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Research article

On the Generalized $\overline{\theta(\mathtt{t})}$-Fibonacci sequences and its bifurcation analysis

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Abstract: This paper introduces a general nabla operator of order two that includes coefficients of various trigonometric functions. We also introduce its inverse, which leads us to derive the second-order $\overline{\theta(\mathtt{t})}$-Fibonacci polynomial, sequence, and its summation. Here, we have obtained the derivative of the $\overline{\theta(\mathtt{t})}$-Fibonacci polynomial using a proportional derivative. Furthermore, this study presents derived theorems and intriguing findings on the summation of terms in the second-order Fibonacci sequence, and we have investigated the bifurcation analysis of the $\overline{\theta(\mathtt{t})}$-Fibonacci generating function. In addition, we have included appropriate examples to demonstrate our findings by using MATLAB.

Keywords: generalized nabla operator variable coefficients; Fibonacci sequence; Fibonacci summation; proportional α-derivative; bifurcation
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1. Introduction

Fractional calculus is a subdivision of classical calculus that concerns itself with derivatives as well as integrals of nearly any fractional order. This mathematical field has gained significant attention due to its ability to model complex phenomena more accurately than integer-order calculus. Fractional calculus offers robust methods for characterizing memory and hereditary features of different materials and processes, rendering it essential in disciplines such as control theory [14], signal processing [4], and differential equations [11–13]. The foundations of fractional calculus were laid by early mathematicians such as Leibniz, Liouville, and Riemann, who explored the concept of generalizing the order of differentiation and integration beyond integers [3, 7]. Modern advancements have further developed these ideas, leading to a robust theoretical framework and numerous practical applications.

The nabla difference operator is a discrete counterpart of the continuous derivative, employed in discrete calculus, specifically in the context of time-scale calculus. This operator, often denoted by ∇, operates on functions defined on discrete domains, such as sequences or time scales. The nabla difference operator plays a crucial role in various fields, including exact analysis, discrete dynamic systems, and operational calculus in [9, 21, 22]. It facilitates the formulation and solution of difference equations, which are the discrete counterparts of differential equations. The papers "On the definitions of nabla fractional operators" and "Discrete fractional calculus includes the nabla operator" go into great detail about what nabla fractional operators are and how they can be used in real life. They stress how important they are in the field of discrete mathematics [1, 2]. Recently, the nabla difference operator has been seen in sequential differences in the nabla fractional calculus, the combined delta-nabla sum operator in discrete fractional calculus, and discrete fractional calculus consisting of the nabla operator in [2, 20]. These studies contribute to a deeper understanding of the interplay between discrete and continuous analysis. In research [15], such as on K*ö*the-Toeplitz duals of generalized difference sequence spaces and Laplace transforms for the nabla-difference operator, investigations into the theoretical underpinnings and applications of these operators in discrete calculus [6].

The Fibonacci sequence is a widely recognized integer sequence defined by the recurrence relation $\mathtt{F}_r = \mathtt{F}_{r-1} + \mathtt{F}_{r-2}$, where $\mathtt{F}_0$ is 0 and $\mathtt{F}_1$ is 1. This sequence appears in various natural phenomena and has numerous applications in computer science, mathematics, and financial modeling. The study of sequence spaces derived from difference operators has gained significant attention, particularly with generalized Fibonacci difference operators. The derivation of sequence spaces arising from triple-band generalized Fibonacci difference operator [18, 24], investigates the structural properties of these spaces, while generalized Fibonacci difference sequence spaces and compact operators [17, 23] explores their impact on the boundedness and compactness of sequences. These works extend classical sequence space theory by integrating the recursive properties of Fibonacci sequences, highlighting new connections in functional analysis.

The motivation for this study arises from the need to further explore and expand the theoretical systems and applications of the nabla difference operator in generating and analyzing Fibonacci sequences. We present the trigonometric nabla difference operator of order 2 and its discrete integral, analyzing the $\overline{\theta(\mathtt{t})}$-sequences, their summation, and the proportional derivative of the $\overline{\theta(\mathtt{t})}$-polynomials.

The contributions of the research are delineated as follows:

1. By utilizing various trigonometric coefficients in the $\overline{\theta(\mathtt{t})}$-Fibonacci equation and its inverses, we formulate new sequences and analyze their characteristics.
2. The nabla difference operator of order 2 facilitates the generation of $\overline{\theta(\mathtt{t})}$-Fibonacci sequences and their exact and numerical solutions.

3. We provide chaotic behavior of the generating function of the second-order Fibonacci sequence with the coefficients of trigonometric functions.
4. The study includes MATLAB examples to demonstrate the practical applications of our theoretical findings.

Throughout this article, we make use of the notations and elucidations from the following Table 1.

Table 1. The notations and elucidations.

ξ	Shift (translation) value (i.e., $\xi \in [0, \infty)$)
$\mathtt{t}^{(\mathtt{x})}$	$\mathtt{t}(\mathtt{t}-\xi)(\mathtt{t}-2\xi)(\mathtt{t}-3\xi)...(\mathtt{t}-(\mathtt{x}-1)\xi)$
$\mathtt{t}-(\mathtt{n}+\mathtt{r})\xi$	$\mathtt{t}_{\mathtt{n},\mathtt{r}}$, where $\mathtt{n}$, $\mathtt{r}$ are integers and $\mathtt{t} \in (-\infty, \infty)$.
E^*-solution	Exact solution
N^*-solution	Numerical solution
$\overline{\theta(\mathtt{t})}$-sequence	Fibonacci sequence derived from general difference equation (Nabla) with trigonometric coefficients of order 2.
$\overline{\theta(\mathtt{t})}$-equation	General difference equation (Nabla) with trigonometric coefficients of order 2.

2. Second-order $\overline{\theta(\mathtt{t})}$-Fibonacci sequence and its sum

Here, we formulate a generic nabla-difference operator that includes a trigonometric co-efficient $\underset{\overline{\theta(\mathtt{t})}}{\nabla} v(\mathtt{t}) = v(\mathtt{t}) - \alpha_1 sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) - \alpha_2 sin(b_2\mathtt{t})v(\mathtt{t}_{0,2})$ which generates second-order $\overline{\theta(\mathtt{t})}$-Fibonacci sequence and its sum.

Definition 2.1. *Let* $\mathtt{t}$ *be any positive real number and* $n \geq 2$*; a second-order generic* $\overline{\theta(\mathtt{t})}$*-sequence is defined recurrently as* $\mathcal{F}_{\mathtt{t},0} = 1$, $\mathcal{F}_{\mathtt{t},1} = \alpha_1 sin(b_1\mathtt{t})$*, and*

$$\mathcal{F}_{\mathtt{t},n} = \alpha_1 sin(b_1\mathtt{t}_{n,1})\mathcal{F}_{\mathtt{t},n-1} + \alpha_2 sin(b_2\mathtt{t}_{n,2})\mathcal{F}_{\mathtt{t},n-2}. \tag{2.1}$$

Definition 2.2. *For any positive real number* $\mathtt{t}$*, a generic nabla difference operator of order two using sine (any trigonometric) function coefficients on* $v(\mathtt{t})$*, denoted as* $\underset{\overline{\theta(\mathtt{t})}}{\nabla} v(\mathtt{t})$*, is defined as*

$$\underset{\overline{\theta(\mathtt{t})}}{\nabla} v(\mathtt{t}) = v(\mathtt{t}) - \alpha_1 sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) - \alpha_2 sin(b_2\mathtt{t})v(\mathtt{t}_{0,2}), \tag{2.2}$$

and the inverse of $\underset{\overline{\theta(\mathtt{t})}}{\nabla} v(\mathtt{t}) = u(\mathtt{t})$ *is defined as*

$$v(\mathtt{t}) = \underset{\overline{\theta(\mathtt{t})}}{\nabla}^{-1} u(\mathtt{t}). \tag{2.3}$$

Definition 2.3. *[16] A proportional* α*- derivative is defined by*

$$\mathtt{D}^{\alpha}\mathtt{f}(\mathtt{t}) = \alpha\mathtt{f}'(\mathtt{t}) + (1-\alpha)\mathtt{f}(\mathtt{t}),\ \alpha \in [0, 1], \tag{2.4}$$

is the proportional α*-derivative of* $\mathtt{f}(\mathtt{t})$*. Here* $\mathtt{D}^{\alpha}$ *is proportional* α*- provided the function* $\mathtt{f}(\mathtt{t})$ *is differentiable at* $\mathtt{t}$*.*

Lemma 2.1. *For any real number* $\mathtt{t}, v(\mathtt{t})$ *is a function. Then it leads*

$$\underset{\theta(\mathtt{t})}{\nabla}^{-1} a^{s\mathtt{t}}\Big[1 - \frac{\alpha_1 sin(b_1\mathtt{t})}{a^{s\xi}} - \frac{\alpha_2 sin(b_2\mathtt{t})}{a^{2s\xi}}\Big] = a^{s\mathtt{t}}. \tag{2.5}$$

Proof: *By doing* $v(\mathtt{t}) = a^{\mathtt{t}_{0,0}}$ *in (2.2), we observe that*

$$\underset{\theta(\mathtt{t})}{\nabla} a^{s\mathtt{t}} = a^{s\mathtt{t}}\Big[1 - \frac{\alpha_1 sin(b_1\mathtt{t})}{a^{s\xi}} - \frac{\alpha_2 sin(b_2\mathtt{t})}{a^{2s\xi}}\Big].$$

Now, the function $v(\mathtt{t})$ *follows from the Definition 2.2.*

Remark 2.1. *When* $\alpha_1 = 1 = \alpha_2$ *in lemma 2.1, then we observe that*

$$\underset{\theta(\mathtt{t})}{\nabla}^{-1} a^{s\mathtt{t}}\Big[1 - \frac{sin(b_1\mathtt{t})}{a^{s\xi}} - \frac{sin(b_2\mathtt{t})}{a^{2s\xi}}\Big] = a^{s\mathtt{t}}. \tag{2.6}$$

Corollary 2.1. *For any real number* $\mathtt{t}$*,* $e^{s\mathtt{t}}$ *is a function. Then we observe*

$$\underset{\theta(\mathtt{t})}{\nabla}^{-1} e^{s\mathtt{t}}\Big[1 - \alpha_1 sin(b_1\mathtt{t})e^{-s\xi} - \alpha_2 sin(b_2\mathtt{t})e^{-2s\xi}\Big] = e^{s\mathtt{t}}. \tag{2.7}$$

Proof: *By doing* $a = e$ *in (2.5), we conclude the proof.*

Corollary 2.2. *For any real number* $\mathtt{t}$*,* $e^{-s\mathtt{t}}$ *is a function. Then we observe*

$$\underset{\theta(\mathtt{t})}{\nabla}^{-1} e^{-s\mathtt{t}}\Big[1 - \alpha_1 sin(b_1\mathtt{t})e^{s\xi} - \alpha_2 sin(b_2\mathtt{t})e^{2s\xi}\Big] = e^{-s\mathtt{t}}. \tag{2.8}$$

Proof: *By doing* $a = e^{-1}$ *in (2.5), we conclude the proof.*

Corollary 2.3. *For any real number* $\mathtt{t}$*,* $e^{-s\mathtt{t}}$ *is a function. Then we observe*

$$\underset{\theta(\mathtt{t})}{\nabla}^{-1} e^{-s\mathtt{t}}\Big[1 - sin(b_1\mathtt{t})e^{s\xi} - sin(b_2\mathtt{t})e^{2s\xi}\Big] = e^{-s\mathtt{t}}. \tag{2.9}$$

Proof: *By doing* $\alpha_1 = \alpha_2 = 1$ *in (2.8), we conclude the proof.*

Propsition 2.1. *If the function* $v(\mathtt{t}) = \overset{-1}{\underset{\theta(\mathtt{t})}{\nabla}} u(\mathtt{t})$ *is a* E^**-solution of (2.2),* $\mathcal{F}_{\mathtt{t},0} = 1, \mathcal{F}_{\mathtt{t},1} = \alpha_1 sin(b_1\mathtt{t})$ *and* $\mathcal{F}_{\mathtt{t},n+1} = \alpha_1 sin(b_1\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n} + \alpha_2 sin(b_2\mathtt{t}_{n,-1})\mathcal{F}_{\mathtt{t},n-1}$*, for* $i = 0, 1, 2, ...$ *then, the* E^**-solution is equal to a* N^**-solution, which is given by*

$$v(\mathtt{t}) - \mathcal{F}_{\mathtt{t},n+1}v(\mathtt{t}_{n,1}) - \alpha_2 sin(b_2\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n}v(\mathtt{t}_{n,2}) = \sum_{i=0}^{n} \mathcal{F}_{\mathtt{t},i}u(\mathtt{t}_{0,-i}). \tag{2.10}$$

Proof: *For the function* $v(\mathtt{t})$*, the Definition 2.2 yields*

$$v(\mathtt{t}) = u(\mathtt{t}) + \alpha_1 sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) + \alpha_2 sin(b_2\mathtt{t})v(\mathtt{t}_{0,2}). \tag{2.11}$$

By changing k by $k_{0,1}$ *and then put the value of* $v(\mathtt{t}_{0,1})$ *into (2.11), we get*
$v(\mathtt{t}) = u(\mathtt{t}) + \mathcal{F}_{\mathtt{t},1}u(\mathtt{t}_{0,1}) + [\mathcal{F}_{\mathtt{t},1}\alpha_1 sin(b_1\mathtt{t}_{0,1}) +$

$$\alpha_2 sin(b_2\mathtt{t})]v(\mathtt{t}_{0,2}) + \mathcal{F}_{\mathtt{t},1}\alpha_2 sin(b_2\mathtt{t}_{0,1})v(\mathtt{t}_{0,3}), \tag{2.12}$$

which leads to $v(\mathtt{t}) = \mathcal{F}_{\mathtt{t},0}u(\mathtt{t}) + \mathcal{F}_{\mathtt{t},1}u(\mathtt{t}_{0,1})+$

$$\mathcal{F}_{\mathtt{t},2}v(\mathtt{t}_{0,2}) + \alpha_2 sin(b_2\mathtt{t}_{0,1})\mathcal{F}_{\mathtt{t},1}v(\mathtt{t}_{0,3}), \tag{2.13}$$

where $\mathcal{F}_{\mathtt{t},0} = 0$, $\mathcal{F}_{\mathtt{t},1} = \alpha_1 sin(b_1\mathtt{t})$ *and* $\mathcal{F}_{\mathtt{t},2} = \mathcal{F}_{\mathtt{t},n+1} = \alpha_1 sin(b_1\mathtt{t}_{1,0})\mathcal{F}_{\mathtt{t},1} + \alpha_2 sin(b_2\mathtt{t}_{1,-1})\mathcal{F}_{\mathtt{t},0}$. *By changing k by* $k_{0,2}$ *in (2.11) and then putting the value of* $v(\mathtt{t}_{0,2})$ *into (2.13), we observe*

$$v(\mathtt{t}) = \mathcal{F}_{\mathtt{t},0}u(\mathtt{t}) + \mathcal{F}_{\mathtt{t},1}u(\mathtt{t}_{0,1}) + \mathcal{F}_{\mathtt{t},2}u(\mathtt{t}_{0,2}) + \mathcal{F}_{\mathtt{t},3}v(\mathtt{t}_{0,3}) + \alpha_2 sin(b_2\mathtt{t}_{0,2})\mathcal{F}_{\mathtt{t},2}v(\mathtt{t}_{0,4}),$$

where $\mathcal{F}_{\mathtt{t},3} = \mathcal{F}_{\mathtt{t},3} = \alpha_1 sin(b_1\mathtt{t}_{2,0})\mathcal{F}_{\mathtt{t},2} + \alpha_2 sin(b_2\mathtt{t}_{2,-1})\mathcal{F}_{\mathtt{t},1}$.
By carrying out this procedure repeatedly (2.10).

Corollary 2.4. *If the function* $\underset{\theta(\mathtt{t})}{\overset{-1}{\nabla}} u(\mathtt{t}) = v(\mathtt{t})$, $\mathcal{F}_{\mathtt{t},0} = 1, \mathcal{F}_{\mathtt{t},1} = sin(b_1\mathtt{t})$ *and* $\mathcal{F}_{\mathtt{t},n+1} = sin(b_1\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n} + sin(b_2\mathtt{t}_{n,1})\mathcal{F}_{\mathtt{t},n-1}$, *for* $i = 0, 1, 2, ...$ *then*

$$v(\mathtt{t}) - \mathcal{F}_{\mathtt{t},n+1}v(\mathtt{t}_{0,-1}) - sin(b_2\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n}v(\mathtt{t}_{0,-2}) = \sum_{i=0}^{n} \mathcal{F}_{\mathtt{t},i}\, u(\mathtt{t}_{0,-i}). \tag{2.14}$$

Proof: *The proof follows by doing* $\alpha_1 = 1 = \alpha_2$ *in Theorem 2.1.*

Corollary 2.5. *Let* $v(\mathtt{t})$ *be any function and*

$$\underset{\theta(\mathtt{t})}{\nabla} v(\mathtt{t}) = a^{s\mathtt{t}}\left[1 - \frac{\alpha_1 sin(b_1\mathtt{t})}{a^{s\xi}} - \frac{\alpha_2 sin(b_2\mathtt{t})}{a^{2s\xi}}\right],$$

then we observe that
$a^{s\mathtt{t}} - \mathcal{F}_{\mathtt{t},n+1}a^{s(\mathtt{t}_{n,-1})} - \alpha_2 sin(b_2\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n}a^{s(\mathtt{t}_{n,-2})}$

$$= \sum_{i=0}^{n}\mathcal{F}_{\mathtt{t},i}a^{s\mathtt{t}_{0,-i}}\Big[1 - \frac{\alpha_1 sin(b_1\mathtt{t}_{0,-i})}{a^{s\xi}} - \frac{\alpha_2 sin(b_2\mathtt{t}_{0,-i})}{a^{2s\xi}}\Big]. \tag{2.15}$$

Proof: *By doing* $v(\mathtt{t}) = a^{s\mathtt{t}}$ *and applying (2.5) in (2.10), we observed the proof.*

The following illustration proves the significance of corollary 2.5.

Corollary 2.6. *Let* $\alpha_1 = \alpha_2 = 1$ *in (2.15). Then we have*
$a^{\mathtt{t}_{0,0}} - \mathcal{F}_{\mathtt{t},n+1}a^{\mathtt{t}_{n,-1}} - sin(b_2\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n}a^{\mathtt{t}_{n,-2}}$

$$= \sum_{i=0}^{n}\mathcal{F}_{\mathtt{t},i}a^{\mathtt{t}_{0,i}}\Big[1 - \frac{sin(b_1\mathtt{t}_{0,-i})}{a^{\xi}} - \frac{sin(b_2\mathtt{t}_{0,-i})}{a^{2\xi}}\Big]. \tag{2.16}$$

Proof: *By doing* $v(\mathtt{t}) = a^{\mathtt{t}_{0,0}}$ *and putting (2.5) in (2.10), we complete the proof.*

The following illustration proves the significance of corollary 2.6.

Corollary 2.7. *Let* $e^{s\mathtt{t}}$ *be a function of* $\mathtt{t} \in (-\infty, \infty)$. *Then*
$e^{s\mathtt{t}} - \mathcal{F}_{\mathtt{t},n+1}e^{s(\mathtt{t}_{n,-1})} - \alpha_2 sin(b_2\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n}e^{s(\mathtt{t}_{n,-2})}$

$$= \sum_{i=0}^{n}\mathcal{F}_{\mathtt{t},i}e^{s\mathtt{t}_{0,-i}}\Big[1 - \alpha_1 sin(b_1\mathtt{t}_{0,-i})e^{-\xi} - \alpha_2 sin(b_2\mathtt{t}_{0,-i})e^{-2\xi}\Big]. \tag{2.17}$$

Proof: *By doing* $v(\mathtt{t}) = e^{s\mathtt{t}}$ *and applying (2.8) in (2.5), we obtain (2.17).*

Propsition 2.2. *Let $x \in \mathbb{N}(0)$. Then, the E^*-solution of $\overline{\theta(\mathtt{t})}$-equation*
$v(\mathtt{t}) - \alpha_1 sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) - \alpha_2 sin(b_2\mathtt{t})v(\mathtt{t}_{0,2}) = \left[\mathtt{t}^x_{0,0} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^x_{0,1} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^x_{0,2}\right]$ *is*

$$\underset{\theta(\mathtt{t})}{\overset{-1}{\nabla}}\left[\mathtt{t}^x_{0,0} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^x_{0,1} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^x_{0,2}\right] = \mathtt{t}^x_{0,0} \tag{2.18}$$

Proof: *By doing $v(\mathtt{t}) = \mathtt{t}^x_{0,0}$ with the equation (2.2) and using (2.3), we obtain (2.18).*

Corollary 2.8. *By doing $x = 2$ with the Theorem 2.2, we observed*

$$\underset{\theta(\mathtt{t})}{\overset{-1}{\nabla}}\left[\mathtt{t}^2_{0,0} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^2_{0,1} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^2_{0,2}\right] = \mathtt{t}^2_{0,0}, \tag{2.19}$$

which is the E^*-solution of the nabla difference equation of order two
$$\underset{\theta(\mathtt{t})}{\nabla} v(\mathtt{t}) = \mathtt{t}^2_{0,0} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^2_{0,1} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^2_{0,2}.$$
Proof: *From (2.18), replacing $x = 2$, we obtain (2.19).*

Corollary 2.9. *If $v(\mathtt{t}) = \underset{\theta(\mathtt{t})}{\overset{-1}{\nabla}}\left[\mathtt{t}^x_{0,0} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^x_{0,1} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^x_{0,2}\right]$ is a E^*-solution of the equation (2.18), then*
$v(\mathtt{t}) - \mathcal{F}_{\mathtt{t},n+1}v(\mathtt{t}_{0,-1}) - \alpha_2 sin(b_2\mathtt{t}_{n,0})\mathcal{F}_{\mathtt{t},n}v(\mathtt{t}_{0,-2})$

$$= \sum_{i=0}^{n}\mathcal{F}_{\mathtt{t},i}[\mathtt{t}^x_{0,i} - \alpha_1 sin(b_1\mathtt{t}_{0,-i})\mathtt{t}^x_{i,-1} - \alpha_1 sin(b_2\mathtt{t}_{0,-i})\mathtt{t}^x_{i,-2}]. \tag{2.20}$$

Proof: *Taking $u(\mathtt{t}) = \mathtt{t}^x_{0,0} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^x_{0,1} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^x_{0,2}$in (2.10), we have (2.20).*

Propsition 2.3. *If $v(\mathtt{t})$ is a E^*-solution of sine-coefficients $\overline{\theta(\mathtt{t})}$-equation*
$v(\mathtt{t}) - \alpha_1 sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) - \alpha_2 sin(b_2\mathtt{t})v(\mathtt{t}_{0,2}) = \mathtt{t}^x_{0,0}a^{\mathtt{t}_{0,0}} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^x_{0,1}a^{\mathtt{t}_{0,1}} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^x_{0,2}a^{\mathtt{t}_{0,2}}$,
then we have
$v(\mathtt{t}) - \mathcal{F}_{\mathtt{t},n+1}(\mathtt{t}_{n,-1}) - \alpha_2 sin(b_2\mathtt{t}_{n,0})v(\mathtt{t}_{n,-2})$

$$= \sum_{i=0}^{n}\mathcal{F}_{\mathtt{t},i}\mathtt{t}^x_{0,i}a^{\mathtt{t}_{0,i}}[1 - \alpha_1 sin(b_1\mathtt{t}_{0,-i})\mathtt{t}^x_{i,-1}a^{-\xi} - \alpha_2 sin(b_2\mathtt{t}_{0,-i})\mathtt{t}^x_{i,-2}a^{-2\xi}]. \tag{2.21}$$

Proof: *By doing $u(\mathtt{t}) = \left[\mathtt{t}^x_{0,0}a^{\mathtt{t}_{0,0}} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^x_{0,1}a^{\mathtt{t}_{0,1}} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^x_{0,2}a^{\mathtt{t}_{0,2}}\right]$ in (2.10) and using (2.5), we get (2.21).*

Corollary 2.10. *A E^*-solution of $\overline{\theta(\mathtt{t})}$-equation is*
$\underset{\theta(\mathtt{t})}{\nabla} v(\mathtt{t}) = \mathtt{t}^3_{0,0}a^{\mathtt{t}_{0,0}} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^3_{0,1}a^{\mathtt{t}_{0,1}} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^3_{0,2}a^{\mathtt{t}_{0,2}}$ *is* $\mathtt{t}^3a^{\mathtt{t}_{0,0}}$
and hence we have
$\mathtt{t}^3_{0,0}a^{\mathtt{t}_{0,0}} - \mathcal{F}_{\mathtt{t},n+1}\mathtt{t}^3_{n,-1}a^{\mathtt{t}_{n,-1}} - \alpha_2 sin(b_2\mathtt{t}_{n,0})\mathtt{t}^3_{n,-2}a^{\mathtt{t}_{n,-2}}$

$$= \sum_{i=0}^{n}\mathcal{F}_{\mathtt{t},i}\mathtt{t}^3_{0,i}a^{\mathtt{t}_{0,i}} - \alpha_1 sin(b_1\mathtt{t}_{0,-i})\mathtt{t}^3_{i,-1}a^{\mathtt{t}_{i,-1}} - \alpha_2 sin(b_2\mathtt{t}_{0,-i})\mathtt{t}^3_{i,-2}a^{\mathtt{t}_{i,-2}}. \tag{2.22}$$

Proof: *The proof is completed by doing $x = 3$ in Theorem 2.3.*

Propsition 2.4. *Let* $v(\mathtt{t})$ *be a solution of* $\overline{\theta(\mathtt{t})}$*-equation*
$v(\mathtt{t}) - \alpha_1 sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) - \alpha_2 sin(b_2\mathtt{t})v(\mathtt{t}_{0,2}) = \mathtt{t}^{(x)}a^{\mathtt{t}_{0,0}} - \alpha_1 sin(b_1\mathtt{t})\mathtt{t}^{(x)}_{0,1}a^{\mathtt{t}_{0,1}} - \alpha_2 sin(b_2\mathtt{t})\mathtt{t}^{(x)}_{0,2}a^{\mathtt{t}_{0,2}}$*, then we have*
$v(\mathtt{t}) - \mathcal{F}_{\mathtt{t},n+1}v(\mathtt{t}_{n,-1}) - \alpha_2 sin(b_2\mathtt{t}_{n,0})v(\mathtt{t}_{n,-2})$

$$= \sum_{i=0}^{n} \mathcal{F}_{\mathtt{t},i}a^{\mathtt{t}_{0,i}}[\mathtt{t}^{(x)}_{0,i} - \alpha_1 sin(b_1\mathtt{t}_{0,-i})\mathtt{t}^{(x)}_{i,-1}a^{-\xi} - \alpha_2 sin(b_2\mathtt{t}_{0,-i})\mathtt{t}^{(x)}_{i,-2}a^{-2\xi}]. \tag{2.23}$$

Proof: *By doing* $v(\mathtt{t}) = \mathtt{t}^{(x)}_{0,0}a^{\mathtt{t}_{0,0}}$ *in (2.10) and using (2.5), we get (2.23).*

Corollary 2.11. *Let* $v(\mathtt{t})$ *be a solution of* $\overline{\theta(\mathtt{t})}$*-equation*
$v(\mathtt{t}) - sin(b_1\mathtt{t})v(\mathtt{t}_{0,1}) - sin(b_2\mathtt{t})v(\mathtt{t}_{0,2}) = \mathtt{t}^{(x)}a^{\mathtt{t}_{0,0}} - sin(b_1\mathtt{t})\mathtt{t}^{(x)}_{0,1}a^{\mathtt{t}_{0,1}} - sin(b_2\mathtt{t})\mathtt{t}^{(x)}_{0,2}a^{\mathtt{t}_{0,2}}$*, then we have*
$v(\mathtt{t}) - \mathcal{F}_{\mathtt{t},n+1}v(\mathtt{t}_{n,-1}) - sin(b_2\mathtt{t}_{n,0})v(\mathtt{t}_{n,-2})$

$$= \sum_{i=0}^{n} \mathcal{F}_{\mathtt{t},i}a^{\mathtt{t}_{0,i}}[\mathtt{t}^{(x)}_{0,i} - sin(b_1\mathtt{t}_{0,-i})\mathtt{t}^{(x)}_{i,-1}a^{-\xi} - sin(b_2\mathtt{t}_{0,-i})\mathtt{t}^{(x)}_{i,-2}a^{-2\xi}]. \tag{2.24}$$

Proof: *By doing* $\alpha_1 = \alpha_2 = 1$ *in (2.23), we obtain (2.24).*

3. Illustrative numerical examples

The objective of this section is to demonstrate the efficacy of the primary findings presented in this paper by employing precise examples from the literature. Also, we have investigated graphical representations of Fibonacci sequences with the co-efficients of the Sine, Cosine and Co-secant functions for different values of $\mathtt{t}$ in the following Figures 1–3.

The validity of the definition 2.1 is confirmed by the subsequent illustrative example 3.1.

Example 3.1. *(i) By doing* $\mathtt{t} = 10$*,* $b_1 = 2$*,* $b_2 = 1$*,* $\xi = 0.4$*,* $\alpha_1 = 2$*, and* $\alpha_2 = 1$ *in (2.1), we obtain a Sine-Fibonacci sequence* $\{1, 1.8259, 0.7097, -0.9351, 1.9327, -3.9778, \cdots\}$.
(ii) When $\mathtt{t} = 15$*,* $\xi = 0.5$*,* $\alpha_1 = 0.2$*,* $\alpha_2 = 0.1$*,* $r1 = 3$*, and* $r2 = 1$ *in (2.1), we have a Cosine-Fibonacci sequence* $\{1.0000, -1.5194, 1.9182, -3.9008, 3.4203, 0.8856, 5.1684, \cdots\}$.
Also, the sine Fibonacci polynomials are observed by
$\mathcal{F}_0(\mathtt{t}) = 1,$
$\mathcal{F}_1(\mathtt{t}) = 2sin(2\mathtt{t}),$
$\mathcal{F}_2(\mathtt{t}) = 4sin(2\mathtt{t})sin(2\mathtt{t} - 0.8) + sin(\mathtt{t}),$
$\mathcal{F}_3(\mathtt{t}) = 8sin(2\mathtt{t})sin(2\mathtt{t} - 1.6)sin(2\mathtt{t} - 0.8) + 2sin(\mathtt{t})sin(2\mathtt{t} - 1.6) + 2sin(\mathtt{t} - 0.4)sin(2\mathtt{t})$*, etc.*
Furthermore, using (2.4), we have the following proportional α*-derivative of the sine Fibonacci polynomials;*
$\mathcal{F}^{\alpha}_0(\mathtt{t}) = 0,$
$\mathcal{F}^{\alpha}_1(\mathtt{t}) = 4\alpha cos(2\mathtt{t}) + 2(1 - \alpha)sin(2\mathtt{t}),$
$\mathcal{F}^{\alpha}_2(\mathtt{t}) = 8\alpha[sin(2\mathtt{t} - 0.8)cos2\mathtt{t} + cos(2\mathtt{t} - 0.8)sin2\mathtt{t} + cos\mathtt{t}]$
$+(1 - \alpha)[4sin(2\mathtt{t} - 0.8)sin2\mathtt{t} + sin\mathtt{t}],$
$\mathcal{F}^{\alpha}_3(\mathtt{t}) = \alpha[16cos(2\mathtt{t} - 1.6)sin(2\mathtt{t} - 0.8)sin(2\mathtt{t}) + 16sin(2\mathtt{t} - 1.6)cos(2\mathtt{t} - 0.8)sin(2\mathtt{t}) +$

$$16sin(2\mathtt{t} - 1.6)sin(2\mathtt{t} - 0.8)cos(2\mathtt{t})2cos(\mathtt{t})sin(2\mathtt{t} - 1.6) + 2sin(\mathtt{t})cos(2\mathtt{t} - 1.6) +$$

$2cos(\mathtt{t}-0.4)sin(2\mathtt{t})+4sin(\mathtt{t}-0.4)cos(2\mathtt{t})]+(1-\alpha)[8sin(2\mathtt{t}-1.6)sin(2\mathtt{t}-0.8)sin(2\mathtt{t})+$ $2sin(\mathtt{t})sin(2\mathtt{t}-1.6)]$, *and etc.*

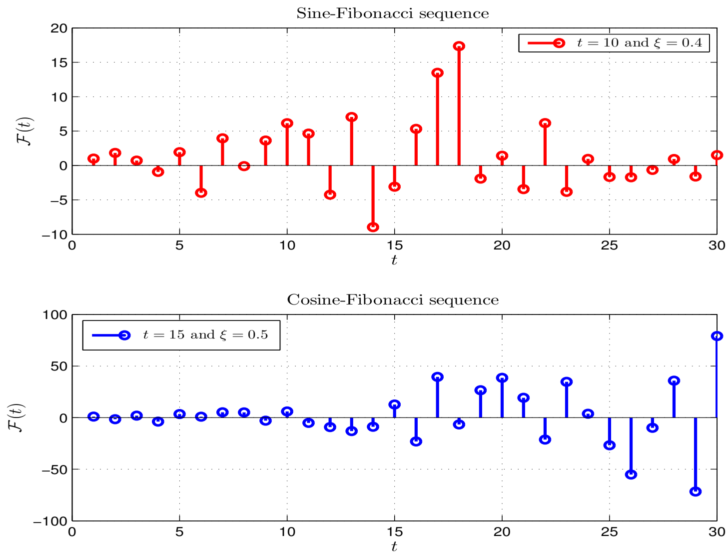

Figure 1. Sine and Cosine-Fibonacci Sequences for $\mathtt{t}=10$, $\xi=0.4$, and $\mathtt{t}=15$, $\xi=0.5$, respectively.

One can derive second-order Fibonacci polynomials and sequences for each pair $\overline{\theta(\mathtt{t})}\in\mathbb{R}^2$. The validity of the result 2.5 is confirmed by the subsequent illustrative example 3.2.

Example 3.2. *Setting* $\mathtt{t}=11$, $b_1=2$, $\alpha_1=3$, $s=2$, $\alpha_2=2$, $n=2$, $\xi=0.3$, $b_2=1$, *and* $a=3$ *in* (2.15)*, we obtain the following:*
(i) The sum of the sine-Fibonacci series

$$3^{22}-\mathcal{F}_{\mathtt{t},3}3^{10.1}-2\mathcal{F}_{\mathtt{t},2}sin(10.4)3^{19.6}$$
$$=\sum_{i=0}^{2}\mathcal{F}_{\mathtt{t},i}3^{(22-0.6i)}\Big[1-\frac{sin(22-0.6i)}{3^{0.6}}-\frac{sin(33-0.9i)}{3^{1.2}}\Big]=48298205761.12,$$

where $\mathcal{F}_{\mathtt{t},0}=1$, $\mathcal{F}_{\mathtt{t},1}=-0.03$, $\mathcal{F}_{\mathtt{t},2}=-2.04$, $\mathcal{F}_{\mathtt{t},3}=-5.65$.
(ii) The sum of the cosine-Fibonacci series

$$3^{22}-\mathcal{F}_{\mathtt{t},3}3^{10.1}-2\mathcal{F}_{\mathtt{t},2}cos(10.4)3^{19.6}$$
$$=\sum_{i=0}^{2}\mathcal{F}_{\mathtt{t},i}3^{(22-0.6i)}\Big[1-\frac{cos(22-0.6i)}{3}^{0.6}-\frac{cos(33-0.9i)}{3^{1.2}}\Big]=78777462484.69,$$

where $\mathcal{F}_{\mathtt{t},0}=1$, $\mathcal{F}_{\mathtt{t},1}=-3.00$, $\mathcal{F}_{\mathtt{t},2}=7.48$, $\mathcal{F}_{\mathtt{t},3}=-6.57$,
and the cosine Fibonacci polynomials are given by
$\mathcal{F}_0(\mathtt{t})=1$,

$\mathcal{F}_1(\mathtt{t}) = 3cos(2\mathtt{t})$,
$\mathcal{F}_2(\mathtt{t}) = 9cos(2\mathtt{t} - 0.6)cos(2\mathtt{t}) + 2cos(\mathtt{t})$
$\mathcal{F}_3(\mathtt{t}) = 27cos(2\mathtt{t} - 1.2)cos(2\mathtt{t} - 0.6)cos(2\mathtt{t}) + 6cos(2\mathtt{t} - 1.2)cos(\mathtt{t}) + 2cos(\mathtt{t} - 0.4)cos(2\mathtt{t})$, *etc. Furthermore, using (2.4), we have the following proportional* α*-derivative of the co-secant Fibonacci polynomials;*
$\mathcal{F}_0^{\alpha}(\mathtt{t}) = 0, \mathcal{F}_1^{\alpha}(\mathtt{t}) = \alpha[-6sin(2\mathtt{t})] + 3(1-\alpha)cos(2\mathtt{t})$,
$\mathcal{F}_2^{\alpha}\mathtt{t}) = \alpha[-18sin(2\mathtt{t} - 0.6)cos(2\mathtt{t}) - 18cos(2\mathtt{t} - 0.6)$
$sin(2\mathtt{t}) - 2sin\mathtt{t}] + (1-\alpha)9cos(2\mathtt{t} - 0.6)cos(2\mathtt{t}) + 2cos(\mathtt{t})]$, *and etc.*

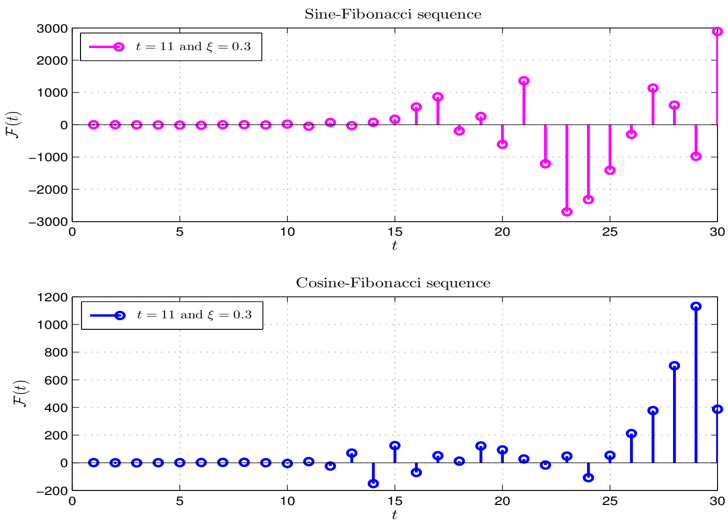

Figure 2. Sine and Cosine-Fibonacci Sequences for $\mathtt{t} = 11,\ \xi = 0.3$.

The validity of the definition 2.7 is confirmed by the subsequent illustrative example 3.3.

Example 3.3. *By doing* $\alpha_1 = 2$, $\mathtt{t} = 10$, $\alpha_2 = 1$, $\xi = 0.4$, $b_1 = 2$, $n = 2$, *and* $b_2 = 1$ *in* (2.16), *we have the following:*
(i) The sine-Fibonacci sum of the series is
$e^{-9} - F_4 e^5 - (0.3)6^2\mathcal{F}_{\mathtt{t},3}e^{-4} = \sum_{i=0}^{3}\mathcal{F}_{\mathtt{t},i}e^{-(9-i)}[1 - (0.8)(9-i)^3 e - (0.3)(9-i)^2 e^2]$
$= 523194317.45$, *where* $\mathcal{F}_{\mathtt{t},0} = 1$, $\mathcal{F}_{\mathtt{t},1} = 1.83$, $\mathcal{F}_{\mathtt{t},2} = 0.71$, *and* $\mathcal{F}_{\mathtt{t},3} = -0.23$.
(ii) The co-secant-Fibonacci sum of the series is
$e^{-9} - F_4 e^5 - (0.3)6^2\mathcal{F}_{\mathtt{t},3}e^{-4} = \sum_{i=0}^{3}\mathcal{F}_{\mathtt{t},i}e^{-(9-i)}\left[1 - (0.8)(9-i)^3 e - (0.3)(9-i)^2 e^2\right] = 1.49$, *where* $\mathcal{F}_{\mathtt{t},0} = 1$, $\mathcal{F}_{\mathtt{t},1} = 5.31$, $\mathcal{F}_{\mathtt{t},2} = 0.71$ *and* $\mathcal{F}_{\mathtt{t},3} = -0.94$.
Also, taking $\alpha_1 = 1$, $b_1 = 1$, $\xi = 0.5$, $\alpha_2 = 2$, *and* $b_3 = 1$ *in* (2.16), *we have the following co-secant Fibonacci polynomials:*
$\mathcal{F}_0(\mathtt{t}) = 1,\ \ \mathcal{F}_1(\mathtt{t}) = cosec(2t),\ \ \mathcal{F}_2(\mathtt{t}) = cosec(t-0.5)cosec(2t) + cosec(3t)$
$\mathcal{F}_3(\mathtt{t}) = cosec(t-1)cosec(t-0.5)cosec(2t) + cosec(t-1)cosec(3t) + cosec(t-0.5)cosec(2t)$, *etc. Further-more, using (2.4), we have the following proportional* α*-derivative of the co-secant Fibonacci polynomials;*
$\mathcal{F}_0^{\alpha}(\mathtt{t}) = 0,\ \ \mathcal{F}_1^{\alpha}(\mathtt{t}) = \alpha[-2cosec(2t)cot(2t)] + (1-\alpha)cosec(2t)$,

$\mathcal{F}_2^{\alpha}(\mathtt{t}) = \alpha[-2cosec(t-0.5)cot(t-0.5)cosec(2t)-2cosec(t-0.5)cosec(2t)cot(2t)-3cosec(3t)cot(3t)] + (1-\alpha)[cosec(t-0.5)cosec(2t) + cosec(3t)]$, *and etc.*

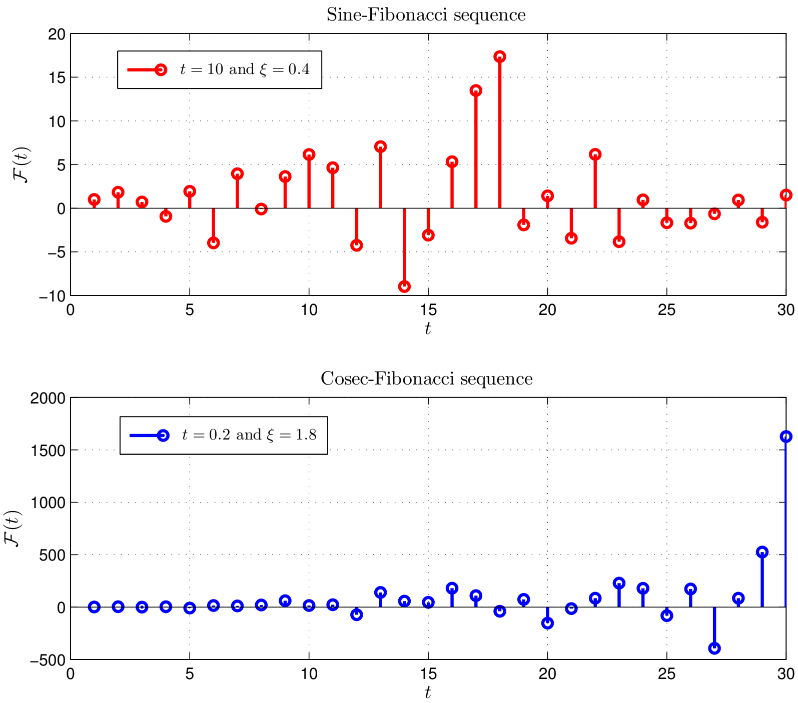

Figure 3. Sine and Co-secant-Fibonacci Sequences for $\mathtt{t} = 10$, $\xi = 0.4$, and $\mathtt{t} = 0.2$, $\xi = 1.8$, respectively.

4. Bifurcation behavior of $\overline{\theta(\mathtt{t})}$-Fibonacci generating function

The objective of this section is to demonstrate the efficacy of the bifurcation analysis of the $\overline{\theta(\mathtt{t})}$-Fibonacci generating function and primary findings presented in this paper by employing analysis from the following precise examples.

A discrete one-dimensional dynamical system is a system subjected to a single equation of this type

$$x(\mathtt{t}+1) = f(\mathtt{t}) \tag{4.1}$$

where $x \in \mathbb{Z}$ and f is a function of x. The variable $\mathtt{t}$ is in general considered as the time, but in discrete systems the time takes only discrete values, so that it is possible to take $\mathtt{t} \in \mathbb{Z}$.
A generalized discrete two-dimensional dynamical system is a system subjected to a single equation of this type

$$x(\mathtt{t}+2\xi) = x(\mathtt{t}+\xi) + x(\mathtt{t}) \tag{4.2}$$

where $\mathtt{t} \in \mathbb{R}$. When we reformulate Eq (4.2), we obtain a two-dimensional discrete system

$$x(\mathtt{t}+\xi) = x_1(\mathtt{t}) + x_2(\mathtt{t})$$

$$x_2(t+\xi) = x_1(t).$$

A trajectory is a set $\{x(t)\}_{t=0}^{\infty}$ of points satisfying the above equation (4.1). It is evident that the initial point $x_0 = x(0)$ determines the entire trajectory. The behaviour of the dynamical system is therefore given by all the trajectories $\{x(t) : x(0) = x_0\}$ for all initial values $x_0 \in I$. When the value of the parameter changes continuously, the behaviour of the system may change in a discontinuous way. One says that a bifurcation occurs for an isolate value of the parameter at which the type of dynamic changes. In bifurcation analysis, the region of stable operation is determined through the search of Hopf bifurcation points. This gives an insight into how the variations in the system parameters influence region of stable operation. This knowledge can be effectively used by the system designers to ensure the stability of the actual system.

A bifurcation diagram is a traditional and visual way to look into how dynamical systems, difference equations, and differential equations change over time [10]. This tool is excellent for looking at how the system reacts to changes in parameters [19]. This diagram illustrates the system's different dynamic patterns and phase transitions by plotting the link between the system reaction and parameters [5, 8]. This section employs bifurcation theory to determine the existence of the period-doubling (flip) bifurcation. We discuss the $\overline{\theta(t)}$-Fibonacci generating function and investigate the bifurcation analysis of the $\overline{\theta(t)}$-Fibonacci sequences.

$\overline{\theta(t)}$-Fibonacci sequences are generated by

$$f(t) = \frac{1}{1-\theta_1(t)t-\theta_2(t)t^2}, \tag{4.3}$$

where $\theta_1(t)$ and $\theta_2(t)$ are any trigonometric and hyperbolic functions. If $\theta_1(t) = sin(b_1 t)$ and $\theta_2(t) = sin(b_2 t)$ in (4.3), then we have the Fibonacci generating function

$$f(t) = \frac{1}{1-sin(b_1 t)t-sin(b_2 t)t^2}. \tag{4.4}$$

(*ii*). If $\theta_1(t) = cos(b_1 t)$ and $\theta_2(t) = cos(b_2 t)$ in (4.3), then we have the Fibonacci generating function

$$f(t) = \frac{1}{1-cos(b_1 t)t-cos(b_2 t)t^2}. \tag{4.5}$$

4.1. Chaotic behavior of the fibonacci generating function

In dynamical systems theory, a period-doubling bifurcation occurs when a slight change in a system's parameters causes a new periodic trajectory to emerge from an existing periodic trajectory. With the doubled period, it takes twice as long (or, in a discrete dynamical system, twice as many iterations) for the numerical values visited by the system to repeat themselves. A period-halving bifurcation occurs when a system switches to a new behavior with half the period of the original system. A period-doubling cascade is an infinite sequence of period-doubling bifurcations. Such cascades are a common route by which dynamical systems develop chaos.

Now, we consider the recurrent form of (4.3)

$$t_r = \frac{1}{1-\theta_1(t_{r-1})t_{r-1}-\theta_2(t_{r-1})t_{r-1}^2}, \tag{4.6}$$

and we consider the recurrent form of (4.4)

$$t_r = \frac{1}{1 - sin(b_1 t_{r-1})t_{r-1} - sin(b_2 t_{r-1})t_{r-1}^2}. \tag{4.7}$$

We consider the recurrent form of (4.5)

$$t_n = \frac{1}{1 - cos(b_1 t_{n-1})t_{n-1} - cos(b_2 t_{n-1})t_{n-1}^2}. \tag{4.8}$$

We examine the orbit $\{t_r\}_{r=0}^{\infty}$ for any point t_0 within the domain of the map.

Figure 4(a) displays a periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *sine* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 2$ and $b_1 \in [-8, 11]$. Figure 4(b) displays periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *cosine* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 2$ and $b_1 \in [-5, 5]$. Figure 4(c) displays periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *tangent* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 1$ and $b_1 \in [-5, 1]$. Figure 4(d) displays periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *cosine* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 1$ and $b_1 \in [-10, 5]$.

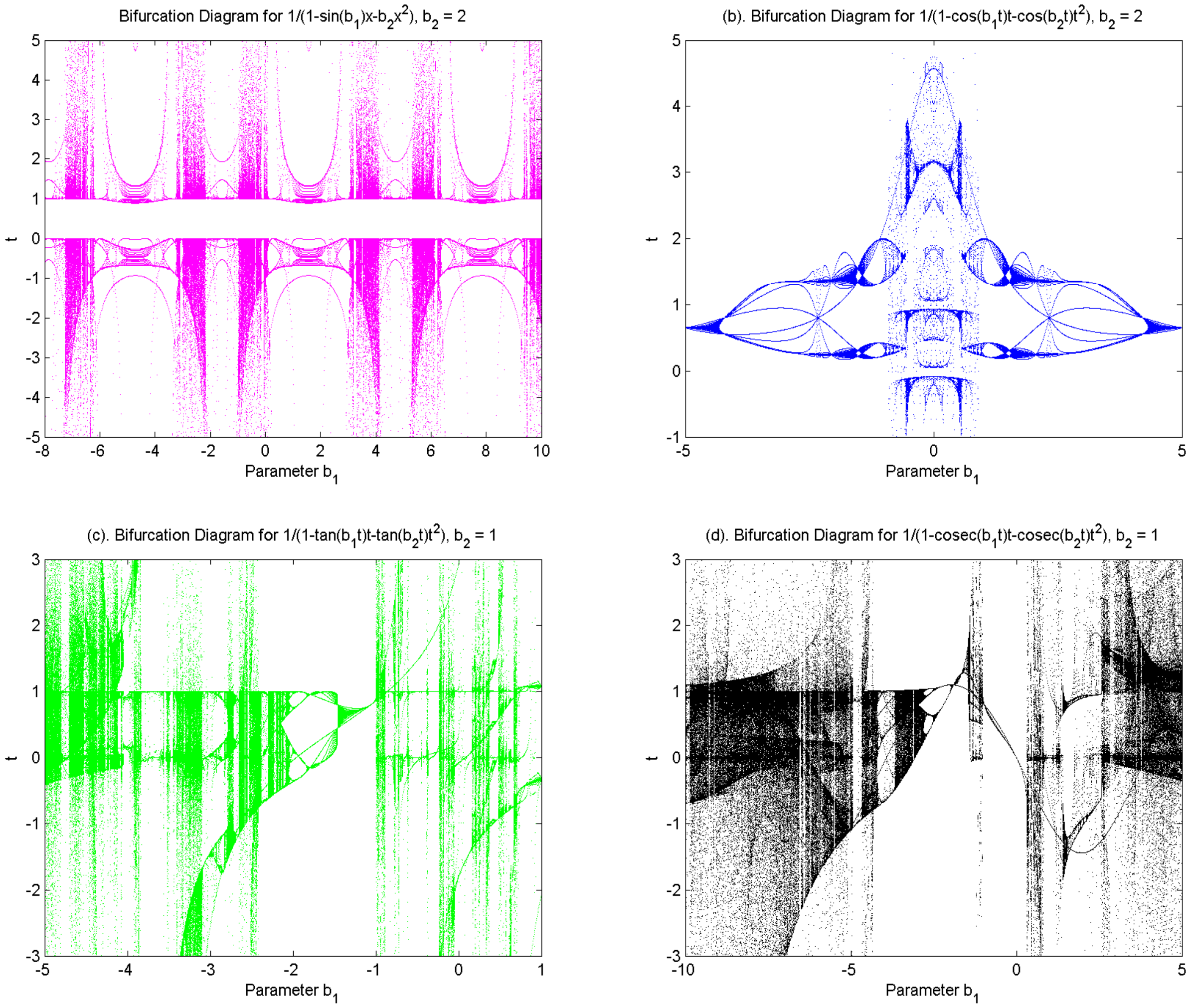

Figure 4. Period doubling bifurcation diagram for $\overline{\theta(\mathtt{t})}$-Fibonacci generating function.

Figure 5(a) displays sub periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *sine* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_{=}2$ and $b_1 \in [0, 0.3]$. Figure 5(b) displays sub periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *cosine* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 2$ and $b_1 \in [0.58, 2]$. Figure 5(c) displays sub periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *tangent* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 1$ and $b_1 \in [-2.2, -1.8]$. Figure 5(d) displays sub periodic doubling bifurcation diagram of the Fibonacci generating function of the co-efficient of the *cosine* function with the initial condition $t_0 = 0.8$ at the intrinsic growth bifurcation parameter $b_2 = 1$ and $b_1 \in [-3, -2.1]$.

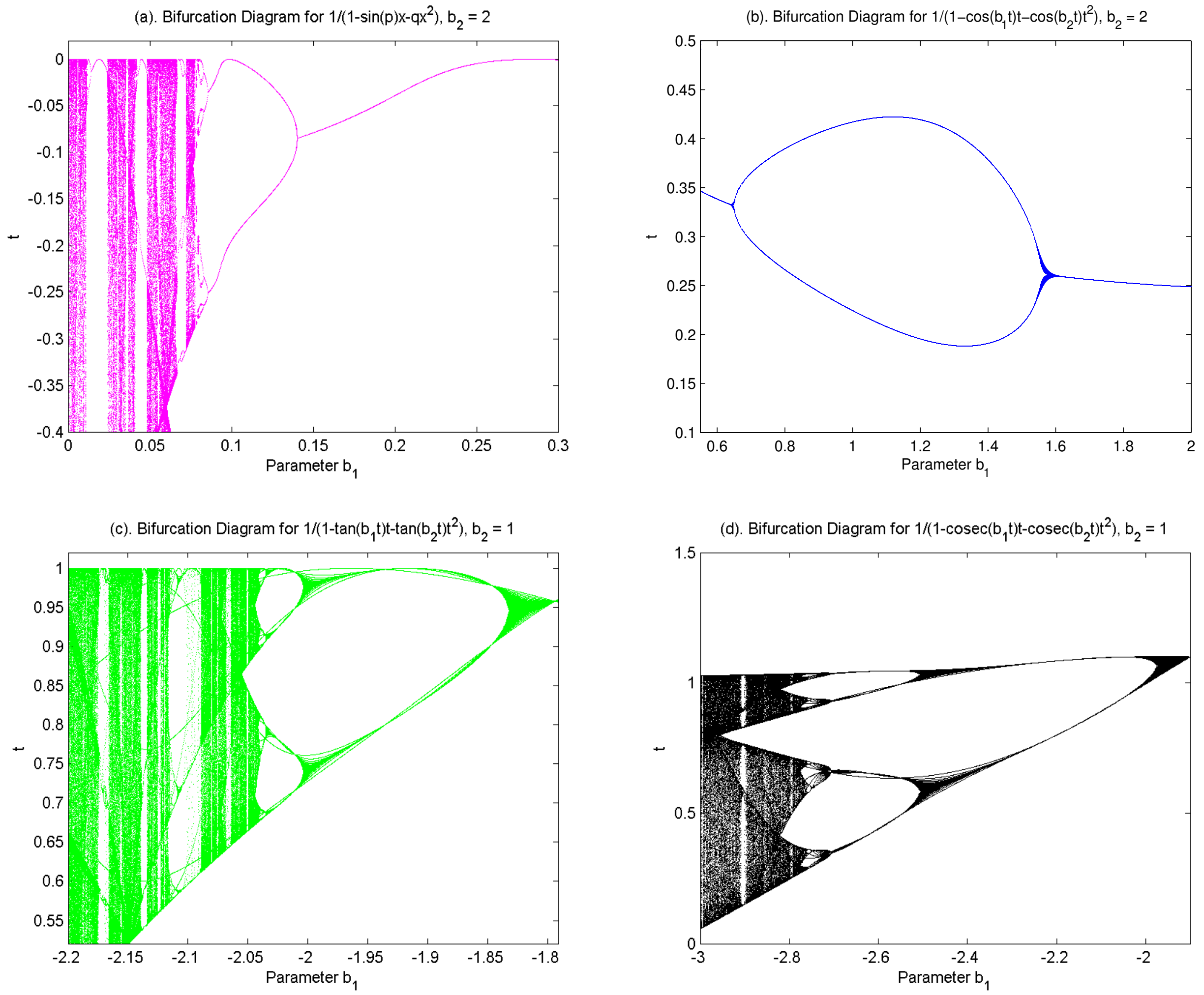

Figure 5. Sub period doubling cascade for $\overline{\theta(\mathtt{t})}$-Fibonacci generating function.

5. Conclusions

This paper deduced an inverse formula for the $\overline{\theta(\mathtt{t})}$-Fibonacci sequence. The inverse of a generic difference (nabla) operator with trigonometric coefficients of order 2 was used to derive this formula. The results we have obtained regarding the E^* and N^* solutions, Fibonacci polynomials, and the proportional derivative of the generic difference equation with trigonometric coefficients of order 2 will be applied to our future research. Additionally, we have conducted a bifurcation analysis of the $\overline{\theta(\mathtt{t})}$-Fibonacci generating function.

Author's contributions

R.P: Conceptualization; R.P, S.T.M.T, and I.K.: Data curation; R.P, S.T.M.T, I.K, and M.V.C.: Formal analysis; P.R, I.K, and M.V.C.: Investigation; R.P, S.T.M.T, and I.K.: Methodology; R.P and S.T.M.T.: Writing-original draft; R.P, I.K and M.V.C.: Writing-review and editing. All authors have read and agreed to the published version of the article.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. T. Abdeljawad, F. M. Atici, On the definitions of nabla fractional operators, *Abstr. Appl. Anal.*, **2012** (2012), 1–13. https://doi.org/10.1155/2012/406757
2. F. M. Atici, P. Eloe, On the definitions of nabla fractional operators, *Electron. J. Qual. Theory Differ. Equ.*, **3** (2009), 1–12. https://doi.org/10.14232/ejqtde.2009.4.3
3. P. L. Butzer, U. Westphal, An introduction to fractional calculus, *Fract. Calc. Appl. Anal.*, (2000), 1–85. https://doi.org/10.1142/9789812817747-0001
4. E. A. Gonzalez, I. Petraˇs, Advances in fractional calculus: Control and signal processing applications, *Proceedings of the 2015 16th International Carpathian Control Conference (ICCC)*, (2015), 147–152.
5. J. Guckenheimer, On the bifurcation of maps of the interval, *Invent. Math.*, **39** (1977), 165–178. https://doi.org/10.1007/BF01390107
6. J. Hein, Z. McCarthy, N. Gaswick, B. McKain, K. Speer, Laplace transforms for the nabla-difference operator, *Panam. Math. J.*, **21** (2011), 79–96.
7. G. Jumarie, Table of some basic fractional calculus formulae derived from a modified riemann liouville derivative for non-differentiable functions, *Appl. Math. Lett.*, **22** (2009), 378–385. https://doi.org/10.1016/j.aml.2008.06.003
8. Y. A. Kuznetsov, Y. A. Kuznetsov, Numerical analysis of bifurcations, Elements of applied bifurcation theory, *Appl. Math. Sci.*, **22** (2023), 505–585. https://doi.org/10.1007/978-3-031-22007-4-10
9. A. L. G. Laura, K. B. Grace, The calculus of proportional α-derivatives, *Rose-Hulman Undergraduate Math. J.*, **18** (2017), 20–42.
10. R. I. Leine, D. V. Campen, B. V. Vrande, Bifurcations in nonlinear discontinuous systems, *Nonlinear Dyn.*, **23** (2000), 105–164. https://doi.org/10.1023/A:1008384928636
11. G. Mani, A. Gnanaprakasam, O. Ege, A. Aloqaily, N. Mlaiki, Fixed point results in $c*$-algebra-valued partial b-metric spaces with related application, *Mathematics*, **11** (2023), 1158. https://doi.org/10.3390/math11051158

12. G. Mani, A. Gnanaprakasam, L. Guran, R. George, Z. D. Mitrovi, Some results in fuzzy b-metric space with b-triangular property and applications to fredholm integral equations and dynamic programming, *Mathematics*, **11** (2023), 4101. https://doi.org/10.3390/math11194101

13. G. Mani, S. Haque, A. Gnanaprakasam, O. Ege, N. Mlaiki, The study of bicomplex-valued controlled metric spaces with applications to fractional differential equations, *Mathematics*, **11** (2023), 2742. https://doi.org/10.3390/math11122742

14. R. Matu, Application of fractional order calculus to control theory, *Math. Models. Methods. Appl. Sci.*, **5** (2011), 1162–1169. https://doi.org/10.1007/978-3-031-22007-4-10

15. E. Mikail, E. Ayhan, On k¨othe-toeplitz duals of generalized difference sequence spaces, *Bull. Malays. Math. Sci. Soc.*, **23** (2000), 25–32. https://math.usm.my/bulletin/html/vol23-1-3.htm

16. P. Rajiniganth, S. Saravanan, M. Rhaima, H. Ghoudi, The generalized discrete proportional derivative and its applications, *Fractal Fract.*, **7** (2023), 838. https://doi.org/10.3390/fractalfract7120838

17. P. Rajiniganth, K. Suresh, S. T. M. Thabet, M. V. Cortez, I. Kedim, $\phi(x)$-tribonnaci polynomial, numbers, and its sum, *J. Math. Comput. Sci.*, **37** (2025), 32–44. https://dx.doi.org/10.22436/jmcs.037.01.03

18. P. Rajiniganth, T. Aparna, I. Khan, K. Suresh, More generalized $k^{(\epsilon)}$-fibonacci sequence, series, and its applications, *AIP Adv.*, **14** (2024), 015233. https://doi.org/10.1063/5.0180627

19. C. Sergent, M. Corazzol, G. Labouret, F. Stockart, M. Wexler, J. R. King, et al., Bifurcation in brain dynamics reveals a signature of conscious processing independent of report, *Nat Commun.*, **12** (2021), 1149. https://doi.org/10.1038/s41467-021-21393-z

20. A. Setniker, Sequential differences in nabla fractional calculus, *PhD thesis, University of Nebraska-Lincoln*, (2019), 1–167.

21. S. Sinha, The fibonacci numbers and its amazing applications, *Int. J. Eng. Sci. Invent.*, **6** (2017), 7–14.

22. H. Wysocki, The nabla difference model of the operational calculus, *Demonstr. Math.*, **46** (2013), 315–326. https://doi.org/10.1515/dema-2013-0461

23. T. Yaying, B. Hazarika, E. Mikail, On generalized fibonacci difference sequence spaces and compact operators, *Asian-Eur. J. Math.*, **15** (2022), 2250116. https://doi.org/10.1515/dema-2013-0461

24. T. Yaying, B. Hazarika, S. A. Mohiuddine, M. Mursaleen, K. J. Ansari, Sequence spaces derived by the triple band generalized fibonacci difference operator, *Adv. Differ. Equ.*, **23** (2020), 1–19. https://doi.org/10.1186/s13662-020-03099-6