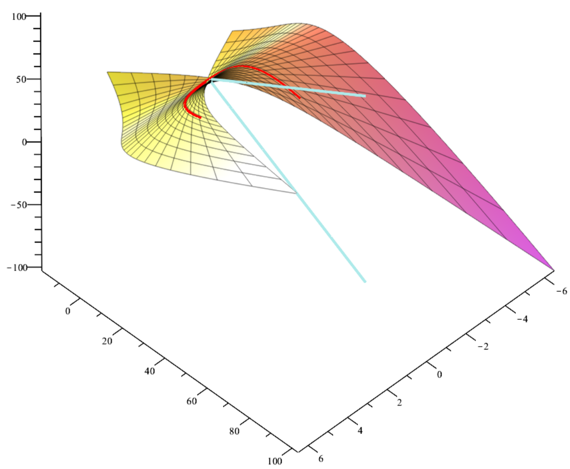

Figure 6. Sweeping surface $\eta_2(s,t)$ generated by $\varphi(s)$ (red) with the spacelike profile curve (turquoise).

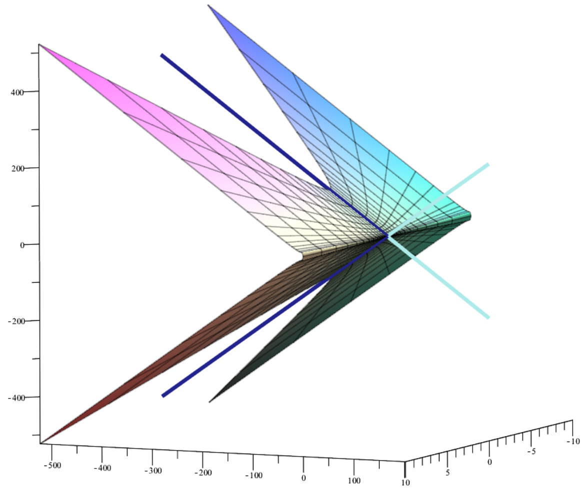

Figure 7. Sweeping surface $\bar{\eta}_2(s,t)$ generated by $\bar{\varphi}(s)$ (blue) with the spacelike profile curve (turquoise).

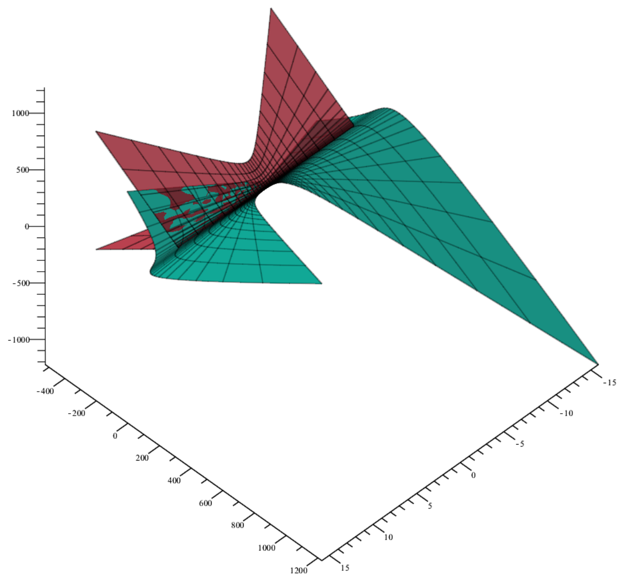

Figure 8. Sweeping surface $\eta_2(s,t)$ generated by $\varphi(s)$ and the surface $\bar{\eta}_2(s,t)$ generated by $\bar{\varphi}(s)$ with the spacelike profile curve.

4. Conclusions

This study analyzes the construction of sweeping surfaces generated by moving a non-null planar profile curve along the involute curve $\bar{\varphi}(s)$ of the main curve $\varphi(s)$ considered in Euclidean 3-space and now defined in Minkowski 3-space, taking into account the causal character of the curves. First, the sweeping surfaces generated by the involutes of spacelike curves with a timelike binormal and spacelike Darboux vector are defined in Minkowski 3-space E_1^3. Then, the singularity, and the mean and Gaussian curvatures of these surfaces are calculated and the conditions for these surfaces to be flat and minimal are examined. Also, the conditions for the parameter curves on the surface to be asymptotic and geodesic are investigated. Finally, the cases where the parameter curves are lines of curvature on the surface are obtained. Additionally, the examples of these surfaces are presented and their graphics are illustrated. Therefore, new surfaces are contributed to the literature of surface theory in Minkowski 3-space.

Author contributions

Özgür Boyacıoğlu Kalkan: Methodology, conceptualization, writing-original draft preparation, supervision, writing, formal analysis, resources; Süleyman Şenyurt: Supervision, formal analysis, validation; Mustafa Bilici: Methodology, investigation, conceptualization, reviewing, editing; Davut Canlı: Investigation, formal analysis, software, validation, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest in this paper.

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