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Research article

Results on some robust multi-cost variational models

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Abstract: In this study, a new family of robust multiobjective fractional variational control problems was introduced and investigated. To this end, a dual model was associated with the class of problems. Further, by considering variants of convexity for the involved functionals (determined by curvilinear integrals that did not depend on the path), we provided some characterization and equivalence results for the considered models. Moreover, an illustrative numerical example was formulated.

Keywords: control; uncertain data; dual model; curvilinear integral functional; variational models
Mathematics Subject Classification: 90C32, 26B25, 90C17, 49J20, 90C46

1. Introduction

In the past decades, scalar and multiobjective variational or programming problems with mixed constraints have been of real importance. We list here the following researchers and their research: Mond and Hanson [16], Hanson [8], Craven and Glover [4], Mond and Smart [18], Mukherjee and Rao [19], and Aggarwal et al. [1]. These researchers studied optimality criteria (necessary and sufficient), the construction of dual variants, and areas of applicability, such as variational control models. Here, we mention the following researchers: Zalmai [30], Zhian and Qingkai [13], Mititelu [15], Chen [3], Hachimi and Aghezzaf [7], Nahak and Nanda [20], Kim and Kim [12], Gulati et al. [6], Jayswal et al. [9], Arana-Jiménez et al. [2], Zhang et al. [33], and Khazafi et al. [11]. Das et al. [5] studied some fractional minimax models with set values and established second-order sufficient optimality conditions. Some duals for the considered problem, under generalized second-order cone convexity and contingent epi-derivative hypotheses, have been formulated. Khan and Al-Solamy [10] formulated duality and sufficiency findings in (H_p, r)-invex non-smooth minimax fractional problems. Sharma [25] presented a duality for higher-order controlled variational problems. Later, Oliveira and Silva [21] investigated sufficient criteria of optimality for multiple-objective control problems. Liu et al. [14] analyzed a special type of variational inequalities with non-local boundary conditions. One year later, Wu et al. [29] investigated the stability of discrete fractional systems with delays. Treanţă and Tareq [26] presented some duality results for a robust optimization problem. Ritu et al. [22], by considering uncertain data in each objective and constraint functional, investigated some multi-dimensional vector variational problems, and stated robust necessary and sufficient efficiency conditions for the problems under consideration. Saeed [23], considering the parametric approach, studied a category of fractional variational models governed by uncertainty in the cost functional. Also, robust Karush-Kuhn-Tucker (KKT)-type necessary and sufficient criteria of optimality have been provided by employing the convexity/concavity hypotheses of the considered functionals. For other excellent contributions to control theory and variational analysis, we mention the papers [31, 32].

In this paper, seen as a natural continuation of the investigations in the referred to articles, we formulate a new class of robust multiobjective fractional variational control problems. Then, a dual model is associated with the above-mentioned class of problems. Further, by considering variants of convexity for the involved functionals (determined by curvilinear integrals that do not depend on the path), we provide characterization and equivalence results on solution sets for the considered models. In addition, a numerical example is formulated. Saeed and Treanţă [24], taking into account convexity assumptions, studied the sufficient conditions of optimality for a family of fractional control problems. Treanţă and Saeed [27] analyzed robust weak, strong, and strict converse dual theorems for multiple objective minimization models driven by multiple integral functionals. The limitations associated with the papers and the principal novelties of this study are: (i) Appearance of mixed-type constraints; (ii) appearance of uncertainty both in the objective- and constraint-type functionals; and (iii) the employment of parametric robust techniques to investigate the considered variational control models. The methodology used in this paper is a combination of techniques from the Lagrange-Hamilton theory, calculus of variations, and control theory. Based on our knowledge, robust dual outcomes associated with such types of variational models are new in the field. Related to some future research directions of the works, let us consider the situations where the partial derivatives of second-order are included, and the functionals are not under (strictly) convexity assumptions (here, we use the ideas formulated in Treanţă [28], by considering concepts of monotonicity, pseudomonotonicity, and hemicontinuity for curvilinear integral-type functionals).

2. Preliminaries

In this paper, we consider $\mathbb{R}^a, \mathbb{R}^s, \mathbb{R}^p$, and $\mathbb{R}^n$ as standard Euclidean spaces. Consider the compact set C in $\mathbb{R}^a$ and a piecewise smooth curve $U \subset C$ that links two different pairs, $t_0 = (t_0^\zeta)$ and $t_1 = (t_1^\zeta)$, in C. Also, let A be the family of state functions of the C^1 class, denoted by $F = (F^i) : C \to \mathbb{R}^s$, and let B be the family of control functions of the C^0 class, denoted by $K = (K^j) : C \to \mathbb{R}^p$. Also, consider the notations: $\Omega := (t, F(t), K(t))$, $F_\zeta(t) := \dfrac{\partial F}{\partial t^\zeta}(t)$, and the relations used for any two pairs x, y $\in \mathbb{R}^n$:

(i) $\mathrm{x} < \mathrm{y} \Leftrightarrow \mathrm{x}^{\mathrm{u}} < \mathrm{y}^{\mathrm{u}},\ \forall \mathrm{u} = \overline{1, \mathrm{n}},$
(ii) $\mathrm{x} = \mathrm{y} \Leftrightarrow \mathrm{x}^{\mathrm{u}} = \mathrm{y}^{\mathrm{u}},\ \forall \mathrm{u} = \overline{1, \mathrm{n}},$
(iii) $\mathrm{x} \leqq \mathrm{y} \Leftrightarrow \mathrm{x}^{\mathrm{u}} \leq \mathrm{y}^{\mathrm{u}},\ \forall \mathrm{u} = \overline{1, \mathrm{n}},$
(iv) $\mathrm{x} \leq \mathrm{y} \Leftrightarrow \mathrm{x}^{\mathrm{u}} \leq \mathrm{y}^{\mathrm{u}},\ \forall \mathrm{u} = \overline{1, \mathrm{n}}$ and $\mathrm{x}^{\mathrm{u}} < \mathrm{y}^{\mathrm{u}}$ for some u.

Next, we formulate the robust multiobjective fractional variational control problem under study:

$$\text{(CP)} \qquad \min_{(F,K)} \left\{ \frac{\int_U v_\xi (\Omega, \tau)\, dt^\xi}{\int_U r_\xi (\Omega, \alpha)\, dt^\xi} := \left(\frac{\int_U v_\xi^1 \left(\Omega, \tau^1\right) dt^\xi}{\int_U r_\xi^1 \left(\Omega, \alpha^1\right) dt^\xi}, \ldots, \frac{\int_U v_\xi^a (\Omega, \tau^a)\, dt^\xi}{\int_U r_\xi^a (\Omega, \alpha^a)\, dt^\xi} \right) \right\}$$

subject to

$$f(\Omega, F_\zeta(t), \pi) \leqq 0,$$
$$g(\Omega, F_\zeta(t), \gamma) := F_\zeta(t) - \Theta^\zeta(\Omega, \gamma) = 0, \quad \zeta = \overline{1, a}$$
$$t \in C,\ F(t_0) = F_0,\ F(t_1) = F_1,$$

where

$$v_\xi^\delta : C \times A \times B \times \mathrm{G}_\delta \to \mathbb{R},\ \delta = \overline{1, a}, \quad v_\xi = (v_\xi^1, \ldots, v_\xi^a),$$
$$r_\xi^\delta : C \times A \times B \times \mathrm{H}_\delta \to \mathbb{R},\ \delta = \overline{1, a}, \quad r_\xi = (r_\xi^1, \ldots, r_\xi^a),$$
$$f^l : J^1\left(C, \mathbb{R}^s\right) \times B \times \mathrm{T}_l \to \mathbb{R},\ l = \overline{1, m}, \quad f = (f^1, \ldots, f^m),$$
$$g^u : J^1\left(C, \mathbb{R}^s\right) \times B \times \mathrm{M}_u \to \mathbb{R}, u = \overline{1, n}, \quad g = (g^1, \ldots, g^n),$$

are of C^1-class, $J^1\left(C, \mathbb{R}^s\right)$ is the bundle of of jets, $\int_U r_\xi^\delta \left(\Omega, \alpha^\delta\right) dt^\xi > 0, \delta = \overline{1, a}$, $\tau = (\tau^\delta), \alpha = (\alpha^\delta), \pi = (\pi^l)$, and $\gamma = (\gamma^u)$ represent parameters of uncertainty in the compact convex sets $\mathrm{G} = (\mathrm{G}_\delta) \subset \mathbb{R}^a, \mathrm{H} = (\mathrm{H}_\delta) \subset \mathbb{R}^a, \mathrm{T} = (\mathrm{T}_l) \subset \mathbb{R}^m$, and $\mathrm{M} = (\mathrm{M}^u) \subset \mathbb{R}^n$, respectively, and all the considered curvilinear integrals are assumed to be path-independent.

The associated counterpart of (CP) is formulated as follows:

$$\text{(RCP)} \qquad \min_{(F,K)} \frac{\int_U \max_{\tau \in \mathrm{G}} v_\xi (\Omega, \tau)\, dt^\xi}{\int_U \min_{\alpha \in \mathrm{H}} r_\xi (\Omega, \alpha)\, dt^\xi}$$

$$:= \min_{(F,K)} \left(\frac{\int_U \max_{\tau^1 \in \mathrm{G}_1} v_\xi^1 \left(\Omega, \tau^1\right) dt^\xi}{\int_U \min_{\alpha^1 \in \mathrm{H}_1} r_\xi^1 \left(\Omega, \alpha^1\right) dt^\xi}, \ldots, \frac{\int_U \max_{\tau^a \in \mathrm{G}_a} v_\xi^a (\Omega, \tau^a)\, dt^\xi}{\int_U \min_{\alpha^a \in \mathrm{H}_a} r_\xi^a (\Omega, \alpha^a)\, dt^\xi} \right),$$

subject to

$$f(\Omega, F_\zeta(t), \pi) \leqq 0, \quad t \in C,\ \pi \in \mathrm{T}$$
$$g(\Omega, F_\zeta(t), \gamma) = 0, \quad t \in C,\ \gamma \in \mathrm{M}$$

$$F(t_0) = F_0,\ F(t_1) = F_1.$$

The *robust feasible solution set* associated to (CP) is written as below

$$\mathrm{D} = \{(F, K) \in A \times B :\ f(\Omega, F_\zeta(t), \pi) \leqq 0,$$

$$g(\Omega, F_\zeta(t), \gamma) = 0,\ F(t_0) = F_0,\ F(t_1) = F_1,\ t \in C,\ \pi \in \mathrm{T},\ \gamma \in \mathrm{M}\}.$$

In the following, we introduce the parametric scalar optimal control problem associated with (CP):

$$(\mathrm{CP_w}) \qquad \min_{(F,K)} \left\{ \int_U v_\xi^w (\Omega, \tau^w)\, dt^\xi - Q_w^0 \int_U r_\xi^w (\Omega, \alpha^w)\, dt^\xi \right\}$$

subject to

$$f(\Omega, F_\zeta(t), \pi) \leqq 0,$$

$$g(\Omega, F_\zeta(t), \gamma) = 0,$$

$$t \in C,\ F(t_0) = F_0,\ F(t_1) = F_1,$$

$$\int_U \left[v_\xi^\delta \left(\Omega, \tau^\delta\right) - Y_\delta^0 r_\xi^\delta \left(\Omega, \alpha^\delta\right)\right] dt^\xi \leq 0, \quad \delta = \overline{1, a},\ \delta \neq w.$$

The associated counterpart of $(\mathrm{CP_w})$ is stated as below

$$(\mathrm{RCP_w}) \qquad \min_{(F,K)} \Big\{ \int_U \max_{\tau^w \in \mathrm{G}_w} v_\xi^w (\Omega, \tau^w)\, dt^\xi - Y_w^0 \int_U \min_{\alpha^w \in \mathrm{H}_w} r_\xi^w (\Omega, \alpha^w)\, dt^\xi \Big\}$$

subject to

$$(F, K) \in \mathrm{D},$$

$$\int_U \left[v_\xi^\delta \left(\Omega, \tau^\delta\right) - Y_\delta^0 r_\xi^\delta \left(\Omega, \alpha^\delta\right)\right] dt^\xi \leq 0, \quad \delta = \overline{1, a},\ \delta \neq w.$$

Definition 2.1 $(\bar{F}, \bar{K}) \in \mathrm{D}$ is said to be *weak robust optimal pair* of $(\mathrm{CP_w})$ if

$$\int_U \max_{\tau^w \in \mathrm{G}_w} v_\xi^w \left(\bar{\Omega}, \tau^w\right) dt^\xi - Y_w^0 \int_U \min_{\alpha^w \in \mathrm{H}_w} r_\xi^w \left(\bar{\Omega}, \alpha^w\right) dt^\xi$$

$$< \int_U \max_{\tau^w \in \mathrm{G}_w} v_\xi^w (\Omega, \alpha^w)\, dt^\xi - Y_w^0 \int_U \min_{\alpha^w \in \mathrm{H}_w} r_\xi^w (\Omega, \alpha^w)\, dt^\xi,$$

for all $(F, K) \in \mathrm{D}$.

Definition 2.2 $(\bar{F}, \bar{K}) \in \mathrm{D}$ is said to be a *robust optimal solution* in $(\mathrm{CP_w})$ if

$$\int_U \max_{\tau^w \in \mathrm{G}_w} v_\xi^w \left(\bar{\Omega}, \tau^w\right) dt^\xi - Y_w^0 \int_U \min_{\alpha^w \in \mathrm{H}_w} r_\xi^w \left(\bar{\Omega}, \alpha^w\right) dt^\xi$$

$$\leq \int_U \max_{\tau^w \in \mathrm{G}_w} v_\xi^w (\Omega, \alpha^w)\, dt^\xi - Y_w^0 \int_U \min_{\alpha^w \in \mathrm{H}_w} r_\xi^w (\Omega, \alpha^w)\, dt^\xi,$$

for all $(F, K) \in \mathrm{D}$.

Definition 2.3 A vector-valued curvilinear integral functional $\int_U v_\xi(\Omega, F_\zeta(t), \tau) dt^\xi$ is called *convex at* $(\bar{F}, \bar{K}) \in A \times B$ if

$$\int_U v_\xi(\Omega, \tau) dt^\xi - \int_U v_\xi(\bar{\Omega}, \tau) dt^\xi \geqq \int_U (F - \bar{F}) v_{\xi,F}(\bar{\Omega}, \tau) dt^\xi + \int_U (K - \bar{K}) v_{\xi,K}(\bar{\Omega}, \tau) dt^\xi$$

$$+ \int_U (F_\zeta - \bar{F}_\zeta) v_{\xi,F_\zeta}(\bar{\Omega}, \tau) dt^\xi$$

holds, for all $(F, K) \in A \times B$.

Definition 2.4 $(\bar{F}, \bar{K}) \in \mathrm{D}$ is said to be a *weak robust efficient solution* for (CP) if $(F, K) \in \mathrm{D}$ does not exist, fulfilling

$$\frac{\int_U \max_{\tau \in \mathrm{G}} v_\xi (\Omega, \tau) \, dt^\xi}{\int_U \min_{\alpha \in \mathrm{H}} r_\xi (\Omega, \alpha) \, dt^\xi} < \frac{\int_U \max_{\tau \in \mathrm{G}} v_\xi \left(\bar{\Omega}, \tau\right) dt^\xi}{\int_U \min_{\alpha \in \mathrm{H}} r_\xi \left(\bar{\Omega}, \alpha\right) dt^\xi}.$$

Definition 2.5 $(\bar{F}, \bar{K}) \in \mathrm{D}$ is said to be a *robust efficient solution* for (CP) if $(F, K) \in \mathrm{D}$ does not exist, satisfying

$$\frac{\int_U \max_{\tau \in \mathrm{G}} v_\xi (\Omega, \tau) \, dt^\xi}{\int_U \min_{\alpha \in \mathrm{H}} r_\xi (\Omega, \alpha) \, dt^\xi} \leq \frac{\int_U \max_{\tau \in \mathrm{G}} v_\xi \left(\bar{\Omega}, \tau\right) dt^\xi}{\int_U \min_{\alpha \in \mathrm{H}} r_\xi \left(\bar{\Omega}, \alpha\right) dt^\xi}.$$

Theorem 2.1 [Robust necessary conditions of efficiency for (CP)] *Let* $(\bar{F}, \bar{K}) \in \mathrm{D}$ *be a weak robust efficient pair of* (CP) *and*

$$\max_{\tau^w \in \mathrm{G}_w} v_\xi^w (\Omega, \tau^w) = v_\xi^w (\Omega, \bar{\tau}^w), \ \min_{\alpha^w \in \mathrm{H}_w} r_\xi^w (\Omega, \alpha^w) = r_\xi^w (\Omega, \bar{\alpha}^w).$$

Then, we have $\bar{\phi} = (\bar{\phi}^w) \in \mathbb{R}^a$, $\bar{\Upsilon} = (\bar{\Upsilon}^l(t)) \in \mathbb{R}_+^m, \bar{\theta} = (\bar{\theta}^u(t)) \in \mathbb{R}^n$, *and* $\bar{\pi} \in \mathrm{T}$, $\bar{\gamma} \in \mathrm{M}$, *fulfilling*

$$\bar{\phi}^T \left[v_{\xi,F} \left(\bar{\Omega}, \bar{\tau}\right) - H^0 r_{\xi,F} \left(\bar{\Omega}, \bar{\alpha}\right) \right] + \bar{\Upsilon}^T f_F(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\pi}) + \bar{\theta}^T g_F(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\gamma})$$

$$-D_\zeta \Big[\bar{\Upsilon}^T f_{F_\zeta}(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\pi}) + \bar{\theta}^T g_{F_\zeta}(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\gamma}) \Big] = 0, \quad \xi = \overline{1, a}$$

$$\bar{\phi}^T \left[v_{\xi,K} \left(\bar{\Omega}, \bar{\tau}\right) - H^0 r_{\xi,K} \left(\bar{\Omega}, \bar{\alpha}\right) \right] + \bar{\Upsilon}^T f_K(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\pi}) + \bar{\theta}^T g_K(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\gamma}) = 0, \ \xi = \overline{1, a}$$

$$\bar{\Upsilon}^T f(\bar{\Omega}, \bar{F}_\zeta(t), \bar{\pi}) = 0, \ \bar{\Upsilon} \geqq 0,$$

$$\bar{\phi} \geq 0,$$

for all $t \in C$, *except the discontinuities.*

Proof. The proof follows the same lines as in Ritu et al. [22], so we omitted it. □

3. Main results

A Wolfe robust dual of (CP) is written as below [see $\Gamma := (t, \mu(t), \nu(t))$]:

$$(\mathrm{W-CP}) \qquad \max_{(\mu(\cdot),\nu(\cdot))} \int_U \{\left[\nu_\xi(\Gamma,\tau) - H^0 r_\xi(\Gamma,\alpha)\right] + \Upsilon^T f(\Gamma,\mu_\zeta,\pi)E + \theta^T g(\Gamma,\mu_\zeta,\gamma)E\} dt^\xi$$

subject to

$$\phi^T\left[\nu_{\xi,F}(\Gamma,\tau) - H^0 r_{\xi,F}(\Gamma,\alpha)\right] + \Upsilon^T f_F(\Gamma,\mu_\zeta,\pi) + \theta^T g_F(\Gamma,\mu_\zeta,\gamma) - D_\zeta\left[\Upsilon^T f_{F_\zeta}(\Gamma,\mu_\zeta,\pi) + \theta^T g_{F_\zeta}(\Gamma,\mu_\zeta,\gamma)\right] = 0,\ \xi = \overline{1,a} \tag{3.1}$$

$$\phi^T\left[\nu_{\xi,K}(\Gamma,\tau) - H^0 r_{\xi,K}(\Gamma,\alpha)\right] + \Upsilon^T f_K(\Gamma,\mu_\zeta,\pi) + \theta^T g_K(\Gamma,\mu_\zeta,\gamma) = 0,\ \xi = \overline{1,a} \tag{3.2}$$

$$\mu(t_0) = F_0, \quad \mu(t_1) = F_1, \tag{3.3}$$

$$\phi \geq 0, \quad E^T\phi = 1, \quad E = (1,....1) \in \mathbb{R}^a. \tag{3.4}$$

The associated counterpart of (W − CP) is given by:

$$(\mathrm{RW-CP}) \qquad \max_{(\mu(.),\nu(.))} \int_U \{\left[\nu_\xi(\Gamma,\bar\tau) - H^0 r_\xi(\Gamma,\bar\alpha)\right] + \Upsilon^T f(\Gamma,\mu_\zeta,\bar\pi)E + \theta^T g(\Gamma,\mu_\zeta,\bar\gamma)E\} dt^\xi$$

subject to

$$\phi^T\left[\nu_{\xi,F}(\Gamma,\bar\tau) - H^0 r_{\xi,F}(\Gamma,\bar\alpha)\right] + \Upsilon^T f_F(\Gamma,\mu_\zeta,\bar\pi) + \theta^T g_F(\Gamma,\mu_\zeta,\bar\gamma) - D_\zeta\left[\Upsilon^T f_{F_\zeta}(\Gamma,\mu_\zeta,\bar\pi) + \theta^T g_{F_\zeta}(\Gamma,\mu_\zeta,\bar\gamma)\right] = 0,\ \xi = \overline{1,a}$$

$$\phi^T\left[\nu_{\xi,K}(\Gamma,\bar\tau) - H^0 r_{\xi,K}(\Gamma,\bar\alpha)\right] + \Upsilon^T f_K(\Gamma,\mu_\zeta,\bar\pi) + \theta^T g_K(\Gamma,\mu_\zeta,\bar\gamma) = 0,\ \xi = \overline{1,a}$$

$$\mu(t_0) = F_0, \quad \mu(t_1) = F_1,$$

$$\phi \geq 0, \quad E^T\phi = 1, \quad E = (1,....1) \in \mathbb{R}^a,$$

for $\tau \in \mathrm{G}$, $\alpha \in \mathrm{H}$, $\pi \in \mathrm{T}$, $\gamma \in \mathrm{M}$.

Consider $\mathrm{D_w} = \{(\mu,\nu,\phi,\Upsilon,\theta,\tau,\alpha,\pi,\gamma)$ fulfilling (3.1)–(3.4)$\}$ is the set of feasible solutions of (RW − CP), called the *robust set of feasible solutions* to (W − CP).

Definition 3.1 $(\bar\mu,\bar\nu,\bar\phi,\bar\Upsilon,\bar\theta,\bar\tau,\bar\alpha,\bar\pi,\bar\gamma) \in \mathrm{D_w}$ is named *weak robust efficient pair* of (W − CP), if there is no $(\mu,\nu,\phi,\Upsilon,\theta,\tau,\alpha,\pi,\gamma) \in \mathrm{D_w}$ with

$$\int_U \{\left[\nu_\xi\left(\bar\Gamma,\bar\tau\right) - H^0 r_\xi\left(\bar\Gamma,\bar\alpha\right)\right] + \Upsilon^T f(\bar\Gamma,\bar\mu_\zeta,\bar\pi)E + \theta^T g(\bar\Gamma,\bar\mu_\zeta,\bar\gamma)E\} dt^\xi$$

$$< \int_U \{\left[\nu_\xi(\Gamma,\bar\tau) - H^0 r_\xi(\Gamma,\bar\alpha)\right] + \Upsilon^T f(\Gamma,\mu_\zeta,\bar\pi)E + \theta^T g(\Gamma,\mu_\zeta,\bar\gamma)E\} dt^\xi,$$

where $\bar\Gamma := (t, \bar\mu(t), \bar\nu(t))$.

The Mond-Weir robust dual (see Mond and Weir [17]) of (CP), is introduced as follows:

$$\text{(MW − CP)} \qquad \max_{(\mu(\cdot),\nu(\cdot))} \int_U \left[v_\xi(\Gamma,\tau) - H^0 r_\xi(\Gamma,\alpha) \right] dt^\xi$$

subject to

$$\phi^T \left[v_{\xi,F}(\Gamma,\tau) - H^0 r_{\xi,F}(\Gamma,\alpha) \right] + \Upsilon^T f_F(\Gamma,\mu_\zeta,\pi) + \theta^T g_F(\Gamma,\mu_\zeta,\gamma)$$
$$-D_\zeta \left[\Upsilon^T f_{F_\zeta}(\Gamma,\mu_\zeta,\pi) + \theta^T g_{F_\zeta}(\Gamma,\mu_\zeta,\gamma) \right] = 0, \; \xi = \overline{1,a} \tag{3.5}$$

$$\phi^T \left[v_{\xi,K}(\Gamma,\tau) - H^0 r_{\xi,K}(\Gamma,\alpha) \right] + \Upsilon^T f_K(\Gamma,\mu_\zeta,\pi) + \theta^T g_K(\Gamma,\mu_\zeta,\gamma) = 0, \; \xi = \overline{1,a} \tag{3.6}$$

$$\Upsilon^T f(\Gamma,\mu_\zeta,\pi) \geqq 0, \tag{3.7}$$

$$g(\Gamma,\mu_\zeta,\gamma) = 0, \tag{3.8}$$

$$\mu(t_0) = F_0, \quad \mu(t_1) = F_1, \tag{3.9}$$

$$\phi \geq 0, \quad E^T\phi = 1, \quad E = (1,....1) \in \mathbb{R}^a. \tag{3.10}$$

The associated counterpart of (MW − CP) is defined as:

$$\text{(RMW − CP)} \qquad \max_{(\mu(.),\nu(.))} \int_U \left[v_\xi(\Gamma,\bar{\tau}) - H^0 r_\xi(\Gamma,\bar{\alpha}) \right] dt^\xi$$

subject to

$$\phi^T \left[v_{\xi,F}(\Gamma,\bar{\tau}) - H^0 r_{\xi,F}(\Gamma,\bar{\alpha}) \right] + \Upsilon^T f_F(\Gamma,\mu_\zeta,\bar{\pi}) + \theta^T g_F(\Gamma,\mu_\zeta,\bar{\gamma})$$
$$-D_\zeta \left[\Upsilon^T f_{F_\zeta}(\Gamma,\mu_\zeta,\bar{\pi}) + \theta^T g_{F_\zeta}(\Gamma,\mu_\zeta,\bar{\gamma}) \right] = 0, \; \xi = \overline{1,a}$$
$$\phi^T \left[v_{\xi,K}(\Gamma,\bar{\tau}) - H^0 r_{\xi,K}(\Gamma,\bar{\alpha}) \right] + \Upsilon^T f_K(\Gamma,\mu_\zeta,\bar{\pi}) + \theta^T g_K(\Gamma,\mu_\zeta,\bar{\gamma}) = 0, \; \xi = \overline{1,a}$$
$$\Upsilon^T f(\Gamma,\mu_\zeta,\pi) \geqq 0,$$
$$g(\Gamma,\mu_\zeta,\gamma) = 0,$$
$$\mu(t_0) = F_0, \quad \mu(t_1) = F_1,$$
$$\phi \geq 0, \quad E^T\phi = 1, \quad E = (1,....1) \in \mathbb{R}^a,$$

for $\tau \in \mathrm{G}$, $\alpha \in \mathrm{H}, \pi \in \mathrm{T}, \gamma \in \mathrm{M}$.

Consider $\mathrm{D_{mw}} = \{(\mu,\nu,\phi,\Upsilon,\theta,\tau,\alpha,\pi,\gamma)$ verifying (3.5)–(3.10)$\}$ is the set of feasible solutions of (RMW − CP), called the *robust set of feasible solutions* of (MW − CP).

Definition 3.2 $(\bar{\mu},\bar{\nu},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma}) \in \mathrm{D_{mw}}$ is said to be a *weak robust efficient solution* to (MW − CP), if $(\mu,\nu,\phi,\Upsilon,\theta,\tau,\alpha,\pi,\gamma) \in \mathrm{D_{mw}}$ does not exist, with

$$\int_U \left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right) \right] dt^\xi < \int_U \left[v_\xi(\Gamma,\bar{\tau}) - H^0 r_\xi(\Gamma,\bar{\alpha}) \right] dt^\xi.$$

Next, we introduce a robust mixed dual associated with (CP), given by:

$$\text{(mixD − CP)} \qquad \max_{(\mu(\cdot),\nu(\cdot))} \int_U \{ \left[v_\xi(\Gamma,\tau) - H^0 r_\xi(\Gamma,\alpha) \right] + \Upsilon^T f(\Gamma,\mu_\zeta,\pi) E$$

$$+\theta^T g(\Gamma,\mu_\zeta,\gamma)E\}dt^\xi$$

subject to

$$\phi^T\left[v_{\xi,F}(\Gamma,\tau)-H^0 r_{\xi,F}(\Gamma,\alpha)\right]+\Upsilon^T f_F(\Gamma,\mu_\zeta,\pi)+\theta^T g_F(\Gamma,\mu_\zeta,\gamma)$$

$$-D_\zeta\Big[\Upsilon^T f_{F_\zeta}(\Gamma,\mu_\zeta,\pi)+\theta^T g_{F_\zeta}(\Gamma,\mu_\zeta,\gamma)\Big]=0,\ \xi=\overline{1,a} \tag{3.11}$$

$$\phi^T\left[v_{\xi,K}(\Gamma,\tau)-H^0 r_{\xi,K}(\Gamma,\alpha)\right]+\Upsilon^T f_K(\Gamma,\mu_\zeta,\pi)+\theta^T g_K(\Gamma,\mu_\zeta,\gamma)=0,\ \xi=\overline{1,a} \tag{3.12}$$

$$\mu(t_0)=F_0,\quad \mu(t_1)=F_1, \tag{3.13}$$

$$\phi\geq 0,\quad E^T\phi=1,\quad E=(1,....1)\in\mathbb{R}^a, \tag{3.14}$$

$$\Upsilon^T f(\Gamma,\mu_\zeta,\pi)\geqq 0, \tag{3.15}$$

$$g(\Gamma,\mu_\zeta,\gamma)=0. \tag{3.16}$$

The robust counterpart associated with (mixD − CP) is defined as:

$$\text{(RmixD − CP)}\qquad \max_{(\mu(.),\nu(.))}\int_U\{\left[v_\xi(\Gamma,\bar{\tau})-H^0 r_\xi(\Gamma,\bar{\alpha})\right]+\Upsilon^T f(\Gamma,\mu_\zeta,\bar{\pi})E$$

$$+\theta^T g(\Gamma,\mu_\zeta,\bar{\gamma})E\}dt^\xi$$

subject to

$$\phi^T\left[v_{\xi,F}(\Gamma,\bar{\tau})-H^0 r_{\xi,F}(\Gamma,\bar{\alpha})\right]+\Upsilon^T f_F(\Gamma,\mu_\zeta,\bar{\pi})+\theta^T g_F(\Gamma,\mu_\zeta,\bar{\gamma})$$

$$-D_\zeta\Big[\Upsilon^T f_{F_\zeta}(\Gamma,\mu_\zeta,\bar{\pi})+\theta^T g_{F_\zeta}(\Gamma,\mu_\zeta,\bar{\gamma})\Big]=0,\ \xi=\overline{1,a}$$

$$\phi^T\left[v_{\xi,K}(\Gamma,\bar{\tau})-H^0 r_{\xi,K}(\Gamma,\bar{\alpha})\right]+\Upsilon^T f_K(\Gamma,\mu_\zeta,\bar{\pi})+\theta^T g_K(\Gamma,\mu_\zeta,\bar{\gamma})=0,\ \xi=\overline{1,a}$$

$$\mu(t_0)=F_0,\quad \mu(t_1)=F_1,$$

$$\phi\geq 0,\quad E^T\phi=1,\quad E=(1,....1)\in\mathbb{R}^a,$$

$$\Upsilon^T f(\Gamma,\mu_\zeta,\pi)\geqq 0,$$

$$g(\Gamma,\mu_\zeta,\gamma)=0,$$

for $\tau\in$ G, $\alpha\in$ H, $\pi\in$ T, $\gamma\in$ M.

Consider $\mathrm{D_m}=\{(\mu,\nu,\phi,\Upsilon,\theta,\tau,\alpha,\pi,\gamma)$ which verifies (3.11)–(3.16)$\}$ is the set of feasible solutions of (RmixD − CP), called the *robust set of feasible solutions* of (mixD − CP).

Definition 3.3 $(\bar{\mu},\bar{\nu},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})\in\mathrm{D_m}$ is named the *weak robust efficient pair* of (mixD − CP), if there is no $(\mu,\nu,\phi,\Upsilon,\theta,\tau,\alpha,\pi,\gamma)\in\mathrm{D_m}$ with

$$\int_U\{\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right)\right]+\Upsilon^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})E+\theta^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})E\}dt^\xi$$

$$<\int_U\{\left[v_\xi(\Gamma,\bar{\tau})-H^0 r_\xi(\Gamma,\bar{\alpha})\right]+\Upsilon^T f(\Gamma,\mu_\zeta,\bar{\pi})E+\theta^T g(\Gamma,\mu_\zeta,\bar{\gamma})E\}dt^\xi.$$

Next, we state a weak robust duality theorem for (CP).

Theorem 3.1 [Weak duality] *Consider that* $(\bar{F},\bar{K})$ *and* $(\bar{\mu},\bar{\nu},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ *are robust feasible pairs of* (CP) *and* (mixD – CP)*, respectively, and* $\max_{\tau\in G} v_{\xi}(\bar{\Omega},\tau) = v_{\xi}(\bar{\Omega},\bar{\tau})$, $\min_{\alpha\in H} r_{\xi}\left(\bar{\Omega},\alpha\right) = r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)$*, and*

$$\int_U \bar{\phi}^T\left[v_{\xi}(.,\bar{\tau}) - H^0 r_{\xi}(.,\bar{\alpha})\right]dt^{\xi},$$

$\int_U \bar{\Upsilon}^T f(.,\bar{\pi})dt^{\xi}$ *and* $\int_U \bar{\theta}^T g(.,\bar{\gamma})dt^{\xi}$ *are convex at* $(\bar{\mu},\bar{\nu})$*. Under the above assumptions, we have*

$$\int_U \left[v_{\xi}(\bar{\Omega},\bar{\tau}) - H^0 r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)\right]dt^{\xi}$$

$$< \int_U \{\left[v_{\xi}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Gamma},\bar{\alpha}\right)\right] + \bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\pi})E + \bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\gamma})E\}dt^{\xi}.$$

Proof. Contrary, let us consider

$$\int_U \{\max_{\tau\in G} v_{\xi}\left(\bar{\Omega},\tau\right) - H^0 \min_{\alpha\in H} r_{\xi}\left(\bar{\Omega},\alpha\right)\}dt^{\xi} < \int_U \{\left[v_{\xi}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Gamma},\bar{\alpha}\right)\right]$$

$$+\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\pi})E + \bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\gamma})E\}dt^{\xi}$$

is valid. Since $\max_{\tau\in G} v_{\xi}\left(\bar{\Omega},\tau\right) - H^0 \min_{\alpha\in H} r_{\xi}\left(\bar{\Omega},\alpha\right) = v_{\xi}\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)$, we obtain

$$\int_U \left[v_{\xi}\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)\right]dt^{\xi} < \int_U \{\left[v_{\xi}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Gamma},\bar{\alpha}\right)\right]$$

$$+\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\pi})E + \bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\gamma})E\}dt^{\xi}$$

is valid. As $(\bar{F},\bar{K})$ is a robust feasible pair of (CP), it follows that

$$\int_U \{\left[v_{\xi}\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)\right] + \bar{\Upsilon}^T f(\bar{\Omega},\bar{F}_{\zeta},\bar{\pi})E + \bar{\theta}^T g(\bar{\Omega},\bar{F}_{\zeta},\bar{\gamma})E\}dt^{\xi}$$

$$< \int_U \{\left[v_{\xi}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Gamma},\bar{\alpha}\right)\right] + \bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\pi})E + \bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\gamma})E\}dt^{\xi}.$$

For $\bar{\phi} \geq 0$ and $\bar{\phi}^T E = 1$, we get

$$\int_U \{\bar{\phi}^T\left[v_{\xi}\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)\right] + \bar{\Upsilon}^T f(\bar{\Omega},\bar{F}_{\zeta},\bar{\pi}) + \bar{\theta}^T g(\bar{\Omega},\bar{F}_{\zeta},\bar{\gamma})\}dt^{\xi}$$

$$< \int_U \{\bar{\phi}^T\left[v_{\xi}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Gamma},\bar{\alpha}\right)\right] + \bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\pi}) + \bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_{\zeta},\bar{\gamma})\}dt^{\xi}. \tag{3.17}$$

Since $\int_U \bar{\phi}^T\left[v_{\xi}(.,\bar{\tau}) - H^0 r_{\xi}(.,\bar{\alpha})\right]dt^{\xi}$, $\int_U \bar{\Upsilon}^T f(.,\bar{\pi})dt^{\xi}$ and $\int_U \bar{\theta}^T g(.,\bar{\gamma})dt^{\xi}$ are convex at $(\bar{\mu},\bar{\nu})$, we get

$$\int_U \{\bar{\phi}^T\left[v_{\xi}\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Omega},\bar{\alpha}\right)\right] - \bar{\phi}^T\left[v_{\xi}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_{\xi}\left(\bar{\Gamma},\bar{\alpha}\right)\right]\}dt^{\xi} \geqq$$

$$\int_U (\bar{F}-\bar{\mu})\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial F}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial F}\left(\bar{\Gamma},\bar{\alpha}\right)\right]dt^\xi$$
$$+\int_U (\bar{K}-\bar{\nu})\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial K}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial K}\left(\bar{\Gamma},\bar{\alpha}\right)\right]dt^\xi, \tag{3.18}$$

$$\int_U\{\bar{\Upsilon}^T f(\bar{\Omega},\bar{F}_\zeta,\bar{\pi})-\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})\}dt^\xi \geqq \int_U (\bar{F}-\bar{\mu})\bar{\Upsilon}^T f_F(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi$$
$$+\int_U (\bar{F}_\zeta-\bar{\mu}_\zeta)\bar{\Upsilon}^T f_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi+\int_U (\bar{K}-\bar{\nu})\bar{\Upsilon}^T f_K(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi \tag{3.19}$$

and

$$\int_U\{\bar{\theta}^T g(\bar{\Omega},\bar{F}_\zeta,\bar{\gamma})-\bar{\Upsilon}^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})\}dt^\xi \geqq \int_U (\bar{F}-\bar{\mu})\bar{\theta}^T g_F(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})dt^\xi$$
$$+\int_U (\bar{F}_\zeta-\bar{\mu}_\zeta)\bar{\theta}^T g_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})dt^\xi+\int_U (\bar{K}-\bar{\nu})\bar{\theta}^T g_K(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})dt^\xi. \tag{3.20}$$

By adding (3.18)–(3.20) and using the dual constraints of $(\bar{\mu},\bar{\nu},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\pi},\bar{\gamma})$, we get

$$\int_U\{\bar{\phi}^T\left[v_\xi\left(\bar{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)\right]+\bar{\Upsilon}^T f(\bar{\Omega},\bar{F}_\zeta,\bar{\pi})+\bar{\theta}^T g(\bar{\Omega},\bar{F}_\zeta,\bar{\gamma})\}dt^\xi$$
$$\geqq \int_U\{\bar{\phi}^T\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right)\right]+\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})+\bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})\}dt^\xi,$$

which contradicts (3.17). □

A robust strong duality for (CP) is established by the following theorem.

Theorem 3.2 [Strong duality] *Let $(\bar{F},\bar{K})$ be a weak robust efficient pair of* (CP), $\max_{\tau\in G}\{v_\xi\left(\bar{\Omega},\tau\right)-H^0\min_{\alpha\in H} r_\xi\left(\bar{\Omega},\alpha\right)\} = v_\xi\left(\bar{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)$, *and the constraint qualification criteris are satisfied. Then, there exist $\bar{\phi}\in\mathbb{R}^a$, $\bar{\Upsilon}=(\bar{\Upsilon}_l(t))\in\mathbb{R}^m_+,\bar{\theta}=(\bar{\theta}_b(t))\in\mathbb{R}^n$, and $\bar{\pi}\in$ T, $\bar{\gamma}\in$ M, $\bar{\tau}\in$ G $\bar{\alpha}\in$ H, such that $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ is a robust feasible pair of* (mixD − CP). *In addition, if Theorem* 3.1 *is valid, the pair $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ becomes a weak robust efficient pair of* (mixD − CP).

Proof. Since $(\bar{F},\bar{K})$ is a weak robust efficient pair of (CP) (see Theorem 2.1), there exist $\bar{\phi}\in\mathbb{R}^a$, $\bar{\Upsilon}=(\bar{\Upsilon}_l(t))\in\mathbb{R}^m_+,\bar{\theta}=(\bar{\theta}_b(t))\in\mathbb{R}^n$, and $\bar{\pi}\in$ T, $\bar{\gamma}\in$ M, $\bar{\tau}\in$ G $\bar{\alpha}\in$ H, such that (3.1)–(3.4) are valid at $(\bar{F},\bar{K})$. Thus, $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ is a robust feasible pair of (mixD − CP) and the values of the objective functionals are equal. Suppose that (3.1)–(3.4) are verified at $(\bar{F},\bar{K})$ and $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$, which is not a weak efficient pair of (mixD − CP). Consequently, there is $(\mu,\nu,\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$, satisfying

$$\int_U\{\left[v_\xi\left(\bar{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)\right]+\Upsilon^T f(\bar{\Omega},\bar{F}_\zeta,\bar{\pi})E+\theta^T g(\bar{\Omega},\bar{F}_\zeta,\bar{\gamma})E\}dt^\xi$$
$$<\int_U\{\left[v_\xi\left(\Gamma,\bar{\tau}\right)-H^0 r_\xi\left(\Gamma,\bar{\alpha}\right)\right]+\Upsilon^T f(\Gamma,\mu_\zeta,\bar{\pi})E+\theta^T g(\Gamma,\mu_\zeta,\bar{\gamma})E\}dt^\xi.$$

Taking into account Theorem 2.1, we obtain

$$\int_U\left[v_\xi\left(\bar{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)\right]dt^\xi$$

$$< \int_U \left[v_\xi(\Gamma,\bar{\tau}) - H^0 r_\xi(\Gamma,\bar{\alpha}) \right] + \Upsilon^T f(\Gamma,\mu_\zeta,\bar{\pi})E + \theta^T g(\Gamma,\mu_\zeta,\bar{\gamma})E\} dt^\xi.$$

Since $\max_{\tau\in \mathrm{G}}\{v_\xi\left(\bar{\Omega},\tau\right) - H^0 \min_{\alpha\in \mathrm{H}} r_\xi\left(\bar{\Omega},\alpha\right)\} = v_\xi\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)$, we have

$$\int_U \{\max_{\tau\in \mathrm{G}} v_\xi\left(\bar{\Omega},\tau\right) - H^0 \min_{\alpha\in \mathrm{H}} r_\xi\left(\bar{\Omega},\alpha\right)\} dt^\xi <$$

$$\int_U \{\left[v_\xi(\Gamma,\bar{\tau}) - H^0 r_\xi(\Gamma,\bar{\alpha}) \right] + \Upsilon^T f(\Gamma,\mu_\zeta,\bar{\pi})E + \theta^T g(\Gamma,\mu_\zeta,\bar{\gamma})E\} dt^\xi,$$

which contradicts Theorem 3.1. Thus, $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ is a weak robust efficient pair of (mixD − CP). □

Finally, we formulate a robust strict converse dual model for (CP).

Theorem 3.3 [Strict converse duality] *Consider that* $(\bar{\mu},\bar{\nu},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ *is a robust feasible pair of* (mixD − CP) *and*

$$\max_{\tau\in \mathrm{G}}\{v_\xi\left(\bar{\Omega},\tau\right) - H^0 \min_{\alpha\in \mathrm{H}} r_\xi\left(\bar{\Omega},\alpha\right)\} = v_\xi\left(\bar{\Omega},\bar{\tau}\right) - H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)$$

and $\int_U \bar{\phi}^T \left[v_\xi(.,\bar{\tau}) - H^0 r_\xi(.,\bar{\alpha}) \right] dt^\xi$, $\int_U \bar{\Upsilon}^T f(.,\bar{\pi}) dt^\xi$ *and* $\int_U \bar{\theta}^T g(.,\bar{\gamma}) dt^\xi$ *are strictly convex at* $(\bar{\mu},\bar{\nu})$. *If* $(\bar{F},\bar{K}) \in \mathrm{D}$ *satisfies*

$$\int_U \left[v_\xi(\bar{\Omega},\bar{\tau}) - H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right) \right] dt^\xi$$

$$= \int_U \{\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right) - H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right) \right] + \bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})E + \bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})E\} dt^\xi.$$

Then, the pair $(\bar{F},\bar{K})$ *is a weak robust efficient pair of* (CP).

Proof. Since $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\alpha},\bar{\pi},\bar{\gamma})$ is a robust feasible pair of (mixD − CP), by multiplying (3.11) and (3.12) by $(\hat{F}-\bar{\mu})$ and $(\hat{K}-\bar{\nu})$, respectively, we get

$$\begin{aligned} &\int_U (\hat{F}-\bar{\mu})\{\bar{\phi}^T \left[\frac{\partial v_\xi}{\partial F}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 \frac{\partial r_\xi}{\partial F}\left(\bar{\Gamma},\bar{\alpha}\right) \right] + \bar{\Upsilon}^T f_F(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\pi}) \\ &+\bar{\theta}^T g_F(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\gamma}) - D_\zeta\left[\bar{\Upsilon}^T f_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\pi}) + \bar{\theta}^T g_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\gamma}) \right]\} dt^\xi \\ &+ \int_U (\hat{K}-\bar{\nu})\{\bar{\phi}^T \left[\frac{\partial v_\xi}{\partial K}\left(\bar{\Gamma},\bar{\tau}\right) - H^0 \frac{\partial r_\xi}{\partial K}\left(\bar{\Gamma},\bar{\alpha}\right) \right] + \bar{\Upsilon}^T f_K(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\pi}) \\ &+\bar{\theta}^T g_K(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\gamma})\} dt^\xi = 0. \end{aligned} \tag{3.21}$$

Next, on the contrary, we assume that $(\bar{F},\bar{K})$ is not a weak robust efficient pair of (CP). Therefore, there exists $(\hat{F},\hat{K}) \in \mathrm{D}$ with

$$\int_U \{\max_{\tau\in \mathrm{G}} v_\xi\left(\hat{\Omega},\tau\right) - H^0 \min_{\alpha\in \mathrm{H}} r_\xi\left(\hat{\Omega},\alpha\right)\} dt^\xi$$

$$< \int_U \{\max_{\tau\in \mathrm{G}} v_\xi\left(\bar{\Omega},\tau\right) - H^0 \min_{\alpha\in \mathrm{H}} r_\xi\left(\bar{\Omega},\alpha\right)\} dt^\xi,$$

or, equivalently,

$$\int_U \left[v_\xi\left(\hat{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\hat{\Omega},\bar{\alpha}\right)\right]dt^\xi < \int_U \left[v_\xi\left(\bar{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)\right]dt^\xi .$$

By considering the assumption

$$\int_U \left[v_\xi(\bar{\Omega},\bar{\tau})-H^0 r_\xi\left(\bar{\Omega},\bar{\alpha}\right)\right]dt^\xi$$

$$=\int_U \{\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right)\right]+\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})E+\bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})E\}dt^\xi ,$$

therefore, the above inequality implies

$$\int_U \left[v_\xi\left(\hat{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\hat{\Omega},\bar{\alpha}\right)\right]dt^\xi < \int_U \{\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right)\right]+$$

$$\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})E+\bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})E\}dt^\xi .$$

Since $\bar{\phi}>0$, we get

$$\int_U \bar{\phi}^T\left[v_\xi\left(\hat{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\hat{\Omega},\bar{\alpha}\right)\right]dt^\xi < \int_U \{\bar{\phi}^T\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right)\right]+$$

$$\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})+\bar{\theta}^T g(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})\}dt^\xi . \tag{3.22}$$

By considering the strict convexity of $\int_U \bar{\phi}^T\left[v_\xi(.,\bar{\tau})-H^0 r_\xi(,,\bar{\alpha})\right]dt^\xi$ at $(\bar{\mu},\bar{\nu})$, we have

$$\int_U \{\bar{\phi}^T\left[v_\xi\left(\hat{\Omega},\bar{\tau}\right)-H^0 r_\xi\left(\hat{\Omega},\bar{\alpha}\right)\right]-\bar{\phi}^T\left[v_\xi\left(\bar{\Gamma},\bar{\tau}\right)-H^0 r_\xi\left(\bar{\Gamma},\bar{\alpha}\right)\right]\}dt^\xi$$

$$>\int_U (\hat{F}-\bar{\mu})\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial F}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial F}\left(\bar{\Gamma},\bar{\alpha}\right)\right]dt^\xi$$
$$+\int_U (\hat{K}-\bar{\nu})\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial K}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial K}\left(\bar{\Gamma},\bar{\alpha}\right)\right]dt^\xi ,$$

and, by (3.22) and feasibility property of $(\bar{\mu},\bar{\nu})$, implies

$$\int_U (\hat{F}-\bar{\mu})\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial F}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial F}\left(\bar{\Gamma},\bar{\alpha}\right)\right]dt^\xi$$

$$+\int_U (\hat{K}-\bar{\nu})\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial K}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial K}\left(\bar{\Gamma},\bar{\alpha}\right)\right]dt^\xi<0. \tag{3.23}$$

Again, using the hypothesis for $\int_U \bar{\Upsilon}^T f(.,\bar{\pi})dt^\xi$ at $(\bar{\mu},\bar{\nu})$, it follows that

$$\int_U \{\bar{\Upsilon}^T f(\hat{\Omega},\bar{F}_\zeta,\bar{\pi})-\bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})\}dt^\xi > \int_U (\hat{F}-\bar{\mu})\bar{\Upsilon}^T f_F(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi$$

$$+\int_U (\hat{F}_\zeta-\bar{\mu}_\zeta)\bar{\Upsilon}^T f_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi+\int_U (\hat{K}-\bar{\nu})\bar{\Upsilon}^T f_K(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi. \tag{3.24}$$

Also, as $(\hat{F},\hat{K})$ and $(\bar{F},\bar{K},\bar{\phi},\bar{\Upsilon},\bar{\theta},\bar{\tau},\bar{\pi},\bar{\gamma})$ are robust feasible pairs of (CP) and (mixD − CP), we get

$$\int_U \bar{\Upsilon}^T f(\hat{\Omega},\bar{F}_\zeta,\bar{\pi})dt^\xi-\int_U \bar{\Upsilon}^T f(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi \leqq 0,$$

and, by using (3.24), we obtain

$$\int_U (\hat{F}-\bar{\mu})\bar{\Upsilon}^T f_F(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi+\int_U (\hat{F}_\zeta-\bar{\mu}_\zeta)\bar{\Upsilon}^T f_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi$$

$$+\int_U (\hat{K}-\bar{\nu})\bar{\Upsilon}^T f_K(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\pi})dt^\xi<0. \tag{3.25}$$

Similarly, since $\int_U \bar{\theta}^T g(.,F_\zeta,\bar{\gamma})dt^\xi$ is also a strictly convex function, we get

$$\int_U (\hat{F}-\bar{\mu})\bar{\theta}^T g_F(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})dt^\xi+\int_U (\hat{F}_\zeta-\bar{\mu}_\zeta)\bar{\theta}^T g_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})dt^\xi$$

$$+\int_U (\hat{K}-\bar{\nu})\bar{\theta}^T g_K(\bar{\Gamma},\bar{\mu}_\zeta,\bar{\gamma})dt^\xi<0. \tag{3.26}$$

By adding the relations (3.23), (3.25), and (3.26), we get the following result

$$\int_U (\hat{F}-\bar{\mu})\{\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial F}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial F}\left(\bar{\Gamma},\bar{\alpha}\right)\right]+\bar{\Upsilon}^T f_F(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\pi})$$

$$+\bar{\theta}^T g_F(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\gamma})-D_\zeta\Big[\bar{\Upsilon}^T f_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\pi})+\bar{\theta}^T g_{F_\zeta}(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\gamma})\Big]\}dt^\xi$$

$$+\int_U (\hat{K}-\bar{\nu})\{\bar{\phi}^T\left[\frac{\partial v_\xi}{\partial K}\left(\bar{\Gamma},\bar{\tau}\right)-H^0\frac{\partial r_\xi}{\partial K}\left(\bar{\Gamma},\bar{\alpha}\right)\right]+\bar{\Upsilon}^T f_K(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\pi})$$

$$+\bar{\theta}^T g_K(\bar{\Gamma},\bar{\mu}_\zeta(t),\bar{\gamma})\}dt^\xi<0,$$

which is a contradiction with (3.21), and this completes the proof. □

Numerical application. Let us highlight the theoretical findings derived in the previous sections of the paper. Suppose we interested in only affine real-valued piecewise differentiable control and state functions, where G = H = [1, 2], $C \subset \mathbb{R}^2$ is a square fixed by the diagonally opposite corners $t_0=(t_0^1,t_0^2)=(0,0)$ and $t_1=(t_1^2,t_1^2)=(\frac{1}{2},\frac{1}{2})\in\mathbb{R}^2$, and $T\subset C$ is a piecewise differentiable curve that links the previous two different pairs. Define the following robust scalar fractional variational control problem:

$$\textit{(CP1)}\quad \min_{(F(\cdot),K(\cdot))}\left\{\frac{\int_U v_\xi(\Omega,\tau)\,dt^\xi}{\int_U r_\xi(\Omega,\alpha)\,dt^\xi}:=\frac{\int_T [K^2+\tau](dt^1+dt^2)}{\int_T [\alpha F e^{2F+\frac{1}{2}}](dt^1+dt^2)}\right\}$$

subject to

$$f(\Omega, F_\zeta(t)) := F^2 + F - 2 \leq 0,$$

$$f(\Omega, F_\zeta(t)) := \frac{\partial F}{\partial t^\zeta} + 2K - 1 = 0, \ \ \zeta = 1, 2$$

$$F(t_0) := F(0,0) = F_0 := 1, \ F(t_1) := F\left(\frac{1}{2}, \frac{1}{2}\right) = F_1 := \frac{1}{3}.$$

The above problem, in fact, extremizes the mechanical work provided by variable forces governed by data uncertainty, namely: $\bar{F}_1 = (K^2 + \tau, K^2 + \tau)$ and $\bar{F}_2 = (\alpha F e^{2F+\frac{1}{2}}, \alpha F e^{2F+\frac{1}{2}})$. The non-fractional optimization problem associated with *(CP1)* is given by:

$$\textit{(NCP1)} \quad \min_{(F(\cdot),K(\cdot))} \Big\{ \int_T [K^2 + \tau](dt^1 + dt^2) - H^0 \int_T [\alpha F e^{2F+\frac{1}{2}}](dt^1 + dt^2) \Big\}$$

subject to

$$F^2 + F - 2 \leq 0,$$

$$\frac{\partial F}{\partial t^\zeta} + 2K - 1 = 0, \ \ \zeta = 1, 2$$

$$F(0,0) = 1, \ F\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3},$$

and the associated counterpart of *(NCP1)* is given as follows:

$$\textit{(RNCP1)} \quad \min_{(F(\cdot),K(\cdot))} \Big\{ \int_T \max_{\tau \in \mathrm{G}} [K^2 + \tau](dt^1 + dt^2) - H^0 \int_T \min_{\alpha \in \mathrm{H}} [\alpha F e^{2F+\frac{1}{2}}](dt^1 + dt^2) \Big\}$$

subject to

$$F^2 + F - 2 \leq 0,$$

$$\frac{\partial F}{\partial t^\zeta} + 2K - 1 = 0, \ \ \zeta = 1, 2$$

$$F(0,0) = 1, \ F\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}.$$

Next, for $\Gamma := (t, \mu(t), \nu(t))$, we introduce a robust mixed dual of *(CP1)*, defined by:

$$(\textit{mixD-CP1}) \qquad \max_{(\mu(\cdot),\nu(\cdot))} \int_U \{\Big[\nu_\xi(\Gamma, \tau) - H^0 r_\xi(\Gamma, \alpha)\Big] + \Upsilon f(\Gamma, \mu_\zeta) + \theta^T g(\Gamma, \mu_\zeta)\} dt^\xi$$

subject to

$$\phi\Big[\nu_{\xi,F}(\Gamma, \tau) - H^0 r_{\xi,F}(\Gamma, \alpha)\Big] + \Upsilon f_F(\Gamma, \mu_\zeta) + \theta^T g_F(\Gamma, \mu_\zeta) - D_\zeta\Big[\Upsilon f_{F_\zeta}(\Gamma, \mu_\zeta) + \theta^T g_{F_\zeta}(\Gamma, \mu_\zeta)\Big] = 0, \quad \xi = \overline{1,2} \tag{3.27}$$

$$\phi\Big[\nu_{\xi,K}(\Gamma, \tau) - H^0 r_{\xi,K}(\Gamma, \alpha)\Big] + \Upsilon f_K(\Gamma, \mu_\zeta) + \theta^T g_K(\Gamma, \mu_\zeta) = 0, \quad \xi = \overline{1,2} \tag{3.28}$$

$$\mu(t_0) = F_0, \quad \mu(t_1) = F_1, \tag{3.29}$$

$$\phi \geq 0, \tag{3.30}$$

$$\Upsilon f(\Gamma, \mu_\zeta) \geq 0, \tag{3.31}$$

$$g(\Gamma, \mu_\zeta) = 0. \tag{3.32}$$

The counterpart associated with *(mixD-CP1)* is defined as below:

$$\textit{(RmixD-CP1)} \qquad \max_{(\mu(.),\nu(.))} \int_U \{\Big[v_\xi(\Gamma, \bar{\tau}) - H^0 r_\xi(\Gamma, \bar{\alpha})\Big] + \Upsilon f(\Gamma, \mu_\zeta)$$

$$+\theta^T g(\Gamma, \mu_\zeta)\} dt^\xi$$

subject to

$$\phi\left[v_{\xi,F}(\Gamma, \bar{\tau}) - H^0 r_{\xi,F}(\Gamma, \bar{\alpha})\right] + \Upsilon f_F(\Gamma, \mu_\zeta) + \theta^T g_F(\Gamma, \mu_\zeta)$$

$$-D_\zeta\Big[\Upsilon f_{F_\zeta}(\Gamma, \mu_\zeta) + \theta^T g_{F_\zeta}(\Gamma, \mu_\zeta)\Big] = 0, \quad \xi = \overline{1,2}$$

$$\phi\left[v_{\xi,K}(\Gamma, \bar{\tau}) - H^0 r_{\xi,K}(\Gamma, \bar{\alpha})\right] + \Upsilon f_K(\Gamma, \mu_\zeta) + \theta^T g_K(\Gamma, \mu_\zeta) = 0, \quad \xi = \overline{1,2}$$

$$\mu(t_0) = F_0, \quad \mu(t_1) = F_1,$$

$$\phi \geq 0,$$

$$\Upsilon f(\Gamma, \mu_\zeta) \geq 0,$$

$$g(\Gamma, \mu_\zeta) = 0,$$

for $\tau \in$ G, $\alpha \in$ H.

The robust set of feasible solutions associated with *(NCP1)* is

$$\text{D} = \Big\{(F, K) \in A \times B : -2 \leq F \leq 1, \ \frac{\partial F}{\partial t^1} = \frac{\partial F}{\partial t^2} = 1 - 2K, \ F(0,0) = 1, \ F\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}\Big\}$$

and, by direct computation, we find $(\overline{F}, \overline{K}) = \left(-\frac{2}{3}(t^1 + t^2) + 1, \frac{5}{6}\right) \in$ D, and at $t^1 = t^2 = 0$ it satisfies the conditions (3.27)–(3.32) with $H^0 = \frac{169}{36e^{\frac{5}{2}}}$, the uncertainty parameters $\overline{\tau} = 2, \overline{\alpha} = 1$, and Lagrange multipliers $\overline{\phi} = \frac{1}{2}, \overline{\Upsilon} = 0, \overline{\theta}_1 = \overline{\theta}_2 = \frac{5}{24}$. Further, it can also be easily verified that the involved functionals $\int_T \bar{\phi}\Big[v_\xi - H^0 r_\xi\Big](dt^1 + dt^2)$, $\int_T \bar{\Upsilon} f(dt^1 + dt^2)$, $\int_T \overline{\theta}^T g(dt^1 + dt^2)$ are strictly convex at $(\overline{\mu}, \overline{\nu}) = \left(1, \frac{5}{6}\right) \in$ D. By direct computation, we obtain

$$\int_U \left[v_\xi(\bar{\Omega}, \bar{\tau}) - H^0 r_\xi\left(\bar{\Omega}, \bar{\alpha}\right)\right] dt^\xi$$

$$= \int_U \{\Big[v_\xi\left(\bar{\Gamma}, \bar{\tau}\right) - H^0 r_\xi\left(\bar{\Gamma}, \bar{\alpha}\right)\Big] + \bar{\Upsilon} f(\bar{\Gamma}, \bar{\mu}_\zeta) + \bar{\theta}^T g(\bar{\Gamma}, \bar{\mu}_\zeta)\} dt^\xi.$$

Hence, all the conditions of Theorem 3.3 are satisfied, ensuring that $(\overline{F}, \overline{K})$ is a weak robust efficient solution to *(CP1)*.

4. Conclusions

In this paper, we formulated a new class of robust multiobjective fractional variational control problems. Then, a dual model was associated with the above-mentioned class of problems. Further, by considering variants of convexity for the involved functionals (determined by curvilinear integrals that do not depend on the path), we have provided some characterization and equivalence results for the models considered. In addition, a numerical example was formulated.

Author contributions

Savin Treanţă: Conceptualization, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization, supervision; Muhammad Uzair Awan: Conceptualization, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization; Muhammad Zakria Javed: Conceptualization, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization; Bandar Bin-Mohsin: Conceptualization, software, validation, formal analysis, investigation, writing-original draft preparation, writing-review and editing, visualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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