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Research article

The Weighted L^p estimates for the fractional Hardy operator and a class of integral operators on the Heisenberg group

Tianyang He[1], **Zhiwen Liu**[2] **and Ting Yu**[1,*]

[1] Research Center for Mathematics and Interdisciplinary Sciences, Frontiers Science Center for Nonlinear Expectations (Ministry of Education), Shandong University, Qingdao, 266237, China
[2] School of Science, Shandong Jianzhu University, Jinan, 250100, China

* **Correspondence:** Email: tingy@sdu.edu.cn.

Abstract: In the setting of a Heisenberg group, we first studied the sharp weak estimate for the n-dimensional fractional Hardy operator from L^p to $L^{q,\infty}$. Next, we studied the sharp bounds for the m-linear n-dimensional integral operator with a kernel on weighted Lebesgue spaces. As an application, the sharp bounds for Hardy, Hardy-Littlewood-Pólya, and Hilbert operators on weighted Lebesgue spaces were obtained. Finally, according to the previous steps, we also found the estimate for the Hausdorff operator on weighted L^p spaces.

Keywords: fractional Hardy operator; m-linear n-dimensional integral operator with a kernel; weighted Lebesgue space; Hardy-Littlewood-Pólya operator; Hilbert operator; sharp bound for the integral operator
Mathematics Subject Classification: Primary 42B25; Secondary 42B20, 47H60, 47B47

1. Introduction

It is well-known that averaging operators play an important role in harmonic analysis. It is often desirable to obtain sharp norm estimates for them in different function spaces. Our starting point is the Hardy operator and its duality form:

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \quad H^*f(x) = \int_x^\infty \frac{f(t)}{t}\mathrm{d}t,$$

where $x > 0$. Hardy [11] established the well-known Hardy integral inequalities

$$\int_0^\infty |Hf(x)|^p\mathrm{d}x \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p\mathrm{d}x, \quad p > 1,$$

and

$$\int_0^\infty |H^* f(x)|^{p'} \mathrm{d}x \leqslant (\frac{p}{p-1})^{p'} \int_0^\infty |f(x)|^{p'} \mathrm{d}x, \quad p > 1,$$

where $p' = p/(p-1)$. He proved that the constant $p/(p-1)$ is sharp. The corresponding higher-dimensional Hardy operator was introduced by Faris [1] in his study of quantum mechanics. Christ and Grafakos [4] gave the following equivalent definition of n-dimensional Hardy operators:

$$\mathscr{H}_n f(x) = \frac{1}{|x|^n} \int_{|y|<|x|} f(y)\mathrm{d}y,$$

where $x \in \mathbb{R}^n \setminus \{\theta\}$, and θ is the origin in $\mathbb{R}^n$. They showed that

$$\| \mathscr{H}_n \|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \frac{p}{p-1}\frac{\omega_n}{n}, \quad 1 < p < \infty.$$

Here ω_n is the superficial area of the unit ball in $\mathbb{R}^n$. The Lebesgue spaces with power weights are another kind of function space to consider the sharp estimates of the Hardy operator. The method in [4] is invalid in this case. Fu et al. [7], by the method of rotation, established the following estimate:

$$\| \mathscr{H}_n \|_{L^p_{|x|^\beta}(\mathbb{R}^n)\to L^p_{|x|^\beta}(\mathbb{R}^n)} = \frac{\omega_n}{\frac{n}{p'} - \frac{\beta}{p}}.$$

For more information about the Hardy operator, we refer to the reader to [14].

Meanwhile, the fractional Hardy operator is also very interesting since it is a useful tool to study the embedding properties of function spaces. Mizuta et al. [15] showed that the optimal bound of the fractional Hardy operator implies the sharp embedding properties of function spaces. There is much literature on function spaces. Let f be a locally integrable function on $\mathbb{R}^n$. Then the n-dimensional fractional Hardy operator and its duality form are

$$H_\alpha f(x) = \frac{1}{|x|^{n-\alpha}} \int_{|y|<|x|} f(y)dy, \quad H^*_\alpha f(x) = \int_{|y|>|x|} \frac{1}{|y|^{n-\alpha}} f(y)dy,$$

where $0 < \alpha < n$, $x \in \mathbb{R}^n \setminus \{0\}$. If $\alpha = 0$, the fractional Hardy operator is the classic Hardy operator. There is much literature on the boundness of these operators [9,13,21,22]. Among them, Lu et al. [21] obtained the following estimates. Suppose

$$0 < \alpha < n, \quad 1 < p \leqslant \frac{n}{\alpha}, \quad \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Then

$$\|H_\alpha f\|_{L^q(\mathbb{R}^n)} \leqslant C \|f\|_{L^p(\mathbb{R}^n)},$$

where

$$\left(\frac{p}{q}\right)^{1/q} \left(\frac{p}{p-1}\right)^{1/q} \left(\frac{q}{q-1}\right)^{1-1/q} \left(1 - \frac{p}{q}\right)^{1/p-1/q} \left(\frac{\omega_n}{n}\right)^{1-\alpha/n} \leqslant C \leqslant \left(\frac{p}{p-1}\right)^{p/q} \left(\frac{\omega_n}{n}\right)^{1-\alpha/n}.$$

If $p = 1$, then

$$\|H_\alpha\|_{L^1(\mathbb{R}^n)\to L^{n/(n-\alpha),\infty}(\mathbb{R}^n)} = \left(\frac{\omega_n}{n}\right)^{1-\frac{\alpha}{n}}.$$

The optimal $L^p \to L^q$ estimate was later obtained in [22]:

$$\|H_\alpha\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \left(\frac{p\prime}{q}\right)^{1/q}\left(\frac{n}{q\alpha}\cdot B\left(\frac{n}{q\alpha},\frac{n}{q'\alpha}\right)\right)^{-\alpha/n}\left(\frac{\omega_n}{n}\right)^{1-\frac{\alpha}{n}},$$

where

$$p' = \frac{p}{p-1}, \quad q' = \frac{q}{q-1},$$

and $B(\cdot,\cdot)$ is the beta function defined by

$$B(z,\omega) = \int_0^1 t^{z-1}(1-t)^{\omega-1}\,dt,$$

where z and ω are complex numbers with positive real parts. Comparing with the complicated bounds in the power weighted spaces, the sharp weak bounds for H_α and H_α^* seem easier to understand. Gao and Zhao [9] set up

$$\left\|H_\alpha^*\right\|_{L^1(\mathbb{R}^n)\to L^{n/(n-\alpha),\infty}(\mathbb{R}^n)} = \left(\frac{\omega_n}{n}\right)^{1-\frac{\alpha}{n}}, \quad \| H_\alpha^* \|_{L^p(\mathbb{R}^n)\to L^{q,\infty}(\mathbb{R}^n)} = \left(\frac{\omega_n}{n}\right)^{\frac{1}{q}+\frac{1}{p'}}(\frac{q}{p'})^{1/p'}.$$

It is then a nature problem to obtain the operator norms of H_α and its dual operator H_α^* in corresponding power weighted spaces. For the latter, Gao et al. [8] set up

$$\left\|H_\alpha^*\right\|_{L^1_{|x|^\rho}(\mathbb{R}^n)\to L^{(n+\beta)/(n-\alpha+\rho),\infty}_{|x|^\beta}(\mathbb{R}^n)} = \left(\frac{\omega_n}{n+\beta}\right)^{(n-\alpha+\rho)/(n+\beta)},$$

and

$$\left\|H_\alpha^*\right\|_{L^p_{|x|^\rho}}(\mathbb{R}^n) \to L^{q,\infty}_{|x|^\beta}(\mathbb{R}^n) = \left(\frac{\omega_n}{n+\beta}\right)^{\frac{1}{q}+\frac{1}{p'}}\left(\frac{q}{p'}\right)^{1/p'}.$$

Then Yu et al. [20] set up

$$\| H_\alpha \|_{L^p_{|x|^\rho}(\mathbb{R}^n)\to L^{q,\infty}_{|x|^\beta}(\mathbb{R}^n)} = (\frac{\omega_n}{n+\beta})^{1/q}\left(\frac{\omega_n}{n-\frac{\rho}{p-1}}\right)^{1/p'},$$

and

$$\| H_\alpha \|_{L^1(\mathbb{R}^n)\to L^{(n+\beta)/(n-\alpha),\infty}_{|x|^\beta}(\mathbb{R}^n)} = (\frac{\omega_n}{n+\beta})^{(n-\alpha)/(n+\beta)}.$$

Inspired by them, we will study the sharp weak bound for the fractional Hardy operator on the Heisenberg group, which plays an important role in several branches of mathematics. Now, allow us to introduce some basic knowledge about the Heisenberg group which will be used later.

The Heisenberg group is a very typical non-commutative group, and research on up-modulation and analytic problems is an extension of Euclidean space upharmonic and analytical problems, which is an important part of non-commutative harmonic analysis [6, 18]. Harmonic analysis on the Heisenberg group has been drawing more and more attention, see [5, 23].

Let us introduce some basic knowledge about the Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is a non-commutative nilpotent Lie group, with the underlying manifold $\mathbb{R}^{2n}\times\mathbb{R}$ and the group law.

Let

$$x = (x_1, \dots, x_{2n}, x_{2n+1}), \quad y = (y_1, \dots, y_{2n}, y_{2n+1}),$$

and then

$$x \times y = (x_1 + y_1, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n}(y_j x_{n+j} - x_j y_{n+j}).$$

The Heisenberg group $\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(x_1, x_2, \dots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, \dots, rx_{2n}, r^2 x_{2n+1}), \quad r > 0.$$

The Haar measure on $\mathbb{H}^n$ coincides with the usual Lebesgue measure on $\mathbb{R}^{2n+1}$. We denote any measurable set $E \subset \mathbb{H}^n$ by $|E|$, and then

$$|\delta_r(E)| = r^Q |E|, \quad d(\delta_r x) = r^Q dx,$$

where $Q = 2n + 2$ is called the homogeneous dimension of $\mathbb{H}^n$.

The Heisenberg distance derived from the norm

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right]^{1/4},$$

where $x = (x_1, x_2, \dots, x_{2n}, x_{2n+1})$, is given by

$$d(p, q) = d(q^{-1}p, 0) = |q^{-1}p|_h.$$

This distance d is left-invariant, meaning that $d(p, q)$ remains constant when both p and q are left shifted by some fixed vector on $\mathbb{H}^n$. Furthermore, d satisfies the trigonometric inequality defined by [12]:

$$d(p, q) \le d(p, x) + d(x, q), \quad p, x, q \in \mathbb{H}^n.$$

For $r > 0$ and $x \in \mathbb{H}^n$, the ball and sphere with center x and radius r on $\mathbb{H}^n$ are given by

$$B(x, r) = \{y \in \mathbb{H}^n : d(x, y) < r\},$$

and

$$S(x, r) = \{y \in \mathbb{H}^n : d(x, y) = r\}.$$

Then we obtain

$$|B(x, r)| = |B(0, r)| = \Omega_Q r^Q,$$

where

$$\Omega_Q = \frac{2\pi^{n+\frac{1}{2}}\Gamma(n/2)}{(n+1)\Gamma(n)\Gamma((n+1)/2)}$$

represents the volume of the unit sphere $B(0, 1)$ on $\mathbb{H}^n$, and $\omega_Q = Q\Omega_Q$. The reader is referred to [16,18] for more details.

Except for the fractional Hardy operator, we also study the sharp bounds for some m-linear n-dimensional integral operators on the Heisenberg group. In 2017, Batbold and Sawano [3] studied one-dimensional m-linear Hilbert-type operators that include the Hardy-Littlewood-Pólya operator on weighted Morrey spaces, and they obtained the sharp bounds. He et al. [10] extended the results in [3] and obtained the sharp bound for the generalized Hardy-Littlewood-Pólya operator on weighted central and noncentral homogenous Morrey spaces. He set up

$$\| T(f_1, \cdots, f_m) \|_{L^{q,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^\gamma)} \leq C_m \prod_{j=1}^{m} \| f_j \|_{L^{q_j,\lambda}(\mathbb{R}^n, |x|^\alpha, |x|^{\frac{q_j \gamma_j}{q}})},$$

where

$$C_m = \int_{\mathbb{R}^{nm}} K(y_1, \cdots, y_m) \prod_{i=1}^{m} |y_i|^{-d(\lambda_i, q_i, \alpha, \frac{q_i \gamma_i}{q})} dy_1 \cdots dy_m < \infty.$$

In 2011, Wu and Fu [19] got the sharp estimate of the m-linear p-adic Hardy operator on Lebesgue spaces with power weights. Zhang et al. [25] obtained the sharp estimate for the m-linear n-dimensional Hausdorff operator on the weighted Morrey space.

Inspired by the above, on the Heisenberg group, we first study the sharp weak estimate for the n-dimensional fractional Hardy operator from L^p to $L^{q,\infty}$. Second, we study a more general operator which includes Hardy, Hardy-Littlewood-Pólya, and Hilbert operators as a special case and consider their operator norm on weighted Lebesgue space. Finally, we also find the sharp bound for the Hausdorff operator on Lebesgue space, which generalizes the previous results.

To get the main conclusion, it is necessary to introduce some fundamental knowledge and definitions. In the setting of the Heisenberg group, these operators and spaces are the fractional Hardy operator, m-linear n-dimensional Hardy operator, m-linear n-dimensional Hardy-Littlewood-Pólya operator, m-linear n-dimensional Hilbert operator, m-linear n-dimensional Hausdorff operator, and weighted L^p and $L^{q,\infty}$.

Definition 1.1. *Let f be nonnegative locally integrable functions on $\mathbb{H}^n$ and $Q = 2n + 1$, $0 < \alpha < Q$. The n-dimensional fractional Hardy operator is defined by*

$$\mathcal{H}_\alpha f(x) = \frac{1}{|x|_h^{Q-\alpha}} \int_{|y|_h < |x|_h} f(y)\, dy, \tag{1.1}$$

where $x \in \mathbb{H}^n \backslash \{0\}$.

Definition 1.2. *Let m be a positive integer and $f_1, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{H}^n$. The m-linear n-dimensional Hardy operator is defined by*

$$\mathcal{H}_1^h(f_1, ..., f_m)(x) = \frac{1}{|x|_h^{mQ}} \int_{|(y_1, ..., y_m)|_h \leqslant |x|_h} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \tag{1.2}$$

where $x \in \mathbb{H}^n \backslash \{0\}$.

Definition 1.3. *Let m be a positive integer and $f_1, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{H}^n$. The m-linear n-dimensional Hardy-Littlewood-Pólya operator is defined by*

$$\mathcal{H}_2^h(f_1, ..., f_m)(x) = \int_{\mathbb{H}^n} \cdots \int_{\mathbb{H}^n} \frac{f_1(y_1) \cdots f_m(y_m)}{[\max(|x|_h^Q, |y_1|_h^Q, ..., |y_m|_h^Q)]^m} dy_1 \cdots dy_m, \tag{1.3}$$

where $x \in \mathbb{H}^n \backslash \{0\}$.

Definition 1.4. *Let m be a positive integer and $f_1, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{H}^n$. The m-linear n-dimensional Hilbert operator is defined by*

$$\mathcal{H}_3^h(f_1, ..., f_m)(x) = \int_{\mathbb{H}^n} \cdots \int_{\mathbb{H}^n} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x|_h^Q + |y_1|_h^Q + \cdots + |y_m|_h^Q)^m} dy_1 \cdots dy_m, \tag{1.4}$$

where $x \in \mathbb{H}^n \setminus \{0\}$.

Definition 1.5. *Let m be a positive integer, $f_1, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{H}^n$, and Φ be a nonnegative function on the Heisenberg group. The m-linear n-dimensional Hausdorff operator is defined by*

$$\mathcal{H}_\Phi^h(f_1, ..., f_m)(x) = \int_{\mathbb{H}^n} \cdots \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y_1|_h^{-1}} x, ..., \delta_{|y_m|_h^{-1}} x)}{|y_1|_h^Q \cdots |y_m|_h^Q} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \tag{1.5}$$

where $x \in \mathbb{H}^n \setminus \{0\}$.

Definition 1.6. *Let $1 \leqslant p < \infty$. The Lebesgue space on the Heisenberg $L^p(\mathbb{H}^n)$ is defined by*

$$L^p(\mathbb{H}^n) = \{f \in L_{loc}^p : \|f\|_{L^p(\mathbb{H}^n)} < \infty\},$$

where

$$\|f\|_{L^p(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |f(x)|^p \, dx \right)^{1/p}. \tag{1.6}$$

Definition 1.7. *Let $\omega : \mathbb{H}^n \to (0, \infty)$ be a positive measurable function, $1 \leqslant p < \infty$. The weighted Lebesgue space on the Heisenberg $L^p(\mathbb{H}^n, \omega)$ is defined by*

$$L^p(\mathbb{H}^n, \omega) = \{f \in L_{loc}^p : \|f\|_{L^p(\mathbb{H}^n, \omega)} < \infty\},$$

where

$$\|f\|_{L^p(\mathbb{H}^n, \omega)} = \left(\int_{\mathbb{H}^n} |f(x)|^p \, \omega(x) \, dx \right)^{1/p}. \tag{1.7}$$

Definition 1.8. *Let $\omega : \mathbb{H}^n \to (0, \infty)$ be a positive measurable function, $1 \leqslant p < \infty$. The weighted weak Lebesgue space on the Heisenberg $L^p(\mathbb{H}^n, \omega)$ is defined by*

$$L^{q,\infty}(\mathbb{H}^n, \omega) = \left\{ f \in L_{loc}^p : \|f\|_{L^{q,\infty}(\mathbb{H}^n, \omega)} < \infty \right\},$$

where

$$\|f\|_{L^{q,\infty}(\mathbb{H}^n, \omega)} = \sup_{\lambda > 0} \lambda \left(\int_{\mathbb{H}^n} \chi_{\{x: f(x) > \lambda\}}(x) \, \omega(x) \, dx \right)^{1/q}. \tag{1.8}$$

Next, we will provide the main results of this article.

2. Sharp estimate for the fractional Hardy operator

In this section, we will study the weighted L^p estimate for the fractional Hardy operator on the Heisenberg group. For the n-dimensional fractional Hardy operator, our results have a restricted condition: $\beta = 0$ when $p = 1$ and $\beta > 0$ when $p > 1$. Removing this restrictive condition requires a more complicated argument, and it will be presented in a future paper.

Theorem 2.1. *Let* $1 < p < \infty$, $1 \leqslant q < \infty$, $\beta < Q(p-1)$, $Q + \gamma > 0$, *and* $0 \leqslant \alpha < \frac{\beta}{p-1}$. *If*

$$\frac{\gamma + Q}{q} + \alpha = \frac{\beta + Q}{p},$$

then

$$\|\mathcal{H}_\alpha\|_{L^p(\mathbb{H}^n,|x|_h^\beta)\to L^{q,\infty}(\mathbb{H}^n,|x|_h^\gamma)} = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}}\left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}}. \tag{2.1}$$

Theorem 2.2. *Let* $Q + \beta > 0$ *and* $0 < \alpha < Q$. *Then*

$$\|\mathcal{H}_\alpha\|_{L^1(\mathbb{H}^n)\to L^{(Q+\beta)/(Q-\alpha),\infty}(\mathbb{H}^n,|x|_h^\gamma)} = \left(\frac{\omega_Q}{Q+\beta}\right)^{(Q-\alpha)/(Q+\beta)}. \tag{2.2}$$

Proof of Theorem 2.1. Noticing $Q - \frac{\beta}{p-1} > Q - \frac{Q(p-1)}{p-1} = 0$, by Hölder's inequality, we have

$$\begin{aligned}
|\mathcal{H}_\alpha f(x)| &= \left|\frac{1}{|x|_h^{Q-\alpha}}\int_{|y|_h<|x|_h} f(y)\,dy\right| = \left|\frac{1}{|x|_h^{Q-\alpha}}\int_{|y|_h<|x|_h} |y|_h^{-\frac{\beta}{p}} f(y)\,|y|_h^{\frac{\beta}{p}}dy\right| \\
&\leqslant \frac{1}{|x|_h^{Q-\alpha}}\left(\int_{|y|_h<|x|_h}|y|_h^{-\frac{\beta p'}{p}}dy\right)^{\frac{1}{p'}}\left(\int_{|y|_h<|x|_h}|f(y)|^p|y|_h^\beta dy\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{|x|_h^{Q-\alpha}}\left(\omega_Q\int_0^{|x|_h} r^{-\frac{\beta}{p}\times\frac{p}{p-1}+Q-1}dr\right)^{\frac{1}{p'}}\left(\int_{\mathbb{H}^n}|f(y)|^p|y|_h^\beta dy\right)^{\frac{1}{p}} \\
&= |x|_h^{\alpha-Q}\times\left(\omega_Q\times\frac{|x|_h^{Q-\frac{\beta}{p-1}}}{Q-\frac{\beta}{p-1}}\right)^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{H}^n,|x|_h^\beta)} \\
&= \left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{H}^n,|x|_h^\beta)}|x|_h^{-\frac{Q}{p}-\frac{\beta}{p}+\alpha} = C_{p,Q,\beta,f}|x|_h^{-\frac{Q}{p}-\frac{\beta}{p}+\alpha},
\end{aligned}$$

where

$$C_{p,Q,\beta,f} = \left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}}\|f\|_{L^p(\mathbb{H}^n,|x|_h^\beta)}.$$

Noticing $|\mathcal{H}_\alpha f(x)| \leqslant C_{p,Q,\beta,f}|x|_h^{-\frac{Q}{p}-\frac{\beta}{p}+\alpha}$, we have $\{x : |\mathcal{H}_\alpha f(x)| > \lambda\} \subset \{x : C_{p,Q,\beta,f}|x|_h^{-\frac{Q}{p}-\frac{\beta}{p}+\alpha} > \lambda\}$. Since

$$Q+\gamma > 0 \ \text{ and } \ \frac{\gamma+Q}{q} + \alpha = \frac{\beta+Q}{p},$$

we have

$$
\begin{aligned}
\|\mathcal{H}_\alpha f\|_{L^{q,\infty}(\mathbb{H}^n,|x|_h^{\gamma})} &= \sup_{\lambda>0} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha f(x)|>\lambda\}}(x)\,|x|_h^{\gamma}\,dx\right)^{\frac{1}{q}} \\
&\leqslant \sup_{\lambda>0} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:C_{p,Q,\beta,f}|x|_h^{-\frac{Q}{p}-\frac{\beta}{p}+\alpha}>\lambda\}}(x)\,|x|_h^{\gamma}\,dx\right)^{\frac{1}{q}} \\
&= \sup_{\lambda>0} \lambda\left(\int_{B(0,(\frac{C_{p,Q,\beta,f}}{\lambda})^{\frac{q}{Q+\gamma}})} |x|_h^{\gamma}\,dx\right)^{\frac{1}{q}} = \sup_{\lambda>0} \lambda\left(\omega_Q \int_0^{(\frac{C_{p,Q,\beta,f}}{\lambda})^{\frac{q}{Q+r}}} r^{Q-1+\gamma}dr\right)^{\frac{1}{q}} \\
&= \sup_{\lambda>0} \lambda\left(\omega_Q \times \frac{(C_{p,Q,\beta,f}/\lambda)^q}{Q+\gamma}\right)^{\frac{1}{q}} = \sup_{\lambda>0} C_{p,Q,\beta,f} \times \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \\
&= \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \times \left(\frac{\omega_Q\,(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{H}^n,|x|_h^{\beta})}.
\end{aligned}
$$

Thus

$$
\|\mathcal{H}_\alpha\|_{L^p\left(\mathbb{H}^n,|x|_h^{\beta}\right)\to L^{q,\infty}\left(\mathbb{H}^n,|x|_h^{\gamma}\right)} \leqslant \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(\frac{\omega_Q\,(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}}.
$$

On the other hand, let

$$
f_0(x) = |x|_h^{-\frac{\beta}{p-1}} \chi_{\{x:|x|_h\leqslant 1\}}(x).
$$

Noticing $Q+\beta(1-\frac{p}{p-1}) = Q-\frac{\beta}{p-1} > 0$, we have

$$
\begin{aligned}
\|f_0\|_{L^p(\mathbb{H}^n,|x|_h^{\beta})} &= \left(\int_{\mathbb{H}^n} \left||x|_h^{-\frac{\beta}{p-1}} \chi_{\{x:|x|_h\leqslant 1\}}(x)\right|^p |x|_h^{\beta}\,dx\right)^{\frac{1}{p}} \\
&= \left(\int_{|x|_h\leqslant 1} |x|_h^{-\frac{\beta p}{p-1}} |x|_h^{\beta} dx\right)^{\frac{1}{p}} = \left(\omega_Q \int_0^1 r^{\beta-\frac{p\beta}{p-1}+Q-1} dr\right)^{\frac{1}{p}} = \left(\frac{\omega_Q\,(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p}} < \infty.
\end{aligned}
$$

So we have proved that $f_0 \in L^p(\mathbb{H}^n, |x|_h^{\beta})$. Then we calculate $\mathcal{H}_\alpha(f_0)(x)$.

$$
\begin{aligned}
\mathcal{H}_\alpha(f_0)(x) &= \frac{1}{|x|_h^{Q-\alpha}} \int_{|y|_h<|x|_h} |y|_h^{-\frac{\beta}{p-1}} \chi_{\{y:|y|_h\leqslant 1\}}(y)\,dy \\
&= \frac{1}{|x|_h^{Q-\alpha}} \begin{cases} \int_{|y|_h<|x|_h} |y|_h^{-\frac{\beta}{p-1}} dy, & |x|_h \leqslant 1 \\ \int_{|y|_h\leqslant 1} |y|_h^{-\frac{\beta}{p-1}} dy, & |x|_h > 1 \end{cases} = \frac{\omega_Q}{|x|_h^{Q-\alpha}} \begin{cases} \int_0^{|x|_h} r^{Q-1-\frac{\beta}{p-1}} dr, & r \leqslant 1 \\ \int_0^1 r^{Q-1-\frac{\beta}{p-1}} dr, & r > 1 \end{cases} \\
&= \frac{\omega_Q\,(p-1)}{pQ-Q-\beta} \begin{cases} |x|_h^{\alpha-\frac{\beta}{p-1}}, & |x|_h \leqslant 1 \\ |x|_h^{\alpha-Q}, & |x|_h > 1 \end{cases}.
\end{aligned}
$$

Denote $C_{p,Q,\beta} = \frac{\omega_Q(p-1)}{pQ-Q-\beta}$ and

$$
\{x : |\mathcal{H}_\alpha(f_0)(x)| > \lambda\} = \{|x|_h \leqslant 1 : C_{p,Q,\beta}\,|x|_h^{\alpha-\frac{\beta}{p-1}} > \lambda\} \cup \{|x|_h > 1 : C_{p,Q,\beta}\,|x|_h^{\alpha-Q} > \lambda\}.
$$

When $0 < \lambda < C_{p,Q,\beta}$, noticing $\alpha < \frac{\beta}{p-1}$ and $\beta < Q(p-1)$, we have $\alpha < Q$ and

$$\{x : |\mathcal{H}_\alpha(f_0)(x)| > \lambda\} = \{|x|_h \leqslant 1\} \cup \left\{|x|_h > 1 : |x|_h \leqslant \left(\frac{C_{p,Q,\beta}}{\lambda}\right)^{\frac{1}{Q-\alpha}}\right\}$$
$$= \left\{x : |x|_h < \left(\frac{C_{p,Q,\beta}}{\lambda}\right)^{\frac{1}{Q-\alpha}}\right\}.$$

When $\lambda \geqslant C_{p,Q,\beta}$, noticing $\alpha < \frac{\beta}{p-1}$ and $\beta < Q(p-1)$, we have $\alpha < Q$ and

$$\{x : |\mathcal{H}_\alpha(f_0)(x)| > \lambda\} = \left\{x : |x|_h < \left(\frac{C_{p,Q,\beta}}{\lambda}\right)^{\frac{1}{\frac{\beta}{p-1}-\alpha}}\right\}.$$

Based on the above analysis, we have

$$\begin{aligned}
&\|\mathcal{H}_\alpha(f_0)\|_{L^{q,\infty}(\mathbb{H}^n,|x|_h^\gamma)} \\
&= \max\left\{\sup_{0<\lambda<C_{p,Q,\beta}} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha(f_0)(x)|>\lambda\}}(x)\,|x|_h^\gamma dx\right)^{\frac{1}{q}}, \sup_{C_{p,Q,\beta}\leqslant\lambda} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha(f_0)(x)|>\lambda\}}(x)\,|x|_h^\gamma dx\right)^{\frac{1}{q}}\right\} \\
&=: \max\{M_1, M_2\}.
\end{aligned}$$

Now we first calculate M_1. Since

$$\|f_0\|_{L^p(\mathbb{H}^n,|x|_h^\beta)} = \left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p}}, \quad \gamma > -Q,$$

and

$$1 - \frac{Q+\gamma}{(Q-\alpha)q} = 1 - \frac{1}{Q-\alpha}(\frac{\beta+Q}{p} - \alpha) = \frac{Q(p-1)-\beta}{p(Q-\alpha)} > 0,$$

we have

$$\begin{aligned}
M_1 &= \sup_{0<\lambda<C_{p,Q,\beta}} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha(f_0)(x)|>\lambda\}}(x)\,|x|_h^\gamma dx\right)^{\frac{1}{q}} = \sup_{0<\lambda<C_{p,Q,\beta}} \lambda\left(\int_{|x|_h<(\frac{C_{p,Q,\beta}}{\lambda})^{\frac{1}{Q-\alpha}}} |x|_h^\gamma dx\right)^{\frac{1}{q}} \\
&= \sup_{0<\lambda<C_{p,Q,\beta}} \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(C_{p,Q,\beta}\right)^{\frac{Q+\gamma}{(Q-\alpha)q}} \lambda^{1-\frac{Q+\gamma}{(Q-\alpha)q}} \\
&= \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} C_{p,Q,\beta} = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \times \left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p}+\frac{1}{p'}} \\
&= \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}} \|f_0\|_{L^p(\mathbb{H}^n,|x|_h^\beta)}.
\end{aligned}$$

Then we calculate M_2, noticing $\|f_0\|_{L^p(\mathbb{H}^n,|x|_h^\beta)} = \left(\frac{\omega_Q(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p}}$, $\gamma > -Q$, and

$$1 - \frac{Q+\gamma}{(\frac{\beta}{p-1}-\alpha)q} = 1 - \frac{1}{\frac{\beta}{p-1}-\alpha}(\frac{\beta+Q}{p} - \alpha) = \frac{\beta - Q(p-1)}{p(\frac{\beta}{p-1}-\alpha)(p-1)} < 0,$$

and we have

$$M_2 = \sup_{C_{p,Q,\beta}\leqslant\lambda} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha(f_0)(x)|>\lambda\}}(x)\,|x|_h^\gamma dx\right)^{\frac{1}{q}} = \sup_{C_{p,Q,\beta}\geqslant\lambda} \lambda\left(\int_{|x|_h<(\frac{C_{p,Q,\beta}}{\lambda})^{\frac{1}{\frac{\beta}{p-1}-\alpha}}} |x|_h^\gamma dx\right)^{\frac{1}{q}}$$

$$= \sup_{C_{p,Q,\beta}\leqslant\lambda} \left(\frac{\omega_Q}{Q+r}\right)^{\frac{1}{q}} \left(C_{p,Q,\beta}\right)^{\frac{Q+\gamma}{(\frac{\beta}{p-1}-\alpha)}} \lambda^{1-\frac{Q+\gamma}{(\frac{\beta}{p-1}-\alpha)}}$$

$$= \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} C_{p,Q,\beta} = \left(\frac{\omega_Q}{Q+r}\right)^{\frac{1}{q}} \times \left(\frac{\omega_Q\,(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p}+\frac{1}{p'}}$$

$$= \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(\frac{\omega_Q\,(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}} \|f_0\|_{L^p(\mathbb{H}^n,|x|_h^\beta)}.$$

Its easy to see that $M_1 = M_2$, and then

$$\|\mathcal{H}_\alpha\|_{L^p\left(\mathbb{H}^n,|x|_h^\beta\right)\to\ \|\mathcal{H}_\alpha\|_{L^{p,\infty}\left(\mathbb{H}^n,|x|_h^\gamma\right)}} = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(\frac{\omega_Q\,(p-1)}{pQ-Q-\beta}\right)^{\frac{1}{p'}}.$$

This finishes the proof of Theorem 2.1.

Proof of Theorem 2.2. It is easy to see that

$$|\mathcal{H}_\alpha f(x)| = \left|\frac{1}{|x|_h^{Q-a}}\int_{|y|_h<|x|_h} f(y)\,dy\right| \leqslant \left|\frac{1}{|x|_h^{Q-a}}\int_{\mathbb{H}^n} f(y)dy\right| = |x|_h^{\alpha-Q}\,\|f\|_{L^1(\mathbb{H}^n)}.$$

Notice $|\mathcal{H}_\alpha f(x)| \leqslant |x|_h^{\alpha-Q}\,\|f\|_{L^1(\mathbb{H}^n)}$, and we have $\{x : |\mathcal{H}_\alpha f(x)| > \lambda\} \subset \{x : |x|_h^{\alpha-Q}\,\|f\|_{L^1(\mathbb{H}^n)} > \lambda\}$. Since $Q-\alpha > 0$ and $Q+\gamma > 0$, we have

$$\|\mathcal{H}_\alpha f\|_{L^{(Q+\gamma)/(Q-\alpha),\infty}\left(\mathbb{H}^n,|x|_h^\gamma\right)} = \sup_{\lambda>0} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha f(x)|>\lambda\}}(x)\,|x|_h^\gamma dx\right)^{\frac{Q-\alpha}{Q+\gamma}}$$

$$\leqslant \sup_{\lambda>0} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|x|_h^{\alpha-Q}\|f\|_{L^1(\mathbb{H}^n)}>\lambda\}}(x)\,|x|_h^\gamma dx\right)^{\frac{Q-\alpha}{Q+\gamma}} = \sup_{\lambda>0} \lambda\left(\int_{|x|_h<(\|f\|_{L^1(\mathbb{H}^n)}/\lambda)^{\frac{1}{Q-\alpha}}} |x|_h^\gamma dx\right)^{\frac{Q-\alpha}{Q+\gamma}}$$

$$= \sup_{\lambda>0} \lambda\left(\omega_Q \int_0^{(\|f\|_{L^1(\mathbb{H}^n)}/\lambda)^{\frac{1}{Q-\alpha}}} r^{Q-1+\gamma} dr\right)^{\frac{Q-\alpha}{Q+\gamma}} = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{Q-\alpha}{Q+\gamma}} \|f\|_{L^1(\mathbb{H}^n)}.$$

Thus

$$\|\mathcal{H}_\alpha f(x)\|_{L^{(Q+\gamma)/(Q-\alpha),\infty}\left(\mathbb{H}^n,|x|_h^\gamma\right)} \leqslant \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{Q-\alpha}{Q+\gamma}} \|f\|_{L^1(\mathbb{H}^n)}.$$

On the other hand, let $f_0(x) = \chi_{\{x:|x|_h\leqslant 1\}}(x)$. Then we have

$$\|f_0\|_{L^1(\mathbb{H}^n)} = \int_{\mathbb{H}^n} \chi_{\{x:|x|_h\leqslant 1\}}(x)\,dx = \frac{\omega_Q}{Q} < \infty,$$

and so $f_0 \in L^1(\mathbb{H}^n)$ and

$$\mathcal{H}_\alpha(f_0)(x) = \frac{1}{|x|_h^{Q-a}} \int_{|y|_h<|x|_h} \chi_{\{y:|y|_h\leqslant 1\}}(y)\,dy$$
$$= \frac{1}{|x|_h^{Q-a}} \begin{cases} \int_{|y|_h<|x|_h} dy, & |x|_h \leqslant 1 \\ \int_{|y|_h\leqslant 1} dy, & |x|_h > 1 \end{cases} = \frac{\omega_Q}{Q} \begin{cases} |x|_h^{\alpha}, & |x|_h \leqslant 1 \\ |x|_h^{\alpha-Q}, & |x|_h > 1 \end{cases}.$$

Denote $C_Q = \frac{\omega_Q}{Q}$ and

$$\{x : |\mathcal{H}_\alpha(f_0)(x)| > \lambda\} = \{|x|_h \leqslant 1 : |x|_h^{\alpha} C_Q > \lambda\} \cup \{|x|_h > 1 : |x|_h^{\alpha-Q} C_Q > \lambda\}.$$

When $\lambda \geqslant C_Q$, noticing $0 < \alpha < Q$, we have $\{x : |\mathcal{H}_\alpha(f_0)(x)| > \lambda\} = \varnothing$.

When $0 < \lambda < C_Q$, noticing $0 < \alpha < Q$, we have

$$\{x : |\mathcal{H}_\alpha(f_0)(x)| > \lambda\} = \left\{x : \left(\frac{\lambda}{C_Q}\right)^{\frac{1}{\alpha}} < |x|_h < \left(\frac{C_Q}{\lambda}\right)^{\frac{1}{Q-\alpha}}\right\}.$$

We have

$$\begin{aligned} &\|\mathcal{H}_\alpha(f_0)(x)\|_{L^{(Q+\gamma)/(Q-\alpha),\infty}\left(\mathbb{H}^n,|x|_h^{\gamma}\right)} \\ &= \max\left\{\sup_{0<\lambda<C_Q} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha f_0(x)|>\lambda\}}(x)\,|x|_h^{\gamma}dx\right)^{\frac{Q-\alpha}{Q+\gamma}}, \sup_{\lambda\geqslant C_Q} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha f_0(x)|>\lambda\}}(x)\,|x|_h^{\gamma}dx\right)^{\frac{Q-\alpha}{Q+\gamma}}\right\} \\ &=: \max\{M_3, M_4\}. \end{aligned}$$

When $\lambda \geqslant C_Q$, then $\{x : |\mathcal{H}_\alpha f_0(x)| > \lambda\} = \varnothing$, and we have $M_4 = 0$. Then, we only need to calculate M_3. In addition, noticing

$$Q + \beta > 0, \quad 0 < \alpha < Q, \quad \|f_0\|_{L^1(\mathbb{H}^n)} = \frac{\omega_Q}{Q},$$

we have

$$\begin{aligned} M_3 &= \sup_{0<\lambda<C_Q} \lambda\left(\int_{\mathbb{H}^n} \chi_{\{x:|\mathcal{H}_\alpha f_0(x)|>\lambda\}}(x)\,|x|_h^{\gamma}dx\right)^{\frac{Q-\alpha}{Q+\gamma}} \\ &= \sup_{0<\lambda<C_Q} \lambda\left(\int_{(\frac{\lambda}{C_Q})^{\frac{1}{\alpha}}<|x|_h<(\frac{C_Q}{\lambda})^{\frac{1}{Q-\alpha}}} |x|_h^{\gamma}dx\right)^{\frac{Q-\alpha}{Q+\gamma}} = \sup_{0<\lambda<C_Q} \lambda\left(\omega_Q \int_{(\frac{\lambda}{C_Q})^{\frac{1}{\alpha}}}^{(\frac{C_Q}{\lambda})^{\frac{1}{Q-\alpha}}} r^{Q-1+\gamma}dr\right)^{\frac{Q-\alpha}{Q+\gamma}} \\ &= \sup_{0<\lambda<C_Q} \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{Q-\alpha}{Q+\gamma}} \left(C_Q^{\frac{Q+\gamma}{Q-\alpha}} - \frac{\lambda^{\frac{Q+\gamma}{\alpha}+\frac{Q+\gamma}{Q-\alpha}}}{C_Q^{\frac{Q+\gamma}{\alpha}}}\right)^{\frac{Q-\alpha}{Q+\gamma}} \\ &= \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{Q-\alpha}{Q+\gamma}} C_Q = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{Q-\alpha}{Q+\gamma}} \|f_0\|_{L^1(\mathbb{H}^n)}. \end{aligned}$$

Then

$$\|\mathcal{H}_\alpha\|_{L^1(\mathbb{H}^n)\to L^{(Q+\gamma)/(Q-\alpha),\infty}\left(\mathbb{H}^n,|x|_h^{\gamma}\right)} = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{Q-\alpha}{Q+\gamma}}.$$

This finishes the proof of Theorem 2.2. Notices that Theorem 2.2 no longer holds when $\alpha = 0$.

3. Sharp weighted L^p estimate for the integral operator with a kernel

In this section, we will study the m-linear n-dimensional integral operator with a kernel on the Heisenberg group. Let $K : \mathbb{H}^n \times \cdots \times \mathbb{H}^n \to (0, \infty)$ be a measurable radial kernel such that $K(y_1, ..., y_m) = K(|y_1|_h, ..., |y_m|_h)$, satisfying

$$C^h = \int_{\mathbb{H}^n} \cdots \int_{\mathbb{H}^n} K(y_1, ..., y_m) \prod_{i=1}^{m} |y_i|_h^{-\frac{\alpha_j}{q} - \frac{Q}{q_j}} dy_1 \cdots dy_m < \infty, \tag{3.1}$$

where α_j, q_j, q, and Q are the pre-defined indicator and some fixed indices, $j = 1, 2, ..., m$. The m-linear n-dimensional integral operator with a kernel is defined by

$$\mathcal{H}^h(f_1, ..., f_m)(x) = \int_{\mathbb{H}^n} \cdots \int_{\mathbb{H}^n} K(y_1, ..., y_m) f_1(\delta_{|x|_h} y_1) \cdots f_m(\delta_{|x|_h} y_m) dy_1 \cdots dy_m, \tag{3.2}$$

where $x \in \mathbb{R}^n \setminus \{0\}$ and f_j is a measurable radial function on $\mathbb{H}^n$ with $j = 1, 2, ..., m$. Note that $\mathcal{H}^h$ is in fact an integral operator having a homogeneous radial K of degree $-mn$.

In this paper, we will give the weighted L^p estimate for the m-linear n-dimensional integral operator with a kernel on the Heisenberg group.

Theorem 3.1. *Let $m \in \mathbb{N}$, $1 < q < \infty$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, $\alpha = \alpha_1 + \cdots + \alpha_m$, $1 < q_j < \infty$ with $j = 1, ..., m$, and f_j be a radial function in $L^{q_j}(\mathbb{H}^n, |x|_h^{\frac{q_j \alpha_j}{q}})$. Assume that the kernel K is the constant defined by (3.1). Then*

$$\left\|\mathcal{H}^h\right\|_{L^{q_1}(\mathbb{H}^n, |x|_h^{\frac{q_1\alpha_1}{q}}) \times \cdots \times L^{q_m}(\mathbb{H}^n, |x|_h^{\frac{q_m\alpha_m}{q}}) \to L^q(\mathbb{H}^n, |x|_h^{\alpha})} = C^h. \tag{3.3}$$

Proof. Consider that

$$g_j(x) = \frac{1}{\omega_Q} \int_{|\xi_j|_{h=1}} f_j(\delta_{|x|_h} \xi_j) d\xi_j, \quad x \in \mathbb{H}^n, \quad j = 1, ..., m.$$

Obviously, g_j satisfies $g_j(x) = g_j(|x|_h)$, and $\mathcal{H}^h(f_1, ..., f_m)(x)$ is equal to

$$\begin{aligned}
&\mathcal{H}^h(g_1, ..., g_m)(x) \\
&= \int_{\mathbb{H}^{mn}} K(y_1, ..., y_m) g_1(\delta_{|x|_h} y_1) \cdots g_m(\delta_{|x|_h} y_m) dy_1 \cdots dy_m \\
&= \int_{\mathbb{H}^{mn}} K(y_1, ..., y_m) \prod_{j=1}^{m} \left(\frac{1}{\omega_Q} \int_{|\xi_j|_h=1} f_j(\delta_{|x|_h |y_j|_h} \xi_j) d\xi_j \right) dy_1 \cdots dy_m \\
&= \frac{1}{\omega_Q^m} \int_{\mathbb{H}^{mn}} K(y_1, ..., y_m) \prod_{j=1}^{m} \left(\int_{|\xi_j|_h=1} f_j(\delta_{|x|_h} \left(\delta_{|y_j|_h} \xi_j\right)) d\xi_j \right) dy_1 \cdots dy_m \\
&= \frac{1}{\omega_Q^m} \int_{\mathbb{H}^{mn}} K(y_1, ..., y_m) \prod_{j=1}^{m} \left(\int_{|z_j|_h=|y_j|_h} f_j(\delta_{|x|_h} z_j) |y_j|_h^{-Q} dz_j \right) dy_1 \cdots dy_m \\
&= \frac{1}{\omega_Q^m} \int_{\mathbb{H}^{mn}} \int_{|y_1|_h=|z_1|_h} \cdots \int_{|y_m|_h=|z_m|_h} K(|y_1|_h, ..., |y_m|_h) \prod_{j=1}^{m} f_j(\delta_{|x|_h} z_j) |y_j|_h^{-Q} dy_1 \cdots dy_m dz_m \cdots dz_1
\end{aligned}$$

$$= \frac{1}{\omega_Q^m}\int_{\mathbb{H}^{mn}}\int_{|t_1|_h=1}\cdots\int_{|t_m|_h=1} K(|z_1|_h,...,|z_m|_h)\prod_{j=1}^{m} f_j(\delta_{|x|_h}z_j)dt_1\cdots dt_m dz_m\cdots dz_1$$
$$= \int_{\mathbb{H}^{mn}} K(z_1,...,z_m)f_1(\delta_{|x|_h}z_1)\cdots f_m(\delta_{|x|_h}z_m)dz_1\cdots dz_m = \mathcal{H}^h(f_1,...,f_m)(x).$$

In the fourth to fifth lines, we let $z_j = \delta_{|y_j|_h}\xi_j$. From the fifth to sixth lines, we perform an integral permutation. In the sixth to seventh lines, we set $y_j = \delta_{|z_j|_h}t_j$. On the other hand, by applying Hölder's inequality, we conclude that

$$\begin{aligned}
\left\|g_j\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})} &= \left(\int_{\mathbb{H}^n}\left|\frac{1}{\omega_Q}\int_{|\xi_j|_h=1} f_j(\delta_{|x|_h}\xi_j)d\xi_j\right|^{q_j}|x|_h^{\frac{q_j\alpha_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&= \frac{1}{\omega_Q}\left(\int_{\mathbb{H}^n}\left|\int_{|\xi_j|_h=1} f_j(\delta_{|x|_h}\xi_j)d\xi_j\right|^{q_j}|x|_h^{\frac{q_j\alpha_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&\leqslant \frac{1}{\omega_Q}\left(\int_{\mathbb{H}^n}\int_{|\xi_j|_h=1}|f_j(\delta_{|x|_h}\xi_j)|^{q_j}d\xi_j\left(\int_{|\xi_j|_h=1}dx\right)^{q_j-1}|x|_h^{\frac{q_j\alpha_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&= \omega_Q^{-\frac{1}{q_j}}\left(\int_{\mathbb{H}^n}\int_{|\xi_j|_h=1}|f_j(\delta_{|x|_h}\xi_j)|^{q_j}d\xi_j\,|x|_h^{\frac{q_j\alpha_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&= \omega_Q^{-\frac{1}{q_j}}\left(\int_{\mathbb{H}^n}\int_{|z_j|_h=|x|_h}|f_j(z_j)|^{q_j}\,|x|_p^{-Q}dz_j\,|x|_h^{\frac{q_j\alpha_j}{q}}dx\right)^{\frac{1}{q_j}}\\
&= \omega_Q^{-\frac{1}{q_j}}\left(\int_{\mathbb{H}^n}\int_{|x|_h=|z_j|_h}|x|_h^{-n}\,|x|_h^{\frac{q_j\alpha_j}{q}}dx|f_j(z_j)|^{q_j}dz_j\right)^{\frac{1}{q_j}}\\
&= \omega_Q^{-\frac{1}{q_j}}\left(\int_{\mathbb{H}^n}\int_{|t_j|_h=1}|z_j|_h^{-Q}|t_j|_h^{-Q}|z_j|_h^{\frac{q_j\alpha_j}{q}}|t_j|_h^{\frac{q_j\alpha_j}{q}}|f_j(z_j)|^{q_j}|z_j|_h^{Q}dt_jdz_j\right)^{\frac{1}{q_j}}\\
&= \left(\int_{\mathbb{H}^n}|f_j(z_j)|^{q_j}|z_j|_h^{\frac{q_j\alpha_j}{q}}dz_j\right)^{\frac{1}{q_j}} = \left\|f_j\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})}.
\end{aligned}$$

From the second to third lines, we apply Hölder's inequality. In the fourth to fifth lines, we let $z_j = \delta_{|x|_h}\xi_j$. From the fifth to sixth lines, we perform an integral permutation. In the sixth to seventh lines, we set $x = \delta_{|z_j|_h}t_j$. Therefore we have

$$\frac{\left\|\mathcal{H}^h(f_1,...,f_m)\right\|_{L^q(\mathbb{H}^n,|x|_h^\alpha)}}{\prod_{j=1}^m\left\|f_j\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})}} \leqslant \frac{\left\|\mathcal{H}^h(g_1,...,g_m)\right\|_{L^q(\mathbb{H}^n,|x|_h^\alpha)}}{\prod_{j=1}^m\left\|g_j\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})}},$$

which implies that the operator $\mathcal{H}^h$ and its restriction to the function g satisfying $g_j(x) = g_j(|x|_h)$ have the same operator norm in $L^q(\mathbb{H}^n,|x|_h^\alpha)$. So without loss of generality, we assume that $f_j \in L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})$ with $j = 1,2,...,m$ satisfies that $f_j(x) = f_j(|x|_h)$ in the rest of the proof. Let q' is the conjugate number of q and $g \in L^{q'}(\mathbb{H}^n,|x|_h^\alpha)$. Using duality and Holder's inequality, and making a change of variables, we obtain the following sequence of inequalities:

$$\left|\left\langle\mathcal{H}^h(f_1,...,f_m),g\right\rangle\right|$$

$$\leqslant \int_{\mathbb{H}^{mn}} |K(y_1,...,y_m)| \int_{\mathbb{H}^n} |g(x)| \left|f_1(\delta_{|x|_h} y_1)\right| \cdots \left|f_m(\delta_{|x|_h} y_m)\right| |x|_h^{\alpha} dx dy_1 \cdots dy_m$$
$$= \int_{\mathbb{H}^{mn}} |K(y_1,...,y_m)| \int_{\mathbb{H}^n} |g(x)||x|_h^{\frac{\alpha}{q'}} |f_1(\delta_{|x|_h} y_1)| \cdots |f_m(\delta_{|x|_h} y_m)||x|_h^{\frac{\alpha}{q}} dx dy_1 \cdots dy_m$$
$$\leqslant \int_{\mathbb{H}^{mn}} |K(y_1,...,y_m)| \left(\int_{\mathbb{H}^n} |g(x)|^{q'} |x|_h^{\alpha} dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{H}^n} \left(|f_1(\delta_{|x|_h} y_1)| \cdots |f_m(\delta_{|x|_h} y_m)||x|_h^{\frac{\alpha}{q}}\right)^q dx\right)^{\frac{1}{q}} dy_1 \cdots dy_m$$
$$\leqslant \|g\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} \int_{\mathbb{H}^{mn}} |K(y_1,...,y_m)| \prod_{j=1}^{m} \left(\int_{\mathbb{H}^n} \left|f_j(\delta_{|y_j|_h} x)\right|^{q_j} |x|_h^{\frac{q_j\alpha_j}{q}} dx\right)^{\frac{1}{q_j}} dy_1 \cdots dy_m$$
$$= \|g\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} \int_{\mathbb{H}^{mn}} |K(y_1,...,y_m)| \prod_{j=1}^{m} |y|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} \prod_{j=1}^{m} \left(\int_{\mathbb{H}^n} |f_j(z_j)|^{q_j} |z_j|_h^{\frac{q_j\alpha_j}{q}} dx\right)^{\frac{1}{q_j}} dy_1 \cdots dy_m$$
$$= \int_{\mathbb{H}^{mn}} K(y_1,...,y_m) \prod_{j=1}^{m} |y|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1 \cdots dy_m \|g\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} \prod_{j=1}^{m} \left\|f_j\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})}$$
$$= C^h \|g\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} \prod_{j=1}^{m} \left\|f_j\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})}.$$

This proves the first part of our theorem.

For the second part, we will show that if the kernel K is nonnegative, then the operator norm $\|\mathcal{H}^h\|$ of $\mathcal{H}^h$ is equal to C^h. For a positive number ε and $i = 1, 2, ..., m$, we define the sequences of functions g_ε and $f_{j,\varepsilon}$ by

$$g_\varepsilon(x) = |x|_h^{-\frac{Q+\alpha}{q'}+\frac{\varepsilon}{q'}} \chi_{B(0,1)}(x), \quad f_{j,\varepsilon} = |x|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}} \chi_{B(0,1)}(x).$$

By a simple computation, we have

$$\|g_\varepsilon\|^{q'}{}_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} = \int_{\mathbb{H}^n} |x|_h^{-Q+\varepsilon} dx = \left(\left(\int_{\mathbb{H}^n} \left(|x|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}}\right)^{q_j} |x|_h^{\frac{q_j\alpha_j}{q}} dx\right)^{\frac{1}{q_j}}\right)^{q_j}$$
$$= \left\|f_{j,\varepsilon}\right\|^{q^j}{}_{L^{q^j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})} = \|g_\varepsilon\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} \prod_{j=1}^{m} \left\|f_{j,\varepsilon}\right\|_{L^{q_j}(\mathbb{H}^n,|x|_h^{\frac{q_j\alpha_j}{q}})} = \frac{\omega_Q}{\varepsilon}.$$

Therefore, we have

$$\left|\left\langle \mathcal{H}^h(f_{1,\varepsilon},...,f_{m,\varepsilon}), g_\varepsilon \right\rangle\right|$$
$$= \int_{B(0,1)} |x|_h^{-\frac{Q+\alpha}{q'}+\frac{\varepsilon}{q'}} |x|_h^{\alpha} \int_{\mathbb{H}^{mn}} |K(y_1,...,y_m)| \prod_{j=1}^{m} f_{j,\varepsilon}(\delta_{|x|_h} y_j) dy_1 \cdots dy_m dx$$
$$= \int_{B(0,1)} |x|_h^{-\frac{Q+\alpha}{q'}+\frac{\varepsilon}{q'}} |x|_h^{\alpha} \int_{B(0,\frac{1}{|x|_h})} \cdots \int_{B(0,\frac{1}{|x|_h})} K(y_1,...,y_m) \prod_{j=1}^{m} \left(|x|_h |y_j|_h\right)^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}} dy_1 \cdots dy_m dx$$
$$= \int_{B(0,1)} |x|_h^{-Q+\varepsilon} \int_{B(0,\frac{1}{|x|_h})} \cdots \int_{B(0,\frac{1}{|x|_h})} K(y_1,...,y_m) \prod_{j=1}^{m} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}} dy_1 \cdots dy_m dx$$

$$= \omega_Q \int_0^1 r^{\varepsilon-1} \int_{B(0,\frac{1}{r})} \cdots \int_{B(0,\frac{1}{r})} K(y_1,...,y_m) \prod_{j=1}^m |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}} dy_1 \cdots dy_m dr$$

$$= \omega_Q \int_1^\infty r^{-1-\varepsilon} \int_{B(0,r)} \cdots \int_{B(0,r)} K(|y_1|_h,...,|y_m|_h) \prod_{j=1}^m |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}} dy_1 \cdots dy_m dr$$

$$= -\frac{\omega_Q}{\varepsilon} \int_1^\infty (r^{-\varepsilon})' \left(\omega_Q^m \int_0^r \cdots \int_0^r K(r_1,...,r_m) \prod_{j=1}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 \cdots dr_m \right) dr$$

$$= \frac{\omega_Q^{m+1}}{\varepsilon} \int_0^1 \cdots \int_0^1 K(r_1,...,r_m) \prod_{j=1}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 \cdots dr_m + \sum_{i=1}^m L_i,$$

where L_i is defined as

$$L_i = \frac{\omega_Q^{m+1}}{\varepsilon} \int_1^\infty r^{-\varepsilon} \int_0^r \cdots \int_0^r K(r_1,...,\overset{(i)}{r},...,r_m) r^{-\frac{\alpha_i}{q}-\frac{Q}{q_i}+\frac{\varepsilon}{q_i}+Q-1}$$
$$\times \prod_{j\neq i}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 ..., \overset{\wedge}{dr_i} \cdots dr_m dr$$
$$= \frac{\omega_Q^{m+1}}{\varepsilon} \int_1^\infty r_i^{-\varepsilon} \int_0^{r_i} \cdots \int_0^{r_i} K(r_1,...,r_m) \prod_{j=1}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 \cdots \overset{\wedge}{dr_i} \cdots dr_m dr_i.$$

Here $\overset{\wedge}{dr_i}$ means that we do not integrate with respect to the variable r_i. The last equality follows from integration by parts and the observation that, if we let

$$F(x,...,x) = \int_0^x \cdots \int_0^x K(r_1,...,r_m) \prod_{j=1}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 \cdots dr_m,$$

then

$$\frac{dF(x,...,x)}{dx} = \sum_{i=1}^m \int_0^x \cdots \int_0^x K(r_1,...,\overset{(i)}{x},...,r_m) x^{-\frac{\alpha_i}{q}-\frac{Q}{q_i}+\frac{\varepsilon}{q_i}+Q-1} \times \prod_{j\neq i}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 \cdots \overset{\wedge}{dr_i} \cdots dr_m,$$

where the upper index (i) means that x replaces the variable r_i in the i-th position. By means of the previous step, we have

$$\frac{\left|\left\langle \mathcal{H}^h(f_{1,\varepsilon},...,f_{m,\varepsilon}), g_\varepsilon \right\rangle\right|}{\|g_\varepsilon\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})} \left\|f_{1,\varepsilon}\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})} \cdots \left\|f_{m,\varepsilon}\right\|_{L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})}}$$
$$= \omega_Q^m \int_0^1 \cdots \int_0^1 K(r_1,...,r_m) \prod_{j=1}^m r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} dr_1 \cdots dr_m + \sum_{i=1}^m \frac{\varepsilon L_i}{\omega_Q}. \tag{3.4}$$

Let E_i denote the domain of integral L_i above (3.4), that is,

$$E_i = \left\{(r_1,...,r_m) \in (0,\infty)^m : 1 \leqslant r_i < \infty, 0 \leqslant r_j \leqslant r_i, j \neq i\right\}.$$

Taking into account that $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$, we can bound the integrand of $\frac{\varepsilon L_i}{\omega_Q}$ on E_i as follows:

$$\begin{aligned} r_i^{-\varepsilon}K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} &\leqslant r_i^{-\varepsilon+\frac{\varepsilon}{q_1}+\cdots+\frac{\varepsilon}{q_m}}K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1} \\ &= r_i^{-\frac{\varepsilon}{q'}}K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1} \\ &\leqslant K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}. \end{aligned}$$

For the integrand of the first term in (3.4) on $[0,1]^m$, we also have that

$$K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1} \leqslant K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}.$$

Since the condition of kernel K (3.1) is equivence to

$$\begin{aligned} C^h &= \int_{\mathbb{H}^{mn}} K(y_1,...,y_m)\prod_{j=1}^{m} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1\cdots dy_m \\ &= \omega_Q^m\int_0^\infty\cdots\int_0^\infty K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}dr_1\cdots dr_m<\infty, \end{aligned} \tag{3.5}$$

using assumption (3.5), we can use the Lebesgue dominated convergence theorem, which implies that

$$\lim_{\varepsilon\to0^+}\frac{\varepsilon L_i}{\omega_Q}=\omega_Q^m\int_1^\infty\int_0^{r_i}\cdots\int_0^{r_i}K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}dr_1\cdots dr_m dr_i, \tag{3.6}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to0^+}\omega_Q^m\int_0^1\cdots\int_0^1 K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+\frac{\varepsilon}{q_j}+Q-1}dr_1\cdots dr_m \\ &= \omega_Q^m\int_0^1\cdots\int_0^1 K(r_1,...,r_m)\prod_{j=1}^{m} r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}dr_1\cdots dr_m. \end{aligned} \tag{3.7}$$

Furthermore, we have

$$[0,1]^m\cup\left(\bigcup_{i=1}^{m}E_i\right)=(0,\infty)^m,$$

and for $i,j=1,...,m$, any of the intersection sets $[0,1]^m\cap E_i$, $E_i\cap E_j$, $i\neq j$, has Lebesgue measure zero in $\mathbb{H}^n$. Consequently, (3.4), (3.6), and (3.7) imply that

$$\begin{aligned} &\left\|\mathcal{H}^h\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})\times\cdots\times L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})\to L^q\left(\mathbb{H}^n,|x|_h^{\alpha}\right)} \\ &= \lim_{\varepsilon\to0^+}\frac{\left|\left\langle\mathcal{H}^h(f_{1,\varepsilon},...,f_{m,\varepsilon}),g_\varepsilon\right\rangle\right|}{\|g_\varepsilon\|_{L^{q'}(\mathbb{H}^n,|x|_h^{\alpha})}\left\|f_{1,\varepsilon}\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})}\cdots\left\|f_{m,\varepsilon}\right\|_{L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})}}=C^h. \end{aligned}$$

This finishes the proof of Theorem 3.1.

4. Application: Sharp weighted L^p estimates for a class of integral operators

By taking a particular kernel K in operator T defined by (3.1), we can obtain sharp weighted L^p estimates for Hardy, Hardy-Littlewood-Pólya, and Hilbert operators on the Heisenberg group. Our results in this section are as follows.

Corollary 4.1. *Assume that the real paramenters q, q_j, α, and α_j with $j = 1, 2, ..., m$ are the same as in Theorem 3.1, and f_j is a radial function in $L^{q_j}(\mathbb{H}^n, |x|_h^{\frac{q_j\alpha_j}{q}})$. Assume also that $-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q > 0$. Then*

$$\left\|\mathcal{H}_1^h\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})\times\cdots\times L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})\to L^q(\mathbb{H}^n,|x|_h^{\alpha})} = \frac{q\omega_Q 2^{1-m}}{mqQ-\alpha-Q}\frac{\prod_{j=1}^m \Gamma((Q-\frac{\alpha_j}{q}-\frac{Q}{q_j})/2)}{\Gamma((mQ-\frac{\alpha}{q}-\frac{Q}{q})/2)}. \tag{4.1}$$

Corollary 4.2. *Assume that the real paramenters q, q_j, α, and α_j with $j = 1, 2, ..., m$ are the same as in Theorem 3.1, f_j is a radial function in $L^{q_j}(\mathbb{H}^n, |x|_h^{\frac{q_j\alpha_j}{q}})$. Assume also that $-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q > 0$ and $-\frac{\alpha}{q} - \frac{Q}{q} < 0$. Then*

$$\left\|\mathcal{H}_2^h\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})\times\cdots\times L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})\to L^q(\mathbb{H}^n,|x|_h^{\alpha})} = \frac{q\omega_Q^m(mQ-\frac{\alpha}{q}-\frac{Q}{q})}{(\alpha+Q)\prod_{j=1}^m(Q-\frac{\alpha_j}{q}-\frac{Q}{q_j})}. \tag{4.2}$$

Corollary 4.3. *Assume that the real paramenters q, q_j, α, and α_j with $j = 1, 2, ..., m$ are the same as in Theorem 3.1, and f_j is a radial function in $L^{q_j}(\mathbb{H}^n, |x|_h^{\frac{q_j\alpha_j}{q}})$. Assume also that $-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q > 0$ and $-\frac{\alpha}{q} - \frac{Q}{q} < 0$. Then*

$$\left\|\mathcal{H}_3^h\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})\times\cdots\times L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})\to L^q(\mathbb{H}^n,|x|_h^{\alpha})} = \frac{\omega_Q^m\Gamma(\frac{\alpha+Q}{qQ})\prod_{j=1}^m\Gamma(1-\frac{\alpha_j}{qQ}-\frac{1}{q_j})}{Q^m\Gamma(m)}. \tag{4.3}$$

Proof of Corollary 4.1. Next, we will use the methods in [7, 17]. If we take the kernel

$$K(y_1, ..., y_m) = \chi_{\{|(y_1,...,y_m)|_h\leqslant 1\}}(y_1, ..., y_m) \tag{4.4}$$

in Theorems 3.1, by a change of variables, it is easy to verify that $\mathcal{H}^h = \mathcal{H}_1^h$, and then $\mathcal{H}_1^h$ can be denoted by

$$\mathcal{H}_1^h = \int_{|(y_1,...,y_m)|_h\leqslant 1} f_1(|x|_h y_1)\cdots f_m(|x|_h y_m)dy_1\cdots dy_m.$$

Then all things reduce to calculating

$$C_1^h = \int_{|(y_1,...,y_m)|_h\leqslant 1}\prod_{i=1}^m |y_i|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1\cdots dy_m.$$

To calculate this integral, employing the polar coordinates $y_j = \rho_j\xi_j$, $j = 1, 2, ..., m$, and Fubini's theorem, we obtain

$$\begin{aligned} C_1^h &= \int_{\mathbb{S}}\cdots\int_{\mathbb{S}}\int_{\sum_{j=1}^m\rho_j^2<1,\rho_j>0,j=1,...,m}\prod_{j=1}^m \rho_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1} d\rho_1\cdots d\rho_m d\sigma(\xi_1)\cdots d\sigma(\xi_m) \\ &= \omega_Q^m\int_{\sum_{j=1}^m\rho_j^2<1,\rho_j>0,j=1,...,m}\prod_{j=1}^m \rho_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1} d\rho_1\cdots d\rho_m. \end{aligned} \tag{4.5}$$

We use the m-dimensional spherical coordinates

$$\begin{aligned}
\rho_1 &= r\cos\varphi_1,\\
\rho_2 &= r\sin\varphi_1\cos\varphi_2,\\
\rho_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3,\\
&\vdots\\
\rho_{m-1} &= r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\cos\varphi_{m-1},\\
\rho_m &= r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\sin\varphi_{m-1},
\end{aligned}$$

where $r \geqslant 0$ is the radial coordinate and φ_j, $j = 1, 2, ..., m-1$, are angular coordinates, $\varphi_j \in [0, \pi]$, $j = 1, 2, ..., m-2$, $\varphi_{m-1} \in [0, 2\pi)$, and the known fact that the associated Jacobian is

$$|J_m| = r^{m-1}\sin^{m-2}\varphi_1\sin^{m-3}\varphi_2\cdots\sin\varphi_{m-2}$$

in (4.5). Since $-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q > 0$, we have

$$\begin{aligned}
C_1^h &= \omega_Q^m \int_0^1 r^{\sum_{j=1}^m(-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1)} r^{m-1} \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}} (\sin\varphi_1)^{m-2}(\sin\varphi_2)^{m-3}\cdots(\sin\varphi_{m-2})^1\\
&(\sin\varphi_1)^{\sum_{j=2}^m(-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1)}(\sin\varphi_2)^{\sum_{j=3}^m(-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1)}\cdots(\sin\varphi_{m-2})^{\sum_{j=m-1}^m(-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1)}(\sin\varphi_{m-1})^{-\frac{\alpha_m}{q}-\frac{Q}{q_m}+Q-1}\\
&(\cos\varphi_1)^{-\frac{\alpha_1}{q}-\frac{Q}{q_1}+Q-1}\cdots(\cos\varphi_{m-2})^{-\frac{\alpha_{m-2}}{q}-\frac{Q}{q_{m-2}}+Q-1}(\cos\varphi_{m-1})^{-\frac{\alpha_{m-1}}{q}-\frac{Q}{q_{m-1}}+Q-1}d\varphi_1\cdots d\varphi_{m-1}dr\\
&= \omega_Q^m\int_0^1 r^{\sum_{j=1}^m(-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q)-1}dr\\
&\times\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\prod_{j=1}^{m-1}(\sin\varphi_j)^{m-(j+1)+\sum_{i=j+1}^m(-\frac{\alpha_i}{q}-\frac{Q}{q_i}+Q-1)}(\cos\varphi_j)^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}d\varphi_1\cdots d\varphi_{m-1}\\
&= \frac{\omega_Q^m}{\sum_{j=1}^m(-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q)}\int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\prod_{j=1}^{m-1}(\sin\varphi_j)^{Q(m-j)-1-\sum_{i=j+1}^m(\frac{\alpha_i}{q}+\frac{Q}{q_i})}(\cos\varphi_j)^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1}d\varphi_1\cdots d\varphi_m\\
&= \frac{\omega_Q^m}{mQ-\frac{\alpha+Q}{q}}\prod_{j=1}^{m-1}\int_0^1 t_j^{Q(m-j)-1-\sum_{i=j+1}^m(\frac{\alpha_i}{q}+\frac{Q}{q_i})}(1-t_j^2)^{\frac{Q-2-\frac{\alpha_j}{q}-\frac{Q}{q_j}}{2}}dt_j\\
&= \frac{q\omega_Q^m 2^{1-m}}{mqQ-\alpha-Q}\prod_{j=1}^{m-1}\int_0^1 s_j^{\frac{Q(m-j)-\sum_{i=j+1}^m(\frac{\alpha_i}{q}+\frac{Q}{q_i})}{2}-1}\left(1-s_j\right)^{\frac{Q-\frac{\alpha_j}{q}-\frac{Q}{q_j}}{2}-1}ds_j\\
&= \frac{q\omega_Q^m 2^{1-m}}{mqQ-\alpha-Q}\prod_{j=1}^{m-1}B\left(\frac{Q(m-j)-\sum_{i=j+1}^m(\frac{\alpha_i}{q}+\frac{Q}{q_i})}{2}, \frac{Q-\frac{\alpha_j}{q}-\frac{Q}{q_j}}{2}\right).
\end{aligned}$$

We also use the fact $\varphi_j \in (0, \pi/2)$, $j = 1, 2, ..., m-1$. From the seventh to eighth lines, we let $t_j = \sin\varphi_j$ for $j = 1, 2, ..., m-1$. From the eighth to ninth lines, we set $s_j = t_j^2$ for $j = 1, 2, ..., m-1$, as well as the definition of the beta function (see, e.g., [24]).

By using the following well-known relation between Euler's beta and gamma functions:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

(see, for example, [24]), after some simple calculations, we see that the following relations hold:

$$\prod_{j=1}^{m-1} B\left(\frac{Q(m-j)-\sum_{i=j+1}^{m}(\frac{\alpha_i}{q}+\frac{Q}{q_i})}{2}, \frac{Q-\frac{\alpha_j}{q}-\frac{Q}{q_j}}{2}\right) = \prod_{j=1}^{m-1}\left(\frac{\Gamma\left(\frac{Q(m-j)-\sum_{i=j+1}^{m}(\frac{\alpha_i}{q}+\frac{Q}{q_i})}{2}\right)\Gamma\left(\frac{Q-\frac{\alpha_j}{q}-\frac{Q}{q_j}}{2}\right)}{\Gamma\left(\frac{Q(m-(j-1))-\sum_{i=j}^{m}(\frac{\alpha_i}{q}+\frac{Q}{q_i})}{2}\right)}\right)$$

$$= \frac{\prod_{j=1}^{m}\Gamma\left(\frac{Q-\frac{\alpha_j}{q}-\frac{Q}{q_j}}{2}\right)}{\Gamma\left(\frac{mQ-\frac{\alpha}{q}-\frac{Q}{q}}{2}\right)}.$$

Then we have

$$C_1^h = \frac{q\omega_Q^m 2^{1-m}}{mqQ-\alpha-Q}\frac{\prod_{j=1}^{m}\Gamma((Q-\frac{\alpha_j}{q}-\frac{Q}{q_j})/2)}{\Gamma((mQ-\frac{\alpha}{q}-\frac{Q}{q})/2)}.$$

This finishes the proof of Corollary 4.1.

Proof of Corollary 4.2. Next, we refer to the methods in [3]. If we take the kernel

$$K(y_1, ..., y_m) = \frac{1}{[\max(1, |y_1|_h^Q, ..., |y_m|_h^Q)]^m} \tag{4.6}$$

in Theorem 2.1, by a change of variables, we have $\mathcal{H}^p = \mathcal{H}_2^h$, and then $\mathcal{H}_2^h$ can be denoted by

$$\mathcal{H}_2^h = \int_{\mathbb{H}^{mn}} \frac{1}{[\max(1, |y_1|_h^Q, ..., |y_m|_h^Q)]^m} f_1(|x|_h y_1)\cdots f_m(|x|_h y_m) dy_1\cdots dy_m.$$

Then, we reduce to calculating

$$C_2^h = \int_{\mathbb{H}^{mn}} \frac{1}{[\max(1, |y_1|_h^Q, ..., |y_m|_h^Q)]^m} \prod_{j=1}^{m} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1\cdots dy_m.$$

To calculate this integral, we divide the integral into m parts. Let

$$E_0 = \{(y_1, ..., y_m) \in \mathbb{H}^n\times\cdots\times\mathbb{H}^n : |y_k|_h \leqslant 1, 1 \leqslant k \leqslant m\};$$
$$E_1 = \{(y_1, ..., y_m) \in \mathbb{H}^n\times\cdots\times\mathbb{H}^n : |y_1|_h > 1, |y_k|_h \leqslant |y_1|_h, 2 \leqslant k \leqslant m\};$$
$$E_i = \{(y_1, ..., y_m) \in \mathbb{H}^n\times\cdots\times\mathbb{H}^n : |y_i|_h > 1, |y_j|_h < |y_i|_h, |y_k|_h \leqslant |y_i|_h, 1 \leqslant j < i < k \leqslant m\};$$
$$E_m = \{(y_1, ..., y_m) \in \mathbb{H}^n\times\cdots\times\mathbb{H}^n : |y_m|_h > 1, |y_j|_h < |y_m|_h, 1 \leqslant j < m\}.$$

It is clear that

$$\bigcup_{j=0}^{m} E_j = \mathbb{H}^n\times\cdots\times\mathbb{H}^n,$$

and $E_i \cap E_j = \varnothing\,(i \neq j)$. Let

$$K_j := \int_{E_j} \frac{1}{[\max(1, |y_1|_h^Q, ..., |y_m|_h^Q)]^m} \prod_{k=1}^{m} |y_k|_h^{-\frac{\alpha_k}{q}-\frac{Q}{q_k}} dy_1\cdots dy_m,$$

and then we have

$$C_2^h = \sum_{j=1}^{m} K_j := \sum_{j=1}^{m} \int_{E_j} \frac{1}{[\max(1, |y_1|_h^Q, ..., |y_m|_h^Q)]^m} \prod_{k=1}^{m} |y_k|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1 \cdots dy_m.$$

Now let us calculate J_j, $j = 1, 2, ..., m$. Since $-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q > 0$, using the polar coordinate transformation, we have

$$\begin{aligned} K_0 &= \int_{E_0} \prod_{j=1}^{m} |y_i|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1 \cdots dy_m = \prod_{j=1}^{m} \int_{|y_i|_h \leqslant 1} |y_i|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_i \\ &= \prod_{j=1}^{m} \omega_Q \int_0^1 r_j^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q-1} dr_j = \frac{\omega_Q^m}{\prod_{j=1}^{m} (Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})}. \end{aligned}$$

For $j = 1$, since $-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q > 0$ and $-\frac{\alpha}{q} - \frac{Q}{q} < 0$, we have

$$\begin{aligned} K_1 &= \int_{E_1} \frac{\prod_{j=1}^{m} |y_i|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}}}{[\max(1, |y_1|_h^Q, ... |y_m|_h^Q)]^m} dy_1 \cdots dy_m = \int_{E_1} |y_1|_h^{-\frac{\alpha_1}{q}-\frac{Q}{q_1}-mQ} \prod_{j=2}^{m} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_2 \cdots dy_m dy_1 \\ &= \int_{|y_1|_h > 1} |y_1|_h^{-\frac{\alpha_1}{q}-\frac{Q}{q_1}-mQ} \prod_{j=2}^{m} \int_{|y_j|_h \leqslant |y_1|_h} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_j dy_1 \\ &= \int_{|y_1|_h > 1} |y_1|_h^{-\frac{\alpha_1}{q}-\frac{Q}{q_1}-mQ} \prod_{j=2}^{m} \omega_Q \frac{|y_1|_h^{Q-\frac{\alpha_j}{q}-\frac{Q}{q_j}}}{Q - \frac{\alpha_j}{q} - \frac{Q}{q_j}} dy_1 \\ &= \frac{\omega_Q^{m-1}}{\prod_{j=2}^{m} (Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})} \int_{|y_1|_h > 1} |y_1|_h^{-\frac{\alpha}{q}-\frac{Q}{q}-Q} dy_1 = \frac{q\omega_Q^m}{(\alpha + Q) \prod_{j=2}^{m} (Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})}. \end{aligned}$$

Similar for $i = 2, ..., m-1$, we have

$$\begin{aligned} K_i &= \int_{E_i} |y_i|_h^{-\frac{\alpha_i}{q}-\frac{Q}{q_i}-mQ} \prod_{j \neq i}^{m} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1 \cdots dy_{i-1} dy_{i+1} dy_m dy_i \\ &= \int_{|y_i|_h > 1} |y_i|_h^{-\frac{\alpha_i}{q}-\frac{Q}{q_i}-mQ} \prod_{j=2}^{i-1} \int_{|y_j|_h \leqslant |y_i|_h} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_j \prod_{k=i+1}^{m} \int_{|y_k|_h \leqslant |y_i|_h} |y_k|_h^{-\frac{\alpha_k}{q}-\frac{Q}{q_k}} dy_k dy_i \\ &= \frac{q\omega_Q^m}{(\alpha + Q) \prod_{1 \leqslant j \leqslant m, j \neq i} (Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})}. \end{aligned}$$

When $i = m$, similar to the previous step, we show that

$$K_m = \int_{|y_i|_h > 1} |y_m|_h^{-\frac{\alpha_m}{q}-\frac{Q}{q_m}-mQ} \prod_{j=1}^{m-1} |y_j|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1 \cdots dy_m = \frac{q\omega_Q^m}{(\alpha + Q) \prod_{j=1}^{m-1} (Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})}.$$

Then, it yields that

$$C_h^2 = K_0 + K_1 + \sum_{i=2}^{m-1} K_i + K_m = \frac{q\omega_Q^m (mQ - \frac{\alpha}{q} - \frac{Q}{q})}{(\alpha + Q) \prod_{j=1}^{m} (Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})}.$$

This finishes the proof of Corollary 4.2.

Proof of Corollary 4.3. If we take the kernel

$$K(y_1, ..., y_m) = \frac{1}{(1 + |y_1|_h^Q + \cdots + |y_m|_h^Q)^m} \tag{4.7}$$

in Theorem 2.1, by a change of variables, we have $\mathcal{H}^p = \mathcal{H}_3^h$, and then $\mathcal{H}_3^h$ can be denoted by

$$\mathcal{H}_3^h = \int_{\mathbb{H}^{mn}} \frac{1}{(1 + |y_1|_h^Q + \cdots + |y_m|_h^Q)^m} f_1(|x|_h y_1) \cdots f_m(|x|_h y_m) dy_1 \cdots dy_m.$$

Then, we reduce to calculating

$$C_3^h = \int_{\mathbb{H}^{mn}} \frac{1}{(1 + |y_1|_h^Q + \cdots + |y_m|_h^Q)^m} \prod_{j=1}^{m} |y_j|_h^{-\frac{\alpha_j}{q} - \frac{Q}{q_j}} dy_1 \cdots dy_m.$$

Actually, this method stems from Benyi and Oh [2], who investigated the one-dimensional case. Following their method, it is easy to find the higher-dimensional case, as well. For completeness, we give the details. Employing the polar coordinates and making a change of variables, we have

$$C_3^h = \omega_Q^m \int_0^\infty \cdots \int_0^\infty \frac{1}{(1 + \rho_1^Q + \cdots + \rho_m^Q)^m} \prod_{j=1}^{m} \rho_j^{-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q - 1} d\rho_1 \cdots d\rho_m.$$

Let $\rho_j^Q = t_j$, and we have

$$C_3^h = \frac{\omega_Q^m}{Q^m} \int_0^\infty \cdots \int_0^\infty \frac{1}{(1 + t_1 + \cdots + t_m)^m} \prod_{j=1}^{m} t_j^{\frac{-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q}{Q} - 1} dt_1 \cdots dt_m.$$

Let us denote the integral on the right by

$$\frac{Q^m}{\omega_Q^m} C_3^h = I_m\left(m, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right).$$

By making the change of variables $t_m = (1 + t_1 + \cdots + t_{m-1})\, t$ and performing integration with respect to dt, we have the following identity:

$$\begin{aligned}
& I_m\left(m, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right) \\
&= \int_0^\infty \cdots \int_0^\infty \frac{[(1 + t_1 + \cdots + t_{m-1})\, t]^{\frac{-\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q}{Q} - 1} (1 + t_1 + \cdots + t_{m-1})}{[(1 + t_1 + \cdots + t_{m-1}) + (1 + t_1 + \cdots + t_{m-1})\, t]^m} \prod_{j=1}^{m-1} t_j^{\frac{-\frac{\alpha_j}{q} - \frac{Q}{q_j} + Q}{Q} - 1} dt_1 \cdots dt_{m-1} dt \\
&= \int_0^\infty (1 + t)^{-m} t^{\frac{-\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q}{Q} - 1} dt I_{m-1}\left(m - \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right).
\end{aligned}$$

Observe that, if we make the change of variables $t + 1 = 1/s$, we obtain

$$\int_0^{\infty} (1+t)^{-a-b} t^{a-1} dt = \int_0^1 s^{a-1} (1-s)^{b-1} ds = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for $a, b > 0$. According to known conditions, obviously, we have $(Q - \frac{\alpha_j}{q} - \frac{Q}{q_j})/Q > 0$. Now, we will show that $m-(Q-\frac{\alpha_m}{q}-\frac{Q}{q_m})/Q-\cdots-(Q-\frac{\alpha_k}{q}-\frac{Q}{q_k})/Q > 0$ with $k = m, m-1, ..., 1$. Since $-\frac{\alpha_j}{q}-\frac{Q}{q_j}+Q > 0$ and $-\frac{\alpha}{q} - \frac{Q}{q} < 0$, we have

$$\begin{aligned} m - \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q} - \cdots - \frac{Q - \frac{\alpha_k}{q} - \frac{Q}{q_k}}{Q} &= \frac{(k-1)Q + (\frac{\alpha_m}{q} + \frac{Q}{q_m}) + \cdots + (\frac{\alpha_k}{q} + \frac{Q}{q_k})}{Q} \\ &= \frac{(k-1)Q + (\frac{\alpha}{q} + \frac{Q}{q}) - (\frac{\alpha_{k-1}}{q} + \frac{Q}{q_{k-1}}) - \cdots - (\frac{\alpha_1}{q} + \frac{Q}{q_1})}{Q} \\ &= \frac{(Q - \frac{\alpha_{k-1}}{q} - \frac{Q}{q_{k-1}}) + \cdots + (Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}) + (\frac{\alpha}{q} + \frac{Q}{q})}{Q} > 0. \end{aligned}$$

Therefore, if we recall the relationship between the beta and gamma functions, we obtain

$$\int_0^{\infty} (1+t)^{-m} t^{(\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q)/Q - 1} dt = \frac{\Gamma((\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q)/Q)\Gamma(m - (\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q)/Q)}{\Gamma(m)},$$

and

$$\begin{aligned} & I_m\left(m, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right) \\ &= \frac{\Gamma\left(\frac{\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q}{Q}\right)\Gamma\left(m - \frac{\frac{\alpha_m}{q} - \frac{Q}{q_m} + Q}{Q}\right)}{\Gamma(m)} I_{m-1}\left(m - \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_{m-1}}{q} - \frac{Q}{q_{m-1}}}{Q}\right). \end{aligned}$$

By a simple induction argument, we obtain from this recurrence that

$$\begin{aligned} I\left(m, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right) &= \frac{\Gamma\left(m - \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q} - \cdots - \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}\right)\Gamma\left(\frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right)\cdots\Gamma\left(\frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}\right)}{\Gamma(m)} \\ &= \frac{\Gamma\left(\frac{\alpha+Q}{qQ}\right)\prod_{j=1}^m \Gamma\left(1 - \frac{\alpha_j}{qQ} - \frac{1}{q_j}\right)}{\Gamma(m)}. \end{aligned}$$

Then

$$C_3^h = \frac{\omega_Q^m}{Q^m} I_m\left(m, \frac{Q - \frac{\alpha_1}{q} - \frac{Q}{q_1}}{Q}, ..., \frac{Q - \frac{\alpha_m}{q} - \frac{Q}{q_m}}{Q}\right) = \frac{\omega_Q^m \Gamma\left(\frac{\alpha+Q}{qQ}\right)\prod_{j=1}^m \Gamma\left(1 - \frac{\alpha_j}{qQ} - \frac{1}{q_j}\right)}{Q^m \Gamma(m)}.$$

This finishes the proof of Corollary 4.3.

5. Further results: Sharp weighted L^p estimate for the Hausdorff operator

In this section, we will use the previous results to give the weighted L^p estimates for the m-linear n-dimensional Hausdorff operator on the Heisenberg group.

Corollary 5.1. *Assume that the real paramenters q, q_j, α, and α_j with $j = 1, 2, ..., m$ are the same as in Theorem 3.1. A nonnegative function Φ on $\mathbb{H}^n$ satisfies*

$$C_\Phi^h = \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{S}^{Q-1}} \cdots \int_{\mathbb{S}^{Q-1}} \frac{\Phi\left(\delta_{r_1} y_1', ..., \delta_{r_m} y_m'\right)}{|r_1|_h^Q \cdots |r_m|_h^Q} \prod_{j=1}^m r_j^{\frac{\alpha_j}{q} + \frac{Q}{q_j} - \frac{\varepsilon}{q_j} - Q + 1} dy_1' \cdots dy_m' dr_1 \cdots dr_m < \infty. \tag{5.1}$$

Then

$$\left\|\mathcal{H}_\Phi^h\right\|_{L^{q_1}(\mathbb{H}^n, |x|_h^{\frac{q_1\alpha_1}{q}}) \times \cdots \times L^{q_m}(\mathbb{H}^n, |x|_h^{\frac{q_m\alpha_m}{q}}) \to L^q(\mathbb{H}^n, |x|_h^\alpha)} = C_\Phi^h. \tag{5.2}$$

Proof. By a change of variables, the m-linear n-dimensional Hausdorff operator become

$$\mathcal{H}_\Phi^h = \int_{\mathbb{H}^n} \cdots \int_{\mathbb{H}^n} \frac{\Phi(y_1, ..., y_m)}{|y_1|_h^n \cdots |y_m|_h^n} f_1(\delta_{|y_1|_h^{-1}} x) \cdots f_m(\delta_{|y_m|_h^{-1}} x) dy_1 \cdots dy_m.$$

We can obtain

$$\begin{aligned} &\left\|\mathcal{H}_\Phi^h\right\|_{L^{q_1}(\mathbb{H}^n, |x|_h^{\frac{q_1\alpha_1}{q}}) \times \cdots \times L^{q_m}(\mathbb{H}^n, |x|_h^{\frac{q_m\alpha_m}{q}}) \to L^q(\mathbb{H}^n, |x|_h^\alpha)} \\ &= \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{S}^{Q-1}} \cdots \int_{\mathbb{S}^{Q-1}} \frac{\Phi\left(\delta_{r_1} y_1', ..., \delta_{r_m} y_m'\right)}{|r_1|_h^Q \cdots |r_m|_h^Q} \prod_{j=1}^m r_j^{\frac{\alpha_j}{q} + \frac{Q}{q_j} - \frac{\varepsilon}{q_j} - Q + 1} dy_1' \cdots dy_m' dr_1 \cdots dr_m \\ &= C_\Phi^h. \end{aligned}$$

This is similar to the proof of Theorem 3.1, so we omit the details. This finishes the proof of Corollary 5.1.

6. Conclusions

First, in the setting of the Heisenberg group, the n-dimensional fractional Hardy operator has a sharp weak estimate from L^p to $L^{q,\infty}$. The weak estimate bound is given by

$$\|\mathcal{H}_\alpha\|_{L^p(\mathbb{H}^n, |x|_h^\beta) \to L^{q,\infty}(\mathbb{H}^n, |x|_h^\gamma)} = \left(\frac{\omega_Q}{Q+\gamma}\right)^{\frac{1}{q}} \left(\frac{\omega_Q (p-1)}{pQ - Q - \beta}\right)^{\frac{1}{p'}}.$$

Additionally, for the L^1 case, we have

$$\|\mathcal{H}_\alpha\|_{L^1(\mathbb{H}^n) \to L^{(Q+\beta)/(Q-\alpha), \infty}(\mathbb{H}^n, |x|_h^\gamma)} = \left(\frac{\omega_Q}{Q+\beta}\right)^{(Q-\alpha)/(Q+\beta)}.$$

Second, we derive the sharp bounds for the m-linear n-dimensional integral operator with a kernel on weighted Lebesgue spaces:

$$\left\|\mathcal{H}^h\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})\times\cdots\times L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})\to L^q(\mathbb{H}^n,|x|_h^{\alpha})} = \int_{\mathbb{H}^n}\cdots\int_{\mathbb{H}^n} K(y_1,...,y_m)\prod_{i=1}^m |y_i|_h^{-\frac{\alpha_j}{q}-\frac{Q}{q_j}} dy_1\cdots dy_m.$$

Finally, as an application, the sharp bounds for Hardy, Hardy-Littlewood-Pólya, and Hilbert operators on weighted Lebesgue spaces are obtained. Moreover, we also find the estimate for the Hausdorff operator on weighted L^p spaces:

$$\begin{aligned}&\left\|\mathcal{H}_\Phi^h\right\|_{L^{q_1}(\mathbb{H}^n,|x|_h^{\frac{q_1\alpha_1}{q}})\times\cdots\times L^{q_m}(\mathbb{H}^n,|x|_h^{\frac{q_m\alpha_m}{q}})\to L^q(\mathbb{H}^n,|x|_h^{\alpha})}\\ &= \int_0^\infty\cdots\int_0^\infty\int_{\mathbb{S}^{Q-1}}\cdots\int_{\mathbb{S}^{Q-1}} \frac{\Phi\left(\delta_{r_1}y_1',...,\delta_{r_m}y_m'\right)}{|r_1|_h^Q\cdots|r_m|_h^Q}\prod_{j=1}^m r_j^{\frac{\alpha_j}{q}+\frac{Q}{q_j}-\frac{\varepsilon}{q_j}-Q+1} dy_1'\cdots dy_m' dr_1\cdots dr_m.\end{aligned}$$

Author contributions

Tianyang He: Conceptualization, methodology; Zhiwen Liu: Writing-original draft; Ting Yu: Writing-review and editing, validation, methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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