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Research article

A comprehensive study of a feedback control problem with a state-dependent implicit pantograph equation of Chandrasekhar type

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Abstract: In this research, we investigate the existence of at least one continuous solution of a problem with feedback control involving implicit pantograph equations of the Chandrasekhar type with state-dependent delay. In addition, we examine the possibility of the uniqueness of the solution under suitable assumptions. Furthermore, we analyze the problem's Hyers-Ulam stability and the continuous dependency of the unique solution on the original data and the parameter. Moreover, we look into this problem in the absence of feedback control. We provided a few instances to indicate our findings.

Keywords: Schauder fixed point theorem; state-dependant; feedback control; pantograph equation; existence of solution; continuous dependence; Hyres-Ulam stability
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1. Introduction

The Chandrasekhar integral equation arises in the context of radiative transfer theory; this type of equation is essential when examining the propagation of radiation through a medium that exhibits scattering [14]. The study of Chandrasekhar's integral equation is crucial in a variety of areas and encountered in several applications, including mathematical and computational methods, astrological applications, radiative transfer theory, and stellar atmospheres (see [5, 11, 12, 48]). Several researchers have focused their attention on this type of equation as seen in [13, 15, 26, 28, 29] and the references therein.

Feedback control system is a fundamental concept in engineering and automation that controls a system's behavior by modifying inputs based on output [1]. A controller is a biological system that regulates the operation of other biological processes. Integrative feedback control is essential for regulation, sensory adaptation, and long-term effects. Control variables are vital in dealing with unanticipated occurrences that disrupt real-world ecosystems, potentially altering biological features.

These difficulties are converted into mathematical models [17, 39, 49]. Several authors investigated feedback problems. In [18], the authors established a necessary condition for a positive periodic solution of a feedback control model for chemostats. A positive periodic solution with feedback control involving a nonlinear neutral delay population problem has been analyzed in [36]. The author of [37] studied the asymptotic stability and solvability of nonlinear functional-integral equations with feedback control.

Stability analysis is an extensive and varied field with deep theoretical roots and numerous applications in engineering, economics, biology, physics, and other disciplines. It is a prevalent topic in the mathematical sciences [6]. An equation or problem can be used to simulate a physical process if a minor change in it causes a corresponding small change in the outcome. Alsina and Ger [4] initially examined the Hyers-Ulam stability of differential equations. Several papers have been devoted to studying the Hyers-Ulam stability of differential and integral equations, for example, (see [2, 3, 42–44]). Another concept in stability theory is continuous dependency [40], which analyzes the behavior of mathematical solutions under different conditions. Hyers-Ulam stability assesses the problem's resilience to disruptions, while continuous dependency examines how even minor parameter changes affect the problem's unique solution. The study of the continuous dependence of the solution has been addressed through many research works [22, 27].

Pantograph differential equations are an effective tool for modeling systems with feedback interactions. From signal processing to finance, control systems, and neural networks, these equations aid in the capture of complicated dynamics in which the present state-depends on past states at scaled time intervals. Solving pantograph differential equations analytically is difficult because of their nonlocal nature; often, numerical methods such as finite difference schemes are used to obtain approximate solutions. Much research has been conducted on pantograph equations due to their significance in various research areas. For instance, in [38], Patade et al. investigated the analytical solution of the pantograph equation with two delays; they analyzed the existence, uniqueness, and stability of the solution. In [23], the authors studied the solvability and the Hyers-Ulam stability of non-local fractional orders of the pantograph equation with a feedback control.

State-dependent (self-reference) differential and integral equations are a special new type of functional differential equations in which the deviation of the argument depends on the time and the state together. In most differential and integral equations with deviating arguments that are found in literature, the deviation of the argument involves only the time, although another important case in theory and practice involves deviating arguments that depend on both the state variable x and the time t (see for instance [9, 10, 20, 34]). This kind of delay is widely utilized in nonlinear analysis and has a wide range of applications, including mechanical models [31], population models [7], infectious disease transmission [41], the two-body problem of classical electrodynamics [19], the dynamics of economical systems [8], and it has many applications in hereditary phenomena [35, 45, 46]. One of the first researches in self-reference differential equations was presented by Eder [20], he classifies the solutions to the differential equation $x'(t) = x(x(t)),\ t \in A \subset R$, and demonstrated the existence and uniqueness of the solution with the condition $x(t_0) = x_0$. Fe'ckan [25] introduced a generalization of Eder's results by studying the functional differential equation of the form $x'(t) = f(x(x(t))),\ t \in A \subset R$, where $f \in C^1(R)$. Buicá [10], examined the uniqueness of solution and data dependence of the problem $x'(t) = f(t, x(x(t))),\ t \in [a, b],\ x(t_0) = x_0$, where $t_0, x_0 \in [a, b]$ and $f \in C([a, b], [a, b])$. Lin and Lu [34] studied the qualitative behavior of a state-dependent functional differential equation. Yang et al. [47] examined an ordinary differential equation with a state-dependent delay. EL-Sayed et al. [24] investigated a nonlocal boundary value problem of a state-dependent differential equation. In [22], the authors analyzed the state-dependent Chandrasekhar integral equation

$$x(t) = b(t) + \lambda x(\int_0^t \frac{t}{t+s} a(s)x(s)ds,), \quad t \in [0, 1],$$

they proved the existence of the solution for this equation and analyzed its continuous dependency on function a. In [30], Humphries et al. examined the state-dependent delay system

$$x'(t) = \alpha x(t) + \beta x(t - a - \eta x(t - b)),$$

where the delay $a + \eta x(t - b)$ depends linearly on the state x with strength η. Lauran in [33], analyzed iterative and non iterative first order differential equations of the form

$$\frac{dx(t)}{dt} = f(t, x(t), x(\lambda t))$$

and

$$\frac{dx(t)}{dt} = f(t, x(t), x(x(t))),$$

respectively, with the initial condition $x(t_0) = x_0$. A generalization of the result in [33] was introduced by Hashem et al. [27], they studied the system of the state dependent functional equations

$$x(t) = f_1(t, y(t), x(y(t))), \quad t \in [0, b],$$

$$y(t) = f_2(t, x(t), y(x(t))), t \quad \in [0, b],$$

the authors proved the existence of a unique solution of the system and discussed the continuous dependence of the solution. In [21], the authors examined the existence of a unique solution of a feedback control problem with an implicit state-dependent pantograph equation; in addition, they investigated the Hyres-Ulam stability of the problem and the continuous dependence of the solution.

Inspired by modern literature, we consider the state-dependent implicit pantograph equation of the Chandrasekhar type

$$\frac{dx}{dt} = b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, \frac{dy}{ds})\, ds), \quad x(0) = x_0, \quad a.e.\ t \in (0, 1] \tag{1.1}$$

with the feedback control

$$\frac{dy}{dt} = b_2(t) + \lambda_2 y(\int_0^1 \frac{t}{t+s} g_2(s, \frac{dx}{ds})\, ds), \quad y(0) = y_0, \quad a.e.\ t \in (0, 1]. \tag{1.2}$$

Where $\lambda_i \in (0, 1)$ and $x_0,\ y_0 \in R$ are the initial data.

Our goal in this work is to investigate the existence and the uniqueness of the solution $(x, y) \in X$ of the problems (1.1) and (1.2). We prove the continuous dependence of the solution on the initial data x_0, y_0 and the parameters λ_i. Furthermore, we establish the Hyres-Ulam stability of the problem. Next, as a particular case of our work, we discuss an issue of the same type without feedback control.

We outline the main contributions of this paper as follows:

- We examine the feedback control problems (1.1) and (1.2) of the state-dependent pantograph equation of the Chandrasekhar type; we study the qualitative properties of the solution of (1.1) under the feedback control (1.2).
- We explore the problem (5.1) of the state-dependent pantograph equation of the Chandrasekhar type; we refer to the qualitative properties of the solution of (5.1) in the absence of the feedback control.

This study enhances the qualitative analysis of a state-dependent pantograph differential equation with feedback control. The article is structured as follows: Section 2 presents the appropriate assumptions and proves the existence results for the solution of the implicit state-dependent problem (1.1) with the feedback control (1.2) using Schauder fixed point theorem. The suitable assumptions and proofs for the uniqueness of the solution will be provided in Section 3. Section 4 investigates the stability analysis of the problem due to the Hyers-Ulam stability. Additionally, we test the possibility of the solution resisting disturbances through the study of the continuous dependency on x_0, y_0 and the parameters λ_i. Furthermore, in Section 5, we examine a special case of our problem without the control variable; we present some results supporting the problem's existence and stability. In Section 6, we give some instances that illustrate the findings. Finally, Section 7 provides a conclusion.

Let $C(I)$ be the class of all continuous functions on $I = [0, 1]$ with the standard norm $||u||_C = \sup_{t\in I} |u(t)|$, and $X = C(I) \times C(I)$ be the Banach space with the norm $||(u, v)||_X = \max\{||u||_C, ||v||_C\}$. Let $L^1(I)$ be the space of real functions defined and Lebesgue integrable on the interval I, which is equipped with the standard norm.

2. Existence results

Consider the problems (1.1) and (1.2) under the assumptions:

(1) $g_i : I \times R \to R$ satisfies Carathèodory condition [16], i.e., it is measurable in $t \in I$ $\forall$ $x \in R$ and continuous in $x \in R$, $\forall$ $t \in I$, and there exist functions $a_i : I \to R^+ = [0, \infty)$, $a_i \in L^1(I)$ and $a_i(t) \leq K_i t^n$, $K_i \in (0, 1)$ where $n \in N$, $\forall$ $t \in I$, such that

$$|g_i(t, x(t))| \leq a_i(t)\, |x(t)|,\ i = 1, 2.$$

(2) $b_i : I \to R$ are continuous functions on I.

(3) There exists a real positive root r of the algebraic equation

$$K\lambda r^2 - r + (B + \lambda A) = 0,$$

where $K\lambda r < 1$, $K = \max\{K_i\}$, $\lambda = \max\{\lambda_i\}$, $B = \max\{||b_i||\}$, $A = \max\{x_0, y_0\}$.

2.1. Formulation of problem

Let $\frac{dx}{dt} = u(t)$, $\frac{dy}{dt} = v(t)$, then

$$x(t) = x_0 + \int_0^t u(s)ds, \tag{2.1}$$

$$y(t) = y_0 + \int_0^t v(s)ds. \tag{2.2}$$

Then the problems (1.1) and (1.2) will be given by

$$u(t) = b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds) \tag{2.3}$$

$$v(t) = b_2(t) + \lambda_2 y(\int_0^1 \frac{t}{t+s} g_2(s, u(s))ds). \tag{2.4}$$

Where

$$\begin{aligned} x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds) &= x_0 + \int_0^{\int_0^1 \frac{t}{t+s} g_1(s,v(s))ds} u(s)ds \\ &\leq x_0 + \|u\| \int_0^1 \frac{t}{t+s} g_1(s, v(s))ds \end{aligned}$$

and

$$\begin{aligned} y(\int_0^1 \frac{t}{t+s} g_2(s, u(s))ds) &= y_0 + \int_0^{\int_0^1 \frac{t}{t+s} g_2(s,u(s))ds} v(s)ds \\ &\leq y_0 + \|v\| \int_0^1 \frac{t}{t+s} g_2(s, u(s))ds. \end{aligned}$$

Define the operator F associated with (2.3) and (2.4) by

$$F(u, v) = (F_1 u, F_2 v),$$

where

$$F_1 u(t) = b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds),$$

$$F_2 v(t) = b_2(t) + \lambda_2 y(\int_0^1 \frac{t}{t+s} g_2(s, u(s))ds).$$

Theorem 2.1. *Let the assumptions (1)–(3) be satisfied. Then the problems (2.3) and (2.4) has at least one solution .*

Proof. Define set $Q_r \subset X$ as

$$Q_r = \{(u, v) \in X : \|u\| \leq r, \ \|v\| \leq r\}.$$

Obviously, Q_r is a closed convex bounded set. Now for $(u, v) \in Q_r, \ t \in I$, we have

$$\begin{aligned} |F_1 u(t)| &= |b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds)| \\ &\leq |b_1(t)| + \lambda_1(|x_0| + \|u\| \int_0^1 \frac{t}{t+s} |g_1(s, v(s))|ds) \\ &\leq \|b_1\| + \lambda_1 |x_0| + \lambda_1 \|u\| \int_0^1 \frac{t}{t+s} |g_1(s, v(s))|ds \\ &\leq \|b_1\| + \lambda_1 |x_0| + \lambda_1 \|u\| K_1 \int_0^1 \frac{t}{t+s} s^n |v(s)|ds \end{aligned}$$

$$
\begin{aligned}
&\leq \|b_1\| + \lambda_1|x_0| + \lambda_1\|u\|K_1\int_0^1 |v(s)|ds \\
&\leq \|b_1\| + \lambda_1|x_0| + \lambda_1\|u\|K_1\|v\| \\
&\leq \|b_1\| + \lambda_1|x_0| + \lambda_1 K_1 r^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|F_2u(t)| &\leq |b_2(t)| + \lambda_2(|y_0| + \|v\|\int_0^1 \frac{t}{t+s} g_2(s,u(s))ds) \\
&\leq \|b_2\| + \lambda_2|y_0| + \lambda_2\|v\|\int_0^1 \frac{t}{t+s}|g_2(s,u(s))|ds \\
&\leq \|b_2\| + \lambda_2|y_0| + \lambda_2\|v\|K_2\int_0^1 \frac{t}{t+s}s^n|u(s)|ds \\
&\leq \|b_2\| + \lambda_2|y_0| + \lambda_2\|v\|K_2\int_0^1 |u(s)|ds \\
&\leq \|b_2\| + \lambda_2|y_0| + \lambda_2\|v\|K_2\|u\| \\
&\leq \|b_2\| + \lambda_2|y_0| + \lambda_2 K_2 r^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|F(u,v)\|_X &= \|(F_1u, F_2v)\|_X = max\{\|F_1u\|_C,\ \|F_2v\|_C\} \\
&\leq \max\{\|b_1\| + \lambda_1|x_0| + \lambda_1 K_1 r^2,\ \|b_2\| + \lambda_2|y_0| + \lambda_2 K_2 r^2\} \\
&\leq B + \lambda A + \lambda K r^2 = r.
\end{aligned}
$$

Then the class of functions $\{F(u,v)\}$ is uniformly bounded on I. Let $(u,v) \in Q_r$ and $t_1,\ t_2 \in I$ with $t_1 < t_2$ such that $|t_2 - t_1| < \delta$, then

$$
\begin{aligned}
&|F_1u(t_2) - F_1u(t_1)| \\
=\ & |b_1(t_2) + \lambda_1 x(\int_0^1 \frac{t_2}{t_2+s} g_1(s,v(s))ds) - b_1(t_1) - \lambda_2 x(\int_0^1 \frac{t_1}{t_1+s} g_1(s,v(s))ds)| \\
\leq\ & |b_1(t_2) - b_1(t_1)| + \lambda_1(x_0 + \|u\|\int_0^1 \frac{t_2}{t_2+s} g_1(s,v(s))ds) - \lambda_1(x_0 + \|u\|\int_0^1 \frac{t_1}{t_1+s} g_1(s,v(s))ds) \\
\leq\ & |b_1(t_2) - b_1(t_1)| + \lambda_1\|u\|\int_0^1 \frac{t_2}{t_2+s}|g_1(s,v(s))|ds - \lambda_1\|u\|\int_0^1 \frac{t_1}{t_1+s}|g_1(s,v(s))|ds \\
\leq\ & |b_1(t_2) - b_1(t_1)| + \lambda_1\|u\|\int_0^1 (\frac{t_2}{t_2+s} - \frac{t_1}{t_1+s})|g_1(s,v(s))|ds \\
\leq\ & |b_1(t_2) - b_1(t_1)| + \lambda_1\|u\|\int_0^1 \frac{(t_2-t_1)s}{(t_2+s)(t_1+s)}|g_1(s,v(s))|ds \\
\leq\ & |b_1(t_2) - b_1(t_1)| + \lambda_1\|u\|\ |t_2 - t_1|\int_0^1 \frac{1}{(t_2+s)}|g_1(s,v(s))|ds \\
\leq\ & |b_1(t_2) - b_1(t_1)| + \lambda_1\|u\|\ |t_2 - t_1|\int_0^1 \frac{K_1 s^n}{t_2+s}|v(s)|ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \ |b_1(t_2)-b_1(t_1)| + \lambda_1\|u\|\,|t_2-t_1|\,K_1\,\|v\| \int_0^1 \frac{s}{t_2+s}s^{n-1}ds \\
&\leq \ |b_1(t_2)-b_1(t_1)| + \lambda K r^2 |t_2-t_1|.
\end{aligned}$$

Then the class of functions $\{F_1 u\}$ is equi-continuous.

Similarly,

$$\begin{aligned}
& |F_2 v(t_2) - F_2 v(t_1)| \\
= \ & |b_2(t_2) + \lambda_2 y(\int_0^1 \frac{t_2}{t_2+s} g_2(s,u(s))ds) - b_2(t_1) - \lambda_2 y(\int_0^1 \frac{t_1}{t_1+s} g_2(s,u(s))ds)| \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2(y_0+\|v\| \int_0^1 \frac{t_2}{t_2+s}|g_2(s,u(s))|ds) - \lambda_2(y_0+\|v\| \int_0^1 \frac{t_1}{t_1+s}|g_2(s,u(s))|ds) \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2\|v\| \int_0^1 \frac{t_2}{t_2+s}|g_2(s,u(s))|ds - \lambda_2\|v\| \int_0^1 \frac{t_1}{t_1+s}|g_2(s,u(s))|ds \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2\|v\| \int_0^1 (\frac{t_2}{t_2+s} - \frac{t_1}{t_1+s})|g_2(s,u(s))|ds \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2\|v\| \int_0^1 \frac{(t_2-t_1)s}{(t_2+s)(t_1+s)}|g_2(s,u(s))|ds \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2\|v\|\,|t_2-t_1| \int_0^1 \frac{1}{(t_2+s)}|g_2(s,u(s))|ds \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2\|v\|\,|t_2-t_1| \int_0^1 \frac{K_2 s^n}{t_2+s}|u(s)|ds \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda_2\|v\|\,|t_2-t_1|\,K_2\,\|u\| \int_0^1 \frac{s}{t_2+s}s^{n-1}ds \\
\leq \ & |b_2(t_2)-b_2(t_1)| + \lambda K r^2 |t_2-t_1|.
\end{aligned}$$

Then the class of functions $\{F_2 u\}$ is equi-continuous. We deduce that $F : Q_r \to Q_r$ and the class functions $\{F(u,v)\}$ is equi-continuous. By Arzela-Theorem [32], $\{F(u,v)\}$ is compact, then F is compact.

Now, let $\{u_n\}$, $\{v_n\} \subset Q_r$ such that $u_n(t) \to u(t)$, $v_n(t) \to v(t)$, where $n \to \infty$, then

$$F_1 u_n(t) = b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s,v_n(s))ds),$$

$$F_2 v_n(t) = b_2(t) + \lambda_2 y(\int_0^1 \frac{t}{t+s} g_2(s,u_n(s))ds)$$

and

$$\lim_{n\to\infty} F_1 u_n(t) = b_1(t) + \lambda_1 \lim_{n\to\infty} x(\int_0^1 \frac{t}{t+s} g_1(s,v_n(s))ds),$$

$$\lim_{n\to\infty} F_2 v_n(t) = b_2(t) + \lambda_2 \lim_{n\to\infty} y(\int_0^1 \frac{t}{t+s} g_2(s,u_n(s))ds).$$

Then, from assumption (1), we have

$$g_1(s, v_n(s)) \to g_1(s, v(s)), \quad g_2(s, u_n(s)) \to g_2(s, u(s))$$

and

$$|g_1(s, v(s))| \leq a_1(t)\ |v(t)| \in L_1[0, 1], \quad |g_2(s, u(s))| \leq a_2(t)\ |u(t)| \in L_1[0, 1].$$

Applying Lebesgue dominated convergence theorem [32], we have

$$\begin{aligned}
\lim_{n\to\infty} F_1 u_n(t) &= b_1(t) + \lambda_1 \lim_{n\to\infty} x(\int_0^1 \frac{t}{t+s} g_1(s, v_n(s)))ds \\
&= b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} \lim_{n\to\infty} g_1(s, v_n(s)))ds) \\
&= b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s)))ds) = F_1 u(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{n\to\infty} F_2 v_n(t) &= b_2(t) + \lambda_2 \lim_{n\to\infty} y(\int_0^1 \frac{t}{t+s} g_2(s, u_n(s)))ds \\
&= b_2(t) + \lambda_2 y(\int_0^1 \frac{t}{t+s} \lim_{n\to\infty} g_2(s, u_n(s)))ds) \\
&= b_2(t) + \lambda_2 y(\int_0^1 \frac{t}{t+s} g_2(s, u(s)))ds) = F_2 v(t).
\end{aligned}$$

Now,

$$\lim_{n\to\infty} F(u_n, v_n) = \lim_{n\to\infty} (F_1 u_n, F_2 v_n) = (F_1 u, F_2 u) = F(u, v).$$

Then $F(u, v)$ is continuous. Now all conditions of Schauder's fixed point theorem [32] are satisfied, then the operator F has at least one fixed point $(u, v) \in Q_r$, Consequently, there exist at least one solution of the problems (2.3) and (2.4). □

Corollary 2.1. *Let the assumptions of Theorem 2.1 be satisfied, by using Eqs (2.1) and (2.2), we deduce that the problems (1.1) and (1.2) has at least one solution $x \in C(I)$.*

3. The uniqueness of the solution

Consider the following additional assumptions:

(1^*) $g_i : I \times R \to R$ satisfies the Lipschitz condition with positive Lipschitz constants M_i such that

$$|g_i(t, x) - g_i(t, y)| \leq M_i\ |x - y|, \ \forall\ t \in I,\ x,\ y \in Q_r\ \max\{M_i\} = M,\ i = 1,\ 2.$$

Theorem 3.1. *Let the assumptions (1)–(3) and* (1^*) *be satisfied, If* $0 < \frac{\lambda r M}{(1-\lambda r K)} < 1$, *then the solution of the problems (1.1) and (1.2) is unique.*

Proof. let $(u_1, v_1), (u_2, v_2)$ be two solutions of (2.3) and (2.4), then

$$\begin{aligned}
|u_1(t) - u_2(t)| &= \Big|\lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v_1(s))ds) - \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds)\Big| \\
&\leq \Big|\lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_1(s))ds - \lambda_1 \|u_2\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds\Big| \\
&\leq \Big|\lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_1(s))ds - \lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds \\
&+ \lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds - \lambda_1 \|u_2\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds\Big| \\
&\leq \Big|\lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_1(s))ds - \lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds\Big| \\
&+ \Big|\lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds - \lambda_1 \|u_2\| \int_0^1 \frac{t}{t+s} g_1(s, v_2(s))ds\Big| \\
&\leq \lambda_1 \|u_1\| \int_0^1 \frac{t}{t+s} |g_1(s, v_1(s)) - g_1(s, v_2(s))| ds \\
&+ \lambda_1 \Big| \|u_1\| - \|u_2\| \Big| \int_0^1 \frac{t}{t+s} |g_1(s, v_2(s))| ds \\
&\leq \lambda_1 r M_1 \int_0^1 \frac{t}{t+s} |v_1(s) - v_2(s)| ds + \lambda_1 \|u_1 - u_2\| \int_0^1 \frac{t}{t+s} |g_1(s, v_2(s))| ds \\
&\leq \lambda_1 r M_1 \|v_1 - v_2\| + \lambda_1 \|u_1 - u_2\| \int_0^1 \frac{t}{t+s} a_1(s) |v_2(s)| ds \\
&\leq \lambda_1 r M_1 \|v_1 - v_2\| + \lambda_1 r K_1 \|u_1 - u_2\| \int_0^1 \frac{t}{t+s} s^n ds \\
&\leq \lambda r M \|v_1 - v_2\| + \lambda r K \|u_1 - u_2\|,
\end{aligned}$$

and

$$(1 - \lambda r K)\|u_1 - u_2\| \leq \lambda r M \|v_1 - v_2\|,$$

then

$$\|u_1 - u_2\| \leq \frac{\lambda r M}{1 - \lambda r K} \|v_1 - v_2\|.$$

Similarly,

$$\|v_1 - v_2\| \leq \frac{\lambda r M}{1 - \lambda r K} \|u_1 - u_2\|.$$

Then

$$\begin{aligned}
\|(u_1, v_1) - (u_2, v_2)\| &= \|(u_1 - u_2), (v_1 - v_2)\| \\
&= \max\{\|u_1 - u_2\|, \|v_1 - v_2\|\} \\
&\leq \max\{\frac{\lambda r M}{1 - \lambda r K} \|v_1 - v_2\|, \frac{\lambda r M}{1 - \lambda r K} \|u_1 - u_2\|\} \\
&\leq \frac{\lambda r M}{1 - \lambda r K} \max\{\|v_1 - v_2\|, \|u_1 - u_2\|\}
\end{aligned}$$

$$\leq \frac{\lambda rM}{1-\lambda rK}\|(u_1-u_2),(v_1-v_2)\|$$
$$\leq \frac{\lambda rM}{1-\lambda rK}\|(u_1,v_1)-(u_2,v_2)\|.$$

Then

$$(1-\frac{\lambda rM}{1-\lambda rK})\|(u_1,v_1)-(u_2,v_2)\| \leq 0.$$

Since $0 < \frac{\lambda rM}{1-\lambda rK} < 1$, then the solution of the problems (2.3) and (2.4) is unique. □

Corollary 3.1. *Let the assumptions of Theorem 3.1 be satisfied; according to the Eqs (2.1) and (2.2), the solution $(x, y) \in X$ of (1.1) and (1.2) is unique.*

4. Stability analysis of the problem

4.1. Hyres-Ulam stability

Definition 4.1. *Let the solution $(x, y) \in X$ of (1.1) and (1.2) be exists, then the problems (1.1) and (1.2) is Hyers-Ulam stable if $\forall \epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any δ-approximate solution $(x_s, y_s) \in X$ of (1.1) and (1.2) satisfies*

$$\max\{|\frac{dx_s}{dt}-b_1(t)-\lambda_1 x_s(\int_0^1 \frac{t}{t+s}g_1(s,\frac{dy_s}{dt})ds)|,\ |\frac{dy_s}{dt}-b_2(t)-\lambda_2 y_s(\int_0^1 \frac{t}{t+s}g_2(s,\frac{dx_s}{dt})ds)|\} < \delta,$$

implies

$$\|(x,y)-(x_s,y_s)\|_X \leq \epsilon.$$

Theorem 4.1. *If the assumptions of Theorem 3.1 are met, then the problems (1.1) and (1.2) is Hyers-Ulam stable.*

Proof. Let

$$\max\{|\frac{dx_s}{dt}-b_1(t)-\lambda_1 x_s(\int_0^1 \frac{t}{t+s}g_1(s,\frac{dy_s}{dt})ds)|,\ |\frac{dy_s}{dt}-b_2(t)-\lambda_2 y_s(\int_0^1 \frac{t}{t+s}g_2(s,\frac{dy_s}{dt})ds)|\} < \delta,$$

then

$$|\frac{dx_s}{dt}-b_1(t)-\lambda_1 x_s(\int_0^1 \frac{t}{t+s}g_1(s,\frac{dy_s}{dt})ds)| < \delta$$

$$-\delta < \frac{dx_s}{dt}-b_1(t)-\lambda_1 x_s(\int_0^1 \frac{t}{t+s}g_1(s,\frac{dy_s}{dt})ds) < \delta.$$

Similarly,

$$|\frac{dy_s}{dt}-b_2(t)-\lambda_2 y_s(\int_0^1 \frac{t}{t+s}g_2(s,\frac{dx_s}{dt})ds)| < \delta$$

$$-\delta < \frac{dy_s}{dt}-b_2(t)-\lambda_2 y_s(\int_0^1 \frac{t}{t+s}g_2(s,\frac{dx_s}{dt})ds) < \delta.$$

Let $\frac{dx_s}{dt} = u_s$ and $\frac{dy_s}{dt} = v_s$, then

$$x_s(t) = x_0 + \int_0^t u_s(s)ds \quad \Rightarrow \quad x_s(\int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds) \le x_0 + \|u_s\| \int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds,$$

$$y_s(t) = y_0 + \int_0^t u_s(s)ds \quad \Rightarrow \quad y_s(\int_0^1 \frac{t}{t+s} g_2(s, u_s(s))ds) \le y_0 + \|v_s\| \int_0^1 \frac{t}{t+s} g_1(s, u_s(s))ds.$$

Hence

$$-\delta < u_s(t) - b_1(t) - \lambda_1 x_s(\int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds) < \delta,$$

$$-\delta < v_s(t) - b_2(t) - \lambda_2 y_s(\int_0^1 \frac{t}{t+s} g_2(s, u_s(s))ds) < \delta.$$

Then,

$$\begin{aligned}
|u(t) - u_s(t)| &= |b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds) - u_s(t)| \\
&= |b_1(t) + \lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds) - \lambda_1 x_s(\int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds) \\
&+ \lambda_1 x_s(\int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds) - u_s(t)| \\
&\le |\lambda_1 x(\int_0^1 \frac{t}{t+s} g_1(s, v(s))ds) - \lambda_1 x_s(\int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds)| \\
&+ |b_1(t) + \lambda_1 x_s(\int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds) - u_s(t)| \\
&\le |\lambda_1 (x_0 + \|u\| \int_0^1 \frac{t}{t+s} g_1(s, v(s))ds) - \lambda_1 (x_0 + \|u_s\| \int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds)| + \delta \\
&\le \lambda_1 \Big| \|u\| \int_0^1 \frac{t}{t+s} g_1(s, v(s))ds - \|u_s\| \int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds \Big| + \delta \\
&\le \lambda_1 \Big| \|u\| \int_0^1 \frac{t}{t+s} g_1(s, v(s))ds - \|u\| \int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds \\
&+ \|u\| \int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds - \|u_s\| \int_0^1 \frac{t}{t+s} g_1(s, v_s(s))ds \Big| + \delta \\
&\le \lambda_1 \|u\| \int_0^1 \frac{t}{t+s} |g_1(s, v(s)) - g_1(s, v_s(s))|ds + \lambda_1 \Big| \|u\| - \|u_s\| \Big| \int_0^1 \frac{t}{t+s} |g_1(s, v_s(s))|ds + \delta \\
&\le \lambda_1 \|u\| M_1 \int_0^1 \frac{t}{t+s} |v(s) - v_s(s)|ds + \lambda_1 \|u - u_s\| \int_0^1 \frac{t}{t+s} a_1(s)|v_s(s)|ds + \delta \\
&\le \lambda_1 r M_1 \|v - v_s\| + \lambda_1 r K_1 \|u - u_s\| \int_0^1 \frac{t}{t+s} s^n ds + \delta \\
&\le \lambda_1 r M_1 \|v - v_s\| + \lambda_1 r K_1 \|u - u_s\| + \delta,
\end{aligned}$$

and

$$(1-\lambda_1 rK_1)\|u-u_s\| \leq \delta + \lambda_1 rM_1\|v-v_s\|.$$

Hence

$$\|u-u_s\| \leq \frac{\delta}{(1-\lambda_1 rK_1)} + \frac{\lambda_1 rM_1}{(1-\lambda_1 rK_1)}\|v-v_s\|.$$

Similarly,

$$\|v-v_s\| \leq \frac{\delta}{(1-\lambda_2 rK_2)} + \frac{\lambda_2 rM_2}{(1-\lambda_2 rK_2)}\|u-u_s\|.$$

Then

$$\begin{aligned}
\|(u,v)-(u_s,v_s)\|_X &= \|((u-u_s),(v-v_s))\|_X = \max\{\|(u-u_s)\|_C,\ \|(v-v_s)\|_C\} \\
&\leq \max\{\frac{\delta}{(1-\lambda_1 rK_1)} + \frac{\lambda_1 rM_1}{(1-\lambda_1 rK_1)}\|v-v_s\|, \\
&\quad \frac{\delta}{(1-\lambda_2 rK_2)} + \frac{\lambda_2 rM_2}{(1-\lambda_2 rK_2)}\|u-u_s\|\} \\
&\leq \frac{\delta}{(1-\lambda rk)} + \max\{\frac{\lambda_1 rM_1}{(1-\lambda_1 rK_1)}\|v-v_s\|,\ \frac{\lambda_2 rM_2}{(1-\lambda_2 rK_2)}\|u-u_s\|\} \\
&\leq \frac{\delta}{(1-\lambda rK)} + \frac{\lambda rM}{(1-\lambda rK)}\max\{\|v-v_s\|,\ \|u-u_s\|\} \\
&\leq \frac{\delta}{(1-\lambda rK)} + \frac{\lambda rM}{(1-\lambda rK)}\|((u-u_s),(v-v_s))\|_X \\
&\leq \frac{\delta}{(1-\lambda rK)} + \frac{\lambda rM}{(1-\lambda rK)}\|(u,v)-(u_s,v_s)\|_X,
\end{aligned}$$

and

$$(1-\frac{\lambda rM}{(1-\lambda rK)})\|(u,v)-(u_s,v_s)\| \leq \frac{\delta}{(1-\lambda rK)},$$

then

$$\|(u,v)-(u_s,v_s)\| \leq \frac{\delta}{1-(\lambda rM+\lambda rK)} = \epsilon.$$

Now,

$$\begin{aligned}
\|(x,y)-(x_s,y_s)\|_X &= \|((x-x_s),(y-y_s))\|_X = \max\{\|(x-x_s)\|_C,\ \|(y-y_s)\|_C \\
&\leq \max\{\|(u-u_s)\|_C, \|(v-v_s)\|_C\} \leq \|((u-u_s),(v-v_s))\| \\
&\leq \|(u,v)-(u_s,v_s)\| \leq \epsilon.
\end{aligned}$$

Then

$$\|(x,y)-(x_s,y_s)\|_X \leq \epsilon.$$

Then the problems (1.1) and (1.2) is Hyers-Ulam stable. □

4.2. Continuous dependence

Definition 4.2. *The solution* $(u,v)\in Q_r$ *of (2.3) and (2.4) depends continuously on* $x_0,\ y_0,\ \lambda$ *if* $\forall\ \epsilon>0$ *there exists* $\delta(\epsilon)>0$ *such that*

$$\max\{|x_0-x_0^*|,\ |y_0-y_0^*|,\ |\lambda_i-\lambda_i^*|\}<\delta\ \Rightarrow\ \|(u,v)-(u^*,v^*)\|_X<\epsilon,\ i=1,2.$$

where

$$u^*(t)=b_1(t)+\lambda_1^*x^*(\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds), \tag{4.1}$$

$$v^*(t)=b_2(t)+\lambda_2^*y^*(\int_0^1\frac{t}{t+s}g_2(s,u^*(s))ds). \tag{4.2}$$

Theorem 4.2. *Let the assumptions of Theorem 3.1 be satisfied, then* (u,v) *depends continuously on the initial data* $x_0,\ y_0$ *and the parameters* $\lambda_i,\ i=1,2$.

Proof. Let $\delta(\epsilon)>0$ be given such that

$$\max\{|x_0-x_0^*|,\ |y_0-y_0^*|,\ |\lambda_i-\lambda_i^*|\}<\delta,\ i=1,2.$$

and let (u^*,v^*) be the solution of (4.1) and (4.2), then

$$\begin{aligned}
&|u(t)-u^*(t)|\\
=\ &|\lambda_1x(\int_0^1\frac{t}{t+s}g_1(s,v(s))ds)-\lambda_1^*x^*(\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds)|\\
\leq\ &|\lambda_1(x_0+\|u\|\int_0^1\frac{t}{t+s}g_1(s,v(s))ds)-\lambda_1^*(x_0^*+\|u^*\|\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds)|\\
\leq\ &|\lambda_1x_0-\lambda_1^*x_0^*|+\left|\lambda_1\|u\|\int_0^1\frac{t}{t+s}g_1(s,v(s))ds-\lambda_1^*\|u^*\|\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds\right|\\
\leq\ &|\lambda_1x_0-\lambda_1x_0^*|+|\lambda_1x_0^*-\lambda_1^*x_0^*|+\left|\lambda_1\|u\|\int_0^1\frac{t}{t+s}g_1(s,v(s))ds-\lambda_1\|u\|\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds\right.\\
+\ &\left.\lambda_1\|u\|\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds-\lambda_1^*\|u^*\|\int_0^1\frac{t}{t+s}g_1(s,v^*(s))ds\right|\\
\leq\ &\lambda_1|x_0-x_0^*|+|x_0^*|\,|\lambda_1-\lambda_1^*|+\lambda_1\|u\|\int_0^1\frac{t}{t+s}|g_1(s,v(s))-g_1(s,v^*(s))|ds\\
+\ &\Big|\lambda_1\|u\|-\lambda_1^*\|u^*\|\Big|\int_0^1\frac{t}{t+s}|g_1(s,v^*(s))|ds\\
\leq\ &\lambda_1\delta+|x_0^*|\delta+\lambda_1\|u\|M_1\int_0^1\frac{t}{t+s}|v(s)-v^*(s)|ds\\
+\ &\Big|\lambda_1\|u\|-\lambda_1\|u^*\|+\lambda_1\|u^*\|-\lambda_1^*\|u^*\|\Big|\int_0^1\frac{t}{t+s}|g_1(s,v^*(s))|ds\\
\leq\ &\lambda_1\delta+|x_0^*|\ \delta+\lambda_1\|u\|M_1\|v-v^*\|+\Big(\lambda_1\Big|\|u\|-\|u^*\|\Big|+\|u^*\|\Big|\lambda_1-\lambda_1^*\Big|\Big)\int_0^1\frac{t}{t+s}a_1(s)\ |v^*(s)|ds\\
\leq\ &\lambda_1\delta+|x_0^*|\ \delta+\lambda_1r\ M_1\|v-v^*\|+(\lambda_1\|u-u^*\|+\|u^*\|\ \delta)\int_0^1\frac{t}{t+s}a_1(s)\ |v^*(s)|ds
\end{aligned}$$

$$\leq \lambda_1\delta + |x_0^*|\,\delta + \lambda_1 r\,M_1\|v-v^*\| + (\lambda_1\|u-u^*\| + \|u^*\|\,\delta) r\,K_1 \int_0^1 \frac{t}{t+s} s^n ds$$
$$\leq (\lambda_1 + |x_0^*| + r^2 K)\,\delta + \lambda_1 r\,M_1\|v-v^*\| + \lambda_1 r\,K_1\|u-u^*\|,$$

and

$$(1-\lambda_1 r K_1)\|u-u^*\| \leq (\lambda_1 + |x_0^*| + r^2 K_1)\,\delta + \lambda_1 r\,M_1\|v-v^*\|.$$

Then

$$\|u-u^*\| \leq \frac{\lambda_1 + |x_0^*| + r^2 K_1}{1-\lambda_1 r K_1}\delta + \frac{\lambda_1 r\,M_1}{1-\lambda_1 r K_1}\|v-v^*\|.$$

Similarly,

$$\|v-v^*\| \leq \frac{\lambda_2 + |y_0^*| + r^2 K_2}{1-\lambda_2 r K_2}\delta + \frac{\lambda_2 r\,M_2}{1-\lambda_2 r K_2}\|u-u^*\|.$$

Then

$$\begin{aligned}
\|(u,v)-(u^*,v^*)\|_X &= \|((u-u^*),(v-v^*))\|_X = \max\{\|(u-u^*)\|_C,\ \|(v-v^*)\|_C\} \\
&\leq \max\{\frac{\lambda_1 + |x_0^*| + r^2 K_1}{1-\lambda_1 r\,K_1}\delta + \frac{\lambda_1 r\,M_1}{1-\lambda_1 r\,K_1}\|v-v^*\|, \\
&\quad \frac{\lambda_2 + |y_0^*| + r^2 K_2}{1-\lambda_2 r\,K_2}\delta + \frac{\lambda_2 r\,M_2}{1-\lambda_2 r\,K_2}\|u-u^*\| \\
&\leq \frac{\lambda + A + r^2 K}{1-\lambda r K}\delta + \max\{\frac{\lambda_1 r\,M_1}{1-\lambda_1 r\,K_1}\|v-v^*\|, \frac{\lambda_2 r\,M_2}{1-\lambda_2 r\,K_2}\|u-u^*\|\} \\
&\leq \frac{\lambda + A + r^2 K}{1-\lambda r K}\delta + \frac{\lambda r\,M}{1-\lambda r\,K}\max\{\|v-v^*\|, \|u-u^*\|\} \\
&= \frac{\lambda + A + r^2 K}{1-\lambda r K}\delta + \frac{\lambda r\,M}{1-\lambda r\,K}\|((u-u^*),(v-v^*))\|_X \\
&= \frac{\lambda + A + r^2 K}{1-\lambda r K}\delta + \frac{\lambda r\,M}{1-\lambda r\,K}\|(u,v)-(u^*,v^*)\|_X,
\end{aligned}$$

and

$$(1-\frac{\lambda r M}{1-\lambda r K})\ \|(u,v)-(u^*,v^*)\|_X \leq \frac{\lambda + A + r^2 K}{1-\lambda r\,K}\delta.$$

Then

$$\|(u,v)-(u^*,v^*)\|_X \leq \frac{\lambda + A + r^2 K}{1-(\lambda r\,M + \lambda r\,K)}\delta = \epsilon.$$

Thus

$$\delta = \frac{1-(\lambda r\,M + \lambda r\,K)}{\lambda + A + r^2 K}\epsilon.$$

Which meas that $\delta = \delta(\epsilon)$. Moreover, we have $\lambda r\,M + \lambda r\,K < 1$ and $A, r, k, \lambda > 0$, then δ is positive. This means that the solution of (2.3) and (2.4) depends continuously on the initial data $x_0,\ y_0$ and the parameters $\lambda_i,\ i = 1, 2$. □

Definition 4.3. *The solution* $(x, y) \in X$ *of (1.1) and (1.2) depends continuously on* $u,\ v$ *if* $\forall\ \epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that*

$$\max\{|u-u^*|,\ |v-v^*|\} < \delta(\epsilon)\ \Rightarrow\ \|(x,y)-(x^*,y^*)\| < \epsilon,$$

where

$$x^*(t) = x_0 + \int_0^t u^*(s)ds, \tag{4.3}$$

$$y^*(t) = y_0 + \int_0^t v^*(s)ds. \tag{4.4}$$

Theorem 4.3. *Let the assumptions of Theorem 4.2 be satisfied, then the solution $(x, y) \in X$ depends continuously on $u,\ v$.*

Proof. Let (x^*, y^*) be the solution of (4.3) and (4.4), then

$$\begin{aligned} |x(t) - x^*(t)| &\leq |\int_0^t u(s)ds - \int_0^t u^*(s)ds| \\ &\leq \int_0^t |u(s) - u^*(s)|ds \leq \|u - u^*\| \leq \delta = \epsilon, \end{aligned}$$

then

$$\|x - x^*\| \leq \epsilon.$$

Similarly,

$$\begin{aligned} |y(t) - y^*(t)| &\leq |\int_0^t v(s)ds - \int_0^t v^*(s)ds| \\ &\leq \int_0^t |v(s) - v^*(s)|ds \leq \|v - v^*\| \leq \delta = \epsilon, \end{aligned}$$

then

$$\|y - y^*\| \leq \epsilon.$$

Now,

$$\|(x, y) - (x^*, y^*)\|_X = \|((x - x^*), (y - y^*))\|_X = \max\{\|x - x^*\|_C,\ \|y - y^*\|_C\} \leq \epsilon.$$

Then

$$\|(x, y) - (x^*, y^*)\|_X \leq \epsilon.$$

□

This means that the solution $(x, y) \in X$ of (1.1) and (1.2) depends continuously on $u,\ v$.

Corollary 4.1. *Let the assumptions of Theorem 4.3 be satisfied, then the solution $(x, y) \in X$ of (1.1) and (1.2) depends continuously on x_0, y_0 and the parameter λ_i.*

5. Special case

In the lack of feedback control; as a special case of our work, we can investigate the following problem of the state-dependent implicit pantograph differential equation of the chandrasekhar type.

$$\frac{dx}{dt} = b(t) + \lambda x(\int_0^1 \frac{t}{t+s}\, g(s, \frac{dx}{dt})\, ds), \quad x(0) = x_0, \quad a.e.\ t \in (0, 1], \tag{5.1}$$

where $\lambda \in (0,1)$, $b : I \to R$ are continuous function and the function $g : I \times R \to R$ satisfies Caratheodory condition.

The existence of the unique solution of (5.1) will be studied. We will also prove the continuous dependence of the solution on the initial data x_0 and the parameter λ. Furthermore, we will establish the Hyres-Ulam stability. This problem can be addressed under the following assumptions:

(i) $g : I \times R \to R$ satisfies Carathèodory condition [16]. (i.e., it is measurable in $t \in I\ \forall x \in R$ and continuous in $x \in R\ \forall t \in I$) and there exists a function $a : I \to R^+ = [0,\infty)$, $a \in L^1(I)$ and $a(t) \le Kt^n$, $K \in (0,1)$, $n \in N\ \forall\ t \in I$, such that

$$|g(t,x(t))| \le a(t)\ |x(t)|.$$

(ii) There exists a real positive root r of the algebraic equation

$$K\lambda r^2 - r + (\|b\| + \lambda|x_0|) = 0.$$

such that $K\lambda r < 1$.

We can formulate the problem as follows: Put $\frac{dx}{dt} = y(t)$, we get

$$y(t) = b(t) + \lambda x(\int_0^1 \frac{t}{t+s} g(s,y(s))ds). \tag{5.2}$$

Using the same techniques, we can derive the following theorems:

Theorem 5.1. *Let the assumptions* (i)–(ii) *be satisfied, then integral equation (5.2) has at least one solution* $y \in C(I)$*. Consequently, the problem (5.1) has at least one solution* $x \in C(I)$.

Theorem 5.2. *Let the assumptions* (i)–(ii) *of Theorem 5.1 be satisfied, if g satisfies lipschetz condition such that* $|g(t,x) - g(t,y)| \le c|x-y|\ \forall t \in I$ *and* $x,\ y \in Q_r$ *where c is a positive constant. If* $\lambda r(c+K) < 1$, *then the solution of the problem (5.1) is unique.*

Theorem 5.3. *Let the assumptions of Theorem (5.2) be satisfied, then the problem (5.1) is Hyers-Ulam stable.*

Theorem 5.4. *Let the assumptions of Theorem 5.2 be satisfied, then* $y \in C(I)$ *depends continuously on the initial data* x_0 *and the parameter* λ.

Theorem 5.5. *Let the assumptions of Theorem 5.2 be satisfied, then the solution* $x \in C(I)$ *depends continuously on y.*

Corollary 5.1. *Let the assumptions of Theorems 5.4 and 5.5 be satisfied, then the solution* $x \in C(I)$ *depends continuously on the initial data* x_0 *and the parameter* λ.

6. Examples

Example 1. Consider the problem

$$\frac{dx}{dt} = \frac{1}{3}(t^2+1) + \frac{1}{2}x(\int_0^1 \frac{t}{t+s}\,\frac{\sin s^3}{5}\,\frac{dy}{dt}ds),\quad x(0) = \frac{1}{7},\quad t \in (0,1]. \tag{6.1}$$

$$\frac{dy}{dt}=\frac{5}{2}\sin t+\frac{1}{3}x(\int_0^1\frac{t}{t+s}\frac{e^{-s}s^2}{7}\frac{dx}{dt}ds),\ \ y(0)=\frac{1}{10},\ \ t\in(0,1]. \tag{6.2}$$

Let $\frac{dx}{dt}=u,\ \ \frac{dy}{dt}=v$ then

$$u(t)=\frac{1}{3}(t^2+1)+\frac{1}{2}x(\int_0^1\frac{t}{t+s}\frac{\sin s^3}{5}v(s)ds),$$

$$v(t)=\frac{5}{2}\sin t+\frac{1}{3}x(\int_0^1\frac{t}{t+s}\frac{e^{-s}s^2}{7}u(s)ds).$$

Set

$$b_1(t)=\frac{1}{3}(t^2+1),\ \ b_2(t)=\frac{5}{2}\sin t$$

and

$$g_1(t,v(t))=\frac{\sin t^3}{5}v(t),\ \ g_2(t,u(t))=\frac{e^{-t}t^2}{7}u(t),$$

then

$$|g_1(t,v(t))|\le\frac{1}{5}t^3\,v(t),\ \ \ |g_2(t,u(t))|\le\frac{1}{7}t^2\,u(t)$$

and

$$a_1(t)\le\frac{1}{5}t^3,\qquad a_2(t)\le\frac{1}{7}t^2.$$

Where $K=\max\{\frac{1}{5},\frac{1}{7}\}=\frac{1}{5},\ \lambda=\max\{\frac{1}{2},\ \frac{1}{3}\}=\frac{1}{2},\ A=\max\{\frac{1}{7},\ \frac{1}{10}\}=\frac{1}{7},\ B=\max\{0.6667,0.043\}=0.6667,\ M=\max\{\frac{1}{5},\frac{1}{7}\}=\frac{1}{5}$. Then we get $r_1=0.8027,\ r_2=9.1973$, such that $\lambda r_1K=0.0803<1$ and $\lambda r_2K=0.9197<1$.

It is clear that all assumptions of Corollary 2.1 are satisfied. Hence there exist at least one solution $(x,y)\in X$ of (6.1) and (6.2). Moreover,we have only r_1 satisfies $\lambda r_1(M+K)=0.1605<1$. Thus all assumptions of Corollary 3.1 are satisfied, then the solution of problems (6.1) and (6.2) is unique.

Example 2. Consider the problem

$$\frac{dx}{dt}=\frac{t^2}{2(4-t)}+\frac{1}{3}x(\int_0^1\frac{t}{t+s}(\frac{t^2}{4}\sin t\,\frac{dx}{dt})ds),\ \ x(0)=\frac{1}{2},\ \ t\in(0,1]. \tag{6.3}$$

Let $\frac{dx}{dt}=y$ then

$$y(t)=\frac{t^2}{2(4-t)}+\frac{1}{3}x(\int_0^1\frac{t}{t+s}(\frac{t^2}{4}\sin t\,y(s))ds),\ \ \ x(0)=\frac{1}{2},\ \ t\in(0,1].$$

Set

$$b(t)=\frac{t^2}{2(4-t)}$$

and

$$g(t,y(t))=\frac{t^2}{4}\sin t\,y(t),$$

thus

$$|g(t, y(t))| \leq \frac{t^2}{4}\, y(t)$$

and

$$a(t) \leq \frac{1}{4}\, t^2.$$

Where $K = \frac{1}{4},\ \lambda = \frac{1}{3},\ x_0 = \frac{1}{2},\ \|b\| = \frac{1}{6},\ c = \frac{1}{4}$. Then we get $r_1 = 0.343,\ r_2 = 11.657,$ such that $\lambda r_1 K = 0.0286 < 1$ and $\lambda r_2 K = 0.9714 < 1$.

It is clear that all assumptions of Theorem 5.1 are satisfied. Hence there exist at least one solution $x \in C[0, 1]$ of (6.3). Moreover, we have only r_1 satisfies $\lambda r_1(c + K) = 0.0572 < 1$.

Thus all assumptions of Theorem 5.2 are satisfied, then the solution of Problem 6.3 is unique.

7. Conclusions

Differential equations with control variables are frequently encountered within several domains such as control theory, optimization, dynamic systems, etc...The existence and uniqueness of solutions of this type of equations is crucial. Sometimes the equations may not have a unique solution, or solutions may need particular conditions to exist. Determining whether a solution is stable or not is often challenging. Stability analysis is critical for control systems to guarantee that small perturbations do not cause significant variations in system behavior. Research on feedback control problems with state-dependent delays has numerous applications in biology, ecology, physics, engineering, and other fields. This type of delay imparts memory effects into the system, which implies intricate and rich dynamics. In this paper, we investigate the existence of at least one continuous solution to a feedback control problem including implicit pantograph equations of the Chandrasekhar type. In addition, we analyzed the uniqueness of the solution in light of appropriate assumptions. Also, we investigated the problem's Hyers-Ulam stability and the continuous dependence of the solution on the original data and parameter. Furthermore, we introduce a brief study for the problem in the absence of feedback control. Finally, we presented few examples in both cases when the problem contains a control variable and in the absence of a control variable.

Author contributions

The authors contributed equally to this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. P. Atkinson, *Feedback control theory for engineers*, Springer, 1968. https://doi.org/10.1007/978-1-4684-7453-4
2. M. R. Abdollahpour, M. T. Rassias, Hyers-Ulam stability of hypergeometric differential equations, *Aequat. Math.*, **93** (2019), 691–698. https://doi.org/10.1007/s00010-018-0602-3
3. M. R. Abdollahpour, R. Aghayari, M. T. Rassias, Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions, *J. Math. Anal. Appl.*, **437** (2016), 605–612. https://doi.org/10.1016/j.jmaa.2016.01.024
4. C. Alsina, R. Ger, On some inequalities and stability results related to the exponential function, *J. Inequal. Appl.*, **2** (1998), 373–380. https://doi.org/10.1155/S102558349800023X
5. I. K. Argyros, Ouadratic equation and applications to Chandrasekhars and related equations, *B. Aust. Math. Sot.*, **32** (1985), 275–292. https://doi.org/10.1017/S0004972700009953
6. R. Bellman, *Stability theory of differential equations*, Dover Publications, 2013.
7. J. Belair, *Population models with state-dependent delays*, CRC Press, 1991. https://doi.org/10.1201/9781003072706-13
8. J. Belair, C. Mackey, Consumer memory and price fluctuations on commodity markets: An integro-differential model, *J. Dyn. Diff. Equat.*, **1** (1989), 299–325. https://doi.org/10.1007/BF01053930
9. V. Berinde, Existence and approximation of solutions of some first order iterative differential equations, *Miskolc Math. Notes*, **11** (2010), 13–26. https://doi.org/10.18514/MMN.2010.256
10. A. Buicá, Existence and continuous dependence of solutions of some functional differential equations, *Semin. Fixed Point Theor.*, **3** (1995), 1–14.
11. J. Caballero, A. B. Mingarelli , K. Sadarangani, Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer, *Electron. J. Differ. Eq.*, **2006** (2006), 1–11.
12. B. Cahlon, M. Eskin, Existence theorems for an integral equation of the Chandrasekhar H-equation with perturbation, *J. Math. Anal. Appl.*, **83** (1981), 159-171. https://doi.org/10.1016/0022-247X(81)90254-7
13. T. Cardinali, S. Matucci, P. Rubbioni, Controllability of nonlinear integral equations of Chandrasekhar type, *J. Fixed Point Theory Appl.*, **24** (2022), 58. https://doi.org/10.1007/s11784-022-00974-5
14. S. Chandrasekhar, *Radiative transfer*, Oxford University Press, 1950. http://doi.org/10.1002/qj.49707633016
15. S. Ya Feng, D. Chen Chang, Exact bounds and approximating solutions to the Fredholm integral equations of Chandrasekhar type, *Taiwanese J. Math.*, **23** (2019), 409–425. https://doi.org/10.11650/tjm/181108
16. R. F. Curtain, A. J. Pritchard, *Functional analysis in modern applied mathematics*, Academic Press, 1977.

17. M. De la Sen, S. Alonso-Quesada, Control issues for the Beverton-Holt equation in ecology by locally monitoring the environment carrying capacity, Non-adaptive and adaptive cases, *Appl. Math. Comput.*, **215** (2009), 2616–2633. https://doi.org/10.1016/j.amc.2009.09.003
18. P. De Leenheer, H. Smith, Feedback control for chemostat models, *J. Math. Biol.*, **46** (2003), 48–70. https://doi.org/10.1007/s00285-002-0170-x
19. R. Driver, A two-body problem of classical electrodynamics: the one-dimensional case, *Ann. Phys.*, **21** (1963), 122–142. https://doi.org/10.1016/0003-4916(63)90227-6
20. E. Eder, The functional differential equation $x'(t) = x(x(t))$, *J. Differ. Equ.*, **54** (1984), 390–400. https://doi.org/10.1016/0022-0396(84)90150-5
21. A. M. A. El-Sayed, E. M. Al-Barg, H. R. Ebead, A Comprehensive View of the Solvability and Stability of a Feedback Control Problem with a State-Dependent Delay Implicit Pantograph Equation, *Int. J. Anal. Appl.*, **22** (2024), 184. https://doi.org/10.28924/2291-8639-22-2024-184
22. A. M. A. El-Sayed, H. H. Hashem, A state-dependent Chandrasekhar integral equation, *Int. J. Nonlinear Anal. Appl.*, **13** (2022), 3049-3056. https://doi.org/10.22075/ijnaa.2022.23027.2461
23. A. M. A. El-Sayed, H. H. Hashem, S. M. A. Al-Issa, comprehensive view of the solvability of non-local fractional orders pantograph equation with a fractal-fractional feedback control, *AIMS Math.*, **9** (2024), 19276–19298. https://doi.org/10.3934/math.2024939
24. A. M. A. El-Sayed, E. M. Hamdallah, H. R. Ebead, On a nonlocal boundary value problem of a state-dependent differential equation, *Mathematics*, **9** (2021), 2800. https://doi.org/10.3390/math9212800
25. M. Féckan, On a certain type of functional differential equations, *Math. Slovaca*, **43** (1993), 39–43.
26. H. H. Hashem, A. M. A. El-Sayed, Stabilization of coupled systems of quadratic integral equations of Chandrasekhar type, *Math. Nachr.*, **290** (2017), 341–348. https://doi.org/10.1002/mana.201400348
27. H. H. G. Hashem, Continuous dependence of solutions of coupled systems of state dependent functional equations, *Adv. Differ. Equ. Contr.*, **22** (2020), 121–135. https://doi.org/10.17654/DE022020121
28. H. H. Hashem, H. O. Alrashidi, Characteristics of solutions of nonlinear neutral integrodifferential equation via Chandrasekhar integral, *J. Math. Comput. Sci.*, **24** (2021), 173–185. https://doi.org/10.22436/jmcs.024.02.08
29. M. A. Hernandez-Veron, E. Martinez, S. Singh, On the Chandrasekhar integral equation, *Comput. Math. Methods*, **3** (2021), e1150. https://doi.org/10.1002/cmm4.1150
30. A. R. Humphries, B. Krauskopf, S. Ruschel, J. Sieber, Nonlinear effects of instantaneous and delayed state dependence in a delayed feedback loop, *Discrete Cont. Dyn. B*, **27** (2022), 7245–7273. https://doi.org/10.3934/dcdsb.2022042
31. R. Johnson, Functional equations, approximations, and dynamic response of systems with variable time-delay, *IEEE T. Automat. Contr.*, **17** (1972), 398–401. https://doi.org/10.1109/TAC.1972.1099999
32. L. V. Kantorovich, G. P. Akilov, *Functional analysis*, translated by howard L. silcock, 1982.
33. M. Lauran, Existence results for some differential equations with deviating argument, *Filomat*, **25** (2011), 21–31. https://doi.org/10.2298/FIL1102021L

34. Y. W. Lin, T. T. Lu, Qualitative behavior of a state-dependent functional differential equation, *J. Appl. Comput. Math.*, **2** (2013), 2–5. https://doi.org/10.4172/2168-9679.1000144
35. M. Miranda, E. Pascali, On a type of evolution of self-referred and hereditary phenomena, *Aequ. Math.*, **71** (2006), 253–268. https://doi.org/10.1007/s00010-005-2821-7
36. P. Nasertayoob, S. M. Vaezpour, Positive periodic solution for a nonlinear neutral delay population equation with feedback control, *J. Nonlinear Sci. Appl.*, **6** (2013), 227–233. https://doi.org/10.22436/JNSA.006.03.08
37. P. Nasertayoob, Solvability and asymptotic stability of a class of nonlinear functional-integral equation with feedback control, *Commun. Nonlinear Anal.*, **5** (2018), 19–27.
38. J. Patade, S. Bhalekar, Analytical Solution of Pantograph Equation with Incommensurate Delay, *Phys. Sci. Rev.*, **3** (2018), 9. https://doi.org/10.1515/psr-2016-9103
39. S. Rezapour, S. Etemad, R. P. Agarwal, K. Nonlaopon, On a Lyapunov-type inequality for control of a y-model thermostat and the existence of its solutions, *Mathematics*, **10** (2022), 4023. https://doi.org/10.3390/math10214023
40. S. L. Ross, *Differential equations*, 1984.
41. P. Waltman, *Deterministic threshold models in the theory of epidemics*, In: Lecture Notes in Biomathematics, Berlin: Springer, 1974. https://doi.org/10.1007/978-3-642-80820-3
42. C. Tunç, E. Biçer, Hyers-Ulam-Rassias stability for a first order functional differential equation, *J. Math. Fund. Sci.*, **47** (2015), 143–153. https://doi.org/10.5614/j.math.fund.sci.2015.47.2.3
43. O. Tunç, C. Tunç, Ulam stabilities of nonlinear iterative integro-differential equations, *RACSAM Rev. R. Acad. A*, **117** (2023), 118. https://doi.org/10.1007/s13398-023-01450-6
44. O. Tunç, C. Tunç, G. Petruçel, J. C. Yao, On the Ulam stabilities of nonlinear integral equations and integro-differential equations, *Math. Method. Appl. Sci.*, **47** (2024), 4014–4028. https://doi.org/10.1002/mma.9800
45. N. M. Tuan, L. T. Nguyen, On solutions of a system of hereditary and self-referred partial-differential equations, *Numer. Algor.*, **55** (2010), 101–113. https://doi.org/10.1007/s11075-009-9360-6
46. U. Van Le, L. T. Nguyen, Existence of solutions for systems of self-referred and hereditary differential equations, *Electron. J. Differ. Eq.*, **2008** (2008), 51.
47. J. Yang, J. Gimeno, R. D. la Llave, Parameterization method for state-dependent delay perturbation of an ordinary differential equation, *SIAM J. Math. Anal.*, **53** (2021), 4. https://doi.org/10.1137/20M1311430
48. M. Yusuf Waziri, W. J.Leong, M. Abu Hassan, M. Monsi, A low memory solver for integral equations of Chandrasekhar type in the radiative transfer problems, *Math. Probl. Eng.*, **2011** (2011), 467017. https://doi.org/10.1155/2011/467017
49. K. Zhao, Stability of a nonlinear Langevin system of ML-Type fractional derivative affected by time-varying delays and differential feedback control, *Fractal Fract.*, **6** (2022), 725. https://doi.org/10.3390/fractalfract6120725