Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(1): 921–931.
DOI: 10.3934/math.2025043
Received: 29 October 2024
Revised: 01 January 2025
Accepted: 08 January 2025
Published: 15 January 2025

Research article

A new perspective on fuzzy mapping theory with invertedly open and closed mappings

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Abstract: In fuzzy mapping theories, we examine fuzzy closedness and fuzzy continuity of a mapping ϕ, characterized respectively by $\overline{\phi(M)} \subseteq \phi(\overline{M})$ and $\phi(\overline{M}) \subseteq \overline{\phi(M)}$, for every fuzzy set M in V. Here, (V, τ) represents a fuzzy topological space (FTs), where $V = \{v\}$ denotes a set of points. This reveals a fundamental symmetry between the two mappings in connection with the closure operator. On the other hand, the fuzzy openness of a mapping ϕ is characterized by $\phi(M^\circ) < (\phi(M))^\circ$ for every fuzzy set M in V. Considering the above statements, it is logical to explore how fuzzy continuity relates to the interior operator. Building on this, we introduce the notion of the invertedly fuzzy open mapping, defined as $(\phi(M))^\circ < \phi(M^\circ)$ for any fuzzy set M in V, and discuss its relationship with fuzzy continuity. In our study, we define and analyze invertedly fuzzy open and invertedly fuzzy closed mappings, along with their respective properties. We also delve into how these mappings connect with fuzzy continuous mappings. Furthermore, we examine a characterization of fuzzy homeomorphism for bijective mappings concerning the interior operator.

Keywords: fuzzy sets; fuzzy topological spaces; fuzzy continuous mappings; induced mappings; invertedly fuzzy open mappings; invertedly fuzzy closed mappings
Mathematics Subject Classification: 54A40, 54C05, 94D05

1. Introduction

In 1965 [21], Zadeh brought in the picture of fuzzy sets (F-sets, in short), a tool to find indefinite answers to real issues. In 1968 [3], the notion of fuzzy topology was initiated by Chang, which was subsequently expanded upon by numerous researchers who applied basic principles from general topology to F-sets like [14, 16], and many more. Chang also developed the theory of fuzzy continuous functions together with its characterizations. Wong originated the concept of fuzzy open and fuzzy closed mappings in [19], where Malghan et al. investigated some of their representations in [11]. The widespread use and discoveries in F-set theory across various fields have been growing rapidly, resulting in significant advancements in both theory and applications [2, 4, 6, 8, 9, 15, 22].

Mappings, crucial for maintaining system structures and enhancing problem-solving models, are adapted across disciplines like mathematics, physics, chemistry, psychology, computer science, and logic to suit specific needs. Fuzzy mappings were also presented by Heilpern [7]. In general, any mathematical structure, a constraint on mappings, and an induced operator can lead to useful induced mappings that are the source of our exploration of the links between various mathematical systems. Several authors have researched diverse induced operators and mappings. Liu and Zhang [10] introduced a method for analyzing the structure of all mappings, induced by mappings, and preserving matrices with rank-1. Induced mappings that preserve the inverse and similarity of a matrix are given by Yang et al [20]. Arkhangel'skii and Ponomarev [1] introduced a novel mapping denoted as $g^{\#} : \wp(U) \to \wp(V)$ given by $g^{\#}(M) = \{v \in V | g^{-1}(v) \subset M\}$ for any subset M of U, where $\wp(U)$ is a collection of all subsets of U. With this induced mapping, open mappings are defined in terms of the inclusion relation between the closures of specific subsets, as discussed in the reference [13]. The above studies are considered to be insufficient in analyzing the induced mappings used in the literature on fuzzy mappings. This study is expected to contribute to filling these apertures in the existing writings with its methodological touch and outcomes.

In the studies of fuzzy mappings, it is proved that for a mapping ϕ from a fuzzy topological space (or FTs) (V, T_1) to an FTs (W, T_2), ϕ is (i) fuzzy closed [11] (fuzzy continuous [18]) if and only if $\overline{\phi(M)} < \phi(\overline{M})$ ($\phi(\overline{M}) < \overline{\phi(M)}$), (ii) fuzzy open [12] if and only if $\phi(M^{\circ}) < (\phi(M))^{\circ}$ for every fuzzy set M in V. Considering (i) and (ii) above, it is also logical to study fuzzy continuity in connection with the interior operator. Taking this into consideration, in [9] we presented a new sight of studying fuzzy continuous mappings by using an induced mapping $\phi^{\#}$ where we have investigated several behaviors of fuzzy continuous functions with respect to their interiors rather than their closures. In view of the above-mentioned characterizations of fuzzy closed and fuzzy continuous mappings in connection with the closure operator showing the complete symmetry between these two mappings, we define and study a mapping ϕ from an FTs (V, T_1) to an FTs (W, T_2) satisfying $(\phi(M))^{\circ} < \phi(M^{\circ})$, for every fuzzy set M in V, which we call an invertedly fuzzy open mapping exhibiting the symmetry between fuzzy open and fuzzy continuous mappings in connection with the interior operator. In this paper, we explore different characterizations of this mapping, particularly in Corollary 1, where we demonstrate that an invertedly fuzzy open map $\phi : V \to W$ ensures that any F-set M in V is a fuzzy open set (FO-set, for short) whenever $\phi(M)$ is an FO-set in W. Some representations of invertedly fuzzy open maps are also given in connection with fuzzy closed subsets and the above-mentioned induced mapping $\phi^{\#}$. On the other hand, the above-stated corollary suggests to us the definition of an invertedly fuzzy closed mapping, as any F-set M in V is a fuzzy closed set (FC-set, in short) whenever $\phi(M)$ is an FC-set in W. The interrelation of these mappings with fuzzy continuous mappings is given and the characterization of fuzzy homeomorphism for bijective mappings in connection with the interior operator is also studied. We also identify fibers in F-sets using fuzzy points (F-points) to illustrate some properties of invertedly fuzzy open and invertedly fuzzy closed mappings. In fuzzy topology, the concept of fuzzy saturated sets plays a crucial role in analyzing the behavior of mappings between fuzzy spaces. These sets are defined through preimages of fuzzy subsets, preserving structural properties under mappings. This work investigates conditions under which fuzzy saturated sets retain fuzzy closedness (FC-sets) when mapped via invertedly fuzzy open and onto mappings, thereby extending foundational results in fuzzy topology.

2. Preliminaries

Throughout the paper, (V, τ) stands for FTs, where $V = \{v\}$ symbolizes a space of points. For V, μ_M from V to $[0, 1]$ denotes the membership function for any F-set M in V. A fuzzy set is described by a membership function that assigns to each $v \in V$ a "grade of membership" $\mu_E(x)$ in E. An empty fuzzy set, denoted by ϕ, is defined such that $\mu_\phi(v) = 0$ for all $v \in V$. Similarly, a fuzzy set V is defined such that $\mu_V(v) = 1$ for all $v \in V$. Some basic definitions related to fuzzy functions and fuzzy topology are given in [3] by Chang. We begin by reviewing some essential definitions and key results that will be utilized in the subsequent sections.

Definition 1. *[3] Let M and N be F-sets in a space $V = \{v\}$, with membership functions $\mu_M(v)$ and $\mu_N(v)$, respectively. Then*

(1) $M = N \Leftrightarrow \mu_M(v) = \mu_N(v)$ for all $v \in V$.

(2) $M < N \Leftrightarrow \mu_M(v) \leq \mu_N(v)$ for all $v \in V$.

(3) $P = M \cup N \Leftrightarrow \mu_P(v) = max\{\mu_M(v), \mu_N(v)\}$ for all $v \in V$.

(4) $P = M \cap N \Leftrightarrow \mu_P(v) = min\{\mu_M(v), \mu_N(v)\}$ for all $v \in V$.

(5) $O = M^c \Leftrightarrow \mu_O(v) = 1 - \mu_M(v)$ for all $v \in V$.

Definition 2. *[3] A family τ of F-sets in X is said to be a fuzzy topology if it satisfies the following conditions:*

(a) ϕ, $V \in \tau$.

(b) If $M, N \in T$ then $M \cap N \in \tau$.

(c) If $M_i \in \tau$ for each $i \in I$, then $\cup M_i \in \tau$.

Every member of τ is called an τ-open F-set (FO-set) and the pair (V, τ) is known as fuzzy topological space or fts for short. Also, an F-set is τ-closed (FC-set) if and only if its complement is τ-open.

Definition 3. *[17] Let (V, τ) be a fuzzy topological space and M be any F-set in V. Then the union of all τ-open F-sets (FO-sets) contained in M is called the interior of M, denoted by M^o. Equivalently, M^o is the largest FO-set contained in M, and $(M^o)^o = M^o$.*
On the other hand, the intersection of all τ-closed fuzzy sets (FC-sts) containing M is called the closure of M, denoted by $\overline{M}$. Clearly, $\overline{M}$ is the smallest FC-set containing M, and $\overline{\overline{M}} = \overline{M}$.

Theorem 1. *[17] In any fuzzy topological space (V, τ), $(\overline{M})^c = (M^c)^o$ and so $\overline{M^c} = (M^o)^c$, for any fuzzy subset M of V where M^c and $\overline{M}$ have usual meanings defined in the definitions above.*

The # image of an F-set in [9] was introduced by Kaur and Goyal to define an induced mapping $\phi^\# : Gz(V) \to Gz(W)$ is associated with any mapping $\phi : V \to W$ where V and W are crisp sets and $Gz(V)$ is a family of all fuzzy subsets of V.

Definition 4. *[9] The # image of an F-set M in V with membership function $\mu_M(v)$, written as $\phi^{\#}(M)$, is an F-set in W with a membership function defined as*

$$\mu_{\phi^{\#}(M)}(w) = \begin{cases} \inf_{z\in\phi^{-1}(w)} \mu_M(z) & \text{if } \phi^{-1}(w) \neq \emptyset \\ 1 & \text{if otherwise} \end{cases}$$

for all $w \in W$ where $\phi^{-1}(w) = \{v \mid \phi(v) = w\}$.

Definition 5. *[9] An F-set $M^{\#}$ in V is described as $M^{\#} = \phi^{-1}(\phi^{\#}(M))$ with membership mapping as*

$$\mu_{M^{\#}}(v) = \mu_{\phi^{-1}(\phi^{\#}(M))}(v) = \mu_{\phi^{\#}(M)}(\phi(v)).$$

The following Lemma 1 outlines several properties of the mapping $\phi^{\#}$.

Lemma 1. *[9] Let $\phi : V \to W$ be any mapping, and M, and N be fuzzy subsets of V and U be a fuzzy subset of W. Then:*

(a) $\phi^{\#}(M) < \phi^{\#}(N)$ *if* $M < N$.

(b) $\phi^{-1}(\phi^{\#}(M)) < M$ *i.e.,* $M^{\#} < M$.

(c) $U < \phi^{\#}(\phi^{-1}(U))$ *and equality holds if ϕ is onto.*

(d) $\phi(M^{\#}) = \phi^{\#}(M) \cap \phi(V)$.

(e) $\phi^{\#}(M \cap N) = \phi^{\#}(M) \cap \phi^{\#}(N)$.

(f) $\phi^{\#}(\phi) = (\phi(V))^c$ *and* $\phi^{\#}(V) = W$.

(g) $\phi(M^{\#}) = \phi^{\#}(M) \cap \phi(M)$.

(h) $\phi^{\#}(\phi^{-1}(\phi^{\#}(M))) = \phi^{\#}(M^{\#}) = \phi^{\#}(M)$.

(i) $\phi^{-1}(\phi^{\#}(\phi^{-1}(U))) = \phi^{-1}(U)$.

Lemma 2. *[9] Let $\phi : V \to W$ be any mapping, and M and U be any fuzzy subsets of V and W respectively. Then:*

(a) $\phi^{-1}(U) < M$ *iff* $U < \phi^{\#}(M)$.

(b) $\phi^{\#}(M^c) = (\phi(M))^c$ *and so* $\phi^{\#}(M) = (\phi(M^c))^c$ *and* $\phi(M) = (\phi^{\#}(M^c))^c$.

Definition 6. *[17] A function $\phi : (V, \tau_1) \to (W, \tau_2)$ is said to be fuzzy continuous (or F-continuous) if, for every τ_2-open fuzzy set in W, its preimage under ϕ is a τ_1-open fuzzy set in V.*

Definition 7. *[9] Let $\phi : V \to W$ be any mapping. A fuzzy subset M of V is named a fuzzy saturated subset of V if $M = \phi^{-1}(U)$ for some fuzzy subset U of W i.e., $\mu_M(v) = \mu_{\phi^{-1}(U)}(v)$ for each $v \in V$.*

Theorem 2. *[9] Let $\phi : V \to W$ be any surjective mapping. In this case, a, b, c, and d are equivalent:*

(a) ϕ is a fuzzy continuous mapping.

(b) $M^{\#}$ is an FO-set in V whenever $\phi(M^{\#})$ is an FO-set in W.

(c) For any fuzzy saturated set M in V, M is an FC-set in V whenever $\phi(M)$ is an FC-set in W.

(d) For any fuzzy saturated set M in V, M is an FO-set in V whenever $\phi^{\#}(M)$ is an FO-set in W.

3. Invertedly fuzzy open mappings

We begin with the definition of an invertedly fuzzy open mapping, followed by Example 1 to illustrate the concept.

Definition 8. A mapping $\phi : V \to W$ is called an *invertedly fuzzy open mapping* if, for any fuzzy set M in V, the interior of its image satisfies $(\phi(M))^0 < \phi(M^0)$.

Remark 1. It is clear from Definition 8 that for a discrete FTs V, every $\phi : V \to W$ is an invertedly fuzzy open mapping.

Example 1. Let V and W be sets of points, where $V = W = \{v_1, v_2, v_3\}$. Consider a function $\phi : V \to W$ described by $\phi(v_1) = v_1, \phi(v_2) = v_3, \phi(v_3) = v_2$ and the fuzzy subset $M = \{(v_1, 0), (v_2, 0), (v_3, 1)\}$ & $N = \{(v_1, 0), (v_2, 1), (v_3, 0)\}$. Also, consider $T_1 = \{0, 1, M, N, M \vee N\}$ and $T_2 = \{0, 1, M\}$ be fuzzy topologies on V and W, respectively, then $\phi : (V, T_1) \to (W, T_2)$ is an invertedly fuzzy open mapping.

Our first result below gives a different representation of an invertedly fuzzy open mapping.

Theorem 3. A mapping $\phi : V \to W$ is an invertedly fuzzy open mapping iff for all fuzzy subsetS M of V, $\phi^\#(\overline{M}) < \overline{\phi^\#(M)}$.

Proof. Invertedly fuzzy openness of ϕ is an equivalent to $(\phi(M))^0 < \phi(M^0)$ for any F-set M in V, where $(\phi(M))^0 < \phi(M^0)$ is an equivalent to $[(\phi^\#(M^c))^c]^0 < [\phi^\#(M^0)^c]^c$ by Lemma 2. Now, $[(\phi^\#(M^c))^c]^0 = [\overline{\phi^\#(M^c)}]^0$ and $[\phi^\#(M^0)^c]^c = [\phi^\#(\overline{M^c})]^0$ by Theorem 1. Therefore, we get $[\overline{\phi^\#(M^c)}]^c < [\phi^\#(\overline{M^c})]^c$, which implies $\phi^\#(\overline{M^c}) < \overline{\phi^\#(M^c)}$, for any arbitrary F-set M in X. Hence, it is proved that ϕ is invertedly fuzzy open mapping if and only if for all fuzzy subsets M of V $\phi^\#(\overline{M}) < \overline{\phi^\#(M)}$. □

We will use the notion of F-points given in [19] to prove the next conclusions.

Definition 9. *[19]* A F-point in V, denoted by q_{v_0}, is an F-set with a membership mapping

$$\mu_q(v) = \begin{cases} w, & \text{for } v = v_0 \\ 0, & \text{otherwise} \end{cases}$$

where $w \in (0, 1)$ and q will have support v_0 with value w.

Remark 2. *(1) The F-point q_{v_0} is named in an F-set M, denoted by $q_{v_0} \in M$, iff $\mu_{q_{v_0}}(v) < \mu_M(v)$ for every $v \in V$.*

(2) Two F-points are equal if their support and values are equal. If either support or values are unequal, then two F-points will be unequal.

Next, we define the definition of fibers in F-sets to prove some characterizations of invertedly fuzzy open mappings and invertedly fuzzy closed mappings.

Definition 10. Let $\phi : V \to W$ be any mapping and q_{w_0} be an F-point with support w_0 and value w. Define $\phi^{-1}(q_{w_0})$, a fiber, which is an F-set in X but not necessarily an F-point, defined by a membership function

$$\mu_{\phi^{-1}(q_{w_0})}(v) = \begin{cases} w, & \text{for every } v \in \phi^{-1}(w_0) \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3. An F-point p_{v_0} belongs to $\phi^{-1}(q_{w_0})$ if F-point p_{v_0} will be defined with a membership function

$$\mu_{p_{v_0}}(v) = \begin{cases} w, & \text{if } v = v_0 \\ 0, & \text{otherwise} \end{cases}$$

where $v_0 \in \phi^{-1}(w_0)$.

The following Theorem 4 yields a property of inverted fuzzy open mappings in relation to the fibers defined above.

Theorem 4. A mapping $\phi : V \to W$ is an invertedly fuzzy open mapping iff an F-set T containing exactly one F-point from every fiber $\phi^{-1}(q_{w_0}), q_{w_0} \in O$ is an FO-set in V, where O is an FO-set in W, contained in $f(V)$.

Proof. Let ϕ be an invertedly fuzzy open mapping and O be any FO-set in W contained in $f(V)$. Let T be an F-set comprising exactly one F-point from every fiber F-set $\phi^{-1}(q_{w_0}), q_{w_0} \in O$. We need to prove that T is an FO-set in V. Now from the definition of fiber, we can now write $\phi(T) = O$. Assume T is not fuzzy open, then $T \nleq T^0$, which implies $\phi(T^0) < O$ where $O \nleq \phi(T^0)$. But since ϕ is an invertedly fuzzy open mapping then $(\phi(T))^0 < \phi(T^0) < O$ where $\phi(T^0) \neq O$, which gives us that $(\phi(T))^0 \neq O$ which is a contradiction.

Since O is an FO-set in W. T is an FO-set in V.

Conversely, suppose M be any F-set in V, we have $(\phi(M))^0 < \phi(M)$, then there exists an F-set $N < M$ such that $\phi(N) = (\phi(M))^0$, which implies $\phi(N)$ is an FO-set in W. To show ϕ is an invertedly fuzzy open mapping, we first prove that N is an FO-set in V.

Now, consider $T_{q_{w_0}} = \phi^{-1}(q_{w_0}) \wedge N$, where $q_{w_0} \in \phi(N)$. Here, $T_{q_{w_0}}$ comprises at least one F-point from fiber $\phi^{-1}(q_{w_0})$ and so, an FO-set in V by assumption. Now, since $N = \vee_{q_{w_0} \in \phi(N)} T_{q_{w_0}}$, then N is an FO-set in V. Therefore, $N < M^0$, which implies $\phi(N) < \phi(M^0)$. Hence, we obtain $(\phi(M))^0 < \phi(M^0)$, that is, ϕ is an invertedly fuzzy open mapping. □

From the proof of the sufficient part of the above Theorem 4, we obtain the following.

Corollary 1. A mapping $\phi : V \to W$ is an invertedly fuzzy open mapping if and only if any F-set M in V is an FO-set whenever $\phi(M)$ is an FO-set in W.

Remark 4. The representation of an invertedly fuzzy open mapping in the above Corollary 1 can also be stated as: An image of every non-FO-set in V under ϕ is a non-fuzzy open mapping in W.

With the following Theorem 5, we show that invertedly fuzzy open mappings can also be characterized in connection with fuzzy closed subsets and induced mapping $\phi^{\#}$.

Theorem 5. A mapping $\phi : V \to W$ is an invertedly fuzzy open mapping iff any F-set N is an FC-set in V whenever $\phi^{\#}(N)$ is an FC-set in W.

Proof. Let N be any F-set in V, where $\phi^{\#}(N)$ is an FC-set in W. Then by Lemma 2, $\phi^{\#}(N) = (\phi(N^c))^c$ and therefore, $(\phi(N^c))^c$ is an FC-set in W, and so, $\phi(N^c)$ is an FO-set in W. Since ϕ is an invertedly fuzzy open mapping, then N^c is an FO-set in V by the above Corollary 1. Hence, N is an FC-set in V.

The proof of the converse part immediately follows from Lemma 2 and the above Corollary 1. □

Remark 5. It can be easily shown that every fuzzy continuous mapping ϕ is an invertedly fuzzy open if ϕ is one-one.

The following Theorem 6 gives a sufficient condition for an invertedly fuzzy open mapping to be fuzzy continuous.

Theorem 6. Let $\phi : V \to W$ be an invertedly fuzzy open mapping. Then ϕ is a fuzzy continuous map if $\phi(V)$ is a FO-set in W. Particularly, ϕ is a fuzzy continuous mapping whenever $(\phi(M))^0 = \phi(M^0)$, for all F-sets M in V.

Proof. Let N be an FO-set in W. The $\phi(\phi^{-1}(N)) = N \wedge \phi(V)$ is an FO-set in V, since $\phi(V)$ is an FO-set in W. Therefore, by Corollary 1, $\phi^{-1}(N)$ is an FO-set in V since ϕ is an invertedly fuzzy open mapping. Hence, ϕ is a fuzzy continuous mapping. □

Note 1. (a) Each fuzzy open mapping and invertedly fuzzy open mapping is a fuzzy continuous mapping.

(b) A mapping $\phi : V \to W$ is a fuzzy continuous iff it is an invertedly fuzzy open mapping, assuming ϕ is bijective.

In the above Theorem 6, we have proved that every invertedly fuzzy open onto mappings is a fuzzy continuous. Therefore, by Theorem 2, we obtain the following result for invertedly fuzzy open mappings in terms of fuzzy saturated sets.

Theorem 7. *Let $\phi : V \to W$ be an invertedly fuzzy open and onto mapping. Then every fuzzy saturated set N of V is an FC-set whenever $\phi(N)$ is an FC-set in W.*

In 1968, Chang [3] discussed fuzzy homeomorphism, a fuzzy continuous bijective mapping from FTs V to FTs W such that the inverse mapping is also a fuzzy continuous. In this study, we find below another equivalent condition for a mapping to be a fuzzy homeomorphism with the help of the Theorem 6.

Theorem 8. *A one-one mapping ϕ from a FTs V onto FTs W is a fuzzy homeomorphism iff $\phi(M^0) = (\phi(M))^0$ for every F-set M of V.*

4. Invertedly fuzzy closed mappings

Corollary 1 above put forward the following definition of invertedly fuzzy closed mappings.

Definition 11. A mapping $\phi : V \to W$ is called invertedly fuzzy closed if, for any F-set M in V, it is an FC-set whenever its image $\phi(M)$ is an FC-set in W.

Remark 6. It is clear from Definition 11 that:

(a) Every $\phi : V \to W$ is an invertedly fuzzy closed mapping if V is a discrete FTs.

(b) Every fuzzy continuous bijective mapping is an invertedly fuzzy closed mapping.

The following Theorem 9 shows a property of an inverted fuzzy closed mapping.

Theorem 9. *Let $\phi : V \to W$ be a mapping. In this case, statements (i), (ii), and (iii) are equivalent:*

(i) ϕ *is an invertedly fuzzy closed mapping.*

(ii) Every F-set M *of* V *is an FO-set in* V *whenever* $\phi^{\#}(M)$ *is an FO-set in* W.

(iii) An F-set S *containing exactly one F-point from fiber* $\phi^{-1}(q_{w_0}), q_{w_0} \in N$*, is an FC-set in* V*, where* N *is an FC-set in* V *contained in* $\phi(V)$.

Proof. (i) $\implies$ (ii). Let ϕ be an invertedly fuzzy open mapping and M be any F-set in V such that $\phi^{\#}(M)$ is an FO-set in W. By the equality of $\phi^{\#}(M) = (\phi(M^c))^c$ in Lemma 2 $\phi(M^c)$ is an FC-set in W, which implies M^c is an FC-set in V, and so, M is an FO-set in V.
(ii) $\implies$ (iii). Let N be an FC-set contained in $\phi(V)$ and S is an F-set containing exactly one F-point from fiber $\phi^{-1}(q_{w_0}), q_{w_0} \in N$. Clearly, $\phi(S) = N$. Therefore, $(\phi^{\#}(S^c))^c = \phi(S) = N$, which is an FC-set and so, $\phi^{\#}(S^c)$ is an FO-set in W. Therefore, by (ii), S^c is an FO-set and so, S is an FC-set in V. Hence (iii) holds.
(iii) $\implies$ (i). Let $\phi(M)$ be an FC-set in W. Consider, $S_{q_{w_0}} = \phi^{-1}(q_{w_0}) \wedge M$, where $q_{w_0} \in \phi(M)$. Then $M = \vee S_{q_{w_0}}, q_{w_0} \in \phi(M)$ is an FC-set in V by (iii).
Therefore, ϕ is an invertedly fuzzy closed mapping. □

Similar to Theorem 7, the result of invertedly fuzzy closed mappings in terms of fuzzy saturated sets is given below as a corollary of the above Theorem 9.

Corollary 2. Let $\phi : V \to W$ be an invertedly fuzzy closed and onto mapping. Then every fuzzy saturated set S is an FO-set in V whenever $\phi(S)$ is an FO-set in W.

The following Example 2 shows that a fuzzy continuous mapping need not be an invertedly fuzzy closed mapping.

Example 2. Let $V = \{a, b, c\}$ and $W = \{p, q, r, s\}$ be spaces of points. Consider a mapping $\phi : V \to W$ is defined by $\phi(a) = q, \phi(b) = r, \phi(c) = q$ and the fuzzy subsets $M = \{(a, 0), (b, 0), (c, 1)\}, N = \{(a, 0), (b, 1), (c, 1)\}$ of X and $P = \{(p, 1), (q, 0), (r, 1), (s, 1)\}$ of Y. Also, take $T_1 = \{0, 1, M, N, M \vee N\}$ and $T_2 = \{0, 1, p\}$ be fuzzy topologies on V and W respectively. Then, the function from FTs (V, T_1) to FTs (W, T_2) is a fuzzy continuous mapping but not an invertedly fuzzy closed mapping since image $\phi(S)$ of F-set $S = \{(a, 0), (b, 0), (c, 1)\}$ is $\phi(S) = \{(p, 0), (q, 1), (r, 0), (s, 0)\}$, which is FC-set in W, but S is not an FC-set in V.

Note 2. Mapping defined in the above example is invertedly fuzzy open also since the image of any F-set in V is not FO-set in W, and hence, ϕ is vacuously invertedly fuzzy open mapping.

5. Conclusions

Fuzzy set topology is a well-liked field for researchers and can be applied in various directions. In the theory of fuzzy topology, fuzzy continuity is a precise observation of the insightful concept of a mapping that differs with unexpected breaks. It is known that fuzzy continuous, fuzzy open, and fuzzy closed mappings can be characterized in many ways and there is a relationship between fuzzy closed and fuzzy continuous maps in terms of the closure operator as mentioned earlier. On the other hand, fuzzy open mappings can be characterized in terms of the interior operator. To find the relationship between fuzzy open and fuzzy continuous mappings in connection with the interior operator, we investigated invertedly fuzzy open and invertedly fuzzy closed mappings and studied their characterizations. Through our exploration, we've discussed symmetries between these concepts and introduced the notion of invertedly fuzzy open and invertedly fuzzy closed mappings, which showcase a new perspective on how maps behave with respect to the interior operator. Also, we have investigated a representation of fuzzy homeomorphism in connection with the interior operator. This work not only deepens our understanding of fuzzy topological spaces but also opens doors to practical applications. Looking ahead, the insights gained from this study offer a foundation for further research and development. Future studies could explore more complex interactions between induced mappings and operators.

Author contributions

Sandeep Kaur and Alkan Özkan: Conceptualization, methodology, formal analysis, investigation, resources, writing-original draft, writing-review and editing, validation; Faizah D. Alanazi: Formal analysis, investigation, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We thank all the reviewers for their valuable suggestions, which helped improve the quality of this paper. The authors extend their appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA, for funding this research work through the project number "NBU-FFR-2025-1102-01" and by scientific research, the TÜBİTAK 2221-Guest or Sabbatical Scientist Support Program under Grant No: 1059B212200415.

Conflict of interest

The authors declare that they have no conflict of interest.

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